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Research Article

Multiplicity Results for p-Laplacian with Critical Nonlinearity of Concave-Convex Type and Sign-Changing Weight Functions

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The multiple results of positive solutions for the following quasilinear elliptic equation: $-\Delta_p u = \lambda f(x)|u|^{q-2}u + g(x)|u|^{p^*-2}u$ in $\Omega, u = 0$ on $\partial\Omega$, are established. Here, $0 \in \Omega$ is a bounded smooth domain in $\mathbb{R}^N, \Delta_p$ denotes the p-Laplacian operator, $1 \le q < p < N, p^* = Np/(N-p), \lambda$ is a positive real parameter, and f, g are continuous functions on $\overline{\Omega}$ which are somewhere positive but which may change sign on Ω. The study is based on the extraction of Palais-Smale sequences in the Nehari manifold.

1. Introduction

In this paper, we study the multiple results of positive solutions for the following quasilinear elliptic equation:

$$\begin{gathered} -\Delta_p u = \lambda f(x)|u|^{q-2}u + g(x)|u|^{p^*-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{$E_{\lambda f,g}$}$$

where $\lambda > 0$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $0 \in \Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $1 < q < p < N$, $p^* = Np/(N-p)$ is the so-called critical Sobolev exponent and the weight functions f, g are satisfying the following conditions:

($f1$) $f \in C(\overline{\Omega})$ and $f^+ = \max\{f, 0\} \not\equiv 0$;

($f2$) there exist β_0, $\rho_0 > 0$ and $x_0 \in \Omega$ such that $B(x_0, 2\rho_0) \subset \Omega$ and $f(x) \ge \beta_0$ for all $x \in B(x_0, 2\rho_0)$. Without loss of generality, we assume that $x_0 = 0$,

($g1$) $g \in C(\overline{\Omega})$ and $g^+ = \max\{g, 0\} \not\equiv 0$;

($g2$) $|g^+|_\infty = g(0) = \max_{x\in\overline{\Omega}} g(x)$;

$(g3)$ $g(x) > 0$ for all $x \in B(0, 2\rho_0)$;

$(g4)$ there exists $\beta > N/(p-1)$ such that

$$g(x) = g(0) + o\left(|x|^{\beta}\right) \quad \text{as } x \longrightarrow 0. \tag{1.1}$$

For the weight functions $f \equiv g \equiv 1$, $(E_{\lambda f,g})$ has been studied extensively. Historically, the role played by such concave-convex nonlinearities in producing multiple solutions was investigated first in the work [1]. They studied the following semilinear elliptic equation:

$$\begin{aligned} -\Delta u &= \lambda u^{q-1} + u^{2^*-1} \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

for $1 < q < 2$ and showed the existence of $\lambda_0 > 0$ such that (1.2) admits at least two solutions for all $\lambda \in (0, \lambda_0)$ and no solution for $\lambda > \lambda_0$. Subsequently, in the work [2, 3], the corresponding quasilinear version has been studied

$$\begin{aligned} -\Delta_p u &= \lambda u^{q-1} + u^{p^*-1} \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

where $1 < p < N$ and $1 < q < p$. They obtained results similar to the results of [1] above, but only for some ranges of the exponents p and q. We summarize their results in what follows.

Theorem 1.1 (see [2, 3]). *Assume that either $2N/(N+2) < p < 3$ or $p > 3, p > q > p^* - 2/(p-1)$. Then there exists $\lambda_0 > 0$ such that (1.3) admits at least two solutions for all $\lambda \in (0, \lambda_0)$ and no solution for $\lambda > \lambda_0$.*

It is possible to get complete multiplicity result for problem (1.3) if Ω is taken to be a ball in $\mathbb{R}^N$. Prashanth and Sreenadh [4] have studied (1.3) in the unit ball $B^N(0;1)$ in $\mathbb{R}^N$ and obtained the following results.

Theorem 1.2 (see [4]). *Let $\Omega = B^N(0;1), 1 < p < N, 1 < q < p$. Then there exists $\lambda_0 > 0$ such that (1.3) admits at least two solutions for all $\lambda \in (0, \lambda_0)$ and no solution for $\lambda > \lambda_0$. Additionally, if $1 < p < 2$, then (1.3) admits exactly two solutions for all small $\lambda > 0$.*

For $p = 2$, Tang [5] has studied the exact multiplicity about the following semilinear elliptic equation:

$$\begin{aligned} -\Delta u &= \lambda u^{q-1} + u^{r-1} \quad \text{in } B^N(0;1), \\ u &> 0 \quad \text{in } B^N(0;1), \\ u &= 0 \quad \text{on } \partial B^N(0;1), \end{aligned} \tag{1.4}$$

where $1 < q < 2 < r \leq 2N/(N-2)$ and $N \geq 3$. We also mention his result below.

Theorem 1.3 (see [5]). *There exists* $\lambda_0 > 0$ *such that* (1.4) *admits exactly two solutions for* $\lambda \in (0, \lambda_0)$*, exactly one solution for* $\lambda = \lambda_0$*, and no solution for* $\lambda > \lambda_0$.

To proceed, we make some motivations of the present paper. Recently, in [6] the author has considered (1.2) with subcritical nonlinearity of concave-convex type, $g \equiv 1$, and f is a continuous function which changes sign in $\overline{\Omega}$, and showed the existence of $\lambda_0 > 0$ such that (1.2) admits at least two solutions for all $\lambda \in (0, \lambda_0)$ via the extraction of Palais-Smale sequences in the Nehair manifold. In a recent work [7], the author extended the results of [6] to the quasilinear case with the more general weight functions f, g but also having subcritical nonlinearity of concave-convex type. In the present paper, we continue the study of [7] by considering critical nonlinearity of concave-convex type and sign-changing weight functions f, g.

In this paper, we use a variational method involving the Nehari manifold to prove the multiplicity of positive solutions. The Nehari method has been used also in [8] to prove the existence of multiple for a singular elliptic problem. The existence of at least one solution can be obtained by using the same arguments as in the subcritical case [7]. The existence of a second solution needs different arguments due to the lack of compactness of the Palais-Smale sequences. For what, we need addtional assumptions $(f2)$ and $(g2)$ to prove the compactness of the extraction of Palais-Smale sequences in the Nehari manifold (see Theorem 4.4). The multiplicity result is proved only for the parameter $\lambda \in (0, (q/p)\Lambda_1)$ (see Theorem 1.5) but for all $1 < p < N$ and $1 \le q < p$. This is not the case in the papers referred [2, 3] where the multiplicity is global but not with the full range of p, q and with the weight functions $f \equiv g \equiv 1$. Finally, we mention a recent contribution on p-Laplacian equation with changing sign nonlinearity by Figuereido et al. [9] which gives the global multiplicity but not with the full range of p and q. The method used in the paper by Figuereido et al. is similar to the method introduced in [1].

In order to represpent our main results, we need to define the following constant Λ_1. Set

$$\Lambda_1 = \left(\frac{p-q}{(p^* - q)|g^+|_\infty} \right)^{(p-q)/(p^*-p)} \left(\frac{p^* - p}{(p^* - q)|f^+|_\infty} \right) |\Omega|^{(q-p^*)/p^*} S^{(N/p)-(N/p^2)q+(q/p)} > 0, \quad (1.5)$$

where $|\Omega|$ is the Lebesgue measure of Ω and S is the best Sobolev constant (see (2.2)).

