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Research Article

Oscillation Criteria for a Class of Second-Order Nonlinear Differential Equations with Damping Term

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A class of second-order nonlinear differential equations with damping term $(r(t)|x'(t)|^{\sigma-1}x'(t))' + p(t)|x'(t)|^{\sigma-1}x'(t)+q(t)f(x(t)) = 0$ are investigated in this paper. By using a new method, we obtain some new sufficient conditions for the oscillation of the above equation, and some references are extended in this paper. Examples are inserted to illustrate this result.

1. Introduction

Consider the following second-order nonlinear differential equations with damping term:

$$\left(r(t)\left|x'(t)\right|^{\sigma-1}x'(t)\right)' + p(t)\left|x'(t)\right|^{\sigma-1}x'(t) + q(t)f(x(t)) = 0, \quad t \geq t_0, \tag{1.1}$$

where $r(t) \in C^1([t_0,\infty);R^+)$, $p(t),q(t) \in C([t_0,\infty);R)$, σ is a positive constant, and f is a continuous real-valued function on the real line R and satisfies $xf(x) \geq 0$ for $x \neq 0$. We restrict our attention to those solutions $x(t)$ of (1.1) which exist on some half line $[t_x,\infty)$ and satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for any $T \geq t_x$.

Recently, there are many authors who have investigated the oscillation for second-order differential equations [1–9], Li [10] and Zhao investigated oscillation criteria for the following equation:

$$\left(r(t)\left(x'(t)\right)^{\sigma}\right)' + p(t)\left(x'(t)\right)^{\sigma} + q(t)f(x(t)) = 0, \quad t \geq t_0, \tag{1.2}$$

where σ is a quotient of odd positive integer. It is obvious that (1.2) is a special case of (1.1). In fact, the conditions of Theorem 3.2 in [10] are too complex.

More recently, Rogovchenko and Tuncay [11] have obtained oscillation criteria of the following:

$$\left(r(t)x'(t)\right)' + p(t)x'(t) + q(t)f(x(t)) = 0, \quad t \geq t_0. \tag{1.3}$$

Motivated by the above discussions, we investigate the oscillation of (1.1) in this paper; our oscillatory conditions and the proof of the main results are more simple than those of Theorem 3.2 in [10].

A solution $x(t)$ of (1.1) is oscillatory if and only if it has arbitrarily large zeros; otherwise, it is nonoscillatory. Equation (1.1) is oscillatory if and only if every solution of (1.1) is oscillatory.

The paper is arranged as follows. In Section 2, we will establish our main results. Finally, examples are given to illustrate our results.

2. Main Results

To obtain our results, we introduce a lemma as follows.

Lemma 2.1 (see [2, 3]). *Let the function $K(t, s, x) : R \times R \times R^+ \to R$ be such that for each fixed t, s, the function $K(t, s, \cdot)$ is nondecreasing. Further, let $h(t)$ be a given function and $u(t)$ satisfies that*

$$u(t) \geq (\leq) h(t) + \int_{t_0}^{t} K(t, s, u(s))ds, \quad t \geq t_0, \tag{2.1}$$

and $v^*(t)$ is the minimal (maximal) solution of

$$v(t) = h(t) + \int_{t_0}^{t} K(t, s, v(s))ds, \quad t \geq t_0. \tag{2.2}$$

Then $u(t) \geq (\leq) v^*(t)$ for all $t \geq t_0$.

Now, we give our main results.

Theorem 2.2. *Assume that $f'(x) \geq 0$, $p(t) \leq 0$, $q(t) > 0$, and $\int_{t_0}^{\infty}(1/r^{1/\sigma}(t))dt = \infty$ hold. Suppose that there exists a positive function $\rho(t)$ such that*

$$\int_{t_0}^{\infty} q(t)\rho(t)dt = \infty, \tag{2.3}$$

$$p(t)\rho(t) \geq r(t)\rho'(t). \tag{2.4}$$

Then every solution of (1.1) *is oscillatory.*

Proof. Assume that (1.1) has a nonoscillatory solution $x(t)$. Without loss of generality, suppose that it is an eventually positive solution (if it is an eventually negative solution, the proof is similar), that is, $x(t) > 0$ for all $t \geq t_0$.

