Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2009, Article ID 976369, 13 pages
doi:10.1155/2009/976369

Research Article

Total Stability in Nonlinear Discrete Volterra Equations with Unbounded Delay

Sung Kyu Choi,[1] Yoon Hoe Goo,[2] Dong Man Im,[3] and Namjip Koo[1]

[1] *Department of Mathematics, Chungnam National University, Daejeon 305-764, South Korea*
[2] *Department of Mathematics, Hanseo University, Seosan 352-820, South Korea*
[3] *Department of Mathematics Education, Cheongju University, Cheongju 360-764, South Korea*

Correspondence should be addressed to Namjip Koo, njkoo@math.cnu.ac.kr

Received 15 December 2008; Revised 20 February 2009; Accepted 24 March 2009

Recommended by Elena Litsyn

We study the total stability in nonlinear discrete Volterra equations with unbounded delay, as a discrete analogue of the results for integrodifferential equations by Y. Hamaya (1990).

1. Introduction

The concepts of stability and asymptotic stability introduced by Lyapunov could be called stabilities under sudden perturbations. The perturbation suddenly moves the systems from its equilibrium state but then immediately disappears. Stability says that the effect of this will not be great if the sudden perturbation is not too great. Asymptotic stability states, in addition, that if the sudden perturbation is not great, the effect of the perturbation will tend to disappear. In practice, however, the perturbations are not simply impulses, and this led Duboshin (1940) and Malkin (1944) to consider what they called stability under constantly acting perturbations, today known as total stability. This says that if the perturbation is not too large and if the system is not too far from the origin initially it will remain near the origin. Total stability can be described roughly as the property that a bounded perturbation has a bounded effect on the solution [1]. Many results have been obtained concerning total stability [1–9].

In [10], Hamaya discussed the relationship between total stability and stability under disturbances from hull for the integrodifferential equation

$$x'(t) = \hat{f}(t, x(t)) + \int_{-\infty}^{0} F(t, s, x(t+s), x(t))ds, \tag{1.1}$$

where $\hat{f} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous and is almost periodic in t uniformly for $x \in \mathbb{R}^d$, and $F : \mathbb{R} \times (-\infty, 0] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous and is almost periodic in t uniformly for $(s, x, y) \in R^* = (-\infty, 0] \times \mathbb{R}^d \times \mathbb{R}^d$. He showed that for a periodic integrodifferential equation, uniform stability and stability under disturbances from hull are equivalent. Also, he showed the existence of an almost periodic solution under the assumption of total stability in [11].

Song and Tian [12] studied periodic and almost periodic solutions of discrete Volterra equations with unbounded delay of the form

$$x(n+1) = f(n, x(n)) + \sum_{j=-\infty}^{0} B(n, j, x(n+j), x(n)), \quad n \in \mathbb{Z}^+, \tag{1.2}$$

where $f : \mathbb{Z} \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous in $x \in \mathbb{R}^d$ for every $n \in \mathbb{Z}$, and for any $j, n \in \mathbb{Z}$ $(j \le n)$, $B : \mathbb{Z} \times \mathbb{Z} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous for $x, y \in \mathbb{R}^d$. They showed that under some suitable conditions, if the bounded solution of (1.2) is totally stable, then it is an asymptotically almost periodic solution of (1.2), and (1.2) has an almost periodic solution. Also, Song [13] proved that if the bounded solution of (1.2) is uniformly asymptotically stable, then (1.2) has an almost periodic solution.

Choi and Koo [9] investigated the existence of an almost periodic solution of (1.2) as a discretization of the results in [10]. The purpose of this paper is to study the total stability for the discrete Volterra equation of the form

$$x(n+1) = f(n, x(n)) + \sum_{j=-\infty}^{0} B(n, j, x(n+j), x(n)) + h(n, x_n). \tag{1.3}$$

To do this, we will employ to change Hamaya's results in [2] for the integrodifferential equation

$$x'(t) = \hat{f}(t, x(t)) + \int_{-\infty}^{0} F(t, s, x(t+s), x(t))ds + \hat{h}(t, x_t), \tag{1.4}$$

into results for the discrete Volterra equation (1.3).

2. Preliminaries

We denote by $\mathbb{Z}, \mathbb{Z}^+, \mathbb{Z}^-$, respectively, the set of integers, the set of nonnegative integers, and the set of nonpositive integers. Let $\mathbb{R}^d$ denote d-dimensional Euclidean space.

Definition 2.1 (see [12]). A continuous function $f : \mathbb{Z} \times \mathbb{R}^d \to \mathbb{R}^d$ is said to be *almost periodic in $n \in \mathbb{Z}$ uniformly for $x \in \mathbb{R}^d$* if for every $\varepsilon > 0$ and every compact set $K \subset \mathbb{R}^d$, there corresponds an integer $N = N(\varepsilon, K) > 0$ such that among N consecutive integers there is one, here denoted by p, such that

$$|f(n+p, x) - f(n, x)| < \varepsilon \tag{2.1}$$

for all $n \in \mathbb{Z}$, uniformly for $x \in \mathbb{R}^d$.

