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Research Article

Finite Dimensional Uniform Attractors for the Nonautonomous Camassa-Holm Equations

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We consider the uniform attractors for the three-dimensional nonautonomous Camassa-Holm equations in the periodic box $\Omega = [0, L]^3$. Assuming $f = f(x,t) \in L^2_{\text{loc}}((0,T); D(A^{-1/2}))$, we establish the existence of the uniform attractors in $D(A^{1/2})$ and $D(A)$. The fractal dimension is estimated for the kernel sections of the uniform attractors obtained.

1. Introduction

We consider the following viscous version of the three-dimensional Camassa-Holm equations in the periodic box $\Omega = [0, L]^3$:

$$\begin{gathered}\frac{\partial}{\partial t}\left(\alpha_0^2 u - \alpha_1^2 \Delta u\right) - \nu \Delta\left(\alpha_0^2 u - \alpha_1^2 \Delta u\right) - u \times \left(\nabla \times \left(\alpha_0^2 u - \alpha_1^2 \Delta u\right)\right) + \frac{1}{\rho_0} \nabla p = f(x,t), \\ \nabla \cdot u = 0, \\ u(x,\tau) = u_\tau(x),\end{gathered} \tag{1.1}$$

where $p/\rho_0 = \pi/\rho_0 + \alpha_0^2 |u|^2 - \alpha_1^2 (u \cdot \Delta u)$ is the modified pressure, while π is the pressure, $\nu > 0$ is the constant viscosity, and $\rho_0 > 0$ is a constant density. The function f is a given body forcing and $\alpha_0 > 0$, $\alpha_1 \geq 0$ are scale parameters. Notice α_0 is dimensionless while α_1 has units of length. Also observe that at the limit $\alpha_0 = 1$, $\alpha_1 = 0$, we obtain the three-dimensional Navier-Stokes equations with periodic boundary conditions.

We consider this equaton in an appropriate space and show that there is an attractor $\mathfrak{A}$ which all solutions approach as $t \to \infty$. The basic idea of our construction, is motivated by the works of [1].

In addition, we assume that the function $f(\cdot,t) =: f(t) \in L^2_{\text{loc}}(\mathbb{R};E)$ is translation bounded, where $E = D(A^{-1/2})$. This property implies that

$$\|f\|^2_{L^2_b} = \|f\|^2_{L^2_b(\mathbb{R};E)} = \sup_{t\in R}\int_t^{t+1} \|f(s)\|^2_E ds < \infty. \tag{1.2}$$

In [1] the authors established the global regularity of solutions of the autonomous Camassa-Holm, or Navier-Stokes-alpha (NS-α) equations, subject to periodic boundary conditions. The inviscid NS-α equations (Euler-α) were introduced in [2] as a natural mathematical generalization of the integrable inviscid 1D Camassa-Holm equation discovered in [3] through a variational formulation. An alternative more physical derivation for the inviscid NS-α equations (Euler-α) was introduced in [4–8].

In the book [9], Haraux considers some special classes of such systems and studies systematically the notion of uniform attractor parallelling to that of global attractor for autonomous systems. Later on, [10] present a general approach, that is, well suited to study equations arising in mathematical physics. In this approach, to construct the uniform (or trajectory) attractors, instead of the associated process $\{U_\sigma(t,\tau),\ t \geq \tau,\ \tau \in \mathbb{R}\}$, one should consider a family of processes $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$, in some Banach space E, where the functional parameter $\sigma_0(s),\ s \in \mathbb{R}$ is called the symbol and Σ is the symbol space including $\sigma_0(s)$. The approach preserves the leading concept of invariance which implies the structure of uniform attractor described by the representation as a union of sections of all kernels of the family of processes. The kernel is the set of all complete trajectories of a process.

In the paper, we study the existence of compact uniform attractor for the nonautonomous three-dimensional Camassa-Holm equations in the periodic box $\Omega = [0,L]^3$. We apply measure of noncompactness method to nonautonomous Camassa-Holm equations with external forces $f(x,t)$ in $L^2_{\text{loc}}(\mathbb{R};E)$ which is normal function (see Definition 4.2). Last, the fractal dimension is estimated for the kernel sections of the uniform attractors obtained.

2. Functional Setting

From (1.1) one can easily see, after integration by parts, that

$$\frac{d}{dt}\int_\Omega \left(\alpha_0^2 u - \alpha_1^2 \Delta u\right) dx = \int_\Omega f dx. \tag{2.1}$$

On the other hand, because of the spatial periodicity of the solution, we have $\int_\Omega \Delta u dx = 0$. As a result, we have $d/dt\int_\Omega \alpha_0^2 u dx = \int_\Omega f dx$, that is, the mean of the solution is invariant provided that the mean of the forcing term is zero. In this paper, we will consider forcing terms and initial values with spatial means that are zero, that is, we will assume $\int_\Omega u_\tau(x)dx = \int_\Omega f dx = 0$ and hence $\int_\Omega u dx = 0$.

Next, let us introduce some notation and background.

(i)We denote $\mathcal{V} = \{\varphi : \varphi$ is a vector-valued trigonometric polynomial defined on Ω, such that $\nabla \cdot \varphi = 0$ and $\int_\Omega \varphi(x)dx = 0\}$, and let H and V be the closures of $\mathcal{V}$ in $L^2(\Omega)^3$

and in $H^1(\Omega)^3$, respectively, observe that $H^\perp$, the orthogonal complement of H in $L^2(\Omega)^3$, is $\{\nabla p : p \in H^1(\Omega)\}$ (cf. [11] or [12]).

