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Research Article

(a, k)-Regularized C-Resolvent Families: Regularity and Local Properties

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We introduce the class of (local) (a, k)-regularized C-resolvent families and discuss its basic structural properties. In particular, our analysis covers subjects like regularity, perturbations, duality, spectral properties and subordination principles. We apply our results in the study of the backwards fractional diffusion-wave equation and provide several illustrative examples of differentiable (a, k)-regularized C-resolvent families.

1. Introduction and Preliminaries

In this review, we will report how a large number of known results concerning (a, k)-regularized resolvents [1–6], C-regularized resolvents [7], and (local) convoluted C-semigroups and cosine functions [8, 9] can be formulated in the case of general (a, k)-regularized C-resolvent families.

The paper is organized as follows. In Theorem 2.2, Remark 2.3, and Theorems 2.5, 2.6, and 2.7, we analyze the properties of subgenerators of (a, k)-regularized C-resolvent families and slightly improve results from [1]. With a view to further study the problem describing heat conduction in materials with memory and the Rayleigh problem of viscoelasticity in L^∞ type spaces, we prove in Theorem 2.8 several different forms of subordination principles [10]. The main objective in Theorems 2.9–2.12, 2.26, 2.28, and 2.32 is to continue the researches raised in [3] and [5, 6]. Our main contributions are Theorems 2.16–2.17, 2.20–2.25, 2.27, and 2.30 clarifying the basic regularity properties of (a, k)-regularized C-resolvent families and a fairly general form of the abstract Weierstrass formula.

It is noteworthy that the complete spectral characterization of subgenerators of (a, k)-regularized C-resolvent families exists only in the exponential case and that it is not clear, with exception of various types of local convoluted C-semigroups and cosine functions [9, 11], in what way one can prove a satisfactory Hille-Yosida theorem for local (a, k)-regularized C-resolvent families.

Throughout this paper E denotes a nontrivial complex Banach space, $L(E)$ denotes the space of bounded linear operators from E into E, E^* denotes the dual space of E, and A denotes a closed linear operator acting on E. The range and the resolvent set of $A$are denoted by $Rang(A)$ and $\rho(A)$, respectively; $[D(A)]$ denotes the Banach space $D(A)$ equipped with the graph norm. From now on, we assume that $L(E) \ni C$ is an injective operator which satisfies $CA \subseteq AC$ and employ the convolution like mapping $*$ which is given by $f * g(t) := \int_0^t f(t-s)g(s)ds$. Recall, the C-resolvent set of A, denoted by $\rho_C(A)$, is defined to be the set of all complex numbers λ satisfying that the operator $\lambda - A$ is injective and that $Rang(C) \subseteq Rang(\lambda - A)$. Let us recall that a linear subspace $Y \subseteq D(A)$ is called a core for A if Y is dense in $D(A)$ with respect to the graph norm. Henceforth we identify a closed linear operator A with its graph $G(A)$; given two closed linear operators A and B on E, the inclusion $A \subseteq B$ means $G(A) \subseteq G(B)$. If X is a closed subspace of E, then A_X denotes the part of A in X, that is, $A_X := \{(x, y) \in A : x \in X,\ y \in X\}$.

We mainly use the following conditions.

(H1): A is densely defined.

(H2): $\rho(A) \neq \emptyset$.

(H3): $\rho_C(A) \neq \emptyset$ and $\overline{Rang(C)} = E$.

(H4): A is densely defined or $\rho_C(A) \neq \emptyset$.

(H5): (H1) $\vee$ (H2) $\vee$ (H3).

(P1): $k(t)$ is Laplace transformable, that is, it is locally integrable on $[0, \infty)$ and there exists $\beta \in \mathbb{R}$ so that $\tilde{k}(\lambda) = \mathcal{L}(k)(\lambda) := \lim_{b\to\infty} \int_0^b e^{-\lambda t} k(t)dt := \int_0^\infty e^{-\lambda t} k(t)dt$ exists for all $\lambda \in \mathbb{C}$ with Re $\lambda > \beta$. Put abs(k) :=inf{Re $\lambda : \tilde{k}(\lambda)$ exists}.

Let us remind that a function $k \in L^1_{\mathrm{loc}}([0, \tau))$ is called a kernel, if for every $\phi \in C([0, \tau))$, the supposition $\int_0^t k(t-s)\phi(s)ds = 0,\ t \in [0, \tau)$, implies $\phi \equiv 0$; due to the famous Titchmarsh's theorem [12], the condition $0 \in \operatorname{supp} k$ implies that $k(t)$ is a kernel. Set $\Theta(t) := \int_0^t k(s)ds,\ t \in [0, \tau)$.

2. (a, k)-Regularized C-Resolvent Families

We start with the following definition.

Definition 2.1. Let $0 < \tau \leq \infty, k \in C([0, \tau)), k \neq 0$, and let $a \in L^1_{\mathrm{loc}}([0, \tau)), a \neq 0$. A strongly continuous operator family $(R(t))_{t\in[0,\tau)}$ is called a (local, if $\tau < \infty$) (a, k)-regularized C-resolvent family having A as a subgeneratorif and only if the following holds:

(i) $R(t)A \subseteq AR(t),\ t \in [0, \tau), R(0) = k(0)C$, and $CA \subseteq AC$,

(ii) $R(t)C = CR(t),\ t \in [0, \tau)$,

(iii) $R(t)x = k(t)Cx + \int_0^t a(t-s)AR(s)x\,ds,\ t \in [0, \tau),\ x \in D(A)$.

In the case $\tau = \infty, (R(t))_{t\geq 0}$ is said to be exponentially bounded if, additionally, there exist $M > 0$ and $\omega \geq 0$ such that $\|R(t)\| \leq Me^{\omega t},\ t \geq 0; (R(t))_{t\in[0,\tau)}$ is said to be nondegenerate if the condition $R(t)x = 0,\ t \in [0, \tau)$ implies $x = 0$.

From now on, we consider only nondegenerate (a, k)-regularized C-resolvent families. Notice that $(R(t))_{t\in[0,\tau)}$ is nondegenerate provided that $k(0) \neq 0$ or that (H5) holds for a subgenerator A of $(R(t))_{t\in[0,\tau)}$.

In the case $k(t) = t^\alpha/\Gamma(\alpha+1)$, where $\alpha > 0$, and $\Gamma(\cdot)$ denotes the Gamma function, it is also said that $(R(t))_{t\in[0,\tau)}$ is an α-times integrated (a, C)-resolvent family; in such a way, we unify the notion of (local) α-times integrated C-semigroups $(a(t) \equiv 1)$ and cosine functions $(a(t) \equiv t)$ [1, 13, 14]. Furthermore, in the case $k(t) := \int_0^t K(s)ds, t \in [0,\tau)$, where $K \in L^1_{\mathrm{loc}}([0,\tau))$ and $K \neq 0$, we obtain the unification concept for (local) K-convoluted C-semigroups and cosine functions [15]. In the case $k(t) \equiv 1, (R(t))_{t\in[0,\tau)}$ is said to be a (local) (a, C)-regularized resolvent family with a subgenerator A (cf. also [16] for the definition which does not include the condition (ii) of Definition 2.1).

Designate by $\wp(R)$ the set which consists of all subgenerators of $(R(t))_{t\in[0,\tau)}$.

Then the following holds.

(i) $A \in \wp(R)$ implies $C^{-1}AC \in \wp(R)$.

(ii) If $A \in \wp(R)$ and $\lambda \in \rho_C(A)$, then

$$R(t)(\lambda - A)^{-1}C = (\lambda - A)^{-1}CR(t), \quad t \in [0,\tau). \tag{2.1}$$

(iii) Assume, additionally, that $a(t)$ is a kernel. Then one can define the integral generator $\widehat{A}$ of $(R(t))_{t\in[0,\tau)}$ by setting

$$\widehat{A} := \left\{ (x, y) \in E \times E : R(t)x - k(t)Cx = \int_0^t a(t-s)R(s)y\,ds,\ t \in [0,\tau) \right\}. \tag{2.2}$$

The integral generator $\widehat{A}$ of $(R(t))_{t\in[0,\tau)}$ is a closed linear operator which satisfies $C^{-1}\widehat{A}C = \widehat{A}$ and extends an arbitrary subgenerator of $(R(t))_{t\in[0,\tau)}$. Furthermore, $\widehat{A} \in \wp(R)$, if $R(t)R(s) = R(s)R(t),\ 0 \le t, s < \tau$.

Recall that in the case of convoluted C-semigroups and cosine functions, the set $\wp(R)$ becomes a complete lattice under suitable algebraic operations and that induced partial ordering coincides with the usual set inclusion. In general, $\wp(R)$ needs not to be finite [9].

Henceforth we assume that the scalar-valued kernels $k,\ k_1, k_2, \ldots$ are continuous on $[0,\tau)$, and that $a \neq 0$ in $L^1_{\mathrm{loc}}([0,\tau))$.

Assume temporarily $\lambda \in \rho_C(A), x \in Rang\,(C), t \in [0,\tau)$, and put $z = (a * R)(t)x$.

Following the proof of [1, Lemma 2.2], we have $z = \lambda(a * R)(t)(\lambda - A)^{-1}x - (a * R)(t)A(\lambda - A)^{-1}x = \lambda(a * R)(t)(\lambda - A)^{-1}x - (R(t)(\lambda - A)^{-1}x - k(t)C(\lambda - A)^{-1}x) = \lambda(\lambda - A)^{-1}C(a * R)(t)C^{-1}x - ((\lambda - A)^{-1}R(t)x - k(t)(\lambda - A)^{-1}Cx)$, where the last two equalities follow on account of $CA \subseteq AC, R(s)A \subseteq AR(s)$ and $R(s)(\lambda - A)^{-1}C = (\lambda - A)^{-1}CR(s),\ s \in [0,\tau)$. Hence, $(\lambda - A)z = \lambda z - (R(t)x - Cx)$,

$$\int_0^t a(t-s)R(s)x\,ds \in D(A), \qquad A\int_0^t a(t-s)R(s)x\,ds = R(t)x - k(t)Cx. \tag{2.3}$$

The closedness of A implies that (2.3) holds for every $t \in [0,\tau)$ and $x \in \overline{Rang\,(C)}$.

Theorem 2.2 (see [1]). (i) *Let A be a subgenerator of an (a, k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$, and let (H5) hold. Then* (2.3) *holds for every $t \in [0,\tau)$ and $x \in E$. If $\rho_C(A) \neq \emptyset$, then* (2.3) *holds for every $t \in [0,\tau)$ and $x \in \overline{Rang\,(C)}$.*

(ii) *Let A be a subgenerator of an (a, k_i)-regularized C-resolvent family $(R_i(t))_{t\in[0,\tau)}, i = 1, 2$. Then $(k_2 * R_1)(t) = (k_1 * R_2)(t),\ t \in [0, \tau)$, whenever (H4) holds.*

(iii) *Let $(R_1(t))_{t\in[0,\tau)}$ and $(R_2(t))_{t\in[0,\tau)}$ be two (a, k)-regularized C-resolvent families having A as a subgenerator. Then $R_1(t)x = R_2(t)x,\ t \in [0, \tau),\ x \in \overline{D(A)}$, and $R_1(t) = R_2(t),\ t \in [0, \tau)$, if (H4) holds.*

(iv) *Let A be a subgenerator of an (a, k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$. If $k(t)$ is absolutely continuous and $k(0) \neq 0$, then A is a subgenerator of an (a, C)-regularized resolvent family on $[0, \tau)$.*

Remark 2.3. (i) Let $(R_i(t))_{t\in[0,\tau)}$ be an (a, k_i)-regularized C-resolvent family with a subgenerator $A,\ i = 1, 2$, and let $D(A) \neq \{0\}$. Then $k_1 = k_2$.

(ii) Let $(R_i(t))_{t\in[0,\tau)}$ be an (a, k_i)-regularized C-resolvent family with a subgenerator $A, i = 1, 2$. Then, for every $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}, (\alpha R_1(t) + \beta R_2(t))_{t\in[0,\tau)}$ is an $(a, \alpha k_1 + \beta k_2)$-regularized C-resolvent family with a subgenerator A.

(iii) Let $(R(t))_{t\in[0,\tau)}$ be an (a, k)-regularized C-resolvent family with a subgenerator A, and let $L^1_{\mathrm{loc}}([0, \tau)) \ni b$ be a kernel. Then A is a subgenerator of an $(a, k * b)$-regularized C-resolvent family $((b * R)(t))_{t\in[0,\tau)}$.

(iv) Let $(R(t))_{t\in[0,\tau)}$ be an (a, C)-regularized resolvent family having A as a subgenerator. Then $((k * R)(t))_{t\in[0,\tau)}$ is an (a, Θ)-regularized C-resolvent family with a subgenerator A.

(v) Suppose $(R(t))_{t\in[0,\tau)}$ is an (a, k)-regularized C-resolvent family with a subgenerator A, (H1) or (H3) holds, and $a(t)$ is a kernel. Then the integral generator $\widehat{A}$ of $(R(t))_{t\in[0,\tau)}$ satisfies $\widehat{A} = C^{-1}AC$. Toward this end, let $(x, y) \in \widehat{A}$. Then $\int_0^t a(t - s)[k(s)Cx + \int_0^s a(s - r)R(r)y\,dr]ds = \int_0^t a(t - s)R(s)x\,ds \in D(A),\ t \in [0, \tau)$, and $A\int_0^t a(t - s)[k(s)Cx + \int_0^s a(s - r)R(r)y\,dr]ds = A\int_0^t a(t - s)R(s)x\,ds = R(t)x - k(t)Cx = \int_0^t a(t - s)R(s)y\,ds, t \in [0, \tau)$. Since $(a * R)(t)y \in D(A),\ (a * a * R)(t)y \in D(A), A(a * a * R)(t)y = (a * (R - kC))(t)y, t \in [0, \tau)$, and $a * k \neq 0$ in $C([0, \tau))$, it follows that $Cx \in D(A), ACx = Cy, x \in D(C^{-1}AC)$, and $C^{-1}ACx = \widehat{A}x = y$. On the other hand, $C^{-1}AC$ is a subgenerator of $(R(t))_{t\in[0,\tau)}$ whenever A is; this implies $C^{-1}AC \subseteq \widehat{A}$ and proves the claim. If (H2) holds, then $\widehat{A} = C^{-1}AC = A$. In what follows, we also assume that $B \in \wp(R)$ and that (H5) holds for B and C. Proceeding as in the proof of [9, Proposition 2.1.1.6], one gets what follows.

(v.1) $C^{-1}AC = C^{-1}BC$ and $C(D(A)) \subseteq D(B)$.

(v.2) A and B have the same eigenvalues.