We consider the following three cases.

Case 1. Suppose that $x'(t)$ is oscillatory. Then there exists $t_1 \geq t_0$ such that $x'(t_1) = 0$. From (1.1), we have

$$\begin{aligned}
&\left(r(t)\left|x'(t)\right|^{\sigma-1}x'(t)\exp\left(\int_{t_0}^{t}\frac{p(s)}{r(s)}ds\right)\right)' \\
&\quad = \left(r(t)\left|x'(t)\right|^{\sigma-1}x'(t)\right)'\exp\left(\int_{t_0}^{t}\frac{p(s)}{r(s)}ds\right) + p(t)\left|x'(t)\right|^{\sigma-1}x'(t)\exp\left(\int_{t_0}^{t}\frac{p(s)}{r(s)}ds\right) \\
&\quad = -q(t)f(x(t))\exp\left(\int_{t_0}^{t}\frac{p(s)}{r(s)}ds\right) < 0,
\end{aligned} \tag{2.5}$$

which means that

$$\begin{aligned}
&r(t)\left|x'(t)\right|^{\sigma-1}x'(t)\exp\left(\int_{t_0}^{t}\frac{p(s)}{r(s)}ds\right) \\
&\quad < r(t_1)\left|x'(t_1)\right|^{\sigma-1}x'(t_1)\exp\left(\int_{t_0}^{t_1}\frac{p(s)}{r(s)}ds\right) = 0, \quad t > t_1,
\end{aligned} \tag{2.6}$$

it follows that $x'(t) < 0$ for all $t > t_1$, which contradicts to the assumption that $x'(t)$ is oscillatory.

Case 2. Suppose that $x'(t) < 0$. From (1.1), we obtain

$$-\left(r(t)\left|x'(t)\right|^{\sigma-1}x'(t)\right)' = \left(r(t)\left(-x'(t)\right)^{\sigma}\right)' = -p(t)\left(-x'(t)\right)^{\sigma} + q(t)f(x(t)) \geq 0, \tag{2.7}$$

then there exists an $M > 0$ and a $t_1 \geq t_0$, such that

$$r(t)\left(-x'(t)\right)^{\sigma} \geq M, \quad t \geq t_1, \tag{2.8}$$

it follows

$$x(t) \leq -\int_{t_1}^{\infty}\frac{1}{r^{1/\sigma}(t)}M^{1/\sigma}dt + x(t_1), \quad t \geq t_1, \tag{2.9}$$

which means that $\lim_{t\to\infty}x(t) = -\infty$, this contradicts the assumption that $x(t) > 0$.

Case 3. Suppose that $x'(t) > 0$. Let $w(t) = \rho(t)r(t)(x'(t))^{\sigma}$, then

$$w'(t) = \left(r(t)\left(x'(t)\right)^{\sigma}\right)'\rho(t) + r(t)\left(x'(t)\right)^{\sigma}\rho'(t), \quad t \geq t_0, \tag{2.10}$$

in view of (1.1), we obtain

$$\frac{w'(t)}{f(x(t))} = -q(t)\rho(t) - \frac{\rho(t)p(t)(x'(t))^{\sigma}}{f(x(t))} + \frac{\rho'(t)r(t)(x'(t))^{\sigma}}{f(x(t))}, \quad t \geq t_0, \tag{2.11}$$

noticing that

$$\begin{aligned}\left(\frac{w(t)}{f(x(t))}\right)' &= \frac{w'(t)f(x(t)) - w(t)f'(x(t))x'(t)}{f^2(x(t))} \\ &= -q(t)\rho(t) - \frac{\rho(t)p(t)(x'(t))^{\sigma}}{f(x(t))} + \frac{\rho'(t)r(t)(x'(t))^{\sigma}}{f(x(t))} - \frac{w(t)f'(x(t))x'(t)}{f^2(x(t))}, \quad t \geq t_0,\end{aligned} \tag{2.12}$$

integrating the above from t_0 to t, we get

$$\frac{w(t)}{f(x(t))} = \frac{w(t_0)}{f(x(t_0))} - \int_{t_0}^{t}\left(q(s)\rho(s) + \frac{\left(\rho(s)p(s) - \rho'(s)r(s)\right)(x'(s))^{\sigma}}{f(x(s))} + \frac{w(s)x'(s)f'(x(s))}{f^2(x(s))}\right)ds. \tag{2.13}$$