Theorem 1.4. *Assume* $(f1)$ *and* $(g1)$ *hold. If* $\lambda \in (0, \Lambda_1)$*, then* $(E_{\lambda f,g})$ *admits at least one positive solution* $u_\lambda \in C^{1,\alpha}(\Omega)$ *for some* $\alpha \in (0, 1)$.

Theorem 1.5. *Assume that* $(f1)$-$(f2)$ *and* $(g1)$–$(g4)$ *hold. If* $\lambda \in (0, (q/p)\Lambda_1)$*, then* $(E_{\lambda f,g})$ *admits at least two positive solutions* $u_\lambda, U_\lambda \in C^{1,\alpha}(\Omega)$ *for some* $\alpha \in (0, 1)$.

This paper is organized as follows. In Section 2, we give some preliminaries and some properties of Nehari manifold. In Sections 3 and 4, we complete proofs of Theorems 1.4 and 1.5.

2. Preliminaries and Nehari Manifold

Throughout this paper, $(f1)$ and $(g1)$ will be assumed. The dual space of a Banach space E will be denoted by E^{-1}. $W_0^{1,p}(\Omega)$ denotes the standard Sobolev space with the following norm:

$$\|u\|^p = \int_\Omega |\nabla u|^p dx. \tag{2.1}$$

$W_0^{1,p}(\Omega)$ with the norm $\|\cdot\|$ is simply denoted by W. We denote the norm in $L^p(\Omega)$ by $|\cdot|_p$ and the norm in $L^p(\mathbb{R}^N)$ by $|\cdot|_{L^p(\mathbb{R}^N)}$. $|\Omega|$ is the Lebesgue measure of Ω. $B(x,r)$ is a ball centered at x with radius r. $O(\varepsilon^t)$ denotes $|O(\varepsilon^t)|/\varepsilon^t \leq C$, $o(\varepsilon^t)$ denotes $|o(\varepsilon^t)|/\varepsilon^t \to 0$ as $\varepsilon \to 0$, and $o_n(1)$ denotes $o_n(1) \to 0$ as $n \to \infty$. C, C_i will denote various positive constants; the exact values of which are not important. S is the best Sobolev embedding constant defined by

$$S = \inf_{u\in W\setminus\{0\}} \frac{|\nabla u|_p^p}{|u|_{p^*}^p}. \tag{2.2}$$

Definition 2.1. Let $c \in \mathbb{R}$, E be a Banach space and $I \in C^1(E,\mathbb{R})$.

(i) $\{u_n\}$ is a $(\mathrm{PS})_c$-sequence in E for I if $I(u_n) = c + o_n(1)$ and $I'(u_n) = o_n(1)$ strongly in E^{-1} as $n \to \infty$.

(ii) We say that I satisfies the $(\mathrm{PS})_c$ condition if any $(\mathrm{PS})_c$-sequence $\{u_n\}$ in E for I has a convergent subsequence.

Associated with $(E_{\lambda f,g})$, we consider the energy functional J_λ in W, for each $u \in W$,

$$J_\lambda(u) = \frac{1}{p}\|u\|^p - \frac{\lambda}{q}\int_\Omega f|u|^q dx - \frac{1}{p^*}\int_\Omega g|u|^{p^*} dx. \tag{2.3}$$

It is well known that J_λ is of C^1 in W and the solutions of $(E_{\lambda f,g})$ are the critical points of the energy functional J_λ (see Rabinowitz [10]).

As the energy functional J_λ is not bounded below on W, it is useful to consider the functional on the Nehari manifold

$$\mathcal{N}_\lambda = \left\{u \in W \setminus \{0\} : \left\langle J_\lambda'(u), u\right\rangle = 0\right\}. \tag{2.4}$$

Thus, $u \in \mathcal{N}_\lambda$ if and only if

$$\left\langle J_\lambda'(u), u\right\rangle = \|u\|^p - \lambda\int_\Omega f|u|^q dx - \int_\Omega g|u|^{p^*} dx = 0. \tag{2.5}$$

Note that $\mathcal{N}_\lambda$ contains every nonzero solution of $(E_{\lambda f,g})$. Moreover, we have the following results.

Lemma 2.2. *The energy functional J_λ is coercive and bounded below on $\mathcal{N}_\lambda$.*

Proof. If $u \in \mathcal{N}_\lambda$, then by $(f1)$, (2.5), and the Hölder inequality and the Sobolev embedding theorem we have

$$J_\lambda(u) = \frac{p^*-p}{p^*p}\|u\|^p - \lambda\left(\frac{p^*-q}{p^*q}\right)\int_\Omega f|u|^q dx \tag{2.6}$$

$$\geq \frac{1}{N}\|u\|^p - \lambda\left(\frac{p^*-q}{p^*q}\right)S^{-q/p}|\Omega|^{(p^*-q)/p^*}\|u\|^q\left|f^+\right|_\infty. \tag{2.7}$$

Thus, J_λ is coercive and bounded below on $\mathcal{N}_\lambda$. □

Define

$$\psi_\lambda(u) = \langle J'_\lambda(u), u\rangle. \tag{2.8}$$

Then for $u \in \mathcal{N}_\lambda$,

$$\langle \psi'_\lambda(u), u\rangle = p\|u\|^p - \lambda q\int_\Omega f|u|^q dx - p^*\int_\Omega g|u|^{p^*} dx \tag{2.9}$$

$$= (p-q)\|u\|^p - (p^*-q)\int_\Omega g|u|^{p^*} dx \tag{2.10}$$

$$= \lambda(p^*-q)\int_\Omega f|u|^q dx - (p^*-p)\|u\|^p. \tag{2.11}$$

Similar to the method used in Tarantello [11], we split $\mathcal{N}_\lambda$ into three parts:

$$\begin{aligned}
\mathcal{N}_\lambda^+ &= \{u \in \mathcal{N}_\lambda : \langle \psi'_\lambda(u), u\rangle > 0\},\\
\mathcal{N}_\lambda^0 &= \{u \in \mathcal{N}_\lambda : \langle \psi'_\lambda(u), u\rangle = 0\},\\
\mathcal{N}_\lambda^- &= \{u \in \mathcal{N}_\lambda : \langle \psi'_\lambda(u), u\rangle < 0\}.
\end{aligned} \tag{2.12}$$

Then, we have the following results.

Lemma 2.3. *Assume that u_λ is a local minimizer for J_λ on $\mathcal{N}_\lambda$ and $u_\lambda \notin \mathcal{N}_\lambda^0$. Then $J'_\lambda(u_\lambda) = 0$ in W^{-1}.*

Proof. Our proof is almost the same as that in Brown and Zhang [12, Theorem 2.3] (or see Binding et al. [13]). □

Lemma 2.4. *One has the following.*

(i) *If $u \in \mathcal{N}_\lambda^+$, then $\int_\Omega f|u|^q dx > 0$.*
(ii) *If $u \in \mathcal{N}_\lambda^0$, then $\int_\Omega f|u|^q dx > 0$ and $\int_\Omega g|u|^{p^*} dx > 0$.*
(iii) *If $u \in \mathcal{N}_\lambda^-$, then $\int_\Omega g|u|^{p^*} dx > 0$.*

Proof. The proof is immediate from (2.10) and (2.11). □

Moreover, we have the following result.

Lemma 2.5. *If $\lambda \in (0, \Lambda_1)$, then $\mathcal{N}_\lambda^0 = \emptyset$ where Λ_1 is the same as in* (1.5).