(v.3) The assumption $A \subseteq B$ implies $\rho_C(A) \subseteq \rho_C(B)$.

(v.4) card $(\wp(R)) = 1$, if $C(D(\widehat{A}))$ is a core for $D(\widehat{A})$.

(v.5) $A \subseteq B \Leftrightarrow D(A) \subseteq D(B)$ and $Ax = Bx,\ x \in D(A) \cap D(B)$; furthermore, the property (v.5) holds whenever $\{A, B\} \subseteq \wp(R)$ and $a(t)$ is a kernel.

We refer the reader to [1, page 283] for the definition of (weak) solutions of the problem

$$u(t) = f(t) + \int_0^t a(t - s)Au(s)ds, \quad t \in [0, \tau), \tag{2.4}$$

where $f \in C([0,\tau) : E)$, and to [1, page 285] for the notion of spaces $C^{n,k}([0,\tau) : E)$, $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ and $C_0^n([0,\tau) : E)$, and $n \in \mathbb{N}$.

Define a subset A^* of $E^* \times E^*$ (the use of symbol $*$ is clear from the context) by $A^* := \{(x^*, y^*) \in E^* \times E^* : x^*(Ax) = y^*(x)$ for all $x \in D(A)\}$. In the case when A is densely defined, A^* is a linear mapping from E^* into E^*.

Lemma 2.4 (see [17]). *Let A be a closed linear operator. Assume $x_0 \in E, y_0 \in E$, and $x^*(y_0) = y^*(x_0)$ for all $(x^*, y^*) \in A^*$. Then $x_0 \in D(A)$, and $Ax_0 = y_0$.*

Define the mapping $K_C : C([0,\tau) : E) \to C([0,\tau) : E)$ by $K_C u := k * Cu$, $u \in C([0,\tau) : E)$. Then K_C is linear, bounded, and injective.

Keeping in mind Lemma 2.4 and the proofs of [1, Theorem 2.7, Corollary 2.9, Remark 2.10, Corollary 2.11, and Corollary 2.13], we have the following.

Theorem 2.5. (i) *Suppose $f \in C([0,\tau) : E)$, A is a subgenerator of a (local) (a,k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$, and (H5) holds. Then* (2.4) *has a unique solution if and only if $R * f \in Rang\,(K_C)$.*

(ii) *(cf. also [18]) Assume $n \in \mathbb{N}, f \in C([0,\tau) : E)$, A is a subgenerator of a (local) n-times integrated (a,C)-resolvent family $(R(t))_{t\in[0,\tau)}$, and (H5) holds. Then* (2.4) *has a unique solution if and only if $C^{-1}(R * f) \in C_0^{n+1}([0,\tau) : E)$.*

(iii) *Let the assumptions of the item (i) of this theorem hold, and let $k \in C_0^n([0,\tau) : E)$. Then $C^{-1}(R * f) \in C^{(n+1)}([0,\tau) : E)$ if and only if $C^{-1}(R * f) \in C_0^{n+1}([0,\tau) : E)$.*

(iv) *Let (H5) hold. Assume that $n \in \mathbb{N}$, A is a subgenerator of an n-times integrated (a,C)-regularized resolvent, and $a \in BV_{\mathrm{loc}}([0,\tau) : E)$, respectively A is a subgenerator of an (a,C)-regularized resolvent family. Assume, further, that $C^{-1}f \in C^{(n+1)}([0,\tau) : E)$, $f^{(k-1)}(0) \in D(A^{n+1-k})$ and $A^{n+1-k}f^{(k-1)}(0) \in Rang\,(C)$, $1 \le k \le n+1$, respectively $C^{-1}f \in AC_{\mathrm{loc}}([0,\tau) : E)$. Then* (2.4) *has a unique solution.*

(v) *Assume that (H5) holds, A is a subgenerator of an (a,k)-regularized C-resolvent family, $k(t)$ is absolutely continuous, and $k(0) \neq 0$. If $C^{-1}f \in C^1([0,\tau) : E)$, then there exists a unique solution of* (2.4).

The proof of following theorem follows from a standard application of Laplace transform techniques.

Theorem 2.6. *Let $k(t)$ and $a(t)$ satisfy (P1), and let $(R(t))_{t\ge 0}$ be a strongly continuous operator family satisfying $\|R(t)\| \le Me^{\omega t}$, $t \ge 0$, for some $M > 0$ and $\omega \ge 0$. Put $\omega_0 := \max(\omega, abs(a), abs(k))$.*

(i) *Suppose A is a subgenerator of the exponentially bounded (a,k)-regularized C-resolvent family $(R(t))_{t\ge 0}$, and (H5) holds. Then, for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega_0$ and $\tilde{k}(\lambda) \neq 0$, the operator $I - \tilde{a}(\lambda)A$ is injective, $Rang(C) \subseteq Rang\,(I - \tilde{a}(\lambda)A)$,*

$$\tilde{k}(\lambda)(I - \tilde{a}(\lambda)A)^{-1}Cx = \int_0^\infty e^{-\lambda t}R(t)x\,dt, \quad x \in E,\ Re\,\lambda > \omega_0,\ \tilde{k}(\lambda) \neq 0, \tag{2.5}$$

$$\left\{\frac{1}{\tilde{a}(\lambda)} : \lambda \in \mathbb{C},\ \operatorname{Re}\lambda > \omega_0,\ \tilde{k}(\lambda)\tilde{a}(\lambda) \neq 0\right\} \subseteq \rho_C(A) \tag{2.6}$$

and $R(s)R(t) = R(t)R(s)$, $t, s \ge 0$.

(ii) *Assume that* (2.5)-(2.6) *hold. Then A is a subgenerator of the exponentially bounded (a,k)-regularized C-resolvent family $(R(t))_{t\geq 0}$.*

The preceding theorem enables one to establish the real and complex characterization of subgenerators of (locally Lipschitz continuous) exponentially bounded (a,k)-regularized C-resolvent families [1, 9, 12].

Theorem 2.7. (i) *Let $k(t)$ and $a(t)$ satisfy (P1), and let $\omega_0 \geq \max(0, abs(a), abs(k))$. Assume that, for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega_0$ and $\tilde{k}(\lambda) \neq 0$, the operator $I - \tilde{a}(\lambda)A$ is injective and that $Rang(C) \subseteq Rang\,(I - \tilde{a}(\lambda)A)$. If there exists an analytic function $\Upsilon : \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega_0\} \to L(E)$ with:*

(i.1) $\Upsilon(\lambda) = \tilde{k}(\lambda)(I - \tilde{a}(\lambda)A)^{-1}C$, $\lambda \in \mathbb{C}, \operatorname{Re}\lambda > \omega_0$,

(i.2) $\|\Upsilon(\lambda)\| \leq M|\lambda|^r$, $\lambda \in \mathbb{C}, \operatorname{Re}\lambda > \omega_0$, *for some $M > 0$ and $r \geq -1$, then, for every $\alpha > 1$, A is a subgenerator of a norm continuous, exponentially bounded $(a, k * t^{\alpha+r-1}/\Gamma(\alpha+r))$-regularized C-resolvent family.*

(ii) *Suppose $k(t)$ and $a(t)$ satisfy (P1) and (H2) or (H3) holds, and A is a subgenerator of an exponentially bounded (a,Θ)-regularized C-resolvent family $(R(t))_{t\geq 0}$ which satisfies the next condition:*

$$\|R(t+h) - R(t)\| \leq Mhe^{\omega(t+h)}, \quad t \geq 0,\ h \geq 0,\ \textit{for some } M > 0,\ \omega \geq 0. \tag{2.7}$$

Then there exists $a \geq \omega_0$ such that

$$\left\{\frac{1}{\tilde{a}(\lambda)} : \lambda > a, \tilde{k}(\lambda)\tilde{a}(\lambda) \neq 0\right\} \subseteq \rho_C(A), \tag{2.8}$$

$$\textit{the mapping } \lambda \mapsto H(\lambda) := \tilde{k}(\lambda)(I - \tilde{a}(\lambda)A)^{-1}C, \quad \lambda > a,\ \tilde{k}(\lambda)\tilde{a}(\lambda) \neq 0 \tag{2.9}$$

is infinitely differentiable and

$$\left\|\frac{d^k}{d\lambda^k}H(\lambda)\right\| \leq \frac{Mk!}{(\lambda-\omega)^{k+1}}, \quad k \in \mathbb{N}_0,\ \lambda > a,\ \tilde{k}(\lambda)\tilde{a}(\lambda) \neq 0. \tag{2.10}$$

(iii) *Suppose $k(t)$ and $a(t)$ satisfy (P1) and* (2.8)–(2.10) *holds. Then A is a subgenerator of an exponentially bounded (a,Θ)-regularized C-resolvent family $(R(t))_{t\geq 0}$ which satisfies* (2.7).

(iv) *Suppose $M > 0$, $\omega \geq 0$, $k(t)$ and $a(t)$ satisfy (P1), and A is densely defined. Then A is a subgenerator of an exponentially bounded (a,k)-regularized C-resolvent family $(R(t))_{t\geq 0}$ which satisfies $\|R(t)\| \leq Me^{\omega t}$, $t \geq 0$ if and only if there exists $a \geq \max(0, abs(a), abs(k))$ such that* (2.8)–(2.10) *hold.*

Denote by a^{*n} the nth convolution power of the kernel $a(t), n \in \mathbb{N}$, and see [10] for the definition of completely positive functions and the notion used in the subsequent theorem and examples. An insignificant technical modification of the proofs of [1, Theorem 3.7] and [10, Theorems 4.1, 4.3, 4.5] (cf. also [7, Lemma 4.2]) implies the next subordination principles.

Theorem 2.8. (i) *Let $a(t)$, $b(t)$, and $c(t)$ satisfy (P1), and let $\int_0^\infty e^{-\beta t}|b(t)|dt < \infty$ for some $\beta \geq 0$. Let*

$$\alpha = \tilde{c}^{-1}\left(\frac{1}{\beta}\right) \quad \text{if} \int_0^\infty c(t)dt > \frac{1}{\beta}, \ \alpha = 0 \ \text{otherwise}, \tag{2.11}$$

and let $\tilde{a}(\lambda) = \tilde{b}(1/\tilde{c}(\lambda)), \lambda \geq \alpha$. Let A be a subgenerator of a (b,k)-regularized C-resolvent family $(R_b(t))_{t\geq 0}$ satisfying $\|R_b(t)\| \leq Me^{\omega_b t}$, $t \geq 0$, for some $M > 0$ and $\omega_b \geq 0$, and let (H2) or (H3) hold. Assume, further, that $c(t)$ is completely positive and that there exists a function $k_1(t)$ satisfying (P1) and

$$\tilde{k}_1(\lambda) = \frac{1}{\lambda\tilde{c}(\lambda)}\tilde{k}\left(\frac{1}{\lambda\tilde{c}(\lambda)}\right), \quad \lambda > \omega_0, \ \tilde{k}\left(\frac{1}{\lambda\tilde{c}(\lambda)}\right) \neq 0, \ \text{for some } \omega_0 > 0. \tag{2.12}$$

Let

$$\omega_a = \tilde{c}^{-1}\left(\frac{1}{\omega_b}\right) \quad \text{if} \int_0^\infty c(t)dt > \frac{1}{\omega_b}, \ \omega_a = 0 \ \text{otherwise}. \tag{2.13}$$

Then A is a subgenerator of an exponentially bounded, locally Lipschitz continuous $(a, 1 * k_1)$-regularized C-resolvent $(R_a(t))_{t\geq 0}$, and there exists $M_a \geq 1$ such that

$$\|S_a(t)\| \leq M_a e^{\omega_a t}, \quad t \geq 0, \ \text{if} \ \ \omega_b = 0 \ \text{or} \ \omega_b\tilde{c}(0) \neq 1, \tag{2.14}$$

respectively, for every $\varepsilon > 0$, there exists $M_\varepsilon \geq 1$ such that

$$\|S_a(t)\| \leq M_\varepsilon e^{\varepsilon t}, \quad t \geq 0, \ \text{if} \ \ \omega_b > 0, \quad \omega_b\tilde{c}(0) = 1. \tag{2.15}$$

Furthermore, if A is densely defined, then A is a subgenerator of an exponentially bounded (a, k_1)-regularized C-resolvent $(R_a(t))_{t\geq 0}$ which fulfills (2.14)*, respectively,* (2.15).

(ii) *Suppose $\alpha \geq 0$, A is a subgenerator of an exponentially bounded α-times integrated C-semigroup, $a(t)$ is completely positive and satisfies (P1), and $k(t)$ satisfies (P1) and $\tilde{k}(\lambda) = \tilde{a}(\lambda)^\alpha$, λ sufficiently large. Then A is a subgenerator of a locally Lipschitz continuous, exponentially bounded $(a, t * k)$-regularized C-resolvent family ($(a, t * a^{*n})$-regularized C-resolvent family if $\alpha = n \in \mathbb{N}$, respectively, (a, t)-regularized C-resolvent family if $\alpha = 0$). If, additionally, A is densely defined, then A is a subgenerator of an exponentially bounded $(a, 1*k)$-regularized C-resolvent family ($(a, 1*a^{*n})$-regularized C-resolvent family if $\alpha = n \in \mathbb{N}$, respectively, (a, C)-regularized resolvent family if $\alpha = 0$).*

(iii) *Suppose $\alpha \geq 0$ and A is a subgenerator of an exponentially bounded α-times integrated C-cosine function. Let $L^1_{\text{loc}}([0,\infty)) \ni c$ be completely positive, and let $a(t) = (c * c)(t), t \geq 0$. (Given $L^1_{\text{loc}}([0,\infty)) \ni a$ in advance, such a function $c(t)$ always exists provided $a(t)$ is completely positive or $a(t) \neq 0$ is a creep function and $a_1(t)$ is log-convex.) Assume that $k(t)$ satisfies (P1), and $\tilde{k}(\lambda) = \tilde{c}(\lambda)^\alpha/\lambda, \lambda$ sufficiently large. Then A is a subgenerator of a locally Lipschitz continuous, exponentially bounded $(a, t * k)$-regularized C-resolvent family ($(a, t * c^{*n})$-regularized C-resolvent family if $\alpha = n \in \mathbb{N}$, resp. (a, t)-regularized C-resolvent family if $\alpha = 0$). If, additionally, A is densely defined, then A is a subgenerator of an exponentially bounded $(a, 1*k)$-regularized C-resolvent family ($(a, 1 * c^{*n})$-regularized C-resolvent family if $\alpha = n \in \mathbb{N}$, resp. (a, C)-regularized resolvent family if $\alpha = 0$).*

Denote by A_p the realization of the Laplacian with Dirichlet or Neumann boundary conditions on $L^p([0,\pi]^n), 1 \leq p < \infty$. By [19, Theorem 4.2], A_p generates an exponentially bounded α-times integrated cosine function for every $\alpha \geq (n-1)|(1/2)-(1/p)|$. Assume further that $c \in BV_{\text{loc}}([0,\infty))$ and that $m(t)$ is a bounded creep function with $m_0 = m(0+) > 0$. Thanks to [10, Proposition 4.4, page 94], we have that there exists a completely positive function $b(t)$ such that $dm * b = 1$. After the usual procedure, the problem [10, (5.34)] describing heat conduction in materials with memory is equivalent to

$$u(t) = \left(a * A_p\right)(t) + f(t), \quad t \geq 0, \tag{2.16}$$

where $a(t) = (b * dc)(t),\ t \geq 0$, and $f(t)$ contains $r * b$ as well as the temperature history. In what follows, we assume that

(i) $p \neq 2$,

(ii) $\Gamma_b = \emptyset$ or $\Gamma_f = \emptyset$,

(iii) there exists a completely positive function $c_1(t)$ such that $a(t) = (c_1 * c_1)(t),\ t \geq 0$.

