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Research Article

The Numerical Range of Toeplitz Operator on the Polydisk

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The numerical range and normality of Toeplitz operator acting on the Bergman space and pluriharmonic Bergman space on the polydisk is investigated in this paper.

1. Introduction

Let H be a Hilbert space with an inner product, and let T be a bounded linear operator on H. The numerical range $W(T)$ of T is the subset of the complex plane $\mathbb{C}$ defined by

$$W(T) = \{\langle Tx, x\rangle : x \in H,\ \langle x, x\rangle = 1\}. \tag{1.1}$$

It is well known that $W(T)$ is a convex set whose closure contains the spectrum of T, which denoted by $\sigma(T)$. If T is a normal operator, then the closure of $W(T)$ is the convex hull of $\sigma(T)$. Moreover, it is also well known that each extreme point of $W(T)$ is an eigenvalue of T. See [1, 2] for more information of the numerical range of a operator.

Brown and Halmos in [3] and Klein in [4] studied the numerical range of arbitrary Toeplitz operator on the Hardy space of the unit disk. Thukral studied the numerical range of Toeplitz operator with harmonic symbol on the Bergman space of the unit disk in [2]. On the Bergman space and pluriharmonic Bergman space of the unit ball, the numerical range and normality of Toeplitz operator was described in [5].

In this paper, we consider the same problem on the Bergman space and pluriharmonic Bergman space of the polydisk. We first study some relations between the numerical range and normality of the Toeplitz operator with n-harmonic function symbols acting on the Bergman space on the polydisk. Next, we consider the same problem on the pluriharmonic Bergman space of the polydisk. Our results show that as case of the ball hold on the polydisk.

2. Toeplitz Operators on the Bergman Space of the Polydisk

Let D be the unit disk in the complex plane. For a fixed integer n, the unit polydisk D^n is the cartesian product of n copies of D. Let $L^2(D^n)$ denote the usual Lebesgue space with respect to the volume measure $V = V_n$ on D^n normalized to have total mass 1. The Bergman space $A^2(D^n)$ is the closed subspace of $L^2(D^n)$ consisting of all holomorphic functions on D^n. Some information on Bergman-type spaces on the polydisk (including Bergman projections) can be found, for example, in [6–10] (see also the reference therein).

Let P be the orthogonal projection from $L^2(D^n)$ onto $A^2(D^n)$. The Toeplitz operator $T_u : A^2(D^n) \to A^2(D^n)$ with symbol $u \in L^\infty(D^n)$ is the linear operator defined by

$$T_u f = P(uf) \tag{2.1}$$

for functions $f \in A^2(D^n)$.

A function $u \in C^2(D^n)$ is called n-harmonic if u is harmonic in each variable separately. More explicitly, u is n-harmonic if

$$\partial_j \overline{\partial}_j u = 0, \quad j = 1, 2, \ldots, n. \tag{2.2}$$

Here ∂_j denotes the complex partial differentiation with respect to the jth variable.

Recall that a complex-valued function $f \in C^2(D^n)$ is said to be pluriharmonic if

$$\partial_j \overline{\partial}_k f = 0, \quad j, k = 1, 2, \ldots, n. \tag{2.3}$$

Note that each pluriharmonic function is n-harmonic function.

In this section, we give characterizations of Toeplitz operator with symbols n-harmonic acting on the Bergman space on the polydisk. For this purpose, we need the following result (see [11]).

Lemma 2.1. *Let $u \in L^\infty(D^n)$. Then u is an n-harmonic function on D^n if and only if*

$$\int_{D^n} u \circ \varphi_a \, dV = u(a) \tag{2.4}$$

for every $a \in D^n$.

Theorem 2.2. *Let u be a bounded n-harmonic function on D^n. Then $u(D^n) \subset W(T_u)$.*

Proof. For each $a \in D^n$, let k_a denote the normalized kernel, namely,

$$k_a(z) = \frac{K_a(z)}{\sqrt{K_a(a)}} = \prod_{j=1}^{n} \frac{\left(1 - |a_j|^2\right)}{\left(1 - \overline{a_j} z_j\right)^2}. \tag{2.5}$$

It is obvious that $k_a \in A^2(D^n)$ and

$$\langle k_a, k_a \rangle = \int_{D^n} |k_a|^2 dV = 1 \tag{2.6}$$

for every $a \in D^n$.