Definition 2.2 (see [12]). Let $B : \mathbb{Z}\times\mathbb{Z}^* \to \mathbb{R}^d$ be continuous for $x, y \in \mathbb{R}^d$, for any $n \in \mathbb{Z}$, $j \in \mathbb{Z}^-$, where $\mathbb{Z}^* = \mathbb{Z}^- \times \mathbb{R}^d \times \mathbb{R}^d$. $B(n, j, x, y)$ is said to be *almost periodic in n uniformly for $(j, x, y) \in Z^*$* if for any $\varepsilon > 0$ and any compact set $K^* \subset Z^*$, there exists a number $l = l(\varepsilon, K^*) > 0$ such that any discrete interval of length l contains a τ for which

$$\left|B(n+\tau, j, x, y) - B(n, j, x, y)\right| \le \varepsilon \tag{2.2}$$

for all $n \in \mathbb{Z}$ and all $(j, x, y) \in K^*$.

For the basic results of almost periodic functions, see [8, 14, 15].

Let $l^-(\mathbb{R}^d)$ denote the space of all $\mathbb{R}^d$-valued bounded functions on $\mathbb{Z}^-$ with

$$\|\phi\| = \sup_{j\in\mathbb{Z}^-}\left|\phi(j)\right| < \infty \tag{2.3}$$

for any $\phi \in l^-(\mathbb{R}^d)$.

Consider the discrete Volterra equation with unbounded delay of the form

$$x(n+1) = f(n, x(n)) + \sum_{j=-\infty}^{0} B(n, j, x(n+j), x(n)) + h(n, x_n), \quad n \in \mathbb{Z}^+, \tag{2.4}$$

under certain conditions for f, B, and h (see below). We assume that, given $\phi \in l^-(\mathbb{R}^d)$, there is a solution x of (1.3) such that $x(n) = \phi(n)$ for $n \in \mathbb{Z}^-$, passing through $(0, \phi^0), \phi^0 \in l^-(\mathbb{R}^d)$.

Let K be any compact subset of $\mathbb{R}^d$ such that $\phi(j) \in K$ for all $j \le 0$ and $x(n) \in K$ for all $n \ge 1$.

For any $\phi, \varphi \in l^-(\mathbb{R}^d)$, we set

$$\rho(\phi, \varphi) = \sum_{q=0}^{\infty} \frac{\rho_q(\phi, \varphi)}{2^q\left[1 + \rho_q(\phi, \varphi)\right]}, \tag{2.5}$$

where $\rho_q(\phi, \varphi) = \max_{-q\le j\le 0}|\phi(j) - \varphi(j)|, q \ge 0$. Then ρ defines a metric on the space $l^-(\mathbb{R}^d)$. Note that the induced topology by ρ is the same as the topology of convergence on any finite subset of $\mathbb{Z}^-$ [12].

In view of almost periodicity, for any sequence $(n_k') \subset \mathbb{Z}^+$ with $n_k' \to \infty$ as $k \to \infty$, there exists a subsequence $(n_k) \subset (n_k')$ such that

$$f(n + n_k, x) \longrightarrow g(n, x) \tag{2.6}$$

uniformly on $\mathbb{Z} \times S$ for any compact set $S \subset \mathbb{R}^d$,

$$B(n + n_k, n + l + n_k, x, y) \longrightarrow D(n, n + l, x, y) \tag{2.7}$$

uniformly on $\mathbb{Z} \times S^*$ for any compact set $S^* \subset Z^*$. We define the *hull*

$$H(f,B) = \{(g,D) : (2.6) \text{ and } (2.7) \text{ hold for some sequence } (n_k) \subset \mathbb{Z}^+ \text{ with } n_k \to \infty \text{ as } k \to \infty\}. \tag{2.8}$$

Note that $(f,B) \in H(f,B)$ and for any $(g,D) \in H(f,B)$, we can assume the almost periodicity of g and D, respectively [12].

3. Main Results

We deal with the discrete Volterra equation with unbounded delay of the form

$$x(n+1) = f(n, x(n)) + \sum_{j=-\infty}^{0} B(n, j, x(n+j), x(n)) + h(n, x_n), \quad n \in \mathbb{Z}^+. \tag{3.1}$$

Throughout this paper we assume the following.

(H1) $f : \mathbb{Z} \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous in $x \in \mathbb{R}^d$ for every $n \in \mathbb{Z}$ and is almost periodic in $n \in \mathbb{Z}$ uniformly for $x \in \mathbb{R}^d$.

(H2) $B : \mathbb{Z} \times \mathbb{Z}^* \to \mathbb{R}^d$ is continuous in $x, y \in \mathbb{R}^d$ for any $n \in \mathbb{Z}$, $j \in \mathbb{Z}^-$ and is almost periodic in $n \in \mathbb{Z}$ uniformly for $(j, x, y) \in \mathbb{Z}^* = \mathbb{Z}^- \times \mathbb{R}^d \times \mathbb{R}^d$. Moreover, for any $\varepsilon > 0$ and any $\tau > 0$ there exists a number $M = M(\varepsilon, \tau) > 0$ such that

$$\sum_{j=-\infty}^{-M} \left| B(n, j, x(n+j), x(n)) \right| \leq \varepsilon \tag{3.2}$$

for all $n \in \mathbb{Z}$ whenever $|x(j)| \leq \tau$ for all $j \in \mathbb{Z}^-$.