We refer the reader to [10, pages 140-141] for the analysis of the problem (2.16) in the case: $p = 2$ and $m,\ c \in \mathcal{BF}$. Applying Theorem 2.8(iii), one gets that A_p is the integral generator of an exponentially bounded $(a, 1 * \mathcal{L}^{-1}(\widetilde{c}_1(\lambda)^{(n-1)|(1/2)-(1/p)|}/\lambda)(t))$-regularized resolvent family, where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform. Notice also that [10, Lemma 4.3, page 105] implies that, for every $\beta \in [0,1]$, the function $\lambda \mapsto (\widetilde{c}_1(\lambda)^{\beta}/\lambda)$ is the Laplace transform of a Bernstein function and that the function $k(t)$ appearing in the formulations of Theorem 2.8(ii)-(iii) always exists. On the other hand, an application of [9, Proposition 2.1.3.12] gives that there exists $\omega > 0$ such that A_p is the integral generator of an exponentially bounded $(\omega - A_p)^{-\lceil (1/2)(n-1)|(1/2)-(1/p)|\rceil}$-regularized cosine function; herein $\lceil s \rceil = \inf\{k \in \mathbb{Z} : s \leq k\}, s \in \mathbb{R}$. Using Theorem 2.8(iii) again, we have that A_p is the integral generator of an exponentially bounded $(a, (\omega - A_p)^{-\lceil (1/2)(n-1)|(1/2)-(1/p)|\rceil})$-regularized resolvent family, and Theorem 2.5(iv) can be applied. In both approaches, regrettably, we must restrict ourselves to the study of pure Dirichlet or Neumann problem. It is also worthwhile to note that Theorem 2.8(iii) can be applied in the analysis of the Rayleigh problem of viscoelasticity in L^∞ type spaces; as a matter of fact, the operator A defined on [10, page 136] generates an exponentially bounded α-times integrated cosine function in $L^\infty((0,\infty))$ for all $\alpha > 0$.

Approximation type theorem for exponentially bounded (a,k)-regularized C-resolvent families follows from Theorem 2.6 and [12, Theorem 1.7.5, page 42], and the representation formulae for exponentially bounded (a,k)-regularized C-resolvent families are consequences of the Post-Widder inversion (the Phragamén-Doetsch inversion). For further information, see [2, 12, 20, 21].

Using the argumentation given in [3, 5], one can prove the following assertions.

Theorem 2.9. (i) *Suppose that the next conditions hold.*

(i.1) *The mapping* $t \mapsto |k(t)|,\ t \in [0,\tau)$, *is nondecreasing.*

(i.2) *There exist* $\varepsilon_{a,k} > 0$ *and* $t_{a,k} \in [0,\tau)$ *such that*

$$\left|\int_0^t a(t-s)k(s)ds\right| \geq \varepsilon_{a,k}\int_0^t |a(t-s)k(s)|ds, \quad t \in [0, t_{a,k}). \tag{2.17}$$

(i.3) *A is a subgenerator of an* (a,k)*-regularized C-resolvent family*$(R(t))_{t\in[0,\tau)}$, *and (H5) holds.*

(i.4) $\limsup_{t\to 0+}\|R(t)\|/|k(t)| < \infty$.

Then

$$D\left(A_{\overline{D(A)}}\right) = \left\{x \in \overline{D(A)} : \lim_{t\to 0+}\frac{R(t)x - k(t)Cx}{(a*k)(t)} \text{ exists}\right\}, \tag{2.18}$$

$$Ax = \lim_{t\to 0+}\frac{R(t)x - k(t)Cx}{(a*k)(t)}, \quad x \in D\left(A_{\overline{D(A)}}\right). \tag{2.19}$$

(ii) *Suppose A is a subgenerator of an* (a,k)*-regularized C-resolvent family* $(R(t))_{t\in[0,\tau)}$ *satisfying* $\|R(t)\| = O(k(t))$, $t \to 0+, \min(a(t),k(t)) > 0$, $t \in (0,\tau)$, *and (H5) holds. Then* (2.18)-(2.19) *hold.*

Theorem 2.10. (i) *Suppose A is a subgenerator of an* (a,k)*-regularized C-resolvent family* $(R(t))_{t\in[0,\tau)}$ *satisfying* $\|R(t)\| = O(k(t))$, $t \to 0+$ *and* $\min(a(t),k(t)) > 0$, $t \in (0,\tau)$. *Then* $\lim_{t\to 0+}(a*R)(t)x/(a*k)(t) = Cx, x \in \overline{D(A)}$.

(ii) *Suppose A is a subgenerator of an* (a,k)*-regularized C-resolvent family* $(R(t))_{t\in[0,\tau)}$ *satisfying* $\|R(t)\| = O(k(t))$, $t \to 0+, \min(a(t),k(t)) > 0$, $t \in (0,\tau)$ *and (H5) holds. If* $x \in \overline{D(A)}, y \in E$ *and* $\lim_{t\to 0+}(R(t)x - k(t)Cx)/(a*k)(t) = y$, *then* $x \in D(A)$ *and* $y = Ax$.

(iii) *Suppose E is reflexive, A is a subgenerator of an* (a,k)*-regularized C-resolvent family* $(R(t))_{t\in[0,\tau)}$ *satisfying* $\|R(t)\| = O(k(t))$, $t \to 0+$, $R(s)R(t) = R(t)R(s)$, $0 \leq t, s < \tau, \min(a(t),k(t)) > 0$, $t \in (0,\tau)$, *and (H5) holds. If* $x \in \overline{D(A)}$ *and* $\lim_{t\to 0+}\|(R(t)x - k(t)Cx)/(a*k)(t)\| < \infty$, *then* $x \in D(A)$.

Theorem 2.11 (cf. also [22]). *Suppose* $\alpha > 0$ *and A is a subgenerator of an* α*-times integrated C-semigroup* $(S_\alpha(t))_{t\in[0,\tau)}$, *respectively, an* α*-times integrated C-cosine function* $(C_\alpha(t))_{t\in[0,\tau)}$, *which satisfies* $\limsup_{t\to 0+}\|S_\alpha(t)\|/t^\alpha < \infty$, *respectively,* $\limsup_{t\to 0+}\|C_\alpha(t)\|/t^\alpha < \infty$. *Then, for every* $x \in D(A)$ *such that* $Ax \in \overline{D(A)}$:

$$Ax = \lim_{t\to 0+}\frac{\Gamma(\alpha+2)}{\Gamma(\alpha+1)}\frac{\Gamma(\alpha+1)S_\alpha(t)x - t^\alpha Cx}{t^{\alpha+1}} \ (\textit{resp.}), \tag{2.20}$$

$$Ax = \lim_{t\to 0+}\frac{\Gamma(\alpha+3)}{\Gamma(\alpha+1)}\frac{\Gamma(\alpha+1)C_\alpha(t)x - t^\alpha Cx}{t^{\alpha+2}}. \tag{2.21}$$

Theorem 2.12. *Suppose* $M > 0, \omega \geq 0$, *A is a densely defined subgenerator of an* (a,k)*-regularized C-resolvent family* $(R(t))_{t\geq 0}$ *which satisfies* $\|R(t)\| \leq Me^{\omega t}, t \geq 0; B \in L(E)$, $Rang(B) \subseteq Rang(C)$, *and* $BCx = CBx$, $x \in D(A)$. *Suppose, further, that there exist a function* $b(t)$ *satisfying (P1) and a number* $\omega_0 \geq \omega$ *such that* $\tilde{b}(\lambda) = \tilde{a}(\lambda)/\tilde{k}(\lambda)$, $\lambda > \omega_0$, $\tilde{k}(\lambda) \neq 0$. *Then the operator* $A + B$ *is a* *subgenerator of an*(a,k)*-regularized* C*-resolvent family* $(R_B(t))_{t\geq 0}$ *which satisfies* $\|R_B(t)\| \leq M/1-\gamma e^{\mu t},\ t\geq 0,$

$$R_B(t) = R(t) + \int_0^t R_B(t-r)C^{-1}B\int_0^r b(r-s)R(s)x\,ds\,dr, \quad t\geq 0,\ x\in D(A). \tag{2.22}$$

Remark 2.13. In order to prove Theorem 2.9(i) and (2.20)-(2.21) in the case of nondensely subgenerators, it is enough to notice that [3, (2.1), page 219] holds for every $z\in\overline{D(A)}$ and that [3, (2.2), page 219] holds for every $x\in D(A)$ such that $Ax\in\overline{D(A)}$. On the other hand, if (2.20), resp. (2.21), holds for some $x\in D(A)$, then it is obvious that $Ax\in\overline{D(A)}$. This implies that the representation formulae (2.20) and (2.21) are best possible in some sense.

Given $\alpha\in(0,\pi]$, set $\Sigma_\alpha := \{\lambda\in\mathbb{C} : \lambda\neq 0,\ |\arg\lambda|<\alpha\}$.

Definition 2.14 (cf. also [23, Definition 5.1]). Let $0<\alpha\leq\pi/2$, and let $(R(t))_{t\geq 0}$ be an (a,k)-regularized C-resolvent family. Then it is said that $(R(t))_{t\geq 0}$ is an analytic (a,k)-regularized C-resolvent family of angle α, if there exists an analytic function $\mathbf{R} : \Sigma_\alpha \to L(E)$ which satisfies

(i) $\mathbf{R}(t) = R(t),\ t>0,$

(ii) $\lim_{z\to 0, z\in\Sigma_\gamma}\mathbf{R}(z)x = k(0)Cx$ for all $\gamma\in(0,\alpha)$ and $x\in E$.

It is said that $(R(t))_{t\geq 0}$ is an exponentially bounded, analytic (a,k)-regularized C-resolvent family, respectively, bounded analytic (a,k)-regularized C-resolvent family, of angle α, if for every $\gamma\in(0,\alpha)$, there exist $M_\gamma>0$ and $\omega_\gamma\geq 0$, resp. $\omega_\gamma=0$, such that $\|\mathbf{R}(z)\|\leq M_\gamma e^{\omega_\gamma \operatorname{Re} z},\ z\in\Sigma_\gamma$.

Since no confusion seems likely, we also write $R(\cdot)$ for $\mathbf{R}(\cdot)$.The next proposition can be proved by means of the arguments given in [7, Section 3] and [10, Chapter 2].

Proposition 2.15. *Suppose* $k(t)$ *and* $a(t)$*satisfy (P1),* $\lim_{\lambda\to+\infty}\lambda\tilde{k}(\lambda) = k(0)\neq 0, A$ *is densely bounded,* $A\notin L(E)$*, and there exists* $\omega_0\geq\max(0, abs(k), abs(a))$ *such that* $\int_0^\infty e^{-\omega t}|a(t)|dt<\infty$*. Assume that* A *is a subgenerator of an exponentially bounded, analytic* (a,k)*-regularized* C*-resolvent family* $(R(t))_{t\geq 0}$ *of angle* $\alpha\in(0,\pi/2]$ *and that there exists* $\omega\geq\omega_0$ *such that*

$$\sup_{z\in\Sigma_\gamma}\left\|e^{-\omega z}R(z)\right\|<\infty, \quad \forall\gamma\in(0,\alpha). \tag{2.23}$$

Then the function $\tilde{a}(\lambda)$ *can be extended to a meromorphic function defined on the sector* $\omega+\Sigma_{\pi/2+\alpha}$.

It is worthwhile to mention that it is not clear, all assumptions of Proposition 2.15 being satisfied, whether A must be a subgenerator of an (a,C)-regularized resolvent family on $[0,\tau)$ (cf. Theorem 2.2(iv)). Further on, let us notice that the assertions (i) and (ii) of [10, Theorem 2.2, page 57] still hold in the case of exponentially bounded, analytic (a,C)-regularized resolvent families.

The subsequent theorem clarifies the basic analytical properties of (a,k)-regularized C-resolvent families. Notice only that the assertion which naturally corresponds to [7, Lemma 3.7] (cf. also [10, Corollary 2.2, page 53]) does not seem attainable in the case of a general (a,k)-regularized C-resolvent family.