Proof. Suppose otherwise that there exists $\lambda \in (0, \Lambda_1)$ such that $\mathcal{N}_\lambda^0 \neq \emptyset$. Then by (2.10) and (2.11), for $u \in \mathcal{N}_\lambda^0$, we have

$$\|u\|^p = \frac{p^* - q}{p - q} \int_\Omega g|u|^{p^*} dx,$$
$$\|u\|^p = \lambda \frac{p^* - q}{p^* - p} \int_\Omega f|u|^q dx. \tag{2.13}$$

Moreover, by $(f1)$, $(g1)$, and the Hölder inequality and the Sobolev embedding theorem, we have

$$\|u\| \geq \left(\frac{p - q}{(p^* - q)|g^+|_\infty} S^{p^*/p} \right)^{1/(p^* - p)},$$
$$\|u\| \leq \left[\lambda \frac{p^* - q}{p^* - p} S^{-q/p} |\Omega|^{(p^* - q)/p^*} |f^+|_\infty \right]^{1/(p-q)}. \tag{2.14}$$

This implies

$$\lambda \geq \left(\frac{p - q}{(p^* - q)|g^+|_\infty} \right)^{(p-q)/(p^*-p)} \left(\frac{p^* - p}{(p^* - q)|f^+|_\infty} \right) |\Omega|^{(q - p^*)/p^*} S^{(N/p) - (N/p^2)q + (q/p)} = \Lambda_1, \tag{2.15}$$

which is a contradiction. Thus, we can conclude that if $\lambda \in (0, \Lambda_1)$, we have $\mathcal{N}_\lambda^0 = \emptyset$. □

By Lemma 2.5, we write $\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-$ and define

$$\alpha_\lambda = \inf_{u \in \mathcal{N}_\lambda} J_\lambda(u), \qquad \alpha_\lambda^+ = \inf_{u \in \mathcal{N}_\lambda^+} J_\lambda(u), \qquad \alpha_\lambda^- = \inf_{u \in \mathcal{N}_\lambda^-} J_\lambda(u). \tag{2.16}$$

Then we get the following result.

Theorem 2.6. (i) *If $\lambda \in (0, \Lambda_1)$ and $u \in \mathcal{N}_\lambda^+$, then one has $J_\lambda(u) < 0$ and $\alpha_\lambda \leq \alpha_\lambda^+ < 0$.*

(ii) *If $\lambda \in (0, (q/p)\Lambda_1)$, then $\alpha_\lambda^- > d_0$ for some positive constant d_0 depending on $\lambda, p, q, N, S, |f^+|_\infty, |g^+|_\infty$, and $|\Omega|$.*

Proof. (i) Let $u \in \mathcal{N}_\lambda^+$. By (2.10), we have

$$\frac{p - q}{p^* - q} \|u\|^p > \int_\Omega g|u|^{p^*} dx, \tag{2.17}$$

and so

$$\begin{aligned} J_\lambda(u) &= \left(\frac{1}{p}-\frac{1}{q}\right)\|u\|^p + \left(\frac{1}{q}-\frac{1}{p^*}\right)\int_\Omega g|u|^{p^*}dx \\ &< \left[\left(\frac{1}{p}-\frac{1}{q}\right) + \left(\frac{1}{q}-\frac{1}{p^*}\right)\frac{p-q}{p^*-q}\right]\|u\|^p \\ &= -\frac{p-q}{qN}\|u\|^p < 0. \end{aligned} \tag{2.18}$$

Therefore, from the definition of α_λ, α_λ^+, we can deduce that $\alpha_\lambda \le \alpha_\lambda^+ < 0$.

(ii) Let $u \in \mathcal{N}_\lambda^-$. By (2.10), we have

$$\frac{p-q}{p^*-q}\|u\|^p < \int_\Omega g|u|^{p^*}dx. \tag{2.19}$$

Moreover, by $(g1)$ and the Sobolev embedding theorem, we have

$$\int_\Omega g|u|^{p^*}dx \le S^{-p^*/p}\|u\|^{p^*}\left|g^+\right|_\infty. \tag{2.20}$$

This implies

$$\|u\| > \left(\frac{p-q}{(p^*-q)\left|g^+\right|_\infty}\right)^{1/(p^*-p)} S^{N/p^2}, \quad \forall u \in \mathcal{N}_\lambda^-. \tag{2.21}$$

By(2.7) in the proof of Lemma 2.2, we have

$$\begin{aligned} J_\lambda(u) &\ge \|u\|^q\left[\frac{p^*-p}{p^*p}\|u\|^{p-q} - \lambda S^{-q/p}\frac{p^*-q}{p^*q}|\Omega|^{(p^*-q)/p^*}\left|f^+\right|_\infty\right] \\ &> \left(\frac{p-q}{(p^*-q)\left|g^+\right|_\infty}\right)^{q/(p^*-p)} S^{qN/p^2} \\ &\quad\times\left[\frac{p^*-p}{p^*p}S^{(p-q)N/p^2}\left(\frac{p-q}{(p^*-q)\left|g^+\right|_\infty}\right)^{(p-q)/(p^*-p)} - \lambda S^{-q/p}\frac{p^*-q}{p^*q}|\Omega|^{(p^*-q)/p^*}\left|f^+\right|_\infty\right]. \end{aligned} \tag{2.22}$$

Thus, if $\lambda \in (0, (q/p)\Lambda_1)$, then

$$J_\lambda(u) > d_0, \quad \forall u \in \mathcal{N}_\lambda^-, \tag{2.23}$$

for some positive constant $d_0 = d_0(\lambda, p, q, N, S, |f^+|_\infty, |g^+|_\infty, |\Omega|)$. This completes the proof. □

For each $u \in W$ with $\int_\Omega g|u|^{p^*} dx > 0$, we write

$$t_{\max} = \left(\frac{(p-q)\|u\|^p}{(p^*-q)\int_\Omega g|u|^{p^*} dx} \right)^{1/(p^*-p)} > 0. \tag{2.24}$$

Then the following lemma holds.

Lemma 2.7. *Let $\lambda \in (0, \Lambda_1)$. For each $u \in W$ with $\int_\Omega g|u|^{p^*} dx > 0$, one has the following:*

(i) *if $\int_\Omega f|u|^q dx \le 0$, then there exists a unique $t^- > t_{\max}$ such that $t^- u \in \mathcal{N}_\lambda^-$ and*

$$J_\lambda(t^- u) = \sup_{t \ge 0} J_\lambda(tu), \tag{2.25}$$

(ii) *if $\int_\Omega f|u|^q dx > 0$, then there exists unique $0 < t^+ < t_{\max} < t^-$ such that $t^+ u \in \mathcal{N}_\lambda^+$, $t^- u \in \mathcal{N}_\lambda^-$, and*

$$J_\lambda(t^+ u) = \inf_{0 \le t \le t_{\max}} J_\lambda(tu); \qquad J_\lambda(t^- u) = \sup_{t \ge 0} J_\lambda(tu). \tag{2.26}$$

Proof. Fix $u \in W$ with $\int_\Omega g|u|^{p^*} dx > 0$. Let

$$k(t) = t^{p-q}\|u\|^p - t^{p^*-q}\int_\Omega g|u|^{p^*} dx \quad \text{for } t \ge 0. \tag{2.27}$$