(H3) $h : \mathbb{Z} \times l^-(\mathbb{R}^d) \to \mathbb{R}^d$ is continuous in $\phi \in l^-(\mathbb{R}^d)$ for every $n \in \mathbb{Z}$. $x_n \in l^-(\mathbb{R}^d)$ is defined as $x_n(j) = x(n+j)$ for $j \in \mathbb{Z}^-$. Furthermore, for any $r > 0$, there exists a function $\alpha_r : \mathbb{Z} \to \mathbb{R}^d$ with the property that $\alpha_r(n) \to 0$ as $n \to \infty$ and

$$\left| h(n, \phi) \right| \leq \alpha_r(n) \tag{3.3}$$

whenever $|\phi(j)| \leq r$ for all $j \in \mathbb{Z}^-$.

(H4) Equation (3.1) has a bounded solution $u(n) = u(n, \phi^0)$ defined on $\mathbb{Z}^+$, through $(0, \phi^0)$, $\phi^0 \in l^-(\mathbb{R}^d)$ such that for some $0 \leq M < \infty$,

$$\left| u\left(n, \phi^0\right) \right| \leq M. \tag{3.4}$$

Note that for any $(g,D) \in H(f,B), D(n,j,x,y)$ satisfies (H2) with $B = D$ [12]. The *limiting equation* of (3.1) is defined as

$$x(n+1) = g(n,x(n)) + \sum_{j=-\infty}^{0} D(n,j,x(n+j),x(n)), \quad n \in \mathbb{Z}^+, \tag{3.5}$$

where $(g,D) \in H(f,B)$. We assume that for any solution $v(n)$ of (3.5), $v(n) \in K$ for all $n \geq 1$, where K is the above-mentioned compact set in $\mathbb{R}^d$.

Theorem 3.1 (see [12]). *Under the assumptions (H1)–(H4), if $u(n) = u(n,\phi^0)$, $\phi^0 \in l^-(\mathbb{R}^d)$ is a bounded solution of* (3.1)*, passing through* $(0,\phi^0)$ *and* $(g,D) \in H(f,B)$*, then the limiting equation* (3.5) *of* (3.1) *has a bounded solution on* $\mathbb{Z}$*.*

Total stability requires that the solution of $x'(t) = \hat{f}(t,x)$ is "stable" not only with respect to the small perturbations of the initial conditions, but also with respect to the perturbations, small in a suitable sense, of the right-hand side of the equation.

Definition 3.2. The bounded solution $u(n)$ of (3.1) is said to be *totally stable* if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that if $n_0 \geq 0$, $\rho(u_{n_0}, x_{n_0}) < \delta$, and $p(n)$ is a function such that $|p(n)| < \delta$ for all $n \geq n_0$, then $\rho(u_n, x_n) < \varepsilon$ for all $n \geq n_0$, where $x(n)$ is any solution of

$$x(n+1) = f(n,x(n)) + \sum_{j=-\infty}^{0} B(n,j,x(n+j),x(n)) + h(n,x_n) + p(n) \tag{3.6}$$

such that $x_{n_0}(j) \in K$ for all $j \in \mathbb{Z}^-$.

Definition 3.3. A function $\phi : \mathbb{Z} \to \mathbb{R}^d$ is called *asymptotically almost periodic* if it is a sum of an almost periodic function ϕ_1 and a function ϕ_2 defined on $\mathbb{Z}$ which tends to zero as $n \to \infty$, that is, $\phi(n) = \phi_1(n) + \phi_2(n)$ for all $n \in \mathbb{Z}$.

It is known [15] that the decomposition $\phi = \phi_1 + \phi_2$ in Definition 3.3 is unique. Moreover, ϕ is asymptotically almost periodic if and only if for any integer sequence (τ'_k) with $\tau'_k \to \infty$ as $k \to \infty$, there exists a subsequence $(\tau_k) \subset (\tau'_k)$ for which $\phi(n+\tau_k)$ converges uniformly for $n \in \mathbb{Z}$ as $k \to \infty$ [15].