Using (2.3), (2.4), and $x'(t) > 0$, we have

$$0 \leq \lim_{t\to\infty}\frac{w(t)}{f(x(t))} = -\infty, \tag{2.14}$$

this is a contradiction, the proof is complete. □

Remark 2.3. If we replace $p(t) \leq 0$, $q(t) > 0$ by $p(t) \leq 0$, $q(t) \leq 0$, $\lim_{t\to\infty}(p(t)/q(t)) = M > 0$, Theorem 2.2 holds also.

Theorem 2.4. *Assume that $f'(x) \geq 0$ holds. Suppose also that*

$$\rho_0(t) = \exp\left(\int_{t_0}^{t}\frac{p(s)}{r(s)}ds\right), \tag{2.15}$$

$$\int_{t_0}^{\infty}\frac{dt}{\left(\rho_0(t)r(t)\right)^{1/\sigma}} = \infty, \tag{2.16}$$

and $\rho_0(t)$ such that (2.3) *holds. Then every solution of* (1.1) *is oscillatory.*

Proof. To the contrary, (1.1) has a nonoscillatory solution $x(t)$. Without loss of generality, we assume that $x(t)$ is an eventually positive solution. Let $w(t) = \rho_0(t)r(t)|x'(t)|^{\sigma-1}x'(t)$, then $w(t)x'(t) = \rho_0(t)r(t)|x'(t)|^{\sigma-1}(x'(t))^2 \geq 0$ for $t \geq t_0$ and

$$w'(t) = \left(r(t)\left|x'(t)\right|^{\sigma-1}x'(t)\right)'\rho_0(t) + r(t)\left|x'(t)\right|^{\sigma-1}x'(t)\rho_0'(t), \quad t \geq t_0, \tag{2.17}$$

in view of (1.1) and (2.15), we obtain

$$\frac{w'(t)}{f(x(t))} = -q(t)\rho_0(t), \quad t \geq t_0, \tag{2.18}$$

since

$$\left(\frac{w(t)}{f(x(t))}\right)' = \frac{w'(t)f(x(t)) - w(t)f'(x(t))x'(t)}{f^2(x(t))} = -q(t)\rho_0(t) - \frac{w(t)f'(x(t))x'(t)}{f^2(x(t))}, \quad t \geq t_0, \tag{2.19}$$

integrating the above from t_0 to t, we have

$$-\frac{w(t)}{f(x(t))} = -\frac{w(t_0)}{f(x(t_0))} + \int_{t_0}^{t} q(s)\rho_0(s)ds + \int_{t_0}^{t} \frac{w(s)x'(s)f'(x(s))}{f^2(x(s))}ds, \quad t \geq t_0. \tag{2.20}$$

In view of (2.3), there exists a constant $m > 0$ and $t_1 \geq t_0$ such that

$$-\frac{w(t_0)}{f(x(t_0))} + \int_{t_0}^{t} q(s)\rho_0(s)ds + \int_{t_0}^{t_1} \frac{w(s)x'(s)f'(x(s))}{f^2(x(s))}ds \geq m, \tag{2.21}$$

which means that

$$-\frac{w(t)}{f(x(t))} \geq m + \int_{t_1}^{t} \frac{w(s)x'(s)f'(x(s))}{f^2(x(s))}ds. \tag{2.22}$$

Because that $x(t)$ is positive, then (2.22) implies $-w(t) > 0$, or equivalently $x'(t) < 0$. Let

$$u(t) = -w(t) = -\rho_0(t)r(t)\left|x'(t)\right|^{\sigma-1}x'(t) = \rho_0(t)r(t)\left(-x'(t)\right)^{\sigma}, \tag{2.23}$$

thus (2.22) can be changed as

$$u(t) \geq mf(x(t)) + \int_{t_1}^{t} \frac{f(x(t))f'(x(s))(-x'(s))}{f^2(x(s))}u(s)ds. \tag{2.24}$$