It is clear that $k(0) = 0$, $k(t) \to -\infty$ as $t \to \infty$. From

$$k'(t) = (p-q)t^{p-q-1}\|u\|^p - (p^*-q)t^{p^*-q-1}\int_\Omega g|u|^{p^*} dx, \tag{2.28}$$

we can deduce that $k'(t) = 0$ at $t = t_{\max}$, $k'(t) > 0$ for $t \in (0, t_{\max})$ and $k'(t) < 0$ for $t \in (t_{\max}, \infty)$. Then $k(t)$ that achieves its maximum at $t_{\max}$ is increasing for $t \in [0, t_{\max})$ and decreasing for $t \in (t_{\max}, \infty)$. Moreover,

$$\begin{aligned}
k(t_{\max}) &= \left(\frac{(p-q)\|u\|^p}{(p^*-q)\int_\Omega g|u|^{p^*} dx} \right)^{(p-q)/(p^*-p)} \|u\|^p \\
&\quad - \left(\frac{(p-q)\|u\|^p}{(p^*-q)\int_\Omega g|u|^{p^*} dx} \right)^{(p^*-q)/(p^*-p)} \int_\Omega g|u|^{p^*} dx \\
&= \|u\|^q \left[\left(\frac{p-q}{p^*-q} \right)^{(p-q)/(p^*-p)} - \left(\frac{p-q}{p^*-q} \right)^{(p^*-q)/(p^*-p)} \right] \left(\frac{\|u\|^{p^*}}{\int_\Omega g|u|^{p^*} dx} \right)^{(p-q)/(p^*-p)} \\
&\ge \|u\|^q \left(\frac{p^*-p}{p^*-q} \right) \left(\frac{p-q}{(p^*-q)|g^+|_\infty} S^{p^*/p} \right)^{(p-q)/(p^*-p)}.
\end{aligned} \tag{2.29}$$

(i) We have $\int_\Omega f|u|^q dx \le 0$. There exists a unique $t^- > t_{\max}$ such that $k(t^-) = \lambda\int_\Omega f|u|^q dx$ and $k'(t^-) < 0$. Now,

$$\begin{aligned}
&(p-q)(t^-)^p\|u\|^p - (p^*-q)(t^-)^p\int_\Omega g|u|^{p^*}dx \\
&\quad = (t^-)^{1+q}\left[(p-q)(t^-)^{p-q-1}\|u\|^p - (p^*-q)(t^-)^{p^*-q-1}\int_\Omega g|u|^{p^*}dx\right] \\
&\quad = (t^-)^{1+q}k'(t^-) < 0, \\
\langle J'_\lambda(t^-u), t^-u\rangle &= (t^-)^p\|u\|^p - (t^-)^{p^*}\int_\Omega g|u|^{p^*}dx - (t^-)^q\lambda\int_\Omega f|u|^q dx \\
&= (t^-)^q\left[k(t^-) - \lambda\int_\Omega f|u|^q dx\right] = 0.
\end{aligned} \tag{2.30}$$

Then we have that $t^-u \in \mathcal{N}_\lambda^-$. For $t > t_{\max}$, we have

$$\begin{aligned}
&(p-q)\|tu\|^p - (p^*-q)\int_\Omega g|tu|^{p^*} < 0, \qquad \frac{d^2}{dt^2}J_\lambda(tu) < 0, \\
\frac{d}{dt}J_\lambda(tu) &= t^{p-1}\|u\|^p - t^{p^*-1}\int_\Omega g|u|^{p^*}dx - t^{q-1}\lambda\int_\Omega f|u|^q dx \\
&= 0 \quad \text{for } t = t^-.
\end{aligned} \tag{2.31}$$

Thus, $J_\lambda(t^-u) = \sup_{t\ge 0} J_\lambda(tu)$.

(ii) We have $\int_\Omega f|u|^q dx > 0$. By (2.29) and

$$\begin{aligned}
k(0) = 0 &< \lambda\int_\Omega f|u|^q dx \\
&\le \lambda S^{-q/p}|\Omega|^{(p^*-q)/p^*}\|u\|^q\left|f^+\right|_\infty \\
&< \|u\|^q\left(\frac{p^*-p}{p^*-q}\right)\left(\frac{p-q}{(p^*-q)\left|g^+\right|_\infty}S^{p^*/p}\right)^{(p-q)/(p^*-p)} \\
&\le k(t_{\max}) \quad \text{for } \lambda \in (0, \Lambda_1),
\end{aligned} \tag{2.32}$$

there are unique t^+ and t^- such that $0 < t^+ < t_{\max} < t^-$,

$$\begin{aligned}
k(t^+) = \lambda\int_\Omega f|u|^q dx = k(t^-), \\
k'(t^+) > 0 > k'(t^-).
\end{aligned} \tag{2.33}$$

We have $t^+u \in \mathcal{N}_\lambda^+$, $t^-u \in \mathcal{N}_\lambda^-$, and $J_\lambda(t^-u) \geq J_\lambda(tu) \geq J_\lambda(t^+u)$ for each $t \in [t^+, t^-]$ and $J_\lambda(t^+u) \leq J_\lambda(tu)$ for each $t \in [0, t^+]$. Thus,

$$J_\lambda(t^+u) = \inf_{0 \leq t \leq t_{\max}} J_\lambda(tu), \qquad J_\lambda(t^-u) = \sup_{t \geq 0} J_\lambda(tu). \tag{2.34}$$

This completes the proof. □

3. Proof of Theorem 1.4

First, we will use the idea of Tarantello [11] to get the following results.

Lemma 3.1. *If $\lambda \in (0, \Lambda_1)$, then for each $u \in \mathcal{N}_\lambda$, there exist $\epsilon > 0$ and a differentiable function $\xi : B(0;\epsilon) \subset W \to \mathbb{R}^+$ such that $\xi(0) = 1$, the function $\xi(v)(u - v) \in \mathcal{N}_\lambda$, and*

$$\left\langle \xi'(0), v \right\rangle = \frac{p\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v\, dx - \lambda q \int_\Omega f|u|^{q-2}uv\, dx - p^* \int_\Omega g|u|^{p^*-2}uv\, dx}{(p-q)\|u\|^p - (p^* - q)\int_\Omega g|u|^{p^*}dx} \tag{3.1}$$

for all $v \in W$.

Proof. For $u \in \mathcal{N}_\lambda$, define a function $F : \mathbb{R} \times W \to \mathbb{R}$ by

$$\begin{aligned} F_u(\xi, w) &= \left\langle J_\lambda'(\xi(u-w)), \xi(u-w) \right\rangle \\ &= \xi^p \int_\Omega |\nabla(u-w)|^p dx - \xi^q \lambda \int_\Omega f|u-w|^q dx \\ &\quad - \xi^{p^*} \int_\Omega g|u-w|^{p^*} dx. \end{aligned} \tag{3.2}$$

Then $F_u(1,0) = \langle J_\lambda'(u), u \rangle = 0$ and

$$\begin{aligned} \frac{d}{d\xi} F_u(1,0) &= p\|u\|^p - \lambda q \int_\Omega f|u|^q dx - p^* \int_\Omega g|u|^{p^*} dx \\ &= (p-q)\|u\|^p - (p^* - q)\int_\Omega g|u|^{p^*} dx \neq 0. \end{aligned} \tag{3.3}$$