(ii)We denote $P : L^2(\Omega)^3 \rightarrow H$ the L^2 orthogonal projection, usually referred as Helmholtz-Leray projector, and by $A = -P\Delta$ the Stokes operator with domain $D(A) = (H^2(\Omega))^3 \cap V$. Notice that in the case of periodic boundary condition, $A = -\Delta|_{D(A)}$ is a self-adjoint positive operator with compact inverse. Hence the space H has an orthonormal basis $\{w_j\}_{j=1}^{\infty}$ of eigenfunctions of A, that is, $Aw_j = \lambda_j w_j$, with

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots, \quad \lambda_j \longrightarrow +\infty, \text{ as } j \longrightarrow \infty, \tag{2.2}$$

in fact, these eigenvalues have the form $|k|^2 4\pi/L^2$ with $k \in Z^3 \setminus \{0\}$.

(iii)We denote $(\cdot,\cdot)$ the L^2-inner product and by $|\cdot|$ the corresponding L^2-norm. By virtue of Poincaré inequality, one can show that there is a constant $c > 0$ such that

$$c|Aw| \leq \|w\|_{H^2} \leq c^{-1}|Aw| \quad \text{for every } w \in D(A) \tag{2.3}$$

and that

$$c\left|A^{1/2}w\right| \leq \|w\|_{H^1} \leq c^{-1}\left|A^{1/2}w\right| \quad \text{for every } w \in V. \tag{2.4}$$

Moreover, one can show that $V = D(A^{1/2})$ (cf. [11, 12]). We denote $((\cdot,\cdot)) = (A^{1/2}\cdot, A^{1/2}\cdot)$ and $\|\cdot\| = |A^{1/2}\cdot|$ the inner product and norm on V, respectively. Notice that, based on the above, the inner product $((\cdot,\cdot))$, restricted to V, is equivalent to the H^1 inner product

$$[u,v] = \alpha_0^2(u,v) + \alpha_1^2((u,v)) \quad \text{for } u,v \in V \tag{2.5}$$

provided $\alpha_1 > 0$. We denote V' is the dual of V.

Hereafter, c will denote a generic scale invariant positive constant, which is independent of the physical parameters in the equation and may be different from line to line and even in the same line.

3. Abstract Results

Let E be a Banach space, and let a two-parameter family of mappings $\{U(t,\tau)\} = \{U(t,\tau)t \geq \tau,\ \tau \in \mathbb{R}\}$ act on E:

$$U(t,\tau) : E \longrightarrow E, \quad t \geq \tau,\ \tau \in \mathbb{R}. \tag{3.1}$$

Definition 3.1. A two-parameter family of mappings $\{U(t,\tau)\}$ is said to be a process in E if

$$\begin{aligned} U(t,s)U(s,\tau) &= U(t,\tau), \quad \forall t \geq s \geq \tau,\ \tau \in \mathbb{R}, \\ U(\tau,\tau) &= Id, \quad \tau \in \mathbb{R}. \end{aligned} \tag{3.2}$$

A family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, acting in E is said to be $(E\times\Sigma, E)$-continuous, if for all fixed t and τ, $t \geq \tau$, $\tau \in \mathbb{R}$, the mapping $(u,\sigma) \mapsto U_\sigma(t,\tau)u$ is continuous from $E\times\Sigma$ into E.

A curve $u(s)$, $s \in \mathbb{R}$ is said to be *a complete trajectory* of the process $\{U(t,\tau)\}$ if

$$U(t,\tau)u(\tau) = u(t), \quad \forall t \geq \tau, \tau \in \mathbb{R}. \tag{3.3}$$

The *kernel*$\mathcal{K}$ of the process $\{U(t,\tau)\}$ consists of all bounded complete trajectories of the process $\{U(t,\tau)\}$:

$$\mathcal{K} = \{u(\cdot) \mid u(\cdot) \text{ satisfies (3.3) and } \|u(s)\|_E \leq M_u \quad \text{for } s \in \mathbb{R}\}. \tag{3.4}$$

The set

$$\mathcal{K}(s) = \{u(s) \mid u(\cdot) \in \mathcal{K}\} \subseteq E \tag{3.5}$$

is said to be the *kernel section* at time $t = s$, $s \in \mathbb{R}$.

For convenience, let $B_t = \cup_{\sigma\in\Sigma}\cup_{s\geq t} U_\sigma(s,t)B$, the closure $\overline{B}$ of the set B and $\mathbb{R}_\tau = \{t \in \mathbb{R} \mid t \geq \tau\}$. Define the uniform (w.r.t. $\sigma \in \Sigma$) ω-limit set $\omega_{\tau,\Sigma}(B)$ of B by $\omega_{\tau,\Sigma}(B) = \cap_{t\geq\tau}\overline{B_t}$ which can be characterized, analogously to that for semigroups, the following:

$$\begin{aligned} y \in \omega_{\tau,\Sigma}(B) \Longleftrightarrow & \text{ there are sequences } \{x_n\} \subset B, \{\sigma_n\} \subset \Sigma, \{t_n\} \subset \mathbb{R}_\tau \\ & \text{such that } t_n \longrightarrow +\infty \text{ and } U_{\sigma_n}(t_n,\tau)x_n \longrightarrow y \quad (n \longrightarrow \infty). \end{aligned} \tag{3.6}$$

We recall characterize the existence of the uniform attractor for a family of processes satisfying (3.6) in term of the concept of measure of noncompactness that was put forward first by Kuratowski (see [13, 14]).