Define

$$K(t,s,u) = \frac{f(x(t))f'(x(s))(-x'(s))}{f^2(x(s))}u. \tag{2.25}$$

Then, for any fixed t and s, $K(t,s,u)$ is nondecreasing in u. Let $v(t)$ be the minimal solution of the equation

$$v(t) = mf(x(t)) + \int_{t_1}^{t} \frac{f(x(t))f'(x(s))(-x'(s))}{f^2(x(s))} v(s)ds. \tag{2.26}$$

Applying Lemma 2.1, we obtain

$$u(t) \geq v(t), \quad t \geq t_0. \tag{2.27}$$

Dividing both sides of (2.26) by $f(x(t))$ and deriving both sides of (2.26), it follows

$$\left(\frac{v(t)}{f(x(t))}\right)' = \left(m + \int_{t_1}^{t} \frac{f'(x(s))(-x'(s))}{f^2(x(s))} v(s)ds\right)' = \frac{f'(x(t))(-x'(t))}{f^2(x(t))} v(t). \tag{2.28}$$

On the other hand,

$$\left(\frac{v(t)}{f(x(t))}\right)' = \frac{v'(t)}{f(x(t))} - \frac{f'(x(t))x'(t)}{f^2(x(t))} v(t). \tag{2.29}$$

Combining (2.28) and (2.29), it means

$$v'(t) \equiv 0. \tag{2.30}$$

So $v(t) = v(t_1) = mf(x(t_1))$, $t \geq t_0$. From (2.27), we obtain

$$-x'(t) \geq \left(mf(x(t_1))\right)^{1/\sigma} \frac{1}{\left(\rho_0(t)r(t)\right)^{1/\sigma}}, \quad t \geq t_1. \tag{2.31}$$

Integrating both sides of the above from t_1 to t, we have

$$-x(t) + x(t_1) \geq \left(mf(x(t_1))\right)^{1/\sigma} \int_{t_1}^{t} \frac{ds}{\left(\rho_0(s)r(s)\right)^{1/\sigma}}. \tag{2.32}$$

Letting $t \to \infty$ in (2.32), and using (2.16), it follows that $\lim_{t\to\infty} x(t) \leq -\infty$, which contradicts to that $x(t)$ is eventually positive. The proof is complete. $\square$

In the following, we always suppose that $H(t) \in C^2(R;R)$ and it satisfies the following two conditions:

(H_1) $H(t) > 0$ for $t \geq t_0$, $H(t)$ is a bounded function;

(H_2) $H'(t) = h(t)$, $h(t)$ is a bounded function.

Theorem 2.5. *Assume that* $f'(x) \geq 0$, $\int_{t_0}^{\infty}(1/r^{1/\sigma}(t))dt = \infty$ *hold, and*

$$p(t) \leq 0, \qquad q(t) > 0, \tag{2.33}$$

or

$$p(t) \leq 0, \quad q(t) \leq 0, \quad \lim_{t\to\infty}\frac{p(t)}{q(t)} = M > 0. \tag{2.34}$$

Suppose further that there exists a function $H(t)$ *that satisfies* (H_1), (H_2), *and such that*

$$\int_{t_0}^{\infty} H(t)\varphi(t)dt = \infty, \tag{2.35}$$

$$\limsup_{t\to\infty} v(t)r(t) < \infty, \tag{2.36}$$

where

$$\varphi(t) = v(t)\big(q(t) - p(t)h(t) - (r(t)h(t))'\big), \tag{2.37}$$

$$v(t) = \exp\left(\int_{t_0}^{t}\left(\frac{p(s)}{r(s)} - \frac{h(s)}{H(s)}\right)ds\right). \tag{2.38}$$

Then every solution of (1.1) *is oscillatory.*

Proof. For the sake of contradiction, (1.1) has a nonoscillatory solution $x(t)$. Without loss of generality, we may assume that $x(t) > 0$ for all $t \geq t_0$.