According to the implicit function theorem, there exist $\epsilon > 0$ and a differentiable function $\xi : B(0;\epsilon) \subset W \to \mathbb{R}$ such that $\xi(0) = 1$,

$$\begin{gathered} \left\langle \xi'(0), v \right\rangle = \frac{p\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v\, dx - \lambda q \int_\Omega f|u|^{q-2}uv\, dx - p^* \int_\Omega g|u|^{p^*-2}uv\, dx}{(p-q)\|u\|^p - (p^* - q)\int_\Omega g|u|^{p^*}dx}, \\ F_u(\xi(v), v) = 0, \quad \forall v \in B(0;\epsilon), \end{gathered} \tag{3.4}$$

which is equivalent to

$$\left\langle J_\lambda'(\xi(v)(u-v)), \xi(v)(u-v)\right\rangle = 0, \quad \forall v \in B(0;\epsilon), \tag{3.5}$$

that is, $\xi(v)(u-v) \in \mathcal{N}_\lambda$. □

Lemma 3.2. *Let $\lambda \in (0, \Lambda_1)$, then for each $u \in \mathcal{N}_\lambda^-$, there exist $\epsilon > 0$ and a differentiable function $\xi^- : B(0;\epsilon) \subset W \to \mathbb{R}^+$ such that $\xi^-(0) = 1$, the function $\xi^-(v)(u-v) \in \mathcal{N}_\lambda^-$, and*

$$\left\langle (\xi^-)'(0), v\right\rangle = \frac{p\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v\, dx - \lambda q \int_\Omega f|u|^{q-2}uv\, dx - p^*\int_\Omega g|u|^{p^*-2}uv\, dx}{(p-q)\|u\|^p - (p^*-q)\int_\Omega g|u|^{p^*}dx} \tag{3.6}$$

for all $v \in W$.

Proof. Similar to the argument in Lemma 3.1, there exist $\epsilon > 0$ and a differentiable function $\xi^- : B(0;\epsilon) \subset W \to \mathbb{R}$ such that $\xi^-(0) = 1$ and $\xi^-(v)(u-v) \in \mathcal{N}_\lambda$ for all $v \in B(0;\epsilon)$. Since

$$\left\langle \psi_\lambda'(u), u\right\rangle = (p-q)\|u\|^p - (p^*-q)\int_\Omega g|u|^{p^*}dx < 0. \tag{3.7}$$

Thus, by the continuity of the function ξ^-, we have

$$\begin{aligned}\left\langle \psi_\lambda'(\xi^-(v)(u-v)), \xi^-(v)(u-v)\right\rangle &= (p-q)\left\|\xi^-(v)(u-v)\right\|^p \\ &\quad - (p^*-q)\int_\Omega g\left|\xi^-(v)(u-v)\right|^p dx < 0,\end{aligned} \tag{3.8}$$

if ϵ sufficiently small, this implies that $\xi^-(v)(u-v) \in \mathcal{N}_\lambda^-$. □

Proposition 3.3. (i) *If $\lambda \in (0, \Lambda_1)$, then there exists a $(PS)_{\alpha_\lambda}$-sequence $\{u_n\} \subset \mathcal{N}_\lambda$ in W for J_λ.*

(ii) *If $\lambda \in (0, (q/p)\Lambda_1)$, then there exists a $(PS)_{\alpha_\lambda^-}$-sequence $\{u_n\} \subset \mathcal{N}_\lambda^-$ in W for J_λ.*

Proof. (i) By Lemma 2.2 and the Ekeland variational principle [14], there exists a minimizing sequence $\{u_n\} \subset \mathcal{N}_\lambda$ such that

$$\begin{aligned} J_\lambda(u_n) &< \alpha_\lambda + \frac{1}{n}, \\ J_\lambda(u_n) &< J_\lambda(w) + \frac{1}{n}\|w - u_n\| \quad \text{for each } w \in \mathcal{N}_\lambda. \end{aligned} \tag{3.9}$$

By $\alpha_\lambda < 0$ and taking n large, we have

$$\begin{aligned} J_\lambda(u_n) &= \left(\frac{1}{p} - \frac{1}{p^*}\right)\|u_n\|^p - \left(\frac{1}{q} - \frac{1}{p^*}\right)\lambda\int_\Omega f|u_n|^q dx \\ &< \alpha_\lambda + \frac{1}{n} < \frac{\alpha_\lambda}{p}. \end{aligned} \tag{3.10}$$

From (2.7), (3.10), $\alpha_\lambda < 0$, and the Hölder inequality, we deduce that

$$|f^+|_\infty \lambda S^{-q/p}|\Omega|^{(p^*-q)/p^*}\|u_n\|^q \geq \lambda\int_\Omega f|u_n|^q dx > \frac{-p^*q}{p(p^*-q)}\alpha_\lambda > 0. \quad (3.11)$$

Consequently, $u_n \neq 0$ and putting together (3.10), (3.11), and the Hölder inequality, we obtain

$$\|u_n\| > \left[\frac{-p^*q}{p\lambda(p^*-q)|f^+|_\infty}\alpha_\lambda S^{q/p}|\Omega|^{(q-p^*)/p^*}\right]^{1/q},$$
$$\|u_n\| < \left[\frac{p(p^*-q)}{q(p^*-p)}\lambda S^{-q/p}|\Omega|^{(p^*-q)/p^*}|f^+|_\infty\right]^{1/(p-q)}. \quad (3.12)$$

Now, we show that

$$\|J'_\lambda(u_n)\|_{W^{-1}} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \quad (3.13)$$

Apply Lemma 3.1 with u_n to obtain the functions $\xi_n : B(0;\epsilon_n) \to \mathbb{R}^+$ for some $\epsilon_n > 0$, such that $\xi_n(w)(u_n - w) \in \mathcal{N}_\lambda$. Choose $0 < \rho < \epsilon_n$. Let $u \in W$ with $u \not\equiv 0$ and let $w_\rho = \rho u/\|u\|$. We set $\eta_\rho = \xi_n(w_\rho)(u_n - w_\rho)$. Since $\eta_\rho \in \mathcal{N}_\lambda$, we deduce from (3.9) that

$$J_\lambda(\eta_\rho) - J_\lambda(u_n) \geq -\frac{1}{n}\|\eta_\rho - u_n\|, \quad (3.14)$$

and by the mean value theorem, we have

$$\langle J'_\lambda(u_n), \eta_\rho - u_n\rangle + o(\|\eta_\rho - u_n\|) \geq -\frac{1}{n}\|\eta_\rho - u_n\|. \quad (3.15)$$

Thus,

$$\langle J'_\lambda(u_n), -w_\rho\rangle + (\xi_n(w_\rho) - 1)\langle J'_\lambda(u_n), (u_n - w_\rho)\rangle \geq -\frac{1}{n}\|\eta_\rho - u_n\| + o(\|\eta_\rho - u_n\|). \quad (3.16)$$

Since $\xi_n(w_\rho)(u_n - w_\rho) \in \mathcal{N}_\lambda$ and (3.16) it follows that

$$-\rho\left\langle J'_\lambda(u_n), \frac{u}{\|u\|}\right\rangle + (\xi_n(w_\rho) - 1)\langle J'_\lambda(u_n) - J'_\lambda(\eta_\rho), (u_n - w_\rho)\rangle \geq -\frac{1}{n}\|\eta_\rho - u_n\| + o(\|\eta_\rho - u_n\|). \quad (3.17)$$