Let $B \in \mathcal{B}(E)$ its Kuratowski measure of noncompactness $\kappa(B)$ is defined by

$$\kappa(B) = \inf\{\delta > 0 \mid B \text{ admits afinite covering by sets of diameter } \leq \delta\}. \tag{3.7}$$

Definition 3.2. A family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, is said to be uniformly (w.r.t. $\sigma \in \Sigma$) ω-limit compact if for any $\tau \in \mathbb{R}$ and $B \in \mathcal{B}(E)$ the set B_t is bounded for every t and $\lim_{t\to\infty}\kappa(B_t) = 0$.

We present now a method to verify the uniform (w.r.t. $\sigma \in \Sigma$) ω-limit compactness (see [15, 16]).

Definition 3.3. A family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, is said to satisfy uniformly (w.r.t. $\sigma \in \Sigma$) Condition (C) if for any fixed $\tau \in \mathbb{R}$, $B \in \mathcal{B}(E)$, and $\varepsilon > 0$, there exist $t_0 = t(\tau,B,\varepsilon) \geq \tau$ and a finite dimensional subspace E_1 of E such that

(i) $P(\cup_{\sigma\in\Sigma}\cup_{t\geq t_0} U_\sigma(t,\tau)B)$ is bounded; and

(ii) $\|(I-P)(\cup_{\sigma\in\Sigma}\cup_{t\geq t_0} U_\sigma(t,\tau)x)\| \leq \varepsilon$, $\forall x \in B$,

where $P : E \to E_1$ is a bounded projector.

Therefore, we have the following results.

Theorem 3.4. *Let Σ be a metric space and let $\{T(t)\}$ be a continuous invariant semigroup $T(t)\Sigma = \Sigma$ on Σ. A family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, acting in E is $(E \times \Sigma, E)$-continuous (weakly) and possesses the compact uniform (w.r.t. $\sigma \in \Sigma$) attractor A_Σ satisfying*

$$\mathcal{A}_\Sigma = \omega_{0,\Sigma}(B_0) = \omega_{\tau,\Sigma}(B_0) = \bigcup_{\sigma\in\Sigma}\mathcal{K}_\sigma(0), \quad \forall\tau \in \mathbb{R}, \tag{3.8}$$

if it

(i) *has a bounded uniformly (w.r.t. $\sigma \in \Sigma$) absorbing set B_0; and*

(ii) *satisfies uniformly (w.r.t. $\sigma \in \Sigma$) Condition (C).*

Moreover, if E is a uniformly convex Banach space, then the converse is true.

4. Uniform Attractor of Nonautonomous Camassa-Holm Equations

This section deals with the existence of the attractor for the three-dimensional nonautonomous Camassa-Holm equations with periodic boundary condition. To this end, we first state some the following results.

Proposition 4.1. *Let $f(x,t) \in L^2_{\text{loc}}((0,T); D(A^{-1/2}))$ and let $u_\tau \in V$. Then problem (1.1) has a unique solution $u(t)$ such that for any $T > \tau$,*

$$u \in C([\tau,T); V) \cap L^2([\tau,T); D(A)), \qquad \frac{du}{dt} \in L^2([\tau,T); H), \tag{4.1}$$

and such that for almost all $t \in [\tau,T)$ and for any $w \in D(A)$,

$$\begin{aligned}&\left\langle \frac{\partial}{\partial t}\left(\alpha_0^2 u + \alpha_1^2\Delta u\right), w\right\rangle_{D(A)'} + \nu\left\langle A\left(\alpha_0^2 u + \alpha_1^2\Delta u\right), w\right\rangle_{D(A)'}\\ &\qquad + \left\langle \widetilde{B}(u, \alpha_0^2 u + \alpha_1^2\Delta u, w\right\rangle_{D(A)'}\\ &\quad = (f, w),\end{aligned} \tag{4.2}$$

here

$$\begin{aligned}\left(\widetilde{B}(u,v), w\right) &= (B(u,v), w) - (B(w,v), u)\\ &= (B(v)u - B^*(v)u, w)\end{aligned} \tag{4.3}$$

for every $u, v, w \in V$.

Proof. We use the Galerkin procedure to prove global existence. The proof of Proposition 4.1 is similar to autonomous Camassa-Holm in [1]. □

If we denote $v = \alpha_0^2 u + \alpha_1^2 Au$, the system (1.1) can be written as

$$\frac{dv}{dt} + \nu Av + B(v)u + B^*(v)u = Pf, \qquad v(x,\tau) = v_\tau(x) \in H. \tag{4.4}$$

In [1] the authors have shown that the semigroup $S(t) : V \to V \quad (t \geq 0)$ associated with the autonomous system (4.4) possesses a global attractor in V and $D(A)$. The main objective of this section is to prove that the nonautonomous system (4.4) have uniform attractors in V and $D(A)$.

Now recall the following facts that can be found in [15].

Definition 4.2. A function $\varphi \in L^2_{\text{loc}}(\mathbb{R}; E)$ is said to be normal if for any $\varepsilon > 0$, there exists $\eta > 0$ such that

$$\sup_{t\in\mathbb{R}} \int_t^{t+\eta} \|\varphi(s)\|_E^2 ds \leq \varepsilon. \tag{4.5}$$

We denote by $L^2_n(\mathbb{R}; E)$ the set of all normal functions in $L^2_{\text{loc}}(\mathbb{R}; E)$.