For each $a = (a_1, a_2, \ldots, a_n) \in D^n$, we let $\varphi_a = (\varphi_{a_1}, \varphi_{a_2}, \ldots, \varphi_{a_n})$, where each φ_{a_i} is the usual Möbius map on D given by

$$\varphi_{a_i}(z_i) = \frac{a_i - z_i}{1 - \overline{a_i} z_i}, \tag{2.7}$$

then φ_a is an automorphism of D^n, and $\varphi_a \circ \varphi_a$ is the identity on D^n. Since the real Jacobian of φ_a is given by $|k_a(z)|^2$, we have

$$\int_{D^n} h \circ \varphi_a dV = \int_{D^n} h |k_a(z)|^2 dV \tag{2.8}$$

whenever the integrals make sense. In particular, by Lemma 2.1, we obtain

$$\int_{D^n} h \circ \varphi_a dV = h(a) \tag{2.9}$$

for function h integrable and holomorphic on D^n. Hence

$$\begin{aligned} \langle T_u k_a, k_a \rangle &= \langle P(u k_a), k_a \rangle \\ &= \langle u k_a, k_a \rangle \\ &= \int_{D_n} u |k_a|^2 dV \\ &= \int_{D_n} u \circ \varphi_a dV = u(a) \end{aligned} \tag{2.10}$$

for every $a \in D^n$. Therefore $u(D^n) \subset W(T_u)$. □

Recall that $T_u \geq 0$ means $\langle T_u f, f \rangle \geq 0$ for every $f \in A^2(D^n)$. Using Theorem 2.2, we obtain the following result.

Theorem 2.3. *Let u be a bounded n-harmonic function on D^n. Then $T_u \geq 0$ if and only if $u \geq 0$.*

Proof. First we assume that $T_u \geq 0$. By the definition of $W(T_u)$, we have $W(T_u) \subset [0, \infty)$. By Theorem 2.2, we see that

$$u(D^n) \subset W(T_u) \subset [0, \infty), \tag{2.11}$$

that is, $u \geq 0$.

Conversely, suppose that $u \geq 0$. For every $f \in A^2(D^n)$, we have

$$\langle T_u f, f\rangle = \langle P(uf), f\rangle = \langle uf, f\rangle = \int_{D^n} u|f|^2 dV \geq 0. \tag{2.12}$$

Hence $T_u \geq 0$ by the arbitrary of f. The proof of the theorem is completed. □

Theorem 2.4. *Let u be a bounded n-harmonic function on D^n. If $W(T_u)$ lies in the upper half-plane and contains 0, then T_u must be self-adjoint.*

Proof. We modify the proof of Theorem 3 in [5]. From the assumption, we have $\mathrm{Im}\langle T_u f, f\rangle \geq 0$ for every $f \in A^2(D^n)$. In addition,

$$\mathrm{Im}\langle T_u f, f\rangle = \mathrm{Im}\langle P(uf), f\rangle = \mathrm{Im}\langle uf, f\rangle = \int_{D^n} (\mathrm{Im}u)|f|^2 dV = \langle T_{\mathrm{Im}u} f, f\rangle \tag{2.13}$$

for every $f \in A^2(D^n)$. Hence $T_{\mathrm{Im}u} \geq 0$ by the arbitrary of f. By Theorem 2.3, we have $\mathrm{Im}u \geq 0$.

On the other hand, since $W(T_u)$ contains 0, there exist some $g \in A^2$ with $\langle g, g\rangle = 1$ such that $\langle T_u g, g\rangle = 0$. Therefore,

$$0 = \mathrm{Im}\langle T_u g, g\rangle = \mathrm{Im}\int_{D^n} u|g|^2 dV = \int_{D^n} (\mathrm{Im}u)|g|^2 dV. \tag{2.14}$$

We obtain $(\mathrm{Im}u)|g|^2 = 0$ by the fact that $\mathrm{Im}u \geq 0$. Because $g \neq 0$, we see that $\mathrm{Im}u = 0$. It follows that u is real and T_u is self-adjoint. □

Theorem 2.5. *Let u be a bounded n-harmonic function on D^n. If $W(T_u)$ is not open in $\mathbb{C}$, then T_u is normal on $A^2(D^n)$.*

Proof. The proof is similar to the proof of Theorem 4 of [5]. We omit the details. □

Since an open convex set is the interior of its closure, we obtain the following corollary.