Thus,

$$\left\langle J'_\lambda(u_n), \frac{u}{\|u\|} \right\rangle \le \frac{\|\eta_\rho - u_n\|}{n\rho} + \frac{o(\|\eta_\rho - u_n\|)}{\rho} + \frac{(\xi_n(w_\rho) - 1)}{\rho} \langle J'_\lambda(u_n) - J'_\lambda(\eta_\rho), (u_n - w_\rho) \rangle. \tag{3.18}$$

Since $\|\eta_\rho - u_n\| \le \rho\xi_n(w_\rho) + |\xi_n(w_\rho) - 1| \|u_n\|$ and

$$\lim_{\rho \to 0} \frac{|\xi_n(w_\rho) - 1|}{\rho} \le \|\xi'_n(0)\|, \tag{3.19}$$

if we let $\rho \to 0$ in (3.18) for a fixed n, then by (3.12) we can find a constant $C > 0$, independent of ρ, such that

$$\left\langle J'_\lambda(u_n), \frac{u}{\|u\|} \right\rangle \le \frac{C}{n}(1 + \|\xi'_n(0)\|). \tag{3.20}$$

The proof will be complete once we show that $\|\xi'_n(0)\|$ is uniformly bounded in n. By (3.1), (3.12), $(f1)$, $(g1)$, and the Hölder inequality and the Sobolev embedding theorem, we have

$$\langle \xi'_n(0), v \rangle \le \frac{b\|v\|}{\left| (p-q)\|u_n\|^p - (p^* - q)\int_\Omega g|u_n|^{p^*} dx \right|} \quad \text{for some } b > 0. \tag{3.21}$$

We only need to show that

$$\left| (p-q)\|u_n\|^p - (p^* - q)\int_\Omega g|u_n|^{p^*} dx \right| > C \tag{3.22}$$

for some $C > 0$ and n large enough. We argue by contradiction. Assume that there exists a subsequence $\{u_n\}$ such that

$$(p-q)\|u_n\|^p - (p^* - q)\int_\Omega g|u_n|^{p^*} dx = o_n(1). \tag{3.23}$$

By (3.23) and the fact that $u_n \in \mathcal{N}_\lambda$, we get

$$\|u_n\|^p = \frac{p^* - q}{p - q}\int_\Omega g|u_n|^{p^*} dx + o_n(1),$$
$$\|u_n\|^p = \lambda \frac{p^* - q}{p^* - p}\int_\Omega f|u_n|^q dx + o_n(1). \tag{3.24}$$

Moreover, by $(f1)$, $(g1)$, and the Hölder inequality and the Sobolev embedding theorem, we have

$$\|u_n\| \geq \left[\frac{p-q}{(p^*-q)|g^+|_\infty} S^{p^*/p}\right]^{1/(p^*-p)} + o_n(1),$$
$$\|u_n\| \leq \left[\lambda \frac{(p^*-q)|f^+|_\infty}{p^*-p} S^{-q/p}|\Omega|^{(p^*-q)/p^*}\right]^{1/(p-q)} + o_n(1). \tag{3.25}$$

This implies $\lambda \geq \Lambda_1$ which is a contradiction. We obtain

$$\left\langle J'_\lambda(u_n), \frac{u}{\|u\|}\right\rangle \leq \frac{C}{n}. \tag{3.26}$$

This completes the proof of (i).

(ii) Similarly, by using Lemma 3.2, we can prove (ii). We will omit detailed proof here. □

Now, we establish the existence of a local minimum for J_λ on $\mathcal{N}_\lambda^+$.

Theorem 3.4. *If $\lambda \in (0, \Lambda_1)$, then J_λ has a minimizer u_λ in $\mathcal{N}_\lambda^+$ and it satisfies that*

(i) $J_\lambda(u_\lambda) = \alpha_\lambda = \alpha_\lambda^+$;

(ii) u_λ *is a positive solution of* $(E_{\lambda f,g})$ *in* $C^{1,\alpha}(\Omega)$ *for some* $\alpha \in (0,1)$.

Proof. By Proposition 3.3(i), there exists a minimizing sequence $\{u_n\}$ for J_λ on $\mathcal{N}_\lambda$ such that

$$J_\lambda(u_n) = \alpha_\lambda + o_n(1), \quad J'_\lambda(u_n) = o_n(1) \qquad \text{in } W^{-1}. \tag{3.27}$$

Since J_λ is coercive on $\mathcal{N}_\lambda$ (see Lemma 2.2), we get that $\{u_n\}$ is bounded in W. Going if necessary to a subsequence, we can assume that there exists $u_\lambda \in W$ such that

$$u_n \rightharpoonup u_\lambda \quad \text{weakly in } W,$$
$$u_n \longrightarrow u_\lambda \quad \text{almost every where in } \Omega, \tag{3.28}$$
$$u_n \longrightarrow u_\lambda \quad \text{strongly in } L^s(\Omega)\ \forall 1 \leq s < p^*.$$

First, we claim that u_λ is a nontrivial solution of $(E_{\lambda f,g})$. By (3.27) and (3.28), it is easy to see that u_λ is a solution of $(E_{\lambda f,g})$. From $u_n \in \mathcal{N}_\lambda$ and (2.6), we deduce that

$$\lambda \int_\Omega f|u_n|^q dx = \frac{q(p^*-p)}{p(p^*-q)}\|u_n\|^p - \frac{p^*q}{p^*-q} J_\lambda(u_n). \tag{3.29}$$

Let $n \to \infty$ in (3.29), by (3.27), (3.28), and $\alpha_\lambda < 0$, we get

$$\int_\Omega f|u_\lambda|^q dx \geq -\frac{p^*q}{p^*-q}\alpha_\lambda > 0. \tag{3.30}$$

Thus, $u_\lambda \in \mathcal{N}_\lambda$ is a nontrivial solution of $(E_{\lambda f,g})$. Now we prove that $u_n \to u_\lambda$ strongly in W and $J_\lambda(u_\lambda) = \alpha_\lambda$. By (3.29), if $u \in \mathcal{N}_\lambda$, then

$$J_\lambda(u) = \frac{p^* - p}{p^* p}\|u\|^p - \frac{p^* - q}{p^* q}\lambda\int_\Omega f|u|^q dx. \tag{3.31}$$

In order to prove that $J_\lambda(u_\lambda) = \alpha_\lambda$, it suffices to recall that $u_\lambda \in \mathcal{N}_\lambda$, by (3.31), and applying Fatou's lemma to get

$$\begin{aligned}
\alpha_\lambda \le J_\lambda(u_\lambda) &= \frac{p^* - p}{p^* p}\|u_\lambda\|^p - \frac{p^* - q}{p^* q}\lambda\int_\Omega f|u_\lambda|^q dx \\
&\le \liminf_{n\to\infty}\left(\frac{p^* - p}{p^* p}\|u_n\|^p - \frac{p^* - q}{p^* q}\lambda\int_\Omega f|u_n|^q dx\right) \\
&\le \liminf_{n\to\infty} J_\lambda(u_n) = \alpha_\lambda.
\end{aligned} \tag{3.32}$$