Define

$$u(t) = v(t)r(t)\left(\frac{|x'(t)|^{\sigma-1}x'(t)}{f(x(t))} + h(t)\right). \tag{2.39}$$

Deriving (2.39), we get

$$\begin{aligned}
u'(t) &= \left(\frac{p(t)}{r(t)} - \frac{h(t)}{H(t)}\right)u(t) \\
&\quad + v(t)\left[-\frac{p(t)|x'(t)|^{\sigma-1}x'(t)}{f(x(t))} - q(t) - \frac{r(t)|x'(t)|^{\sigma-1}(x'(t))^2 f'(x(t))}{f^2(x(t))} + (r(t)h(t))'\right] \\
&\leq -\frac{h(t)}{H(t)}u(t) + p(t)v(t)h(t) - v(t)q(t) + v(t)(r(t)h(t))' \\
&= -\frac{h(t)}{H(t)}u(t) - \varphi(t).
\end{aligned} \tag{2.40}$$

Multiplying (2.40) by $H(t)$, it follows

$$\varphi(t)H(t) \le -H(t)u'(t) - h(t)u(t). \tag{2.41}$$

We consider the following three cases.

Case 1 ($u(t)$ is oscillatory). Then there exists a sequence $\{t_n\}$, $n = 1, 2, \dots, t_n \to \infty$ as $n \to \infty$ and such that $u(t_n) = 0$, $n = 1, 2, \dots$. Integrating both sides of (2.41) from t_0 to t_n, we obtain

$$\begin{aligned}\int_{t_0}^{t_n} H(t)\varphi(t)dt &\le -\int_{t_0}^{t_n} H(t)u'(t)dt - \int_{t_0}^{t_n} h(t)u(t)dt \\ &= -H(t)u(t)\big|_{t_0}^{t_n} - \int_{t_0}^{t_n}\left(-H'(t)u(t) + h(t)u(t)\right)dt \\ &= H(t_0)u(t_0) - H(t_n)u(t_n) = H(t_0)u(t_0),\end{aligned} \tag{2.42}$$

that is

$$\lim_{t_n\to\infty}\int_{t_0}^{t_n} H(t)\varphi(t)dt \le H(t_0)u(t_0), \tag{2.43}$$

which contradicts (2.35).

Case 2 ($u(t)$ is eventually positive). Integrating both sides of (2.41) from t_0 to ∞, we obtain

$$\int_{t_0}^{\infty} H(t)\varphi(t)dt \le H(t_0)u(t_0) - \lim_{t\to\infty} H(t)u(t) \le H(t_0)u(t_0), \tag{2.44}$$

which also contradicts to (2.35).

Case 3 ($u(t)$ is eventually negative). If $\limsup_{t\to\infty} u(t) > -\infty$, then there exists a sequence $\{\bar{t}_n\}$, $n = 1, 2, \dots$, that satisfies $\{\bar{t}_n\} \to \infty$ as $n \to \infty$ and such that $\lim_{\bar{t}_n\to\infty} u(\bar{t}_n) = \limsup_{t\to\infty} u(t) = M_1 > -\infty$. Because $H(t)$ is a bounded function, then there exists a $M_2 > 0$ such that $H(\bar{t}_n) \le M_2$, $n = 1, 2, \dots$. According to (2.41), we obtain

$$\int_{t_0}^{\bar{t}_n} H(t)\varphi(t)dt \le H(t_0)u(t_0) - H\left(\bar{t}_n\right)u\left(\bar{t}_n\right) \le H(t_0)u(t_0) - M_2 u\left(\bar{t}_n\right). \tag{2.45}$$

Using (2.35) and taking limit as $\bar{t}_n \to \infty$, it is easy to show that

$$\begin{aligned}\infty = \lim_{\bar{t}_n\to\infty}\int_{t_0}^{\bar{t}_n} H(t)\varphi(t)dt &\le H(t_0)u(t_0) - \lim_{\bar{t}_n\to\infty} H\left(\bar{t}_n\right)u\left(\bar{t}_n\right) \\ &\le H(t_0)u(t_0) - M_1M_2 < \infty,\end{aligned} \tag{2.46}$$

which is obviously a contradiction.