Theorem 2.16. *Suppose $\alpha \in (0,\pi/2], k(t)$ and $a(t)$ satisfy (P1), (H5) holds, and $\widetilde{k}(\lambda)$can be analytically continued to a function $g : \omega + \Sigma_{\pi/2+\alpha} \to \mathbb{C}$,where $\omega \geq \max(0, abs(k), abs(a))$. Suppose, further, that A is a subgenerator of an analytic (a,k)-regularized C-resolvent family $(R(t))_{t\geq 0}$ of angle α and that (2.23) holds. Set*

$$N := \{\lambda \in \omega + \Sigma_{\pi/2+\alpha} : g(\lambda) \neq 0\}. \tag{2.24}$$

Then N is an open connected subset of $\mathbb{C}$. Assume that there exists an analytic function $\widehat{a} : N \to \mathbb{C}$ such that $\widehat{a}(\lambda) = \widetilde{a}(\lambda)$, $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda > \omega$. Then the operator$I - \widehat{a}(\lambda)A$is injective for every $\lambda \in N$, $Rang(C) \subseteq Rang(I - \widehat{a}(\lambda)C^{-1}AC)$ for every $\lambda \in N_1 := \{\lambda \in N : \widehat{a}(\lambda) \neq 0\}$,

$$\sup_{\lambda\in N_1\cap(\omega+\Sigma_{\pi/2+\gamma_1})} \left\| (\lambda-\omega)g(\lambda)\left(I-\widehat{a}(\lambda)C^{-1}AC\right)^{-1}C \right\| < \infty, \quad \gamma_1 \in (0,\alpha), \tag{2.25}$$

$$\text{the mapping } \lambda \mapsto \left(I-\widehat{a}(\lambda)C^{-1}AC\right)^{-1}C, \quad \lambda \in N_1 \text{ is analytic}, \tag{2.26}$$

$$\lim_{\lambda\to+\infty,\ \widetilde{k}(\lambda)\neq 0} \lambda\widetilde{k}(\lambda)(I-\widetilde{a}(\lambda)A)^{-1}Cx = k(0)Cx, \quad x \in E. \tag{2.27}$$

Proof. By Theorem 2.6(i), it follows that, for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$, the operator $I - \widetilde{a}(\lambda)A$ is injective and that $Rang(C) \subseteq Rang(I - \widetilde{a}(\lambda)A)$. Since (2.23) holds, one yields that the function $q : \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega\} \to L(E)$ given by $q(\lambda) = \int_0^\infty e^{-\lambda t}R(t)dt$, $\lambda \in \mathbb{C}, \operatorname{Re}\lambda > \omega$ has an analytic extension $\widetilde{q}(\lambda) : \omega + \Sigma_{\pi/2+\alpha} \to L(E)$ such that $\sup_{\lambda\in\omega+\Sigma_{\pi/2+\gamma}} \|(\lambda-\omega)\widetilde{q}(\lambda)\| < \infty$ for all $\gamma \in (0,\alpha)$ [12]. The set N is open and connected ([9], Subsection 2.1.4), and clearly, the mapping $F(\lambda) := \widetilde{q}(\lambda)/g(\lambda), \lambda \in N$, is analytic. Denote by V the set which consists of all complex numbers $\lambda \in N$ such that $I - \widehat{a}(\lambda)A$ is injective, $Rang(C) \subseteq Rang(I - \widehat{a}(\lambda)A)$, and $F(\lambda) = (I - \widehat{a}(\lambda)A)^{-1}C$. Let $\rho_C(A) \ni \mu$ satisfy $\widehat{a}(\mu) \neq 0$. Then

$$F(\lambda)(I-\widehat{a}(\lambda)A)x = Cx, \quad \lambda \in V,\ x \in D(A), \tag{2.28}$$

$$F(\lambda)Cy = CF(\lambda)y, \quad \lambda \in V,\ y \in E, \tag{2.29}$$

$$\begin{aligned} F(\lambda)Cy &= \frac{1}{\widehat{a}(\lambda)}\left(\frac{1}{\widehat{a}(\lambda)}-A\right)^{-1}C^2y \\ &= \frac{1}{\widehat{a}(\lambda)}\left(\frac{1}{\widehat{a}(\mu)}-A\right)^{-1}C^2y - \left(\frac{1}{\widehat{a}(\lambda)}-\frac{1}{\widehat{a}(\mu)}\right)\left(\frac{1}{\widehat{a}(\mu)}-A\right)^{-1}CF(\lambda)y, \\ &\quad \lambda \in V,\ \widehat{a}(\lambda) \neq 0, \end{aligned} \tag{2.30}$$

and the uniqueness theorem for analytic functions implies that (2.28)-(2.29) hold for every $\lambda \in N$ and that (2.30) holds for every $\lambda \in N$ such that $\widehat{a}(\lambda) \neq 0$. Let $(I - \widehat{a}(\lambda)A)x = 0$ for some $\lambda \in N$ and $x \in D(A)$. Thanks to (2.28), $Cx = 0, x = 0$, and $I - \widehat{a}(\lambda)A$ is injective. Assume, for the time being, $\lambda \in N$ and $\widehat{a}(\lambda) \neq 0$. Then (2.29)-(2.30) hold, and one gets $(I-\widehat{a}(\lambda)A)CF(\lambda)y = (I - \widehat{a}(\lambda)A)F(\lambda)Cy = C^2y - (1/\widehat{a}(\lambda) - 1/\widehat{a}(\mu))(1/\widehat{a}(\mu) - A)^{-1}CF(\lambda)y + \widehat{a}(\lambda)(1/\widehat{a}(\lambda) - 1/\widehat{a}(\mu))[-CF(\lambda)y + 1/\widehat{a}(\mu)(1/\widehat{a}(\mu) - A)^{-1}CF(\lambda)y] - 1/\widehat{a}(\mu)(1/\widehat{a}(\mu) - A)^{-1}C^2y$. Then (2.30) implies $(I-\widehat{a}(\lambda)A)CF(\lambda)y = C^2y-1/\widehat{a}(\mu)(1/\widehat{a}(\mu)-A)^{-1}C^2y+1/\widehat{a}(\mu)(1/\widehat{a}(\mu)-A)^{-1}CF(\lambda)y-\widehat{a}(\lambda)/\widehat{a}(\mu)[-CF(\lambda)y+1/\widehat{a}(\mu)(1/\widehat{a}(\mu)-A)^{-1}CF(\lambda)y] := C^2y+R_{\lambda,\mu}$. Clearly, $R_{\lambda,\mu}=0$ if and only if $C^2y-CF(\lambda)y-\widehat{a}(\lambda)(1/\widehat{a}(\mu)-A)CF(\lambda)y+(\widehat{a}(\lambda)/\widehat{a}(\mu))CF(\lambda)y=0$. In order to see that the last equality is true, one can again apply (2.30). Thereby, $(I-\widehat{a}(\lambda)A)CF(\lambda)y=C^2y$, $\lambda\in N$, $\widehat{a}(\lambda)\neq 0$, and as an outcome, we obtain that the operator $I-\widehat{a}(\lambda)A$ is injective for all $\lambda\in N$ and that $Rang(C)\subseteq Rang(I-\widehat{a}(\lambda)C^{-1}AC)$ for all $\lambda\in N$ with $\widehat{a}(\lambda)\neq 0$, as required. The estimates (2.25)–(2.27) follow by using the argumentation given in the proof of [9, Theorem 2.1.4.4]. □

Vice versa, we have the following theorem which can be proved as in the case of convoluted C-semigroups [9].

Theorem 2.17. *Assume $k(t)$ and $a(t)$ satisfy (P1), $\omega\geq\max(0,abs(k),\ abs(a))$ and $\alpha\in(0,\pi/2]$. Assume, further, that A is a closed linear operator and that, for every $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda>\omega$ and $\widetilde{k}(\lambda)\neq 0$, we have that the operator $I-\widetilde{a}(\lambda)A$ is injective and that $Rang(C)\subseteq Rang(I-\widetilde{a}(\lambda)A)$. If there exists an analytic function $q:\omega+\Sigma_{\pi/2+\alpha}\to L(E)$such that*

$$q(\lambda)=\widetilde{k}(\lambda)(I-\widetilde{a}(\lambda)A)^{-1}C,\quad \lambda\in\mathbb{C},\ \operatorname{Re}\lambda>\omega,\ \widetilde{k}(\lambda)\neq 0, \tag{2.31}$$

$$\sup_{\lambda\in\omega+\Sigma_{\pi/2+\gamma}}\left\|(\lambda-\omega)q(\lambda)\right\|<\infty,\quad \forall\gamma\in(0,\alpha), \tag{2.32}$$

$$\lim_{\lambda\to+\infty}\lambda q(\lambda)x=k(0)Cx,\quad x\in E,\ \text{if}\ \overline{D(A)}\neq E, \tag{2.33}$$

then A is a subgenerator of an exponentially bounded, analytic (a,k)-regularized C-resolvent family of angle α.

Example 2.18 (cf. also [9, Theorem 2.1.4.7]). Let $\beta\in(0,2),\alpha>0,k(t)=t^{\alpha}/\Gamma(\alpha+1)$, and let $a(t)=t^{\beta-1}/\Gamma(\beta)$. Let A be densely defined. Then A is a subgenerator of an exponentially bounded, analytic (a,k)-regularized C-resolvent family of angle γ if and only if for every $\delta\in(0,\gamma)$, there exist $M_\delta>0$ and $\omega_\delta\geq 0$ such that

$$\begin{gathered}(\omega_\delta+\Sigma_{\pi/2+\delta})^{1/\beta}\subseteq\rho_C(A),\\ \left\|\left(\lambda^{\beta}-A\right)^{-1}C\right\|\leq M_\delta(1+|\lambda|)^{\alpha-\beta},\quad \lambda\in(\omega_\delta+\Sigma_{\pi/2+\delta})^{1/\beta},\end{gathered} \tag{2.34}$$

the mapping $\lambda\mapsto(\lambda^{\beta}-A)^{-1}C,\lambda\in(\omega_\delta+\Sigma_{\pi/2+\delta})^{1/\beta}$ is analytic (continuous).

Let (M_p) be a sequence of positive real numbers such that $M_0=1$ and that

(M.1) $M_p^2\leq M_{p+1}M_{p-1},\ p\in\mathbb{N}$,

(M.2) $M_n\leq AH^n\min_{p,\,q\in\mathbb{N},p+q=n}M_pM_q,\ n\in\mathbb{N},$ for some $A>1$, and $H>1$,

(M.3) $\sum_{p=1}^{\infty}(M_{p-1}/M_p)<\infty$.

The Gevrey sequences $(p!^s),(p^{ps})$, and $(\Gamma(1+ps))$ satisfy the above conditions, where $s>1$. Put $m_p := (M_p/M_{p-1})$, $p\in\mathbb{N}$; by (M.1), (m_p) is increasing, and (M.3)$'$ implies $\sum_{p=1}^{\infty}(1/m_p)<\infty$. The associated function of (M_p) is defined by $M(\lambda):=$

$\sup_{p\in\mathbb{N}_0} \ln(|\lambda|^p/M_p), \lambda \in \mathbb{C} \setminus \{0\}, M(0) := 0$. As is known, the function $t \mapsto M(t),\ t \geq 0$, is increasing, absolutely continuous, $\lim_{t\to\infty} M(t) = +\infty$ and $\lim_{t\to\infty}(M(t)/t) = 0$. For consistency of terminology with [24], we also employ the sequence $(L_p := M_p^{1/p})$ and set $\omega_L(t) := \sum_{p=0}^{\infty}(t^p/L_p^p),\ t \geq 0$.

We need the following family of kernels. Define, for every $l > 0$, the next entire function of exponential type zero $\omega_l(\lambda) := \prod_{p=1}^{\infty} 1 + l\lambda/m_p,\ \lambda \in \mathbb{C}$. Then

$$|\omega_l(\lambda)| \geq \sup_{k\in\mathbb{N}} \prod_{p=1}^{k} \left|1 + (l\lambda/m_p)\right| \geq \sup_{k\in\mathbb{N}} \prod_{p=1}^{k} \frac{l|\lambda|}{m_p} \geq \sup_{k\in\mathbb{N}} \frac{(l|\lambda|)^k}{M_p}, \quad \lambda \in \mathbb{C},\ \mathrm{Re}\ \lambda \geq 0, \tag{2.35}$$

and this implies that $|\omega_l(\lambda)| \geq e^{M(l|\lambda|)},\ \lambda \in \mathbb{C},\ \mathrm{Re}\ \lambda \geq 0$. It is noteworthy that, for every $\alpha \in (0, \pi/2), p \in \mathbb{N}_0$ and $\lambda \in \Sigma_{\pi/2+\alpha}, |1 + l\lambda/m_p| \geq l|Im\ \lambda|/m_p \geq l(1 + \tan\alpha)^{-1}|\lambda|/m_p$.

This yields

$$|\omega_l(\lambda)| \geq e^{M\left(l(1+\tan\alpha)^{-1}|\lambda|\right)}, \quad \alpha \in \left(0, \frac{\pi}{2}\right),\ l > 0,\ \lambda \in \Sigma_{(\pi/2)+\alpha}. \tag{2.36}$$

Put now

$$k_l(t) := \mathcal{L}^{-1}\left(\frac{1}{\omega_l(\lambda)}\right)(t), \quad t \geq 0,\ l > 0. \tag{2.37}$$

Then, for every $l > 0, 0 \in\ \mathrm{supp}\ k_l$ and k_l is infinitely differentiable in $t \geq 0$.

Definition 2.19. Let $(R(t))_{t\in[0,\tau)}$ be a (local) (a,k)-regularized C-resolvent family having A as a subgenerator, and let the mapping $t \mapsto R(t),\ t \in (0,\tau)$, be infinitely differentiable (in the uniform operator topology). Then it is said that $(R(t))_{t\in[0,\tau)}$ is of class C^L, resp. of class C_L, if and only if for every compact set $K \subseteq (0,\tau)$ there exists $h_K > 0$, resp. for every compact set $K \subseteq (0,\tau)$ and for every $h > 0$:

$$\sup_{t\in K,\ p\in\mathbb{N}_0} \left\| \frac{h_K^p (d^p/dt^p) R(t)}{L_p^p} \right\| < \infty, \quad (\text{resp.},) \sup_{t\in K,\ p\in\mathbb{N}_0} \left\| \frac{h^p (d^p/dt^p) R(t)}{L_p^p} \right\| < \infty, \tag{2.38}$$

$(R(t))_{t\in[0,\tau)}$ is said to be *ρ-hypoanalytic*,$1 \leq \rho < \infty$, if $(R(t))_{t\in[0,\tau)}$ is of class C^L with $L_p = p!^{\rho/p}$.

By the proof of the scalar-valued version of the Pringsheim theorem, it follows that the mapping $t \mapsto R(t),\ t \in (0,\tau)$ is real analytic if and only if $(R(t))_{t\in[0,\tau)}$ is ρ-hypoanalytic with $\rho = 1$.

The main objective in Theorems 2.20–2.24 is to enquire into the basic differential properties of (a,k)-regularized C-resolvent families.

Theorem 2.20 ([25]). *Suppose$A$ is a closed linear operator, $k(t)$ and $a(t)$ satisfy (P1), $r \geq -1$, and there exists $\omega \geq \max(0, abs(k), abs(a))$ such that, for every $z \in \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > \omega,\ \tilde{k}(\lambda) \neq 0\}$, we have that the operator $I - \tilde{a}(z)A$ is injective and that $Rang(C) \subseteq Rang(I - \tilde{a}(z)A)$. If, additionally, for every $\sigma > 0$, there exist $C_\sigma > 0$, $M_\sigma > 0$ and an open neighborhood $\Omega_{\sigma,\omega}$ of the region*

$$\Lambda_{\sigma,\omega} := \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \leq \omega,\ \operatorname{Re} \lambda \geq -\sigma \ln|\ Im\ \lambda| + C_\sigma\} \cup \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq \omega\}, \tag{2.39}$$

and an analytic mapping $h_\sigma : \Omega_{\sigma,\omega} \to L(E)$ such that $h_\sigma(\lambda) = \tilde{k}(\lambda)(I - \tilde{a}(\lambda)A)^{-1}C, \operatorname{Re} \lambda > \omega$, $\tilde{k}(\lambda) \neq 0$ and that $\|h_\sigma(\lambda)\| \leq M_\sigma |\lambda|^r\ \lambda \in \Lambda_{\sigma,\omega}$, then, for every $\zeta > 1$, A is a subgenerator of a norm continuous, exponentially bounded $(a, k * t^{\zeta+r-1}/\Gamma(\zeta + r))$-regularized C-resolvent family $(R(t))_{t\geq 0}$ satisfying that the mapping $t \mapsto R(t),\ t > 0$ is infinitely differentiable.