Remark 4.3. Obviously, $L^2_n(\mathbb{R}; E) \subset L^2_b(\mathbb{R}; E)$. Denote by $L^2_c(\mathbb{R}; E)$ the class of translation compact functions $f(s),\ s \in \mathbb{R}$, whose family of $\mathscr{H}(f)$ is precompact in $L^2_{\text{loc}}(\mathbb{R}; E)$. It is proved in [15] that $L^2_n(\mathbb{R}; E)$ and $L^2_c(\mathbb{R}; E)$ are closed subspaces of $L^2_b(\mathbb{R}; E)$, but the latter is a proper subset of the former (for further details see [15]).

We now define the *symbol space* $\mathscr{H}(\sigma_0)$ for (4.4). Let a fixed symbol $\sigma_0(s) = f_0(s) = f_0(\cdot, s)$ be normal functions in $L^2_{\text{loc}}(\mathbb{R}; E)$, that is, the family of translation $\{f_0(s+h),\ h \in \mathbb{R}\}$ forms a normal function set in $L^2_{\text{loc}}([T_1, T_2]; E)$, where $[T_1, T_2]$ is an arbitrary interval of the time axis $\mathbb{R}$. Therefore,

$$\mathscr{H}(\sigma_0) = \mathscr{H}(f_0) = \left[f_0(x, s+h) \mid h \in \mathbb{R}\right]_{L^2_{\text{loc}}(\mathbb{R};E)}. \tag{4.6}$$

Now, for any $f(x,t) \in \mathscr{H}(f_0)$, problem (4.4) with f instead of f_0 possesses a corresponding process $\{U_f(t,\tau)\}$ acting on V. As is proved in [10], the family $\{U_f(t,\tau) \mid f \in \mathscr{H}(f_0)\}$ of processes is $(V \times \mathscr{H}(f_0); V)$-continuous.

Let

$$\mathcal{K}_f = \left\{v_f(x,t) \text{ for } t \in \mathbb{R} \mid v_f(x,t) \text{ is solution of (4.4) satisfying} \|v_f(\cdot,t)\|_H \leq M_f \ \forall t \in \mathbb{R}\right\} \tag{4.7}$$

be the so-called kernel of the process $\{U_f(t,\tau)\}$.

Proposition 4.4. *The process $\{U_f(t,\tau)\}$ associated with* (4.4) *possesses absorbing sets*

$$\mathcal{B}_0 = \{v \mid \|v\| \leq r_0\}, \qquad \mathcal{B}_1 = \{v \mid |Av| \leq r_1\}. \tag{4.8}$$

Proof. The proof of Proposition 4.4 is similar to autonomous Camassa-Holm equation. □

The main results in this section are as follows.

Theorem 4.5. *If $f_0(x,s)$ is a normal function in $L^2_{\mathrm{loc}}(\mathbb{R};V')$, then the processes $\{U_{f_0}(t,\tau)\}$ corresponding to problem (1.1) possess compact uniform (w.r.t. $\tau \in \mathbb{R}$) attractor $\mathfrak{A}_0$ in V which coincides with the uniform (w.r.t. $f \in \mathscr{H}(f_0)$) attractor $\mathfrak{A}_{\mathscr{H}(f_0)}$ of the family of processes $\{U_f(t,\tau)\}$, $f \in \mathscr{H}(f_0)$:*

$$\mathfrak{A}_0 = \mathfrak{A}_{\mathscr{H}(f_0)} = \omega_{0,\mathscr{H}(f_0)}(\mathcal{B}_0) = \bigcup_{f\in\mathscr{H}(f_0)} \mathcal{K}_f(0), \tag{4.9}$$

where $\mathcal{B}_0$ is the uniformly (w.r.t. $f \in \mathscr{H}(f_0)$) absorbing set in V, and $\mathcal{K}_f$ is the kernel of the process $\{U_f(t,\tau)\}$. Furthermore, the kernel $\mathcal{K}_f$ is nonempty for all $f \in \mathscr{H}(f_0)$.

Proof. As in the previous section, for fixed N, let H_1 be the subspace spanned by $w_1,\dots,w_N$, and H_2 the orthogonal complement of H_1 in H. We write

$$u = u_1 + u_2; \quad u_1 \in H_1,\ u_2 \in H_2 \text{ for any } \ u \in H. \tag{4.10}$$

Now, we only have to verify Condition (C). Namely, we need to estimate $|u_2(t)|$, where $u(t) = u_1(t) + u_2(t)$ is a solution of (4.4) given in Proposition 4.1.

Letting $w = u_2$ in (4.2), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\left(\alpha_0^2|u_2|^2 + \alpha_1^2\|u_2\|^2\right) + \nu\left(\alpha_0^2\|u_2\|^2 + \alpha_1^2|Au_2|^2\right)\\ &+ \left(\widetilde{B}\left(u,\alpha_0^2 u + \alpha_1^2 Au\right), u_2\right)\\ &= (Pf, u_2),\end{aligned} \tag{4.11}$$

Notice that

$$\left|(Pf,u_2)\right| \le |f|_{V'}\|u_2\| \le \frac{|f|_{V'}^2}{\nu\alpha_0^2} + \frac{\nu}{4}\alpha_0^2\|u_2\|^2. \tag{4.12}$$