Corollary 2.6. *Let u be a bounded n-harmonic function on D^n. If T_u is not normal on $A^2(D^n)$, then $W(T_u)$ is the interior of its closure.*

Lemma 2.7 (See [1]). *If $W(T)$ is a line segment, then T is normal.*

We will consider the problem of when the converse of this fact is also true. First, we prove the following three results.

Proposition 2.8. *Let u be bounded real n-harmonic on D^n. If u is nonconstant, then $m, M \notin W(T_u)$, where $m = \inf u$ and $M = \sup u$.*

Proof. If $m \in W(T_u)$, then m is an extreme point of $W(T_u)$ and hence is an eigenvalue of T_u. Therefore, there exists a nonzero $f \in A^2(D^n)$ such that $T_u f = mf$, that is, $P(uf - mf) = 0$. We obtain

$$0 = \langle P(uf - mf), f \rangle = \langle (u - m)f, f \rangle = \int_{D^n} (u - m)\left|f\right|^2 dV. \tag{2.15}$$

Since $u - m \geq 0$ on D^n, we get $(u - m)|f|^2 = 0$. Because f is nonzero, we obtain $u = m$. Therefore u is a constant, which is a contradiction. So $m \notin W(T_u)$.

Similarly to the above proof we get $M \notin W(T_u)$. □

Theorem 2.9. *Let u be a bounded real function on D^n. Then $\sigma(T_u) \subset [m, M]$, where $m = \inf u$ and $M = \sup u$.*

Proof. Suppose that $\lambda \notin [m, M]$, then either $u - \lambda > 0$ or $u - \lambda < 0$ on D^n. First we assume that $u - \lambda > 0$ and choose $\epsilon > 0$ such that

$$\sup_{z \in D^n} |\epsilon(u(z) - \lambda) - 1| < 1. \tag{2.16}$$

We obtain

$$\|T_{\epsilon(u-\lambda)} - I\| = \|T_{\epsilon(u-\lambda)-1}\| \leq \|\epsilon(u - \lambda) - 1\|_\infty < 1. \tag{2.17}$$

It follows from the last inequality that $T_{\epsilon(u(z)-\lambda)}$ and $T_{u-\lambda}$ are invertible. Because $T_{u-\lambda} = T_u - \lambda$, we get $\lambda \notin \sigma(T_u)$.

Now we assume that $u - \lambda < 0$ on D^n. From the above proof and the facts that $-u + \lambda > 0$ and

$$T_u - \lambda = T_{u-\lambda} = -T_{-u+\lambda}, \tag{2.18}$$

we get the desired result. □

Theorem 2.10. *Let u be a bounded nonconstant real n-harmonic function on D^n. Then*

$$W(T_u) = (m, M), \quad \textit{where } m = \inf u, \ M = \sup u. \tag{2.19}$$

Proof. Using Proposition 2.8, Theorems 2.2 and 2.9, similarly to the proof of Theorem 8 of [5], we get the desired result. We omit the details. □

Lemma 2.11. *Let u be a bounded pluriharmonic function on D^n. Then T_u is normal on $A^2(D^n)$ if and only if $u(D^n)$ is a part of a line in $\mathbb{C}$.*

Proof. The proof is similar to [12, Proposition 13]. We omit the details. □

Theorem 2.12. *Let u be a bounded nonconstant pluriharmonic function on D^n. If T_u is normal on $A^2(D^n)$, then $W(T_u)$ is an open line segment.*

Proof. By Lemma 2.11, $u(D^n)$ is a part of a line in $\mathbb{C}$ when T_u is normal. Therefore, there exist constants $s, t \in \mathbb{C}$ and a nonconstant bounded real pluriharmonic function v such that $u = sv + t$ on D^n. Since each pluriharmonic function is n-harmonic function, by Theorem 2.10, we have $W(T_v) = (m, M)$, where $m = \inf v$ and $M = \sup v$. For a given bounded linear operator T on a Hilbert space, we note that

$$W(\alpha T + \beta) = \alpha W(T) + \beta, \quad \alpha, \beta \in \mathbb{C}. \tag{2.20}$$