If $\limsup_{t\to\infty} u(t) = -\infty$, $\lim_{t\to\infty} u(t) = -\infty$. From the definition of $h(t)$, combining (2.36) and (2.39), it follows that $x'(t) < 0$ and $\lim_{t\to\infty}(|x'(t)|^{\sigma-1}x'(t)/f(x(t))) = -\infty$, which means that $\lim_{t\to\infty}((-x'(t))^{\sigma}/f(x(t))) = \infty$. Owing to $p(t) \le 0$, $q(t) \ge 0$, or $p(t) \le 0$, $q(t) \le 0$ and $\lim_{t\to\infty}(p(t)/q(t)) = M > 0$, using the similar method of the proof of Case 2 in Theorem 2.2, we will derive a contradiction. Then the proof is complete. □

Theorem 2.6. *Assume that* (2.36) *holds,* $f'(x) \ge 0$, $\int_{t_0}^{\infty}(1/r^{1/\sigma}(t))dt = \infty$, *and*

$$p(t) \le 0, \qquad q(t) > 0, \tag{2.47}$$

or

$$p(t) \le 0, \quad q(t) \le 0, \quad \lim_{t\to\infty}\frac{p(t)}{q(t)} = M > 0. \tag{2.48}$$

Suppose further that there exists a function $H(t)$ *that satisfies* (H_1), (H_2) *and such that*

$$\int_{t_0}^{\infty} H(t)\overline{\varphi}(t)dt = \infty, \tag{2.49}$$

where

$$\overline{\varphi}(t) = v(t)\big(q(t) + p(t)h(t) + (r(t)h(t))'\big), \tag{2.50}$$

and $v(t)$ *is defined in* (2.38). *Then every solution of* (1.1) *is oscillatory.*

Proof. For the sake of contradiction, (1.1) has a nonoscillatory solution. Without loss of generality, we may assume that (1.1) has an eventually positive solution (if it has an eventually negative solution, the proof is similar), then there exists a $t_1 > t_0$ such that $x(t) > 0$ for all $t \ge t_1$. Define

$$u(t) = v(t)r(t)\left(\frac{|x'(t)|^{\sigma-1}x'(t)}{f(x(t))} - h(t)\right). \tag{2.51}$$

The rest of the proof is similar to Theorem 2.5. The proof is complete. □

Theorem 2.7. *Assume that* (2.36) *holds,* $f'(x) \ge 0$, $\int_{t_0}^{\infty}(1/r^{1/\sigma}(t))dt = \infty$, *and*

$$p(t) \le 0, \qquad q(t) > 0, \tag{2.52}$$

or

$$p(t) \le 0, \quad q(t) \le 0, \quad \lim_{t\to\infty}\frac{p(t)}{q(t)} = M > 0. \tag{2.53}$$

Suppose further that there exists a function $H(t)$ *that satisfies* (H_1), (H_2) *and such that*

$$\int_{t_0}^{\infty} H(t)\phi(t)dt = \infty, \tag{2.54}$$

where

$$\phi(t) = v(t)\left(-p(t)h(t) - (r(t)h(t))'\right), \tag{2.55}$$

where $v(t)$ *is defined in* (2.38). *Then every solution of* (1.1) *is oscillatory.*

Proof. For the sake of contradiction, (1.1) has a nonoscillatory solution $x(t)$. Without loss of generality, we may assume that $x(t) > 0$ for all $t \geq t_0$.

Define

$$u(t) = v(t)r(t)\left(\frac{|x'(t)|^{\sigma-1}x'(t)}{x(t)} + h(t)\right). \tag{2.56}$$

Noting that $xf(x) \geq 0$ for $x \neq 0$, so $f(x)/x \geq 0$ for $x \neq 0$. Deriving (2.56), we obtain

$$\begin{aligned} u'(t) = &\left(\frac{p(t)}{r(t)} - \frac{h(t)}{H(t)}\right)u(t) \\ &+ v(t)\left[-\frac{p(t)|x'(t)|^{\sigma-1}x'(t)}{x(t)} - \frac{q(t)f(x(t))}{x(t)} - \frac{r(t)|x'(t)|^{\sigma-1}(x'(t))^2}{x^2(t)} + (r(t)h(t))'\right] \\ &\leq -\frac{h(t)}{H(t)}u(t) + p(t)v(t)h(t) + v(t)(r(t)h(t))' \\ &= -\frac{h(t)}{H(t)}u(t) - \phi(t). \end{aligned} \tag{2.57}$$