Theorem 3.4. *Under the assumptions (H1)–(H4), if $u(n)$ is a bounded and totally stable solution of* (3.1)*, then it is asymptotically almost periodic.*

Proof. Let (n_k) be an integer sequence with $n_k \to \infty$ as $k \to \infty$. Set $u^k(n) = u(n+n_k)$, $k = 1,2,\ldots$. Then $u^k(n)$ is a solution of

$$x(n+1) = f(n+n_k,x(n)) + \sum_{j=-\infty}^{0} B(n+n_k,j,x(n+j),x(n)) + h(n+n_k,x_n) \tag{3.7}$$

and $u^k(n) \in K$ for $k = 1,2,\ldots$. Clearly, $u^k(n)$ is totally stable with the same number (ε,δ) for the total stability of $u(n)$. We can assume that $(u^k(n))$ converges uniformly on any finite set in $\mathbb{Z}^-$ as $k \to \infty$ by taking a subsequence if necessary. Then there exists a number $k_1(\varepsilon) > 0$ such that $\rho(u_0^k, u_0^m) < \delta(\varepsilon)$ whenever $k, m \geq k_1(\varepsilon)$. Put

$$\begin{aligned} p(n) = {} & f(n + n_m, u^m(n)) + \sum_{j=-\infty}^{0} B(n + n_m, j, u^m(n + j), u^m(n)) \\ & + h(n + n_m, u_n^m) - f(n + n_k, u^m(n)) \\ & - \sum_{j=-\infty}^{0} B(n + n_k, j, u^m(n + j), u^m(n)) - h(n + n_k, u_n^m). \end{aligned} \tag{3.8}$$

Then $u^m(n) = u(n + n_m)$ is a solution of

$$x(n + 1) = f(n + n_k, x(n)) + \sum_{j=-\infty}^{0} B(n + n_k, j, x(n + j), x(n)) + h(n + n_k, x_n) + p(n) \tag{3.9}$$

and $u^m(n) \in K$ for all $n \in \mathbb{Z}$. We will show that there exists a number $k_2(\varepsilon) > 0$ such that $|p(n)| < \delta(\varepsilon)$ for all $n \geq 0$ whenever $k, m \geq k_2(\varepsilon)$.

Note that for all $x \in K$, there exists a number $c > 0$ such that $|x| \leq c$. It is clear that $|u^k(n)| \leq c$ and $|u^m(n)| \leq c$ for all $n \in \mathbb{Z}$.

In view of (H2), there exists a number $M = M(c, \varepsilon) > 0$ such that

$$\begin{aligned} & \sum_{j=-\infty}^{-M} \left| B(n + n_m, j, u^m(n + j), u^m(n)) \right| \leq \frac{1}{5}\delta(\varepsilon), \quad n \in \mathbb{Z}, \\ & \sum_{j=-\infty}^{-M} \left| B(n + n_k, j, u^m(n + j), u^m(n)) \right| \leq \frac{1}{5}\delta(\varepsilon), \quad n \in \mathbb{Z}. \end{aligned} \tag{3.10}$$

From the almost periodicity of $f(n, x)$ and $B(n, j, x, y)$, respectively, there exists a number $k_2(\varepsilon) \geq k_1(\varepsilon)$ for which

$$\begin{aligned} & \left| B(n + n_m, j, u^m(n + j), u^m(n)) - B(n + n_k, j, u^m(n + j), u^m(n)) \right| \\ & \quad < \frac{\delta(\varepsilon)}{5M}, \quad n \in \mathbb{Z},\ j \in [-M + 1, 0], \end{aligned} \tag{3.11}$$

$$\left| f(n + n_m, u^m(n)) - f(n + n_k, u^m(n)) \right| < \frac{1}{5}\delta(\varepsilon), \quad n \in \mathbb{Z}, \tag{3.12}$$

whenever $k, m \geq k_2(\varepsilon)$. Since $h(n, \phi) \to 0$ as $n \to \infty$, we obtain that if $k, m \geq k_2(\varepsilon)$, then

$$|h(n + n_m, u_n^m) - h(n + n_k, u_n^m)| < \frac{1}{5}\delta(\varepsilon), \quad n \in \mathbb{Z}^+. \tag{3.13}$$

Then, by (3.10), and (3.11), we have

$$\begin{aligned}
&\sum_{j=-\infty}^{0} \left| B(n+n_m, j, u^m(n+j), u^m(n)) - B(n+n_k, j, u^m(n+j), u^m(n)) \right| \\
&\quad \le \sum_{j=-\infty}^{-M} \left| B(n+n_m, j, u^m(n+j), u^m(n)) \right| \\
&\qquad + \sum_{j=-\infty}^{-M} \left| B(n+n_k, j, u^m(n+j), u^m(n)) \right| \\
&\qquad + \sum_{j=-M+1}^{0} \left| B(n+n_m, j, u^m(n+j), u^m(n)) - B(n+n_k, j, u^m(n+j), u^m(n)) \right| \\
&\quad < \frac{1}{5}\delta(\varepsilon) + \frac{1}{5}\delta(\varepsilon) + \frac{\delta(\varepsilon)}{5M} M \\
&\quad = \frac{3}{5}\delta(\varepsilon).
\end{aligned} \tag{3.14}$$

Therefore we have

$$\begin{aligned}
\left| p(n) \right| &\le \left| f(n+n_m, u^m(n)) - f(n+n_k, u^m(n)) \right| \\
&\quad + \sum_{j=-\infty}^{0} \left| B(n+n_m, j, u^m(n+j), u^m(n)) - B(n+n_k, j, u^m(n+j), u^m(n)) \right| \\
&\quad + \left| h(n+n_m, u_n^m) - h(n+n_k, u_n^m) \right| \\
&< \frac{1}{5}\delta(\varepsilon) + \frac{3}{5}\delta(\varepsilon) + \frac{1}{5}\delta(\varepsilon) \\
&= \delta(\varepsilon),
\end{aligned} \tag{3.15}$$