From the above inequalities we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\left(\alpha_0^2|u_2|^2 + \alpha_1^2\|u_2\|^2\right) + \frac{3\nu}{4}\left(\alpha_0^2\|u_2\|^2 + \alpha_1^2|Au_2|^2\right)\\ &+ \left(\widetilde{B}\left(u,\alpha_0^2 u + \alpha_1^2 Au\right), u_2\right)\\ \le\ &\frac{|f|_{V'}^2}{\nu\alpha_0^2}.\end{aligned} \tag{4.13}$$

Since $\widetilde{B}$ satisfies the following inequality (see [1, 12]):

$$\left|\left(\widetilde{B}(u,v),w\right)\right| \leq c\|u\|\|v\||w|^{1/2}\|w\|^{1/2}, \quad \forall u,v,w \in V, \tag{4.14}$$

then by Young's inequality,

$$\begin{aligned}
&\left|\left(\widetilde{B}\left(u,\alpha_0^2 u + \alpha_1^2 Au\right),u_2\right)\right| \\
&\quad\leq c\left(\alpha_0^2\|u\|^2 + \alpha_1^2\|u\|\|Au\|\right)|u_2|^{1/2}\|u_2\|^{1/2} \\
&\quad\leq \frac{\nu}{4}\left(\alpha_0^2\|u_2\|^2 + \alpha_1^2|Au_2|^2\right) + M_1(\alpha_0,\alpha_1,r_0,r_1).
\end{aligned} \tag{4.15}$$

Thus, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\alpha_0^2|u_2|^2 + \alpha_1^2\|u_2\|^2\right) + \frac{\nu}{2}\left(\alpha_0^2\|u_2\|^2 + \alpha_1^2|Au_2|^2\right) \\
&\quad\leq M_1(\alpha_0,\alpha_1,r_0,r_1) + \frac{|f|_{V'}^2}{\nu\alpha_0^2}.
\end{aligned} \tag{4.16}$$

Therefore, we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\alpha_0^2|u_2|^2 + \alpha_1^2\|u_2\|^2\right) + \frac{\nu}{2}\lambda_{m+1}\left(\alpha_0^2|u_2|^2 + \alpha_1^2\|u_2\|^2\right) \\
&\quad\leq M_1 + \frac{c}{\nu}|f|_{V'}^2.
\end{aligned} \tag{4.17}$$

Here, $M_1 = M_1(\alpha_0,\alpha_1,r_0,r_1)$ depends on λ_{m+1}, and is not increasing as λ_{m+1} increasing.

By the Gronwall inequality, the above inequality implies

$$\begin{aligned}
&\alpha_0^2|u_2|^2 + \alpha_1^2\|u_2\|^2 \\
&\quad\leq \left(\alpha_0^2|u_2(t_0+1)|^2 + \alpha_1^2\|u_2(t_0+1)\|^2\right)e^{-\nu\lambda_{m+1}(t-(t_0+1))} \\
&\qquad + \frac{2M_1}{\nu\lambda_{m+1}} + \frac{2c}{\nu}\int_{t_0+1}^{t} e^{-\nu\lambda_{m+1}(t-s)}|f|_{V'}^2 ds.
\end{aligned} \tag{4.18}$$

Applying [10, Definition 4.1 and Lemma II 1.3] for any $\varepsilon_1 > 0$, $\varepsilon = \varepsilon_1/\alpha_1^2$,

$$\frac{2c}{\nu}\int_{t_0+1}^{t} e^{-\nu\lambda_{m+1}(t-s)}|f|_{V'}^2 ds < \frac{\varepsilon_1}{3}. \tag{4.19}$$

Using (2.2) and letting $t_1 = t_0 + 1 + 1/\nu\lambda_{m+1} \ln 3r_0^2/\varepsilon_1$, then $t \geq t_1$ implies

$$\frac{2M_1}{\nu\lambda_{m+1}} < \frac{\varepsilon_1}{3},$$
$$\left(\alpha_0^2|u_2(t_0+1)|^2 + \alpha_1^2\|u_2(t_0+1)\|^2\right)e^{-\nu\lambda_{m+1}(t-(t_0+1))} \leq r_0^2 e^{-\nu\lambda_{m+1}(t-(t_0+1))} < \frac{\varepsilon_1}{3}. \tag{4.20}$$

Therefore, we deduce from (4.18) that

$$\|u_2\|^2 \leq \varepsilon, \quad \forall t \geq t_1, \ f \in \mathscr{H}(f_0), \tag{4.21}$$

which indicates $\{U_f(t,\tau)\}$, $f \in \mathscr{H}(f_0)$ satisfying uniform ($w.r.t.$ $f \in \mathscr{H}(f_0)$) Condition (C) in V. Applying Theorem 3.4, the proof is complete. □

According to Propositions 4.1 and 4.4, we can now regard that the families of processes $\{U_f(t,\tau)\}$, $f \in \mathscr{H}(f_0)$ for (1.1) are defined in $D(A)$ and $\mathcal{B}_1$ is a uniformly (w. r. t. $f \in \mathscr{H}(f_0)$) absorbing set in $D(A)$.