Theorem 2.21. *Suppose $k(t)$ and $a(t)$ satisfy (P1), (H5) hold and A is a subgenerator of an (a, k)-regularized C-resolvent family $(R(t))_{t\geq 0}$ satisfying $\|R(t)\| \leq Me^{\omega' t},\ t \geq 0$ for appropriate constants $\omega' \geq \max(0, abs(k), abs(a))$, and $M > 0$. If there exists $\omega > \omega'$ such that, for every $\sigma > 0$, there exist $C_\sigma > 0$ and $M_\sigma > 0$ so that*

- (i) *there exist an open neighborhood $\Omega_{\sigma,\omega}$ of the region $\Lambda_{\sigma,\omega}$, and the analytic mappings $f_\sigma : \Omega_{\sigma,\omega} \to \mathbb{C}, g_\sigma : \Omega_{\sigma,\omega} \to \mathbb{C}$, and $h_\sigma : \Omega_{\sigma,\omega} \to L(E)$ such that $f_\sigma(\lambda) = \tilde{k}(\lambda),\ \lambda \in \mathbb{C},\ \operatorname{Re} \lambda \geq \omega$ and $g_\sigma(\lambda) = \tilde{a}(\lambda),\ \lambda \in \mathbb{C},\ \operatorname{Re} \lambda \geq \omega$,*
- (ii) *for every $\lambda \in \Lambda_{\sigma,\omega}$ with $\operatorname{Re} \lambda \leq \omega$, the operator $I - \tilde{a}(\lambda)A$ is injective and $Rang\ (C) \subseteq Rang\ (I - \tilde{a}(\lambda)A)$,*
- (iii) *$h_\sigma(\lambda) = f_\sigma(\lambda)(I - g_\sigma(\lambda)A)^{-1}C,\ \lambda \in \Lambda_{\sigma,\omega}$,*
- (iv) *$\|h_\sigma(\lambda)\| \leq M_\sigma |Im\ \lambda|,\ \lambda \in \Lambda_{\sigma,\omega}, \operatorname{Re} \lambda \leq \omega$, and $\max(|f_\sigma(\lambda)|, |g_\sigma(\lambda)|) \leq M_\sigma,\ \lambda \in \Lambda_{\sigma,\omega}$,*

then the mapping $t \mapsto R(t)x,\ t > 0$ is infinitely differentiable for every fixed $x \in D(A^2)$. Furthermore, if $D(A^2)$ is dense in E, then the mapping $t \mapsto R(t),\ t > 0$, is infinitely differentiable.

Proof. Assume $\sigma > 0, \varsigma > 0, \omega_0 > \omega$, and put $\Gamma^1_\sigma := \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda = 2C_\sigma - \sigma \ln(-\operatorname{Im} \lambda),\ -\infty < \operatorname{Im} \lambda \leq -e^{(2C_\sigma/\sigma)}\}, \Gamma^2_\sigma := \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda = \omega_0,\ -e^{(2C_\sigma/\sigma)} \leq \operatorname{Im} \lambda \leq e^{(2C_\sigma/\sigma)}\}, \Gamma^3_\sigma := \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda = 2C_\sigma - \sigma \ln(\operatorname{Im} \lambda), e^{(2C_\sigma/\sigma)} \leq \operatorname{Im} \lambda < +\infty\}, \Gamma_\sigma := \Gamma^1_\sigma \cup \Gamma^2_\sigma \cup \Gamma^3_\sigma$, and $\Gamma_{k,\sigma} := \{\lambda \in \Gamma_\sigma : |\lambda| \leq k\}, k \in \mathbb{N}$. The curves Γ_σ and $\Gamma_{k,\sigma}$ are oriented so that $\operatorname{Im}\lambda$ increases along Γ_σ and $\Gamma_{k,\sigma},\ k \in \mathbb{N}$. Set, for a sufficiently large $k_0 \in \mathbb{N}, S^k_\sigma(t) := (1/2\pi i)\int_{\Gamma_{k,\sigma}} e^{\lambda t}(h_\sigma(\lambda)/\lambda^2)d\lambda,\ t \geq 0,\ k \geq k_0$. One can simply prove that $(d^j/dt^j)S^k_\sigma(t) = (1/2\pi i)\int_{\Gamma_{k,\sigma}} e^{\lambda t}\lambda^{j-2}h_\sigma(\lambda)d\lambda,\ t \geq 0,\ k \geq k_0,\ j \in \mathbb{N}$. Let $k_0 < k < l$. Then (iv) implies

$$\begin{aligned}\left\|\frac{d^j}{dt^j}S^k_\sigma(t) - \frac{d^j}{dt^j}S^l_\sigma(t)\right\| &= \frac{1}{2\pi}\left\|\int_{\Gamma_{l,\sigma}\ \cap\ \{\lambda\in\mathbb{C}:k\leq|\lambda|\leq l\}} e^{\lambda t}\lambda^{j-2}h_\sigma(\lambda)d\lambda\right\| \\ &\leq \frac{M_\sigma}{2\pi}e^{2C_\sigma t}\int_{\Gamma_{l,\sigma}\cap\{\lambda\in\mathbb{C}:k\leq|\lambda|\leq l\}} |\operatorname{Im} \lambda|^{1-\sigma t}|\lambda|^{j-2}|d\lambda|,\end{aligned} \tag{2.40}$$

for all $j \in \mathbb{N}_0$. Since $|\operatorname{Im} \lambda|^{1-\sigma t}|\lambda|^{j-2} \sim |\operatorname{Im} \lambda|^{j-1-\sigma t},\ |\lambda| \to \infty,\ \lambda \in \Gamma_\sigma$, one gets that, for every $j \in \mathbb{N}_0$ and $t > j/\sigma$, the sequence $((d^j/dt^j)S^k_\sigma(t))_k$ is convergent in $L(E)$ and that the convergence is uniform on every compact subset of $[j/\sigma + \varsigma, \infty)$. Put $S_{j,\sigma}(t) := \lim_{k\to\infty}(d^j/dt^j)S^k_\sigma(t),\ j \in \mathbb{N}_0,\ t > j/\sigma$. Then it is obvious that $(d/dt)S_{j,\sigma}(t) = S_{(j+1),\sigma}(t),\ j \in \mathbb{N}_0,\ t > (j+1)/\sigma + \varsigma$.

This implies that the mapping $t \mapsto S_{0,\sigma}(t),\ t > (j+1/\sigma)+\varsigma$ is j-times differentiable and that $(d^j/dt^j)S_{0,\sigma}(t) = S_{j,\sigma}(t), t > (j+1)/\sigma)+\varsigma$. On the other hand, it is clear that, for every $\sigma > 0, x \in D(A^2)$, and $\lambda \in \{z \in \Omega_{\sigma,\omega} : g_\sigma(z) \neq 0\}$,

$$(I - g_\sigma(\lambda)A)^{-1}Cx = Cx + g_\sigma(\lambda)CAx + g_\sigma(\lambda)(I - g_\sigma(\lambda)A)^{-1}CA^2x. \tag{2.41}$$

By (2.41), we get that, for every $x \in D(A^2)$ and $t > 0$,

$$S_{0,\sigma}(t)x = \frac{1}{2\pi i}\int_{\Gamma_\sigma} e^{\lambda t} f_\sigma(\lambda)\frac{Cx + g_\sigma(\lambda)CAx + g_\sigma(\lambda)(I - g_\sigma(\lambda)A)^{-1}CA^2x}{\lambda^2}d\lambda. \tag{2.42}$$

With (iv) and the residue theorem in view, it follows that, for every $t > 0$ and $x \in D(A^2)$,

$$\begin{aligned} S_{0,\sigma}(t)x &= \int_0^t (t-s)k(s)dsCx + (t * k * a)(t)CAx \\ &\quad + \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} e^{\lambda t}\frac{f_\sigma(\lambda)g_\sigma(\lambda)(I - g_\sigma(\lambda)A)^{-1}CA^2x}{\lambda^2}d\lambda. \end{aligned} \tag{2.43}$$

Put $R_2(t) := \int_0^t (t-s)R(s)x\,ds,\ x \in E,\ t \geq 0$. By Theorem 2.6(i), we get that

$$\frac{\tilde{k}(\lambda)}{\lambda^2}(I - \tilde{a}(\lambda)A)^{-1}Cx = \int_0^\infty e^{-\lambda t}R_2(t)x\,dt, \quad x \in E,\ \text{Re}\ \lambda > \omega,\ \tilde{k}(\lambda) \neq 0. \tag{2.44}$$

This implies that the function $\lambda \mapsto h_\sigma(\lambda)/\lambda^2$ is bounded on some right half plane. Taking into account (2.41), we have that, for every $t \geq 0$ and $x \in D(A^2)$,

$$\begin{aligned} R_2(t) &= \mathcal{L}^{-1}\left(\frac{\tilde{k}(\lambda)}{\lambda^2}(I - \tilde{a}(\lambda)A)^{-1}Cx\right)(t) = \int_0^t (t-s)k(s)dsCx + (t * k * a)(t)CAx \\ &\quad + \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} e^{\lambda t}\frac{f_\sigma(\lambda)g_\sigma(\lambda)(I - g_\sigma(\lambda)A)^{-1}CA^2x}{\lambda^2}d\lambda. \end{aligned} \tag{2.45}$$

By (2.43)-(2.45), $S_{0,\sigma}(t) = R_2(t),\ t > 0$. The arbitrariness of σ implies that the mapping $t \mapsto R_2(t),\ t > 0$ is infinitely differentiable, finishing the proof. □

Using the argumentation given in [25], one can prove the following theorems.

Theorem 2.22. *Suppose $k(t)$ and $a(t)$ satisfy (P1), A is a subgenerator of a (local) (a,k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}, \omega \geq \max(0, abs(k), abs(a))$, and $m \in \mathbb{N}$. Denote, for every $\varepsilon \in (0,1)$ and a corresponding $K_\varepsilon > 0$,*

$$F_{\varepsilon,\omega} =: \{\lambda \in \mathbb{C} : \text{Re}\ \lambda \geq -\ln \omega_L(K_\varepsilon|Im\,\lambda|) + \omega\}. \tag{2.46}$$

Assume that, for every $\varepsilon \in (0,1)$, *there exist* $C_\varepsilon > 0, M_\varepsilon > 0$, *an open neighborhood* $O_{\varepsilon,\omega}$ *of the region* $G_{\varepsilon,\omega} := \{\lambda \in \mathbb{C} : \text{Re}\,\lambda \geq \omega, \widetilde{k}(\lambda) \neq 0\} \cup \{\lambda \in F_{\varepsilon,\omega} : \text{Re}\,\lambda \leq \omega\}$, *and analytic mappings* $f_\varepsilon : O_{\varepsilon,\omega} \to \mathbb{C}, g_\varepsilon : O_{\varepsilon,\omega} \to \mathbb{C}$ *and* $h_\varepsilon : O_{\varepsilon,\omega} \to L(E)$ *such that*

(i) $f_\varepsilon(\lambda) = \widetilde{k}(\lambda),\ \text{Re}\ \lambda > \omega; g_\varepsilon(\lambda) = \widetilde{a}(\lambda),\ \text{Re}\ \lambda > \omega$,

(ii) *for every* $\lambda \in F_{\varepsilon,\omega}$, *the operator* $I - g_\varepsilon(\lambda)A$ *is injective and* $Rang(C) \subseteq Rang(I - g_\varepsilon(\lambda)A)$,

(iii) $h_\varepsilon(\lambda) = f_\varepsilon(\lambda)(I - g_\varepsilon(\lambda)A)^{-1}C,\ \lambda \in G_{\varepsilon,\omega}$,

(iv) $\|h_\varepsilon(\lambda)\| \leq M_\varepsilon(1+|\lambda|)^m e^{\varepsilon|\text{Re}\ \lambda|},\ \lambda \in F_{\varepsilon,\omega}, \text{Re}\ \lambda \leq \omega$ *and* $\|h_\varepsilon(\lambda)\| \leq M_\varepsilon(1+|\lambda|)^m,\ \lambda \in \mathbb{C},\ \text{Re}\ \lambda \geq \omega$.

Then $(R(t))_{t\in[0,\tau)}$ *is of class* C^L.

Theorem 2.23. *Suppose* $k(t)$ *and* $a(t)$ *satisfy (P1), A is a subgenerator of a (local)* (a,k)*-regularized C-resolvent family* $(R(t))_{t\in[0,\tau)}, \omega \geq \max(0,\ abs(k),\ abs(a))$, *and* $m \in \mathbb{N}$. *Denote, for every* $\varepsilon \in (0,1), \rho \in [1,\infty)$ *and a corresponding* $K_\varepsilon > 0$,

$$F_{\varepsilon,\omega,\rho} =: \left\{\lambda \in \mathbb{C} : \text{Re}\ \lambda \geq -K_\varepsilon |Im\ \lambda|^{1/\rho} + \omega\right\}. \tag{2.47}$$

Assume that, for every $\varepsilon \in (0,1)$, *there exist* $C_\varepsilon > 0, M_\varepsilon > 0$, *an open neighborhood* $O_{\varepsilon,\omega}$ *of the region* $G_{\varepsilon,\omega,\rho} := \{\lambda \in \mathbb{C} : \text{Re}\,\lambda \geq \omega,\ \widetilde{k}(\lambda) \neq 0\} \cup \{\lambda \in F_{\varepsilon,\omega,\rho} : \text{Re}\,\lambda \leq \omega\}$, *and analytic mappings* $f_\varepsilon : O_{\varepsilon,\omega} \to \mathbb{C}, g_\varepsilon : O_{\varepsilon,\omega} \to \mathbb{C}$ *and* $h_\varepsilon : O_{\varepsilon,\omega} \to L(E)$ *such that the conditions (i)–(iv) of Theorem 2.22 hold with* $F_{\varepsilon,\omega}$, *resp.* $G_{\varepsilon,\omega}$, *replaced by* $F_{\varepsilon,\omega,\rho}$, *respectively,* $G_{\varepsilon,\omega,\rho}$. *Then* $(R(t))_{t\in[0,\tau)}$ *is of class* C^L.