This implies that $J_\lambda(u_\lambda) = \alpha_\lambda$ and $\lim_{n\to\infty}\|u_n\|^p = \|u_\lambda\|^p$. Let $v_n = u_n - u_\lambda$, then Brézis and Lieb lemma [15] implies that

$$\|v_n\|^p = \|u_n\|^p - \|u_\lambda\|^p + o_n(1). \tag{3.33}$$

Therefore, $u_n \to u_\lambda$ strongly in W. Moreover, we have $u_\lambda \in \mathcal{N}_\lambda^+$. On the contrary, if $u_\lambda \in \mathcal{N}_\lambda^-$, then by Lemma 2.7, there are unique t_0^+ and t_0^- such that $t_0^+ u_\lambda \in \mathcal{N}_\lambda^+$ and $t_0^- u_\lambda \in \mathcal{N}_\lambda^-$. In particular, we have $t_0^+ < t_0^- = 1$. Since

$$\frac{d}{dt}J_\lambda(t_0^+ u_\lambda) = 0, \qquad \frac{d^2}{dt^2}J_\lambda(t_0^+ u_\lambda) > 0, \tag{3.34}$$

there exists $t_0^+ < \bar{t} \le t_0^-$ such that $J_\lambda(t_0^+ u_\lambda) < J_\lambda(\bar{t} u_\lambda)$. By Lemma 2.7,

$$J_\lambda(t_0^+ u_\lambda) < J_\lambda\left(\bar{t} u_\lambda\right) \le J_\lambda(t_0^- u_\lambda) = J_\lambda(u_\lambda), \tag{3.35}$$

which is a contradiction. Since $J_\lambda(u_\lambda) = J_\lambda(|u_\lambda|)$ and $|u_\lambda| \in \mathcal{N}_\lambda^+$, by Lemma 2.3 we may assume that u_λ is a nontrivial nonnegative solution of $(E_{\lambda f,g})$. Moreover, from $f, g \in L^\infty(\Omega)$, then using the standard bootstrap argument (see, e.g., [16]) we obtain $u_\lambda \in L^\infty(\Omega)$; hence by applying regularity results [17, 18] we derive that $u_\lambda \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1)$ and finally, by the Harnack inequality [19] we deduce that $u_\lambda > 0$. This completes the proof. □

Now, we begin the proof of Theorem 1.4. By Theorem 3.4, we obtain $(E_{\lambda f,g})$ that has a positive solution u_λ in $C^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1)$.

4. Proof of Theorem 1.5

Next, we will establish the existence of the second positive solution of $(E_{\lambda f,g})$ by proving that J_λ satisfies the $(\mathrm{PS})_{\alpha_\lambda^-}$ condition.

Lemma 4.1. *Assume that* $(f1)$ *and* $(g1)$ *hold. If* $\{u_n\} \subset W$ *is a (PS)*$_c$*-sequence for* J_λ*, then* $\{u_n\}$ *is bounded in* W*.*

Proof. We argue by contradiction. Assume that $\|u_n\| \to \infty$. Let $\widehat{u}_n = u_n/\|u_n\|$. We may assume that $\widehat{u}_n \rightharpoonup \widehat{u}$ in W. This implies that $\widehat{u}_n \to \widehat{u}$ strongly in $L^s(\Omega)$ for all $1 \le s < p^*$ and

$$\frac{\lambda}{q}\int_\Omega f|\widehat{u}_n|^q dx = \frac{\lambda}{q}\int_\Omega f|\widehat{u}|^q dx + o_n(1). \tag{4.1}$$

Since $\{u_n\}$ is a $(\mathrm{PS})_c$-sequence for J_λ and $\|u_n\| \to \infty$, there hold

$$\begin{gathered}\frac{1}{p}\int_\Omega |\nabla \widehat{u}_n|^p dx - \frac{\lambda\|u_n\|^{q-p}}{q}\int_\Omega f|\widehat{u}_n|^q dx - \frac{\|u_n\|^{p^*-p}}{p^*}\int_\Omega g|\widehat{u}_n|^{p^*} dx = o_n(1),\\ \int_\Omega |\nabla \widehat{u}_n|^p dx - \lambda\|u_n\|^{q-p}\int_\Omega f|\widehat{u}_n|^q dx - \|u_n\|^{p^*-p}\int_\Omega g|\widehat{u}_n|^{p^*} dx = o_n(1).\end{gathered} \tag{4.2}$$

From (4.1)-(4.2), we can deduce that

$$\int_\Omega |\nabla \widehat{u}_n|^p dx = \frac{p(p^*-q)}{q(p^*-p)}\|u_n\|^{q-p}\lambda\int_\Omega f|\widehat{u}|^q dx + o_n(1). \tag{4.3}$$

Since $1 \le q < 2$ and $\|u_n\| \to \infty$, (4.3) implies

$$\int_\Omega |\nabla \widehat{u}_n|^p dx \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \tag{4.4}$$

which is contrary to the fact $\|\widehat{u}_n\| = 1$ for all n. □

Lemma 4.2. *Assume that* $(f1)$ *and* $(g1)$ *hold. If* $\{u_n\} \subset W$ *is a (PS)*$_c$*-sequence for* J_λ *with* $c \in (0, (1/N)|g^+|_\infty^{-(N-p)/p} S^{N/p})$*, then there exists a subsequence of* $\{u_n\}$ *converging weakly to a nontrivial solution of* $(E_{\lambda f,g})$*.*

Proof. Let $\{u_n\} \subset W$ be a $(\mathrm{PS})_c$-sequence for J_λ with $c \in (0, (1/N)|g^+|_\infty^{-(N-p)/p} S^{N/p})$. We know from Lemma 4.1 that $\{u_n\}$ is bounded in W, and then there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$ and $u_0 \in W$ such that

$$\begin{gathered}u_n \rightharpoonup u_0 \quad \text{weakly in } W,\\ u_n \longrightarrow u_0 \quad \text{almost every where in } \Omega,\\ u_n \longrightarrow u_0 \quad \text{strongly in } L^s(\Omega)\ \forall 1 \le s < p^*.\end{gathered} \tag{4.5}$$

It is easy to see that $J'_\lambda(u_0) = 0$ and

$$\lambda\int_\Omega f(x)|u_n|^q dx = \lambda\int_\Omega f(x)|u_0|^q dx + o_n(1). \tag{4.6}$$

Next we verify that $u_0 \not\equiv 0$. Arguing by contradiction, we assume $u_0 \equiv 0$. Setting

$$l = \lim_{n\to\infty} \int_\Omega g|u_n|^{p^*} dx. \tag{4.7}$$

Since $J'_\lambda(u_n) = o_n(1)$ and $\{u_n\}$ is bounded, then by (4.6), we can deduce that

$$0 = \left\langle \lim_{n\to\infty} J'_\lambda(u_n), u_n \right\rangle = \lim_{n\to\infty} \left(\|u_n\|^p - \int_\Omega g|u_n|^{p^*} \right) = \lim_{n\to\infty} \|u_n\|^p - l, \tag{4.8}$$

that is,

$$\lim_{n\to\infty} \|u_n\|^p = l. \tag{4.9}$$

If $l = 0$, then we get $c = \lim_{n\to\infty} J_\lambda(u_n) = 0$, which contradicts with $c > 0$. Thus we conclude that $l > 0$. Furthermore, the Sobolev inequality implies that

$$\|u_n\|^p \geq S\left(\int_\Omega |u_n|^{p^*}\right)^{p/p^*} \geq S\left(\int_\Omega \frac{g}{|g^+|_\infty}|u_n|^{p^*}\right)^{p/p^*} = S|g^+|_\infty^{-(N-p)/N}\left(\int_\Omega g|u_n|^{p^*}\right)^{p/p^*}. \tag{4.10}$$