$k, m \ge k_2(\varepsilon)$, by (3.12), (3.13), and (3.14). Since $u^k(n)$ is totally stable, we obtain that $\rho(u_n^k, u_n^m) < \varepsilon$ for all $n \ge 0$ if $k, m \ge k_2(\varepsilon)$. This implies that for all $k, m \ge k_2(\varepsilon)$,

$$|u(n+n_k) - u(n+n_m)| \le \sup_{-M \le s \le 0} |u(n+n_k+s) - u(n+n_m+s)| < 4\varepsilon \tag{3.16}$$

for all $\varepsilon \le (1/4)$ and all $n \ge 0$. It follows that for any $(n_k) \subset \mathbb{Z}$ with $n_k \to \infty$ as $k \to \infty$ there exists a subsequence $(n_{k_j}) \subset (n_k)$ such that $(u(n+n_{k_j}))$ converges uniformly on $\mathbb{Z}^+$ as $j \to \infty$, that is, $u(n)$ is asymptotically almost periodic. This completes the proof. □

Remark 3.5. Hino et al. [5] showed that for the functional differential equation

$$x'(t) = \hat{f}(t, x_t), \tag{3.17}$$

the solution $v(t)$ of the limiting equation

$$x'(t) = \widehat{g}(t, x_t), \qquad (v, \widehat{g}) \in H\left(u, \widehat{f}\right) \tag{3.18}$$

of (3.17) is asymptotically almost periodic if $v(t)$ is totally stable. Here $(v, \widehat{g}) \in H(u, \widehat{f})$ means that there exists a sequence $(t_k), t_k \to \infty$ as $k \to \infty$, such that $\widehat{f}(t + t_k, \phi) \to \widehat{g}(t, \phi) \in H(\widehat{f})$ uniformly on any compact set in B and $u(t + t_k) \to v(t)$ uniformly on any compact interval in the set of nonnegative real numbers, where the space B is the fading memory space by Hale and Kato [16].

Theorem 3.6. *Assume that (H1)–(H4). If the solution $v(n)$ of (3.5) satisfying $(v, g, D) \in H(u, f, B)$ is totally stable, then the bounded solution $u(n)$ of (3.1) is also totally stable.*

Proof. From $(v, g, D) \in H(u, f, B)$, there exists a sequence $(n_k) \subset \mathbb{Z}, n_k \to \infty$ as $k \to \infty$, such that $f(n + n_k, x) \to g(n, x)$ uniformly on $\mathbb{Z} \times K$, $B(n + n_k, j, x, y) \to D(n, j, x, y)$ uniformly on $\mathbb{Z} \times S^* \times K \times K$ for any compact set $S^* \subset \mathbb{Z}^-$, and $u(n + n_k) \to v(n)$ uniformly on any compact set in $\mathbb{R}$ as $k \to \infty$. Set $u^k(n) = u(n + n_k), k = 1, 2, \ldots$. Then it is clear that $u^k(n)$ is a solution of

$$x(n+1) = f(n + n_k, x(n)) + \sum_{j=-\infty}^{0} B(n + n_k, j, x(n + j), x(n)) + h(n + n_k, x_n), \tag{3.19}$$

such that $u_0^k(j) \in K$ for all $j \leq 0$, where $u_0^k(j) = u^k(0 + j) = u(j + n_k)$. Note that for all $x \in K, |x| \leq c$ for some $c > 0$. Let $x(\tau)$ be a function such that $x(\tau) \in K$ for all $\tau \leq n$. By (H2), there exists a number $M = M(c, \varepsilon) > 0$ such that

$$\sum_{j=-\infty}^{-M} |B(n + n_k, j, x(n + j), x(n))| \leq \frac{1}{5}\delta\left(\frac{1}{2}\delta\left(\frac{\varepsilon}{2}\right)\right), \tag{3.20}$$

where $\delta(\cdot)$ is the number for the total stability of $v(n)$. Also, we have

$$\sum_{j=-\infty}^{-M} |D(n, j, x(n + j), x(n))| \leq \frac{1}{5}\delta\left(\frac{1}{2}\delta\left(\frac{\varepsilon}{2}\right)\right) \tag{3.21}$$

for the same M since $B(n + n_k, j, x, y) \to D(n, j, x, y)$. Hence, by the same argument as in the proof of Theorem 3.4, there exists a positive integer $k_0(\varepsilon)$ such that if $k \geq k_0(\varepsilon)$, then

$$\begin{aligned} &\Bigg| f(n + n_k, x(n)) + \sum_{j=-\infty}^{0} B(n + n_k, j, x(n + j), x(n)) + h(n + n_k, x_n) \\ &\quad - g(n, x(n)) - \sum_{j=-\infty}^{0} D(n, j, x(n + j), x(n)) \Bigg| < \delta\left(\frac{1}{2}\delta\left(\frac{\varepsilon}{2}\right)\right), \end{aligned} \tag{3.22}$$

$$\rho\left(u_0^k, v_0\right) < \delta\left(\frac{1}{2}\delta\left(\frac{\varepsilon}{2}\right)\right). \tag{3.23}$$