Theorem 2.24. *Suppose* $\alpha > 0, j \in \mathbb{N}$, *and* $(R(t))_{t\in[0,\tau)}$ *is a (local)* (a,k)*-regularized C-resolvent family with a subgenerator A. Set*

$$R_\alpha(t)x := \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} R(s)x\,ds, \quad t \in [0,\tau),\ x \in E. \tag{2.48}$$

Then $(R(t))_{t\in[0,\tau)}$ *is an* $(a, k*(t^{\alpha-1}/\Gamma(\alpha)))$*-regularized C-resolvent family with a subgenerator A. Furthermore, if the mapping* $t \mapsto R(t),\ t \in (0,\tau)$ *is j-times differentiable, then the mapping* $t \mapsto R_\alpha(t),\ t \in (0,\tau)$ *is likewise j-times differentiable. If this is the case, then we have, for every* $t \in [0,\tau), b \in (0,t)$, *and* $x \in E$:

$$\begin{aligned}\frac{d^j}{dt^j}R_\alpha(t)x = \int_0^b \frac{(t-s)^{\alpha-1-j}}{\Gamma(\alpha)} \prod_{i=1}^{j}(\alpha - i)R(s)x\,ds + \sum_{i=0}^{j} \frac{(t-b)^{\alpha+i-j}}{\Gamma(\alpha+i+1)} \\ \times \prod_{k=0}^{j-1}(\alpha + i - k)R^{(i)}(b)x + \int_b^t \frac{(t-s)^\alpha}{\Gamma(\alpha+1)} \frac{d^j}{ds^j}R(s)x\,ds,\end{aligned} \tag{2.49}$$

and we have the following.

(i) *If* $(R(t))_{t\in[0,\tau)}$ *is of class* C^L, *resp. of class* C_L, *then* $(R_\alpha(t))_{t\in[0,\tau)}$ *is likewise of class* C^L, *resptivley, of class* C_L.

(ii) *If* $(R(t))_{t\in[0,\tau)}$ *is* ρ*-hypoanalytic,* $1 \leq \rho < \infty$, *then* $(R_\alpha(t))_{t\in[0,\tau)}$ *is likewise* ρ*-hypoanalytic.*

Before going further, notice that we can slightly reformulate Theorem 2.16 and Theorems 2.21–2.23 in the case when the functions $\widetilde{k}(\lambda)$ and $\widetilde{a}(\lambda)$ possess the meromorphic extensions on the corresponding regions defined in formulation of mentioned theorems. Having in mind [9, Theorem 2.1.1.11, Theorem 2.1.1.14], we have the following interesting analogue of [25, Theorem 2.8] which cannot be so easily interpreted in the case of general (a,k)-regularized C-resolvent families.

Theorem 2.25. (i) *Let* A *be a subgenerator of a local* K*-convoluted* C*-cosine function* $(C_K(t))_{t\in[0,\tau)}, 0 \in \operatorname{supp} K, K \in C^\infty((0,\tau))$ *(*$K \in C^j((0,\tau)), j \in \mathbb{N}$*) resp.* K *is of class* C^L *(*C_L*), and let* $K = K_{1|[0,\tau)}$ *for an appropriate complex-valued function* $K_1 \in L^1_{loc}([0,2\tau))$. *(Put* $\Theta_1(t) = \int_0^t K_1(s)ds$; *since it makes no misunderstanding, we will also write* K *and* Θ, *for* K_1 *and* Θ_1, *respectively, and denote by* $K * K$ *the restriction of this function to any subinterval of* $[0,2\tau)$.*) Let the mapping* $t \mapsto C_K(t),\ t \in (0,\tau)$, *be infinitely differentiable (*j*-times differentiable,* $j \in \mathbb{N}$*), respectively, and let* $(C_K(t))_{t\in[0,\tau)}$ *be of class* C^L *(*C_L*). Then* A *is a subgenerator of a local* $(K * K)$*-convoluted* C^2*-cosine function* $(C_{K*K}(t))_{t\in[0,2\tau)}$ *satisfying that the mapping* $t \mapsto C_{K*K}(t),\ t \in (0,2\tau)$, *is infinitely differentiable ((*$j-1$*)-times differentiable), resp.* $(C_{K*K}(t))_{t\in[0,2\tau)}$ *is of class* C^L *(*C_L*). Furthermore, the suppositions* $j \in \mathbb{N}$ *and* $K \in C^j((0,\tau)) \cap C^{j-1}([0,\tau))$ *imply the following: if the mapping* $t \mapsto C_K(t),\ t \in (0,\tau)$, *is* j*-times differentiable, then the mapping* $t \mapsto C_{K*K}(t),\ t \in (0,2\tau)$ *is likewise* j*-times differentiable.*

(ii) *Suppose* $\alpha \geq 0,\ \ j \in \mathbb{N}$, *and* A *is a subgenerator of a local* α*-times integrated* C*-cosine function* $(C_\alpha(t))_{t\in[0,\tau)}$. *Then* A *is a subgenerator of a local* (2α)*-times integrated* C^2*-cosine function* $(C_{2\alpha}(t))_{t\in[0,2\tau)}$ *and the following holds.*

(ii.1) *If the mapping* $t \mapsto C_\alpha(t),\ t \in (0,\tau)$ *is infinitely differentiable (*j*-times differentiable,* $j \in \mathbb{N}$*), then the mapping* $t \mapsto C_{2\alpha}(t),\ t \in (0,2\tau)$ *is infinitely differentiable ((*$j-1$*)-times differentiable;* j*-times differentiable, provided* $\alpha \geq j$*).*

(ii.2) *If* $(C_\alpha(t))_{t\in[0,\tau)}$ *is of class* C^L, *resp.* C_L, *then* $(C_{2\alpha}(t))_{t\in[0,2\tau)}$ *is likewise of class* C^L, *resp.* C_L.

(ii.3) *Assume* $\alpha \in \mathbb{N}_0, j \in \mathbb{N}$, *and the mapping* $t \mapsto C_\alpha(t),\ t \in (0,\tau)$ *is infinitely differentiable (*j*-times differentiable). Then the mapping* $t \mapsto C_{2\alpha}(t),\ t \in (0,2\tau)$, *is* j*-times differentiable.*

Proof. The first part of (i) can be proved by passing to the theory of semigroups (see [9, Theorem 2.1.1.11] and [25, Theorem 2.8]). So, let us assume $j \in \mathbb{N}, K \in C^j((0,\tau)) \cap C^{j-1}([0,\tau)), \tau_0 \in (0,\tau)$, and let the mapping $t \mapsto C_K(t),\ t \in (0,\tau)$ be j-times differentiable. By

[9, Theorem 2.1.1.14], A is a subgenerator of a local $(K * K)$-convoluted C^2-cosine function $(C_{K*K}(t))_{t\in[0,2\tau)}$, which is given by

$$C_{K*K}(t)x = \begin{cases} \displaystyle\int_0^t K(t-s)C_K(s)Cx\,ds, \quad t \in [0,\tau_0], \\ 2C_K(\tau_0)C_K(t-\tau_0)x + \left(\displaystyle\int_0^{t-\tau_0} + \int_0^{\tau_0}\right)K(t-r)C_K(r)Cx\,dr \\ \quad -\displaystyle\int_{2\tau_0-t}^{\tau_0} K(r+t-2\tau_0)C_K(r)Cx\,dr \\ \quad -\displaystyle\int_0^{t-\tau_0} K(r+2\tau_0-t)C_K(r)Cx\,dr, \quad t \in (\tau_0, 2\tau_0), \quad x \in E. \end{cases} \tag{2.50}$$

Since the mapping $t \mapsto C_K(t),\ t \in (0,\tau)$ is j-times differentiable and $K \in C^j((0,\infty))$, we have that the mapping $t \mapsto C_{K*K}(t),\ t \in (0,\tau)$ is also j-times differentiable. Arguing as in [25, Theorem 2.8], one gets that the mappings $t \mapsto C_K(\tau_0)C_K(t-\tau_0),\ t \in (\tau_0, 2\tau_0), t \mapsto (\int_0^{t-\tau_0} + \int_0^{\tau_0})K(t-r)C_K(r)C\,dr,\ t \in (\tau_0, 2\tau_0)$ and $t \mapsto \int_0^{t-\tau_0} K(r+2\tau_0-t)C_K(r)C\,dr,\ t \in (\tau_0, 2\tau_0)$ are j-times differentiable.

Let $f(t) := \int_{2\tau_0-t}^{\tau_0} K(r+t-2\tau_0)C_K(r)C\,dr,\ t \in (\tau_0, 2\tau_0)$. Using the fact that $K \in C^1((0,\tau)) \cap C([0,\tau))$, we have $f'(t) = \int_{2\tau_0-t}^{\tau_0} K'(r+t-2\tau_0)C_K(r)C\,dr + K(0)C_K(2\tau_0-t)C, t \in (\tau_0, 2\tau_0)$. Repeating this procedure leads us to the fact that the mapping $t \mapsto f(t), t \in (\tau_0, 2\tau_0)$ is j-times differentiable, and this completes the proof of (i). The proof of (ii) in the case $\alpha \in \mathbb{N}$ follows immediately from (i) with $K(t) = (t^{\alpha-1}/\Gamma(\alpha))$ while the proof of (ii) in the case $\alpha = 0$ is much easier [16]. □

Suppose that $\min(a(t), k(t)) > 0,\ t \in (0,\tau)$ and that A is a subgenerator of an (a,k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$. We define the Favard class $F_{a,k}$ by setting

$$F_{a,k} := \left\{ x \in E : \sup_{t\in(0,\tau)} \frac{\|R(t)x - k(t)Cx\|}{(a*k)(t)} < \infty \right\}. \tag{2.51}$$

Equipped with the norm $|\cdot|_{a,k} := \|\cdot\| + \sup_{t\in(0,\tau)}(\|R(t)\cdot - k(t)C\cdot\|/(a*k)(t)), F_{a,k}$ becomes a Banach space, and in the case when $\|R(t)\| = O(k(t)),\ t \in [0,\tau)$, we have $D(A) \subseteq F_{a,k}$. The proof of [5, Theorem 3.4] immediately implies the following assertion.

Theorem 2.26. *Assume* $\min(a(t), k(t)) > 0,\ t \in (0,\tau),\ abs(k) = abs(a) = 0, A$ *is a subgenerator of an* (a,k)*-regularized* C*-resolvent family* $(R(t))_{t\geq 0}$ *satisfying* $\|R(t)\| = O(1),\ t \geq 0$ *and (H5) holds.*

(i) *Let* $x \in F_{a,k}$. *Then*

$$\sup_{\lambda>0,\ \tilde{k}(\lambda)\neq 0} \left\| A(I - \tilde{a}(\lambda)A)^{-1}Cx \right\| < \infty. \tag{2.52}$$

(ii) *Assume, in addition, that the mapping* $\tilde{a} : (0,\infty) \to (0,\infty)$ *is surjective and that* $\sup_{t>0}(1*a)(t)/(a*k)(t) < \infty$. *Then* (2.52) *implies* $Cx \in F_{a,k}$.

The assertion (ii) of the next theorem improves [5, Theorem 4.6].

Theorem 2.27 (cf. [26, Theorem 4.2] and Proposition 2.12.7). (i) *Suppose A is a subgenerator of a (local, global exponentially bounded) (a,k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$, $D(A)$ and $Rang(C)$ are dense in E and $\alpha > 0$. Then A^* is a subgenerator of a (local, global exponentially bounded) $(a, k*_0(t^{\alpha-1}/\Gamma(\alpha)))$-regularized C^*-resolvent family $(R^*_\alpha(t))_{t\in[0,\tau)}$, which is given by*

$$R_\alpha(t)x^* := \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} R(s)^* x^* ds, \quad t \in [0,\tau),\ x^* \in E^*. \tag{2.53}$$

(ii) *Suppose A is a subgenerator of a (local, global exponentially bounded) (a,k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$, and $D(A)$ and $Rang$ (C) are dense in E. Then the part of A^* in $\overline{D(A^*)}$ is a subgenerator of a (local, global exponentially bounded) (a,k)-regularized $C^*_{\overline{D(A^*)}}$-resolvent family in E^*.*

(iii) *Suppose E is reflexive, $D(A)$ and $Rang$ (C) are dense in E, $k(t)$ and $a(t)$ satisfy (P1), and A is a subgenerator of a (local, global exponentially bounded) (a,k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$. Then A^* is a subgenerator of a (local, global exponentially bounded) (a,k)- regularized C^*-resolvent family (of the same exponential type, in the second case).*

Suppose, for the time being, that $a \in C([0,\tau))$ and denote, for every $\lambda \in \mathbb{C}$, by $s(t,\lambda)$ the unique continuous solution of the equation

$$s(t,\lambda) = a(t) + \lambda \int_0^t a(t-v)s(v,\lambda)dv, \quad t \in [0,\tau). \tag{2.54}$$

Put $r(t,\lambda) := k(t)+\lambda\int_0^t s(t-v,\lambda)k(v)dv,\ t \in [0,\tau)$. Arguing as in [5, Section 5], one can simply verify the validity of the next theorem.

Theorem 2.28. (i) *Let A be a subgenerator of an (a,k)- regularized C-resolvent family$(R(t))_{t\in[0,\tau)}$, and let (H5) hold. If the operator $r(t,\lambda)C - R(t)$ is bijective for some $t \in [0,\tau)$ and $\lambda \in \mathbb{C}$, then $\lambda \in \rho(A)$.*

(ii) *Let A be a densely defined subgenerator of an (a,k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$, and let (H5) hold. If $\overline{Rang\ (\lambda - A)} \neq E$ for some $\lambda \in \mathbb{C}$, then, for every $t \in [0,\tau)$, $\overline{Rang\ (r(t,\lambda)C - R(t))} \neq E$.*

(iii) *Let A be a subgenerator of an (a,k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$. Then the assumption $Ax = \lambda x$, for some $x \in E$ and $\lambda \in \mathbb{C}$, implies $r(t,\lambda)Cx = R(t)x,\ t \in [0,\tau)$.*

For further information concerning duality and spectral properties of (a,k)-regularized resolvent families, we refer to [5].