Multiplying (2.57) by $H(t)$, we get

$$H(t)\phi(t) \leq -H(t)u'(t) - h(t)u(t). \tag{2.58}$$

The rest of the proof is similar to Theorem 2.5; the proof is complete. □

Theorem 2.8. *Assume that* (2.36) *holds,* $f'(x) \geq 0$, $\int_{t_0}^{\infty}(1/r^{1/\sigma}(t))dt = \infty$, *and*

$$p(t) \leq 0, \qquad q(t) > 0, \tag{2.59}$$

or

$$p(t) \leq 0, \quad q(t) \leq 0, \quad \lim_{t\to\infty}\frac{p(t)}{q(t)} = M > 0. \tag{2.60}$$

Suppose further that there exists a function $H(t)$ satisfies (H_1), (H_2) and such that

$$\int_{t_0}^{\infty} H(t)\overline{\phi}(t)dt = \infty, \tag{2.61}$$

where

$$\overline{\phi}(t) = v(t)\left(p(t)h(t) + (r(t)h(t))'\right), \tag{2.62}$$

where $v(t)$ is defined in (2.38). *Then every solution of* (1.1) *is oscillatory.*

Proof. For the sake of contradiction, (1.1) has a nonoscillatory solution $x(t)$. Without loss of generality, we may assume that $x(t) > 0$ for all $t \geq t_0$.

Define

$$u(t) = v(t)r(t)\left(\frac{|x'(t)|^{\sigma-1}x'(t)}{x(t)} - h(t)\right). \tag{2.63}$$

The rest of the proof is similar to Theorem 2.5; the proof is complete. □

3. Examples

Example 3.1. Consider the following delay differential equation:

$$(tx'(t))' - 2x'(t) + \left(t + \frac{3}{4t}\right)x(t) = 0. \tag{3.1}$$

It is obvious that $\sigma = 1$, $q(t) = (t + 3/4t) > 0$, $p(t) = -2 < 0$, $r(t) = t$, and $\int_{t_0}^{\infty}(1/t)dt = \infty$. It is difficult to distinguish whether every solution of (3.1) is oscillatory by Theorem 3.2 of [10].

By taking $\rho(t) = 1/t^2$, then

$$\begin{gathered}\int_{t_0}^{\infty} q(t)\rho(t)dt = \int_{t_0}^{\infty}\left(t + \frac{3}{4t}\right)\frac{1}{t^2}dt = \infty, \\ p(t)\rho(t) = -\frac{2}{t^2} = r(t)\rho'(t).\end{gathered} \tag{3.2}$$

From Theorem 2.2 or Theorem 2.4, it is easy to show that (3.1) is oscillatory.

In fact, $x(t) = t^{1/2}\cos t$ is such an oscillatory solution.

Example 3.2. Consider the following differential equation:

$$(tx'(t))' - x'(t) + tx(t) = 0. \tag{3.3}$$

It is obvious that $\sigma = 1$, $r(t) = t$, $p(t) = -1 < 0$, $q(t) = t > 0$, and $\int_{t_0}^{\infty}(1/r(t))dt = \int_{t_0}^{\infty}(1/t)dt = \infty$.

We are taking $H(t) = C > 0$, $h(t) = 0$. By a simple calculation, it is easy to show that $v(t) = \exp(\int_{t_0}^{t}(p(s)/r(s) - h(s)/H(s))ds) = \exp(\int_{t_0}^{t}(-1/s - 0/C)ds) = t_0/t$, $\limsup_{t\to\infty} v(t)r(t) = \limsup_{t\to\infty} t_0 < \infty$, $\varphi(t) = v(t)(q(t) - p(t)h(t) - (r(t)h(t))') = t_0$, and $\int_{t_0}^{\infty} H(t)\varphi(t)dt = \int_{t_0}^{\infty} t_0 C dt = \infty$.

From Theorem 2.5 or Theorem 2.6, it follows that (3.3) is oscillatory.

In fact, $x(t) = \sin t$ is such an oscillatory solution.

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