Theorem 4.6. *If $f_0(x,s)$ is normal function in $L^2_{\rm loc}(\mathbb{R}; V')$, then the processes $\{U_{f_0}(t,\tau)\}$ corresponding to problem* (1.1) *possesses compact uniform* ($w.r.t.$ $\tau \in \mathbb{R}$) *attractor $\mathfrak{A}_1$ in $H^2_{per} = D(A)$ which coincides with the uniform* ($w.r.t.$ $f \in \mathscr{H}(f_0)$) *attractor $\mathfrak{A}_{\mathscr{H}(f_0)}$ of the family of processes $\{U_f(t,\tau)\}$, $f \in \mathscr{H}(f_0)$:*

$$\mathfrak{A}_1 = \mathfrak{A}_{\mathscr{H}(f_0)} = \omega_{0,\mathscr{H}(f_0)}(\mathcal{B}_1) = \bigcup_{f \in \mathscr{H}(f_0)} \mathcal{K}_f(0), \tag{4.22}$$

where $\mathcal{B}_1$ is the uniformly ($w.r.t.$ $f \in \mathscr{H}(f_0)$) *absorbing set in $D(A)$ and $\mathcal{K}_f$ is the kernel of the process $\{U_f(t,\tau)\}$. Furthermore, the kernel $\mathcal{K}_f$ is nonempty for all $f \in \mathscr{H}(f_0)$.*

Proof. Using Proposition 4.4, we have the family of processes $\{U_f(t,\tau)\}$, $f \in \mathscr{H}(f_0)$ corresponding to (4.4) possesses the uniformly (w.r.t. $f \in \mathscr{H}(f_0)$) absorbing set in $D(A)$.

Now we testify that the family of processes $\{U_f(t,\tau)\}$, $f \in \mathscr{H}(f_0)$ corresponding to (4.4) satisfies uniform ($w.r.t.$ $f \in \mathscr{H}(f_0)$) Condition (C).

Letting $w = Au_2$ in (4.2), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\alpha_0^2\|u_2\|^2 + \alpha_1^2|Au_2|^2\right) + \nu\left(\alpha_0^2|Au_2|^2 + \alpha_1^2\left|A^{3/2}u_2\right|^2\right) \\
&\quad + \left(\widetilde{B}\left(u, \alpha_0^2 u + \alpha_1^2 Au\right), Au_2\right) \\
&= (Pf, Au_2).
\end{aligned} \tag{4.23}$$

Notice that

$$\left|(Pf, Au_2)\right| \leq |f|_{V'}\left|A^{3/2}u_2\right| \leq \frac{|f|_{V'}^2}{\nu\alpha_1^2} + \frac{\nu}{4}\alpha_1^2\left|A^{3/2}u_2\right|^2. \tag{4.24}$$

Therefore, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\alpha_0^2\|u_2\|^2+\alpha_1^2|Au_2|^2\right)+\frac{3\nu}{4}\left(\alpha_0^2|Au_2|^2+\alpha_1^2\left|A^{3/2}u_2\right|^2\right)&\\ +\left(\widetilde{B}\left(u,\alpha_0^2u+\alpha_1^2Au\right),u_2\right)&\\ \leq\frac{|f|_{V'}^2}{\nu\alpha_1^2}.&\end{aligned}\tag{4.25}$$

To estimate $(\widetilde{B}(u,\alpha_0^2u+\alpha_1^2Au)),Au_2)_g$, we recall some inequalities ([1, 11, 12, 14]): for every $u,v\in D(A_g)$,

$$|B(u,v)|\leq c\begin{cases}|u|^{1/2}\|u\|^{1/2}\|v\|^{1/2}|Av|^{1/2}\\ |u|^{1/2}|Av|^{1/2}\|v\|;\end{cases}\tag{4.26}$$

and [12]

$$|w|_{L^\infty(\Omega)}\leq c\|w\|\left(1+\log\frac{|Aw|}{\lambda_1\|w\|^2}\right)^{1/2}\tag{4.27}$$

from which we deduce that

$$|B(u,v)|\leq c|u|_{L^\infty(\Omega)}|\nabla v||u||\nabla v|_{L^\infty(\Omega)},\tag{4.28}$$

and using (4.27),

$$|B(u,v)|\leq c\begin{cases}\|u\|\|v\|\left(1+\log\dfrac{|Au|^2}{\lambda_1\|u\|^2}\right)^{1/2}\\ |u||Av|\left(1+\log\dfrac{\left|A^{3/2}v\right|^2}{\lambda_1\|Av\|^2}\right)^{1/2}\end{cases}.\tag{4.29}$$

Expanding and using Young's inequality, together with the first one of (4.29) and the second one of (4.26), we have

$$\begin{aligned}&\left|\left(\widetilde{B}\left(u,\alpha_0^2u+\alpha_1^2Au\right)\right),Au_2\right)\right|\\ &\quad\leq\left|\left(\widetilde{B}\left(u_1,\alpha_0^2u+\alpha_1^2Au\right),Au_2\right)\right|+\left|\left(\widetilde{B}\left(u_2,\alpha_0^2u+\alpha_1^2Au\right),Au_2\right)\right|\\ &\quad\leq cL^{1/2}\|u_1\||Au_2|\left(\alpha_0^2\|u\|+\alpha_1^2|Au|\right)+c|u_2|^{1/2}|Au_2|^{3/2}\left(\alpha_0^2\|u\|+\alpha_1^2|Au|\right)\\ &\quad\leq\frac{\nu}{4}\left(\alpha_0^2|Au_2|^2+\alpha_1^2\left|A^{3/2}u_2\right|^2\right)+M_2(\alpha_0,\alpha_1,r_0,r_1),\quad t\geq t_0+1,\end{aligned}\tag{4.30}$$

where we use

$$|Au_1|^2 \le \lambda_m \|u_1\|^2, \tag{4.31}$$

and set

$$L = 1 + \log \frac{\lambda_{m+1}}{\lambda_1}. \tag{4.32}$$

Thus, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\alpha_0^2\|u_2\|^2 + \alpha_1^2|Au_2|^2\right) + \frac{\nu}{2}\left(\alpha_0^2|Au_2|^2 + \alpha_1^2\left|A^{3/2}u_2\right|^2\right) \\ &\quad \le M_2 + \frac{|f|_{V'}^2}{\nu\alpha_1^2}. \end{aligned} \tag{4.33}$$

Here $M_2 = M_2(\alpha_0, \alpha_1, r_0, r_1)$ depends on λ_{m+1}, and is not increasing as λ_{m+1} increasing.