Proposition 2.29 (cf. [9, Proposition 2.1.1.17]). *Suppose $\pm A$ are subgenerators of (local, global exponentially bounded) (a,k)- regularized C- resolvent families $(R^{\pm}(t))_{t\in[0,\tau)}$, and A^2 is closed. Then A^2 is a subgenerator of a (local, global exponentially bounded) $(a*a,k)$-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$, which is given by $R(t)x := 1/2(R^+(t)x + R^-(t)x),\ x \in E,\ t \in [0,\tau)$.*

Proof. Clearly, $(R(t))_{t\in[0,\tau)}$ is a strongly continuous operator family, $CA^2 \subseteq A^2C$, $R(0) = k(0)C$, $R(t)C = CR(t)$ and $R(t)A^2 \subseteq A^2R(t)$, $t \in [0,\tau)$. Let $x \in D(A^2)$. Then we have

$$\begin{aligned}(a*a*R)(t)A^2x &= \frac{1}{2}(a*a*R^{+}A)(t)Ax + \frac{1}{2}\left(a*a*R^{-}A\right)(t)Ax \\ &= \frac{1}{2}(a*(R^{+}-kC))(t)Ax + \frac{1}{2}\left(a*\left(R^{-}-kC\right)\right)(t)Ax \\ &= \frac{1}{2}(a*R^{+})(t)Ax + \frac{1}{2}\left(a*R^{-}\right)(t)Ax \\ &= \frac{1}{2}(R^{+}(t)x - k(t)Cx) + \frac{1}{2}\left(R^{-}(t)x - k(t)Cx\right) \\ &= R(t)x - k(t)Cx, \quad t \in [0,\tau).\end{aligned} \tag{2.55}$$

This completes the proof. □

The next version of the abstract Weierstrass formula extends [15, Theorem 11].

Theorem 2.30. (i) *Assume that $k(t)$ and $a(t)$ satisfy (P1), and there exist $M > 0$ and $\omega > 0$ such that $|k(t)| \leq Me^{\omega t}, t \geq 0$. Assume, further, that there exist a number $\omega' \geq \omega$ and a function $a_1(t)$ satisfying (P1) and $\tilde{a}_1(\lambda) = \tilde{a}(\sqrt{\lambda}), \lambda \in \mathbb{C}$,* Re *$\lambda > \omega'$. Let A be a subgenerator of an exponentially bounded (a,k)-regularized C-resolvent family $(C(t))_{t\geq 0}$, and let (H5) hold. Then A is a subgenerator of an exponentially bounded, analytic (a_1,k_1)-regularized C-resolvent family $(R(t))_{t\geq 0}$ of angle $(\pi/2)$, where*

$$k_1(t) := \int_0^\infty \frac{e^{-s^2/4t}}{\sqrt{\pi t}} k(s)ds, \quad t > 0,\ k_1(0) := k(0), \tag{2.56}$$

$$R(t)x := \int_0^\infty \frac{e^{-s^2/4t}}{\sqrt{\pi t}} C(s)xds, \quad t > 0,\ x \in E,\ R(0) := k(0)C. \tag{2.57}$$

(ii) *Assume $k(t)$ satisfy (P1), $\beta > 0$, and there exist $M > 0$ and $\omega > 0$ such that $|k(t)| \leq Me^{\omega t}$, $t \geq 0$. Let A be a subgenerator of an exponentially bounded $(t^{2\beta-1}/\Gamma(2\beta), k)$-regularized C-resolvent family $(C(t))_{t\geq 0}$, and let (H5) hold. Then A is a subgenerator of an exponentially bounded, analytic $(t^{\beta-1}/\Gamma(\beta), k_1)$-regularized C-resolvent family $(R(t))_{t\geq 0}$ of angle $\pi/2$, where $k_1(t)$ and $R(t)$ are defined through* (2.56) *and* (2.57).

Proof. Since $k(t)$ is continuous and exponentially bounded, one can use the substitution $r = s/\sqrt{t}$ and the dominated convergence theorem after that to deduce that, for every $s \geq 0$,

$$k_1(t) = \int_0^\infty \frac{e^{-r^2/4}}{\sqrt{\pi}} k\left(r\sqrt{t}\right)dr \longrightarrow k_1(s), \quad t \longrightarrow s. \tag{2.58}$$

This implies $k_1 \in C([0,\infty))$. Moreover, $k_1(t)$ is a kernel since

$$\limsup_{\lambda \to +\infty} \frac{\ln\left|\tilde{k}_1(\lambda)\right|}{\lambda} = \limsup_{\lambda \to +\infty} \frac{\ln\left|\tilde{k}\left(\sqrt{\lambda}\right)/\sqrt{\lambda}\right|}{\lambda} = 0. \tag{2.59}$$

Let $x \in E$ be fixed. Then, for every $s \geq 0$,

$$R(t)x = \int_0^\infty \frac{e^{-r^2/4}}{\sqrt{\pi}} C\left(r\sqrt{t}\right)x\,dr \longrightarrow R(s)x, \quad t \longrightarrow s. \tag{2.60}$$

By (2.60), $(R(t))_{t\geq 0}$ is a strongly continuous, exponentially bounded operator family. Furthermore, one can employ Theorem 2.6(i) and [12, Proposition 1.6.8] to obtain that, for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \beta^2$ and $\tilde{k}_1(\lambda) \neq 0$,

$$\begin{aligned}
\int_0^\infty e^{-\lambda t} R(t)x\,dt &= \int_0^\infty e^{-\lambda t} \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-s^2/4t} C(s)x\,ds\,dt \\
&= \frac{1}{\sqrt{\lambda}} \int_0^\infty e^{-\sqrt{\lambda}s} C(s)x\,ds \\
&= \frac{1}{\sqrt{\lambda}} \tilde{k}\left(\sqrt{\lambda}\right)\left(I - \tilde{a}\left(\sqrt{\lambda}\right)A\right)^{-1} Cx \\
&= \tilde{k}_1(\lambda)(I - \tilde{a}_1(\lambda)A)^{-1}Cx.
\end{aligned} \tag{2.61}$$

By Theorem 2.6(ii), we get that $(R(t))_{t\geq 0}$ is an exponentially bounded (a_1,k_1)-regularized C-resolvent family with a subgenerator A, and the remnant of the proof of (i) may be carried out by modifying the corresponding part of the proof of [15, Theorem 11]; the assertion (ii) follows from (i) with $a(t) = t^{2\beta-1}/\Gamma(2\beta)$. □

Notice that $a_1(t) = \int_0^\infty s(e^{-s^2/4t}/2\sqrt{\pi}t^{3/2})a(s)ds,\ t > 0$, whenever the function $a(t)$ is exponentially bounded.

Example 2.31. (i)(Reference [15]) Let $E = L^p(\mathbb{R}),\ 1 \leq p \leq \infty, \alpha \in (-1,1)$, and $a(t) = (t^\alpha/\Gamma(\alpha+1))$. Consider the next multiplication operator with maximal domain in E:

$$Af(x) := \left(1 + x + ix^2\right)^{\alpha+1} f(x), \quad x \in \mathbb{R},\ f \in E. \tag{2.62}$$

Assume $s \in (1,2), \delta = 1/s, M_p = p!^s$, and $K_\delta(t) := \mathcal{L}^{-1}(e^{-\lambda^\delta})(t),\ t \geq 0$. Then A generates a global (not exponentially bounded) (a,K_δ)-regularized resolvent family since, for every $\tau \in (0,\infty)$, A generates a local (a,K_δ)-regularized resolvent family on $[0,\tau)$. In order to show this, designate by $M(t)$ the associated function of the sequence (M_p) and put $\Lambda_{\alpha,\beta,\gamma} := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq (M(\alpha\lambda)/\gamma) + \beta\},\ \alpha > 0,\ \beta > 0,\ \gamma > 0$. Clearly, there exists a constant $C_s > 0$ such that $M(\lambda) \leq C_s|\lambda|^{1/s},\ \lambda \in \mathbb{C}$. Given $\tau > 0$, choose $\alpha > 0$ and $\beta > 0$ such that $\tau \leq \cos(\delta\pi/2)/C_s\alpha^{1/s}$ as well as that $\Lambda_{\alpha,\beta,1} \subseteq \rho(A)$ and that the resolvent of A is bounded on the set

$\{\lambda^{\alpha+1} : \lambda \in \Lambda_{\alpha,\beta,1}\}$. Put $\Gamma := \partial(\Lambda_{\alpha,\beta,1})$, and assume that the curve Γ is upward oriented. Define, for every $f \in E, x \in \mathbb{R}$ and $t \in [0, \cos(\delta\pi/2)/C_s\alpha^{1/s})$,

$$(R_\delta(t)f)(x) := \frac{1}{2\pi i}\int_\Gamma \frac{\lambda^{\alpha+1}e^{\lambda t-\lambda^\delta}}{\lambda^{\alpha+1}-(1+x+ix^2)^{\alpha+1}}d\lambda f(x). \tag{2.63}$$

Then one can straightforwardly check that $(R_\delta(t))_{t\in[0,\tau)}$ is a local (a, K_δ)-regularized resolvent family generated by A. Arguing in the same way, we get that there exists $\tau_0 > 0$ such that A generates a local $(a, K_{1/2})$-regularized resolvent family on $[0, \tau_0)$, where $K_{1/2}(t) := \mathcal{L}^{-1}(e^{-\lambda^{1/2}})(t), t \geq 0$.

(ii) [References [9, 27]] Let $A_{(p!^s)}$ and $E_{(p!^s)}$ be as in [27, Example 1.6] with $M_p = p!^s (s > 1)$. Let $\beta \in (0, 1)$, and let, for every $l > 0, k_l(t) = \mathcal{L}^{-1}(1/\prod_{p=1}^{\infty}(1 + l\lambda/p^{s/\beta}))(t),\ t \geq 0$, (see (2.36)-(2.37)) and $a(t) = t^{\beta-1}/\Gamma(\beta)$. Then it is obvious that there exist $l' > 0$ and $K > 0$ such that $\|\lambda\tilde{k}_{l'}(\lambda)(I - \tilde{a}(\lambda)A)^{-1}\| \leq K,\ \lambda \in \Sigma_{\pi/2\beta}$. This in combination with Theorem 2.17 implies that, for every $l > l'$, the operator $A_{(p!^s)}$ generates an analytic (a, k_l)-regularized resolvent of angle $(\pi/2)((1/\beta) - 1)$. In the meantime, $A_{(p!^s)}$ does not generate an exponentially bounded $(a, t^\alpha/\Gamma(\alpha + 1))$-regularized resolvent $(\alpha \geq 0)$ since $A_{(p!^s)}$ is not stationary dense.

(iii) [References [9, 28]; cf. also [29, Example 2.20]] Suppose $E := L^2[0, \pi]$ and $A := -\Delta$ with the Dirichlet or Neumann boundary conditions, $\beta \in [1/2, 1), \alpha > 1+\beta, a(t) = (t^{\beta-1}/\Gamma(\beta))$, and

$$h_{\alpha,\beta}(\lambda) := \frac{1}{\lambda^\alpha}\prod_{n=0}^{\infty}\frac{n^2-\lambda^\beta}{n^2+\lambda^\beta}, \quad \operatorname{Re}\lambda > 0,\ \lambda \neq n^{2/\beta},\ n \in \mathbb{N}. \tag{2.64}$$

Define $\mathbf{h}_{\alpha,\beta} : \Sigma_{\pi/2\beta} \to \mathbb{C}$ by setting: $\mathbf{h}_{\alpha,\beta}(\lambda) = h_{\alpha,\beta}(\lambda),\ \lambda \in \Sigma_{\pi/2\beta},\ \lambda \neq n^{2/\beta},\ n \in \mathbb{N}$, and $\mathbf{h}_{\alpha,\beta}(n^{2/\beta}) = 0,\ n \in \mathbb{N}$. Then the function $\mathbf{h}_{\alpha,\beta}(\lambda)$ is analytic, and there exists a constant $C > 0$ such that

$$\left\|\mathbf{h}_{\alpha,\beta}(\lambda)(I - \tilde{a}(\lambda)A)^{-1}\right\| \leq \frac{1 + C|\lambda|^\beta}{|\lambda|^\alpha}, \quad \lambda \in \Sigma_{\pi/2\beta}. \tag{2.65}$$

Let $k(t) = \mathcal{L}^{-1}(\mathbf{h}_{\alpha,\beta}(\lambda))(t), t \geq 0$. By Theorem 2.17, it follows that A generates an exponentially bounded, analytic (a, k)-regularized resolvent $(R(t))_{t\geq0}$ of angle $(\pi/2)(1/\beta - 1)$. Using the inverse Laplace transform, one can simply prove that $\|R(t)\| = O(t^{\alpha-1} + t^{\alpha+\beta-1}),\ t \geq 0$. Since Δ generates a cosine function, we are in a position to apply Theorem 2.17 to deduce that Δ generates an exponentially bounded, analytic (a, k)-regularized resolvent of angle $(\pi/2)(1/\beta - 1)$. By Proposition 2.29, we have that the biharmonic operator Δ^2, equipped with the suitable boundary conditions, generates an exponentially bounded, analytic $(a * a, k)$-regularized resolvent of angle $(\pi/2)(1/\beta - 1)$. Then the use of Theorem 2.30(ii) enables one to see that there exists a continuous kernel $k_1(t)$ such that Δ^2 generates an exponentially bounded, analytic (a, k_1)-regularized resolvent family of angle $\pi/2$. Keeping in mind the fact that $-\Delta^{2n}$ generates an analytic C_0-semigroup of angle $\pi/2$ (cf. for example [30, page 215]), one can prove that, for every $n \in \mathbb{N}$, there exists an exponentially bounded kernel $k_n(t)$ such that the polyharmonic operator Δ^{2^n} generates an exponentially bounded, (a, k_n)-regularized resolvent family of angle $\pi/2$ [15].