It follows from $T_u = sT_v + t$ that

$$W(T_u) = sW(T_v) + t = (sm + t, sM + t). \tag{2.21}$$

Therefore $W(T_u)$ is an open line segment. □

3. Toeplitz Operators on the Pluriharmonic Bergman Space

In this section, we consider the same problem for Toeplitz operators acting on the pluriharmnoic Bergman space in the polydisk. The pluriharmonic Bergman space $b^2(D^n)$ is the space of all pluriharmonic functions in $L^2(D^n)$. It is well known that $b^2(D^n)$ is a closed subspace of L^2 and hence is a Hilbert space. Hence, for each $z \in D^n$, there exists a unique function $R_z \in b^2(D^n)$ called the pluriharmonic Bergman kernel, which has the following reproducing property:

$$f(z) = \int_{D^n} f(w)\overline{R_z(w)}dV(w) \tag{3.1}$$

for every $f \in b^2$. From this reproducing formula, it follows that the orthogonal projection Q from $L^2(D^n)$ onto $b^2(D^n)$ is realized as an integral operator

$$Q(\varphi)(z) = \int_{D^n} \varphi(w)\overline{R_z(w)}dV_n(w), \quad z \in D^n \tag{3.2}$$

for $\varphi \in L^2$.

It is well known that a function $f \in C^2(D^n)$ is pluriharmonic if and only if it admits a decomposition $f = g + \overline{h}$, where g and h are holomorphic. Furthermore, if $f \in b^2(D^n)$, then it is not hard to see $g, h \in A^2(D^n)$. Hence

$$b^2(D^n) = A^2(D^n) + \overline{A^2(D^n)}. \tag{3.3}$$

Therefore

$$R_z = K_z + \overline{K_z} - 1, \tag{3.4}$$

where K_z is the well-known holomorphic Bergman kernel. By (3.2) and (3.4), we see that Q admits the following integral representation:

$$Q(\varphi)(z) = \int_{D^n} \varphi(w)\left(\overline{K_z(w)} + K_z(w) - 1\right) dV(w), \quad z \in D^n \tag{3.5}$$

for $\varphi \in L^2(D^n)$.

Let $u \in L^2(D^n)$. The Toeplitz operator t_u with symbol u is defined by

$$t_u f = Q(uf) \tag{3.6}$$

for $f \in b^2(D^n)$. The operator t_u is densely defined. In fact, we have $Q(uf) \in b^2(D^n)$ for any $f \in H^\infty(D^n)$. Using the the same arguments as the Section 2, we have the following results.

Theorem 3.1. *Let u be a bounded n-harmonic function on D^n. Then $u(D^n) \subset W(t_u)$.*

Theorem 3.2. *Let u be bounded n-harmonic function on D^n. Then $t_u \geq 0$ if and only if $u \geq 0$.*

Theorem 3.3. *Let u be bounded n-harmonic function on D_n. If $W(t_u)$ lies in the upper half-plane and contains 0, then t_u must be self-adjoint.*

Theorem 3.4. *Let u be bounded n-harmonic function in $b^2(D^n)$. If $W(t_u)$ is not open in $\mathbb{C}$, then t_u is normal on $b^2(D^n)$.*

Theorem 3.5. *Let u be nonconstant real n-harmonic function on D_n. Then one has $W(t_u) = (m, M)$ where $m = \inf u$ and $M = \sup u$.*

We also need a corresponding result of Lemma 2.11 (see [13, Theorem 1.2]).

Lemma 3.6. *Let u be a bounded pluriharmonic function on D_n. Then t_u is normal on $b^2(D^n)$ if and only if $u(D_n)$ is a part of a line in $\mathbb{C}$.*

Theorem 3.7. *Let u be bounded nonconstant pluriharmonic function on D_n. If T_u is normal on $b^2(D^n)$, then $W(t_u)$ is an open line segment.*

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