Put

$$r(n) = f\left(n+n_k, u^k(n)\right) + \sum_{j=-\infty}^{0} B\left(n+n_k, j, u^k(n+j), u^k(n)\right) + h\left(n+n_k, u_n^k\right) - g\left(n, u^k(n)\right) - \sum_{j=-\infty}^{0} D\left(n, j, u^k(n+j), u^k(n)\right). \tag{3.24}$$

Then $u^k(n)$ is a solution of

$$x(n+1) = g(n, x(n)) + \sum_{j=-\infty}^{0} D(n, j, x(n+j), x(n)) + r(n) \tag{3.25}$$

such that $u_0^k(j) \in K$ for $j \le 0$. Note that $|r(n)| < \delta((1/2)\delta(\varepsilon/2))$ for $n \ge 0$ by (3.22). From (3.22) and the fact that $v(n)$ is totally stable, we have

$$\rho\left(u_n^k, v_n\right) < \frac{1}{2}\delta\left(\frac{\varepsilon}{2}\right) \tag{3.26}$$

for all $n \ge 0$.

Let $m = k_0(\varepsilon)$. To show that $u(n)$ is totally stable we will show that if $n_0 \ge 0$, $\rho(u_{n_0}, y_{n_0}) < (1/2)\delta(\varepsilon/2)$, and $|p(n)| < (1/2)\delta(\varepsilon/2)$ for $n \ge n_0$, then $\rho(u_n, y_n) < \varepsilon$ for all $n \ge n_0$, where $y(n)$ is a solution of

$$x(n+1) = f(n, x(n)) + \sum_{j=-\infty}^{0} B(n, j, x(n+j), x(n)) + h(n, x_n) + p(n) \tag{3.27}$$

such that $y_{n_0}(j) \in K$ for all $j \le 0$. Suppose that this is not the case. Then there exists an integer $\sigma > n_0$ such that

$$\rho(u_\sigma, y_\sigma) = \varepsilon \quad \text{for } \sigma > n_0, \qquad \rho(u_n, y_n) < \varepsilon \quad \text{for } n_0 \le n < \sigma. \tag{3.28}$$

We set $z(n) = y(n+n_m)$. Then $z(n)$ is a solution of

$$x(n+1) = f(n+n_m, x(n)) + \sum_{j=-\infty}^{0} B(n+n_m, j, x(n+j), x(n)) + h(n+n_m, x_n) + p(n+n_m) \tag{3.29}$$

defined on $[n_0 - n_m, \sigma - n_m]$ such that $z_{n_0-n_m}(j) = y_{n_0}(j) \in K$ for all $j \le 0$. Also, $z(n)$ is a solution of

$$x(n+1) = g(n, x(n)) + \sum_{j=-\infty}^{0} D(n, j, x(n+j), x(n)) + q(n), \tag{3.30}$$

where

$$q(n) = f(n+n_m, z(n)) + \sum_{j=-\infty}^{0} B(n+n_m, j, z(n+j), z(n)) \\ + h(n+n_m, z_n) + p(n+n_m) - g(n, z(n)) - \sum_{j=-\infty}^{0} D(n, j, z(n+j), z(n)). \tag{3.31}$$

Note that $|z(n)| \le c$ for all $n \le \sigma - n_m$, and $|p(n+n_m)| < (1/2)\delta(\varepsilon/2)$ for $n \ge n_0 - n_m$. Thus $|q(n)| < \delta(\varepsilon/2)$ for $n_0 - n_m \le n \le \sigma - n_m$. Also, we have

$$\rho(u_{n_0}, v_{n_0-n_m}) < \frac{1}{2}\delta\left(\frac{\varepsilon}{2}\right), \qquad \rho(u_{n_0}, z_{n_0-n_m}) = \rho(u_{n_0}, y_{n_0}) < \frac{1}{2}\delta\left(\frac{\varepsilon}{2}\right) \tag{3.32}$$

from (3.26). Thus we obtain

$$\rho(v_{n_0-n_m}, z_{n_0-n_m}) \le \rho(v_{n_0-n_m}, u_{n_0}) + \rho(u_{n_0}, z_{n_0-n_m}) < \delta\left(\frac{\varepsilon}{2}\right). \tag{3.33}$$

Since $v(n)$ is totally stable, we have

$$\rho(v_{\sigma-n_m}, z_{\sigma-n_m}) < \frac{\varepsilon}{2}. \tag{3.34}$$

On the other hand, (3.26) implies that

$$\rho(u_n, v_{n-n_m}) < \frac{1}{2}\delta\left(\frac{\varepsilon}{2}\right), \quad n \ge n_0. \tag{3.35}$$