It has recently been proved that, in the case $\beta = 1$, there exists an exponentially bounded continuous kernel $K(t)$ such that A generates an exponentially bounded, analytic K-convoluted semigroup of angle $\pi/2$ [25]. Let us consider now the case $\beta \in (1,2)$ and $a(t) = (t^{\beta-1}/\Gamma(\beta))$. Choose a number $a \in (1/2, 1/\beta)$ and after that a number $s \in (1, 1/\beta a)$. Put $k_l(t) := \mathcal{L}^{-1}(1/\prod_{p=1}^{\infty} 1 + l\lambda/p^s)(t),\ t \geq 0$. Arguing as in [25], one yields that the function $\mathbf{h}(\lambda) = \prod_{n=0}^{\infty}(n^2 - \lambda)/(n^2 + \lambda),\ \lambda \in \mathbb{C} \setminus \{\pm n^2 : n \in \mathbb{N}_0\};\ \mathbf{h}(n^2) = 0,\ n \in \mathbb{N}$, is analytic, and that there exists $M > 0$ such that, for every $\gamma \in (0, \pi(1/\beta - 1/2))$, there exist $M_\gamma > 0$ and $c_\gamma > 0$ such that, for every $\lambda \in \Sigma_\gamma$,

$$\begin{aligned}
&\left\|\mathbf{h}\left(\lambda^b\right)(I - \tilde{a}(\lambda)A)^{-1}\right\| \\
&\quad \leq |\lambda|^b \max\left(\left(M + \frac{1}{|\lambda|^\beta}\right), M_\gamma\left(\frac{\left[\sqrt{2|\lambda|^\beta} + 1\right]}{|\lambda|^\beta} + \frac{\pi^2}{3}\right)\exp\left(c_\gamma|\lambda|^{\beta a}\right)\right).
\end{aligned} \tag{2.66}$$

Furthermore, there exists an exponentially bounded continuous kernel $k(t)$ such that $\tilde{k}(\lambda) = \tilde{k}_l(\lambda)\mathbf{h}(\lambda^\beta),\ \lambda \in \mathbb{C}, \operatorname{Re}\lambda > 0$. By (2.66), it follows that A generates an exponentially bounded, analytic (a,k)-regularized resolvent of angle $\pi(1/\beta - 1/2)$. Furthermore, an application of Theorem 2.17 gives that Δ generates an exponentially bounded, analytic (a,k)-regularized resolvent of angle $\pi(1/\beta - 1/2)$. By Proposition 2.29, we have that Δ^2 generates an exponentially bounded, analytic $(a * a, k)$-regularized resolvent of angle $\pi(1/\beta - 1/2)$. Arguing as in the case $\beta \in [1/2, 1)$, we have that, for every $n \in \mathbb{N}$, there exists an exponentially bounded kernel $k_n(t)$ such that the polyharmonic operator Δ^{2^n} generates an exponentially bounded, (a, k_n)-regularized resolvent family of angle $\pi/2$. In the case $\beta = 2$, it is known that A cannot be the generator of any exponentially bounded convoluted cosine function [15]; the case $\beta \in (0, 1/2)$ requires an additional analysis. Finally, it is worth noting that we can incorporate the above results in the study of the equation

$$\mathbf{D}_t^\beta u(t,x) = (-\Delta)^{2^n} u(t,x), \quad x \in (0,\pi),\ t > 0\ (n \in \mathbb{N}_0), \tag{2.67}$$

where $\mathbf{D}_t^\beta$ denotes the Caputo fractional derivative [29].

The next theorem generalizes [6, Theorem 3.6, Corollary 3.8] (cf. also [31, Theorem 2.1] and [32, Theorem 3]).

Theorem 2.32. (i) *Assume $C([0,\infty)) \ni a$ satisfies (P1), (H5) holds, $B \in L(E)$, $Rang\,(B) \subseteq Rang(C)$ that and A is a subgenerator of an exponentially bounded (a,a)-regularized C-resolvent family $(R(t))_{t\geq 0}$. Assume, further, that there exists $\omega \geq 0$ such that, for every $h \geq 0$ and for every function $f \in C([0,\infty) : E)$,*

- (Ma) $\int_0^h R(h-s)C^{-1}Bf(s)\,ds \in D(A)$,
- (Mb) $\|A\int_0^h R(h-s)C^{-1}Bf(s)\,ds\| \leq e^{\omega t}\mu_B(h)\|f\|_{[0,h]}, t \geq 0$, *where* $\|f\|_{[0,h]} := \sup_{t\in[0,h]}\|f(t)\|, \mu_B(t) : [0,\infty) \to [0,\infty)$ *is continuous, nondecreasing and satisfies* $\mu_B(0) = 0$,
- (Mc) *there exists an injective operator $C_1 \in L(E)$ such that $Rang(C_1) \subseteq Rang(C)$ and that $C_1A(I+B) \subseteq A(I+B)C_1$.*

Then $A(I+B)$ is a subgenerator of an exponentially bounded (a,a)-regularized C_1-resolvent family $(S(t))_{t\geq 0}$ which satisfies the following integral equation

$$S(t)x = R(t)C^{-1}C_1x + A\int_0^t R(t-s)C^{-1}BS(s)x\,ds, \quad t\geq 0,\ x\in E. \tag{2.68}$$

(ii) *Let A be a subgenerator of an exponentially bounded, once integrated C-cosine function and let ω, B, and C_1 be as in (i). Then $A(I+B)$ is a subgenerator of an exponentially bounded, once integrated C_1-cosine function.*

Remark 2.33. (i) Assume that A is a subgenerator of an exponentially bounded (a,a)-regularized C-resolvent family $(R(t))_{t\geq 0}$ and that a Banach space $(Z, |\cdot|_Z)$ satisfies the conditions (Za), (Zb), and (Zc) given in the formulation of [6, Definition 4.1]. (In particular, these conditions hold for $[D(A)]$.) Then (Ma) and (Mb) are fulfilled if $C^{-1}B\in L(X,Z)$.

(ii) (References [32, 33]) Let $B\in L(E)$, and let $BC=CB$.

(ii.1) Assume that BA is a subgenerator of a (local) (a,k)-regularized C-resolvent family, and (H5) holds for BA and C. Then AB is a subgenerator of an (a,k)-regularized C-resolvent family.

(ii.2) Assume that AB is a subgenerator of a (local) (a,k)-regularized C-resolvent family and (H5) holds for AB and C. Then BA is a subgenerator of an (a,k)-regularized C-resolvent family, provided $\rho(BA)\neq\emptyset$.

The proof of the next generalization of [15, Proposition 3] is provided for the sake of completeness.

Theorem 2.34. *Assume that $\tau\in(0,\infty]$, $L^1_{\mathrm{loc}}([0,\tau))\ni a_1$ is a kernel, $L^1_{\mathrm{loc}}([0,\tau))\ni k$ is a kernel, $a(t)=(a_1*a_1)(t)$, $t\in[0,\tau)$, and $k_1(t)=(k*a_1)(t)$, $t\in[0,\tau)$. Put $\mathcal{A}\equiv\begin{pmatrix}0 & I\\ A & 0\end{pmatrix}$, $\mathcal{C}\equiv\begin{pmatrix}C & 0\\ 0 & C\end{pmatrix}$, and assume that (H5) holds. Then A is a subgenerator of an (a,k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$ if and only if $\mathcal{A}$ is a subgenerator of an (a_1,k_1)-regularized $\mathcal{C}$-resolvent family $(S(t))_{t\in[0,\tau)}$. If this is the case, then we have*

$$S(t)=\begin{pmatrix}(a_1*R)(t) & (a*R)(t)\\ R(t)-k(t)C & (a_1*R)(t)\end{pmatrix}, \quad 0\leq t<\tau, \tag{2.69}$$

and the integral generators of $(R(t))_{t\in[0,\tau)}$ and $(S(t))_{t\in[0,\tau)}$, denoted respectively by B and $\mathcal{B}$, satisfy $\mathcal{B}=\begin{pmatrix}0 & I\\ B & 0\end{pmatrix}$.

Proof. It is immediately verified that $(S(t))_{t\in[0,\tau)}$ is a nondegenerate, strongly continuous operator family in $E\times E$ which satisfies $S(t)\mathcal{A}\subseteq\mathcal{A}S(t)$ and $S(t)\mathcal{C}=\mathcal{C}S(t)$, $0\leq t<\tau$. Furthermore, the function $k_1(t)$ is a continuous kernel, $S(0)=0=k_1(0)\mathcal{C}$, and $\mathcal{C}\mathcal{A}\subseteq\mathcal{A}\mathcal{C}$.

Let $x \in D(A)$, and let $y \in E$. Then a simple computation involving (H5) shows that, for every $t \in [0,\tau)$,

$$\begin{aligned}
\int_0^t a_1(t-s)\mathcal{A}S(s)\begin{pmatrix} x \\ y \end{pmatrix} ds &= \int_0^t a_1(t-s)\begin{pmatrix} R(s)x - k(s)Cx + (a_1 * R)(s)y \\ A(a_1 * R)(s)x + A(a * R)(s)y \end{pmatrix} ds \\
&= \begin{pmatrix} (a_1 * R)(t)x - (a_1 * k)(t)Cx + (a_1 * a_1 * R)(t)y \\ A(a * R)(t)x + (a_1 * (R - kC))(t)y \end{pmatrix} \\
&= \begin{pmatrix} (a_1 * R)(t)x - (a_1 * k)(t)Cx + (a_1 * a_1 * R)(t)y \\ R(t)x - k(t)Cx + (a_1 * (R - kC))(t)y \end{pmatrix} \\
&= S(t)\begin{pmatrix} x \\ y \end{pmatrix} - k_1(t)\mathcal{C}\begin{pmatrix} x \\ y \end{pmatrix}.
\end{aligned} \tag{2.70}$$

Assume now that $\mathcal{A}$ is a subgenerator of an (a_1, k_1)-regularized $\mathcal{C}$-resolvent family $(S(t))_{t\in[0,\tau)}$. Put $S(t) = \begin{pmatrix} S^1(t) & S^2(t) \\ S^3(t) & S^4(t) \end{pmatrix}_{t\in[0,\tau)}$, where $S^i(t) \in L(E), i \in \{1,2,3,4\}$, and $0 \le t < \tau$. A simple consequence of $S(t)\mathcal{C} = \mathcal{C}S(t),\ t \in [0,\tau)$ is $S^i(t)C = CS^i(t),\ t \in [0,\tau),\ i \in \{1,2,3,4\}$. Since $S(t)\mathcal{A} \subseteq \mathcal{A}S(t),\ t \in [0,\tau)$, one gets

$$\begin{aligned}
&S^1(t)x + S^2(t)y \in D(A), \\
&S^1(t)y + S^2(t)Ax = S^3(t)x + S^4(t)y, \\
&S^3(t)y + S^4(t)Ax = A\left(S^1(t)x + S^2(t)y\right), \quad 0 \le t < \tau,\ x \in D(A),\ y \in E.
\end{aligned} \tag{2.71}$$

Hence, $S^3(t)x = S^2(t)Ax, x \in D(A)$, and $S^3(t)y = AS^2(t)y, y \in E, 0 \le t < \tau$. This implies that, for every $x \in D(A), S^3(t)Ax = AS^2(t)Ax = AS^3(t)x,\ t \in [0,\tau)$. Thereby, $S^3(t)A \subseteq AS^3(t), t \in [0,\tau)$, and $(R(t) \equiv S^3(t) + k(t)C)_{t\in[0,\tau)}$ is a strongly continuous operator family in E satisfying $R(0) = k(0)C, R(t)C = CR(t)$ and $R(t)A \subseteq AR(t),\ 0 \le t < \tau$. Since, for every $\lambda \in \mathbb{C}, \lambda \in \rho_{\mathcal{C}}(\mathcal{A})$ if and only if $\lambda^2 \in \rho_C(A)$ [25], we have that (H5) holds for $\mathcal{A}$ and $\mathcal{C}$. Since, for every $x \in E$ and $y \in E$,

$$\mathcal{A}\begin{pmatrix} (a_1 * S_1)(t) & (a_1 * S_2)(t) \\ (a_1 * S_3)(t) & (a_1 * S_4)(t) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} S_1(t) & S_2(t) \\ S_3(t) & S_4(t) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - k_1(t)\mathcal{C}\begin{pmatrix} x \\ y \end{pmatrix}, \tag{2.72}$$

one gets $(a_1 * S_3)(t)x = S_1(t)x - k_1(t)Cx,\ (a_1 * S_4)(t)x = S_2(t)x, A(a_1 * S_3)(t)x = S_3(t)x$ and $A(a_1 * S_2)(t)x = S_4(t)x - k_1(t)Cx,\ 0 \le t < \tau$. Hence, $A(a * R)(t)x = A(a * (S_3 + kC))(t)x = A(a_1 * a_1 * (S_3 + kC))(t)x = A(a_1 * (S_1 - k_1C + (a_1 * k)C))(t)x = S_3(t)x = R(t)x - k(t)Cx,\ t \in [0,\tau)$. This implies that $(R(t))_{t\in[0,\tau)}$ is a nondegenerate operator family, and we finally get that $(R(t))_{t\in[0,\tau)}$ is an (a,k)-regularized C-resolvent family with a subgenerator A. The remnant of the proof follows from a slight technical modification of the final part of the proof of [15, Proposition 3]. □

Remark 2.35. (i) Let $\tau = \infty$, and let $k(t)$ and $a_1(t)$ be exponentially bounded. Then $(R(t))_{t\in[0,\tau)}$ is exponentially bounded if and only if $(S(t))_{t\in[0,\tau)}$ is exponentially bounded.

(ii) Let $j \in \mathbb{N}, \alpha > 0,\ a_1(t) = t^{\alpha-1}/\Gamma(\alpha)$ and $k \in C^j((0,\tau))$, respectivley, $k \in C^\infty((0,\tau))$, and let the mapping $t \mapsto R(t),\ t \in (0,\tau)$ be j-times differentiable, respectivley infinitely differentiable. Then the mapping $t \mapsto S(t), t \in (0,\tau)$ is also j-times differentiable, resp. infinitely differentiable. Furthermore, if $k(t)$ is of class C^L, resp. C_L (ρ-hypoanalytic, $1 \leq \rho < \infty$) and $(R(t))_{t\in[0,\tau)}$ is of class C^L, resp. C_L (ρ-hypoanalytic), then $(S(t))_{t\in[0,\tau)}$ is also of class C^L, resp. C_L (ρ-hypoanalytic).

(iii) Let $a_1(t) = 1/\sqrt{\pi t}$ and $k_1(t) = t^{n-(1/2)}/\Gamma(n-(1/2)), n \in \mathbb{N}$. Then Theorem 2.34 enables one to discuss the maximal interval of existence of a local (a_1, k_1)-regularized C-resolvent family and to construct an example of a local (a_1, k_1)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$ which cannot be extended beyond the interval $[0,\tau)$; combining with [25, Examples 1, 3, 5] and [34, Theorem 3.1], it is possible to construct examples of infinitely differentiable, nonanalytic (a_1, k_1)-regularized C-resolvent families and examples of (pseudo)differential operators generating (a_1, k_1)-regularized C-resolvent families of class C^L.

(iv) Assume $a_1(t) = 1/\sqrt{\pi t}, A$ is a subgenerator of a (local) K-convoluted C-semigroup $(S_K(t))_{t\in[0,\tau)}, k(t) = \int_0^t K(s)ds,\ t \in [0,\tau)$, and (H3) holds (see Theorem 2.2(iii)). Let $k_1(t)$ possess the same meaning as in Theorem 2.34. Then, for every $x \in D(A)$ and $y \in E$, the system of integral equations

$$
\begin{aligned}
& u \in C([0,\tau) : [D(A)]), \quad v \in C([0,\tau) : E), \\
& u(t) = k_1(t)Cx + \int_0^t \frac{v(s)ds}{\sqrt{\pi(t-s)}}, \quad t \in [0,\tau), \\
& v(t) = k_1(t)Cy + \int_0^t \frac{Au(s)ds}{\sqrt{\pi(t-s)}}, \quad t \in [0,\tau),
\end{aligned}
\tag{2.73}
$$

has a unique solution.

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