By the Gronwall inequality, the above inequality implies

$$\begin{aligned} &\alpha_0^2\|u_2\|^2 + \alpha_1^2|Au_2|^2 \\ &\quad \le \left(\alpha_0^2\|u_2(t_0+1)\|^2 + \alpha_1^2|Au_2(t_0+1)|^2\right)e^{-\nu\lambda_{m+1}(t-(t_0+1))} \\ &\qquad + \frac{2M_2}{\nu\lambda_{m+1}} + \frac{2c}{\nu}\int_{t_0+1}^{t} e^{-\nu\lambda_{m+1}(t-s)}|f|_{V'}^2 ds. \end{aligned} \tag{4.34}$$

Applying [10, Definition 4.1 and Lemma II 1.3] for any $\varepsilon_1 > 0$, $\varepsilon = \varepsilon_1/\alpha_1^2$,

$$\frac{2c}{\nu}\int_{t_0+1}^{t} e^{-\nu\lambda_{m+1}(t-s)}|f|_{V'}^2 ds < \frac{\varepsilon_1}{3}. \tag{4.35}$$

Using (2.2) and let $t_1 = t_0 + 1 + 1/\nu\lambda_{m+1} \ln 3r_1^2/\varepsilon_1$, then $t \ge t_1$ implies

$$\begin{gathered} \frac{2M_2}{\nu\lambda_{m+1}} < \frac{\varepsilon_1}{3}; \\ \left(\alpha_0^2\|u_2(t_0+1)\|^2 + \alpha_1^2|Au_2(t_0+1)|^2\right)e^{-\nu\lambda_{m+1}(t-(t_0+1))} \le r_1^2 e^{-\nu\lambda_{m+1}(t-(t_0+1))} < \frac{\varepsilon_1}{3}. \end{gathered} \tag{4.36}$$

Therefore, we deduce from (4.34) that

$$|Au_2|^2 \le \varepsilon, \quad \forall t \ge t_1,\ f \in \mathscr{H}(f_0), \tag{4.37}$$

which indicates $\{U_f(t,\tau)\}$, $f \in \mathscr{H}(f_0)$ satisfying uniform (w.t.r. $f \in \mathscr{H}(f_0)$) Condition (C) in $D(A)$. $\square$

5. Dimension of the Uniform Attractor

In this section, we estimate the fractal dimension (for definition see, e.g., [10, 12, 17, 18]) of the kernel sections of the uniform attractors obtained in Section 4 by applying the methods in [19].

Process $\{U(t,\tau)\}$ is said to be uniformly quasidifferentiable on $\{\mathcal{K}(s)\}_{\tau\in\mathbb{R}}$ if there is a family of bounded linear operators $\{L(t,\tau;u) \mid u \in \mathcal{K}(s), t \geq \tau, \tau \in \mathbb{R}\}$, $L(t,\tau;u) : E \to E$ such that

$$\lim_{\varepsilon\to 0}\sup_{\tau\in\mathbb{R}} \sup_{\substack{u,v\in\mathcal{K}(s)\\ 0<|u-v|\leq\varepsilon}} \frac{|U(t,\tau)v - U(t,\tau)u - L(t,\tau;u)(v-u)|}{|v-u|} = 0. \tag{5.1}$$

We want to estimate the fractal dimension of the kernel sections $\mathcal{K}(s)$ of the process $\{U(t,\tau)\}$ generated by the abstract evolutionary equation (4.4). Assume that $\{L(t,\tau;u)\}$ is generated by the variational equation corresponding to (4.4),

$$\partial_t w = F'(u,t)w, \quad w|_{t=\tau} = w_\tau \in E, \quad t \geq \tau, \ \ \tau \in \mathbb{R}, \tag{5.2}$$

that is, $L(t,\tau;u_\tau)w_\tau = w(t)$ is the solution of (5.2), and $u(t) = U(t,\tau)u_\tau$ is the solution of (1.1) with initial value $u_\tau \in \mathcal{K}(\tau)$. For natural number $j \in N$, we set

$$\widetilde{q}_j = \lim_{T\to+\infty} \sup_{\tau\in R} \sup_{u_\tau\in\mathcal{K}(\tau)} \left(\frac{1}{T}\int_\tau^{\tau+T} \operatorname{Tr}\bigl(F'(u(s),s)\bigr)ds \right), \tag{5.3}$$

where Tr is trace of the operator.

We will need the following [10, Theorem VIII.3.1].