Hence, if $n_0 \ge 0, \rho(u_{n_0}, y_{n_0}) < (\frac{1}{2})\delta(\frac{\varepsilon}{2})$, and $|p(n)| < (1/2)\delta(\varepsilon/2)$ for $n \ge n_0$, then we obtain

$$\rho(u_\sigma, y_\sigma) \le \rho(u_\sigma, v_{\sigma-n_m}) + \rho(v_{\sigma-n_m}, z_{\sigma-n_m}) < \varepsilon. \tag{3.36}$$

This contradicts (3.28). Therefore $\rho(u_n, y_n) < \varepsilon$ for all $n \ge n_0$ when $n_0 \ge 0$, $\rho(u_{n_0}, y_{n_0}) < \delta^*(\varepsilon)$ and $|p(n)| < \delta^*(\varepsilon)$ for all $n \ge n_0$, where $\delta^*(\varepsilon) = (1/2)\delta(\varepsilon/2)$. Consequently, $u(n)$ is totally stable. □

The following definitions are the discrete analogues of Hamaya's definitions in [2].

Definition 3.7. The bounded solution $u(n)$ of (3.1) is said to be *attracting* in $H(f, B)$ if there exists a $\delta_0 > 0$ such that for any $n_0 \ge 0$ and any $(v, g, D) \in H(u, f, B), \rho(v_{n_0}, x_{n_0}) < \delta_0$ implies $\rho(v_n, x_n) \to 0$ as $n \to \infty$, where $x(n)$ is a solution of (3.5) such that $x_{n_0}(j) \in K$ for all $j \le 0$.

Definition 3.8. The bounded solution $u(n)$ of (3.1) is said to be *totally asymptotically stable* if it is totally stable and there exists a $\delta_0 > 0$ and for any $\varepsilon > 0$ there exists an $\eta(\varepsilon) > 0$ and a $T(\varepsilon) > 0$

such that if $n_0 \geq 0, \rho(u_{n_0}, x_{n_0}) < \delta_0$ and $p(n)$ is any function which satisfies $|p(n)| < \eta(\varepsilon)$ for $n \geq n_0$, then $\rho(u_n, x_n) < \varepsilon$ for all $n \geq n_0 + T(\varepsilon)$, where $x(n)$ is a solution of

$$x(n+1) = f(n, x(n)) + \sum_{j=-\infty}^{0} B(n, j, x(n+j), x(n)) + h(x, x_n) + p(n) \tag{3.37}$$

such that $x_{n_0}(j) \in K$ for all $j \leq 0$.

Note that the total asymptotic stability is equivalent to the uniform asymptotic stability whenever $p(n) \equiv 0$.

Theorem 3.9. *Under the assumptions (H1)–(H4), if the bounded solution $u(n)$ of* (3.1) *is attracting in $H(f, B)$ and totally stable, then it is totally asymptotically stable.*

Proof. Let δ_0 be the number for the attracting of $u(n)$ in $H(f, B)$ and let $\delta_0^* = \delta(\delta_0/2)$, where $\delta(\cdot)$ is the number for the total stability of $u(n)$. Suppose that $u(n)$ is not totally asymptotically stable. Then there exists a number $\varepsilon > 0$ with $\varepsilon \leq (\delta_0/4)$ and exist sequences $(j_k), (n_k), (p_k)$, and (x^k) such that $j_k \geq 0,\ n_k \geq j_k + 2k,\ \rho(u_{j_k}, x_{j_k}^k) < \delta_0^*$ and $\rho(u_{n_k}, x_{n_k}^k) \geq \varepsilon$ for all $k = 1, 2, \ldots$, where $x^k(n)$ is a solution of

$$x(n+1) = f(n, x(n)) + \sum_{j=-\infty}^{0} B(n, j, x(n+j), x(n)) + h(x, x_n) + p_k(n). \tag{3.38}$$

such that $x_{j_k}^k(j) \in K$ for all $j \leq 0$ and $p_k : \mathbb{Z} \to \mathbb{R}^d$ with $|p_k(n)| < \min\{1/k, \delta_0^*\}$ for $n \geq j_k$. Note that $\rho(u_{j_k}, x_{j_k}^k) < \delta_0^*$ and $|p_k(n)| < \delta_0^*$ for $n \geq j_k$. Then we have

$$\rho\left(u_n, x_n^k\right) < \frac{1}{2}\delta_0 \tag{3.39}$$

for all $n \geq j_k$ and $k = 1, 2, \ldots$, since $u(n)$ is totally stable. Also, there exists an integer number $k_0(\varepsilon) > 0$ such that if $k \geq k_0(\varepsilon)$, then $|p_k(n)| < 1/k < \delta(\varepsilon)$ for all $n \geq j_k$.

We claim that $\rho(u_n, x_n^k) \geq \delta(\varepsilon)$ on $[j_k + k, j_k + 2k]$ if $n \geq n_0$. If we assume that $\rho(u_n, x_n^k) < \delta(\varepsilon)$ on $[j_k + k, j_k + 2k]$, then $\rho(u_n, x_n^k) < \varepsilon$ for $n \geq j_k + 2k$ since $u(n)$ is totally stable. This contradicts $\rho(u_{n_k}, x_{n_k}^k) \geq \varepsilon,\ k = 1, 2, \ldots$, because $n_k \geq j_k + 2k$.

Now, for the sequence $(j_k + k)$, taking a subsequence if necessary, there exists a $(v, g, D) \in H(u, f, B)$.

If we set $y^k(n) = x^k(n + j_k + k)$, then $y^k(n)$ is the defined on $[-k, k]$. There exists a subsequence of $(y^k(n))$, which we denote by $(y^k(n))$ again, and a function $y(n)$ such that $y^k(n) \to y(n)$ uniformly on any compact set in $\mathbb{R}$ as $k \to \infty$ such that $y_0(j) \in K$ for all $j \leq 0$. Moreover, we can show that $y(n)$ is a solution of

$$x(n+1) = g(n, x(n)) + \sum_{j=-\infty}^{0} D(n, j, x(n+j), x(n)) \tag{3.40}$$

such that $y_0(j) \in K$ for all $j \leq 0$, by the same method as in [13, Theorem 3.1] . We have

$$\delta(\varepsilon) < \rho\left(u_{n+j_k+k}, y_n^k\right) < \frac{\delta_0}{2}, \quad 0 \leq n \leq k, \ k \geq k_0. \tag{3.41}$$

Then, by letting $k \to \infty$, we obtain

$$\delta(\varepsilon) < \rho(v_n, y_n) \leq \frac{\delta_0}{2}, \quad n \geq 0. \tag{3.42}$$

Since $u(n)$ is attracting in $H(f, B)$, we have $\rho(v_n, y_n) \to 0$ as $n \to \infty$. This contradicts $\rho(v_n, y_n) \geq \delta(\varepsilon)$. Hence $u(n)$ is totally asymptotically stable. This completes the proof of the theorem. □

Acknowledgment

The authors are thankful to the anonymous referees for their valuable comments and corrections to improve this paper.

References

[1] Z. S. Athanassov, "Total stability of sets for nonautonomous differential systems," *Transactions of the American Mathematical Society*, vol. 295, no. 2, pp. 649–663, 1986.

[2] Y. Hamaya, "Total stability property in limiting equations of integrodifferential equations," *Funkcialaj Ekvacioj*, vol. 33, no. 2, pp. 345–362, 1990.

[3] Y. Hino and S. Murakami, "Total stability and uniform asymptotic stability for linear Volterra equations," *Journal of the London Mathematical Society*, vol. 43, no. 2, pp. 305–312, 1991.

[4] Y. Hino, "Total stability and uniformly asymptotic stability for linear functional-differential equations with infinite delay," *Funkcialaj Ekvacioj*, vol. 24, no. 3, pp. 345–349, 1981.

[5] Y. Hino and T. Yoshizawa, "Total stability property in limiting equations for a functional-differential equation with infinite delay," *Československá Akademie Věd. Časopis Pro Pěstování Matematiky*, vol. 111, no. 1, pp. 62–69, 1986.

[6] Y. Hino and S. Murakami, "Total stability in abstract functional differential equations with infinite delay," in *Proceedings of the 6th Colloquium on the Qualitative Theory of Differential Equations (Szeged, 1999)*, Proc. Colloq. Qual. Theory Differ. Equ., pp. 1–9, Electron. J. Qual. Theory Differ. Equ., Szeged, Hungary, 2000.

[7] P. Anderson and S. R. Bernfeld, "Total stability of scalar differential equations determined from their limiting functions," *Journal of Mathematical Analysis and Applications*, vol. 257, no. 2, pp. 251–273, 2001.

[8] T. Yoshizawa, *Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions*, vol. 14 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1975.

[9] S. K. Choi and N. Koo, "Almost periodic solutions of nonlinear discrete Volterra equations with unbounded delay," *Advances in Difference Equations*, vol. 2008, Article ID 692713, 15 pages, 2008.

[10] Y. Hamaya, "Stability property for an integrodifferential equation," *Differential and Integral Equations*, vol. 6, no. 6, pp. 1313–1324, 1993.

[11] Y. Hamaya, "Periodic solutions of nonlinear integro-differential equations," *The Tohoku Mathematical Journal*, vol. 41, no. 1, pp. 105–116, 1989.

[12] Y. Song and H. Tian, "Periodic and almost periodic solutions of nonlinear discrete Volterra equations with unbounded delay," *Journal of Computational and Applied Mathematics*, vol. 205, no. 2, pp. 859–870, 2007.

[13] Y. Song, "Asymptotically almost periodic solutions of nonlinear Volterra difference equations with unbounded delay," *Journal of Difference Equations and Applications*, vol. 14, no. 9, pp. 971–986, 2008.

[14] C. Corduneanu, *Almost Periodic Functions*, Chelsea, New York, NY, USA, 2nd edition, 1989.

[15] C. Zhang, *Almost Periodic Type Functions and Ergodicity*, Science Press, Beijing, China; Kluwer Academic Publishers, Dordrecht, The Netherlands, 2003.

[16] J. K. Hale and J. Kato, "Phase space for retarded equations with infinite delay," *Funkcialaj Ekvacioj*, vol. 21, no. 1, pp. 11–41, 1978.