Theorem 5.1. *Under the assumptions above, let us suppose that $U_{\tau\in\mathbb{R}}\mathcal{K}(\tau)$ is relatively compact in E, and there exists q_j, $j = 1,2,\ldots$, such that*

$$\begin{gathered} \widetilde{q}_j \leq q_j, \quad \text{for any } j \geq 1, \\ q_{n_0} \geq 0, \quad q_{n_0+1} < 0, \text{ for some } n_0 \geq 1, \\ q_j \leq q_{n_0} + \bigl(q_{n_0} - q_{n_0+1}\bigr)\bigl(n_0 - j\bigr), \quad \forall j = 1,2,\ldots. \end{gathered} \tag{5.4}$$

Then,

$$d_F(\mathcal{K}(\tau)) \leq d_0 := n_0 + \frac{q_{n_0}}{q_{n_0} - q_{n_0+1}}, \quad \forall\tau \in \mathbb{R}. \tag{5.5}$$

We now consider (4.4) with $f \in L^2_n(\mathbb{R};V')$. The equations possess a compact uniform (*w.r.t.* $f \in \mathcal{H}(f)$) attractor $\mathcal{A}_{\mathcal{H}(f)}$ and $\bigcup_{\tau\in\mathbb{R}} \mathcal{K}_f(\tau) \subset \mathcal{A}_{\mathcal{H}(f)}$. By [10, 12], we know that the associated process $\{U_f(t,\tau)\}$ is uniformly quasidifferentiable on $\{\mathcal{K}_f(\tau)\}_{\tau\in\mathbb{R}}$ and the quasidifferential is Hölder-continuous with respect to $u_\tau \in \mathcal{K}_f(\tau)$. The corresponding variational equation is

$$\partial_t w = -Aw - \widetilde{B}(u,w) - \widetilde{B}(w,u) \equiv F'(u(t),t)w, \qquad w|_{t=\tau} = w_\tau \in E, \quad \tau \in \mathbb{R}, \tag{5.6}$$

where $\widetilde{B}(u,w) = -P(u \times (\nabla \times w))$, one can easily show that

$$\left(\widetilde{B}(u,w),\theta\right) = (B(u,w),\theta) - (B(\theta,w),u), \quad \text{for every } \; u,w,\theta \in V. \tag{5.7}$$

We have the main results in this section.

Theorem 5.2. *Let $f \in L_n^2(\mathbb{R};V')$. Then the fractal dimension of the kernel sections of the uniform attractor, $d_F\mathcal{K}_f(s)$, satisfy*

$$d_F\mathcal{K}_f(s) \le c \;\max\left\{ \frac{1}{\nu^{4/3}\lambda_1}\left(\frac{G}{\alpha_1^2}\right)^{2/3}, \frac{r_0^{3/10}G^{3/5}}{\nu^{3/2}\lambda_1^{3/5}\alpha_0^{3/4}\alpha_1^{9/20}} \right\}, \tag{5.8}$$

where

$$G = \frac{1}{\nu^2\gamma}\limsup_{T\to\infty}\sup_{\tau\in R}\frac{1}{T}\int_\tau^{\tau+T}|f|_{V'}^2 ds, \qquad \frac{1}{\gamma} = \min\left\{\frac{1}{\nu\alpha_0^2}, \frac{1}{\nu\lambda_1\alpha_1^2}\right\}. \tag{5.9}$$

Proof. Now we estimate the m-dimensional trace of $F'(u(t),t)$. Let Q_m is the orthogonal projector from H to Q_mH with orthonormal basis $\varphi_1,\ldots,\varphi_m \in V$. Similar to that of [1, 19], we have

$$\begin{aligned} &\operatorname{Tr}\left(F'(u(t),t)\cdot Q_m\right) \\ &\quad = \sum_{i=1}^m \left(F'(u(t),t)\varphi_i,\varphi_i\right) \le -\frac{c\nu}{2}\lambda_1 m^{5/3} \\ &\qquad + \frac{c}{\nu^{3/2}}\frac{\|u\|^{1/2}}{\alpha_0^{5/4}\alpha_1^{3/4}}\left(\alpha_0^2\|u\|^2 + \alpha_1^2|Au|^2\right) + c|Au|^{4/3}\left(\frac{m^2}{\nu}\right)^{1/3}. \end{aligned} \tag{5.10}$$

It follows from (4.2) that

$$\limsup_{T\to\infty}\sup_{\tau\in\mathbb{R}}\frac{1}{T}\int_\tau^{\tau+T}\left(\alpha_0^2\|u\|^2 + \alpha_1^2|Au|^2\right)ds \le \frac{1}{\nu^2\gamma}\limsup_{T\to\infty}\sup_{\tau\in\mathbb{R}}\frac{1}{T}\int_\tau^{\tau+T}|f|_{V'}^2 ds. \tag{5.11}$$

By HÖlder's inequality we have

$$\limsup_{T\to\infty}\sup_{\tau\in\mathbb{R}}\frac{1}{T}\int_\tau^{\tau+T}|Au|^{4/3}ds \le \limsup_{T\to\infty}\sup_{\tau\in\mathbb{R}}\left(\frac{1}{T}\int_\tau^{\tau+T}|Au|^2 ds\right)^{2/3}, \tag{5.12}$$

and we get

$$\limsup_{T\to\infty}\sup_{\tau\in\mathbb{R}}\frac{1}{T}\int_{\tau}^{\tau+T}|Au|^{4/3}ds \le \left(\frac{G}{\alpha_1^2}\right)^{2/3}. \tag{5.13}$$

Therefore, from the above, we have

$$d_F\mathcal{K}_f(s) \le c\ \max\left\{\frac{1}{\nu^{4/3}\lambda_1}\left(\frac{G}{\alpha_1^2}\right)^{2/3}, \frac{r_0^{3/10}G^{3/5}}{\nu^{3/2}\lambda_1^{3/5}\alpha_0^{3/4}\alpha_1^{9/20}}\right\}, \tag{5.14}$$

which concludes our proof. □

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