Then as $n \to \infty$ we have

$$l = \lim_{n\to\infty} \|u_n\|^p \geq S|g^+|^{-(N-p)/N} \lim_{n\to\infty} \left(\int_\Omega g|u_n|^{p^*}\right)^{p/p^*} = S|g^+|_\infty^{-(N-p)/N} l^{p/p^*}, \tag{4.11}$$

which implies that

$$l \geq |g^+|_\infty^{-(N-p)/p} S^{N/p}. \tag{4.12}$$

Hence, from (4.6) to (4.12) we get

$$\begin{aligned} c &= \lim_{n\to\infty} J_\lambda(u_n) \\ &= \frac{1}{p}\lim_{n\to\infty} \|u_n\|^p - \frac{\lambda}{q}\lim_{n\to\infty}\int_\Omega f|u_n|^q dx - \frac{1}{p^*}\lim_{n\to\infty}\int_\Omega g|u_n|^{p^*} dx \\ &= \left(\frac{1}{p} - \frac{1}{p^*}\right) l \\ &\geq \frac{1}{N}|g^+|_\infty^{-(N-p)/p} S^{N/p}. \end{aligned} \tag{4.13}$$

This is a contradiction to $c < (1/N)|g^+|_\infty^{-(N-p)/p} S^{N/p}$. Therefore u_0 is a nontrivial solution of $(E_{\lambda f,g})$. □

Lemma 4.3. *Assume that* $(f1)$-$(f2)$ *and* $(g1)$–$(g4)$ *hold. Then for any* $\lambda > 0$*, there exists* $v_\lambda \in W$ *such that*

$$\sup_{t\geq 0} J_\lambda(tv_\lambda) < \frac{1}{N}\left|g^+\right|_\infty^{-(N-p)/p} S^{N/p}. \tag{4.14}$$

In particular, $\alpha_\lambda^- < (1/N)|g^+|_\infty^{-(N-p)/p} S^{N/p}$ *for all* $\lambda \in (0, \Lambda_1)$ *where* Λ_1 *is as in* (1.5).

Proof. For convenience, we introduce the following notations:

$$\begin{gathered} I(u) = \int_\Omega \left\{\frac{1}{p}|\nabla u|^p - \frac{1}{p^*} g|u|^{p^*}\right\} dx, \\ \chi_{B(0,2\rho_0)} = \begin{cases} 1 & \text{if } x \in B(0,2\rho_0), \\ 0 & \text{if } x \notin B(0,2\rho_0), \end{cases} \\ Q(u) = \frac{|\nabla u|_p^p}{\left|\left(g\chi_{B(0,2\rho_0)}\right)^{1/p^*} u\right|_{p^*}^p}. \end{gathered} \tag{4.15}$$

From $(g3)$ to $(g4)$, we know that there exists $\delta_0 \in (0, \rho_0)$ such that for all $x \in B(0, 2\delta_0)$,

$$g(x) = g(0) + o\left(|x|^\beta\right) \quad \text{for some } \beta > \frac{N}{p-1}. \tag{4.16}$$

Motivated by some ideas of selecting cut-off functions in [20, Lemma 4.1], we take such cut-off function $\eta(x)$ that satisfies $\eta(x) \in C_0^\infty(B(0, 2\delta_0))$, $\eta(x) = 1$ for $|x| < \delta_0$, $\eta(x) = 0$ for $|x| > 2\delta_0$, $0 \leq \eta \leq 1$, and $|\nabla \eta| \leq C$. Define, for $\varepsilon > 0$,

$$u_\varepsilon(x) = \frac{\varepsilon^{(N-p)/p^2}\eta(x)}{\left(\varepsilon + |x|^{p/(p-1)}\right)^{(N-p)/p}}. \tag{4.17}$$

Step 1. Show that $\sup_{t\geq 0} I(tu_\varepsilon) \leq (1/N)|g^+|_\infty^{-(N-p)/p} S^{N/p} + O(\varepsilon^{(N-p)/p})$.

On that purpose, we need to establish the following estimates (as $\varepsilon \to 0$):

$$\left|\left(g\chi_{B(0,2\rho_0)}\right)^{1/p^*} u_\varepsilon\right|_{p^*}^p = \left|g^+\right|_\infty^{-(N-p)/N} |U|_{L^{p^*}(\mathbb{R}^N)}^p + O\left(\varepsilon^{N/p}\right), \tag{4.18}$$

$$|\nabla u_\varepsilon|_p^p = |\nabla U|_{L^p(\mathbb{R}^N)}^p + O\left(\varepsilon^{(N-p)/p}\right), \tag{4.19}$$

where $U(x)=(1+(x)^{p/(p-1)})^{-(N-p)/p}\in W^{1,p}(\mathbb{R}^N)$ is a minimizer of $\{|\nabla u|_p^p/|u|_{p^*}^p\}_{u\in W^{1,p}(\mathbb{R}^N)\setminus\{0\}}$, that is,

$$\frac{|\nabla U|^p_{L^p(\mathbb{R}^N)}}{|U|^p_{L^{p^*}(\mathbb{R}^N)}}=S=\inf_{u\in W^{1,p}(\mathbb{R}^N)\setminus\{0\}}\frac{|\nabla u|^p_{L^p(\mathbb{R}^N)}}{|u|^p_{L^{p^*}(\mathbb{R}^N)}},\tag{4.20}$$

and $\omega_N=2\pi^{N/2}/N\Gamma(N/2)$ which is the volume of the unit ball $B(0,1)$ in $\mathbb{R}^N$. We only show that equality (4.18) is valid; proofs of (4.19) are very similar to [20]. In view of (4.17), we get that

$$\left|\left(g\chi_{B(0,2\rho_0)}\right)^{1/p^*}u_\varepsilon\right|_{p^*}^{p^*}=\int_{B(0,2\delta_0)}g(x)|u_\varepsilon|^{p^*}dx=\int_{\mathbb{R}^N}\frac{\varepsilon^{N/p}\eta^{p^*}(x)g(x)}{\left(\varepsilon+|x|^{p/(p-1)}\right)^N}dx.\tag{4.21}$$

On the other hand, let $x=\varepsilon^{(p-1)/p}y$, we can deduce that

$$\int_{\mathbb{R}^N}\frac{1}{\left(\varepsilon+|x|^{p/(p-1)}\right)^N}dx=\varepsilon^{-N/p}\int_{\mathbb{R}^N}\frac{1}{\left(1+|y|^{p/(p-1)}\right)^N}dy=\varepsilon^{-N/p}|U|^{p^*}_{L^{p^*}(\mathbb{R}^N)}.\tag{4.22}$$

Combining with $g(0)=g^+|_\infty$ and the equalities above, we have

$$\begin{aligned}&\varepsilon^{-N/p}\left|g^+\right|_\infty|U|^{p^*}_{L^{p^*}(\mathbb{R}^N)}-\varepsilon^{-N/p}\left|\left(g\chi_{B(0,2\rho_0)}\right)^{1/p^*}u_\varepsilon\right|_{p^*}^{p^*}\\&\quad=\int_{\mathbb{R}^N\setminus B(0,\delta_0)}\frac{g(0)-\eta^{p^*}(x)g(x)}{\left(\varepsilon+|x|^{p/(p-1)}\right)^N}dx+\int_{B(0,\delta_0)}\frac{g(0)-g(x)}{\left(\varepsilon+|x|^{p/(p-1)}\right)^N}dx,\end{aligned}\tag{4.23}$$

hence

$$\begin{aligned}0&\le\varepsilon^{-N/p}\left|g^+\right|_\infty|U|^{p^*}_{L^{p^*}(\mathbb{R}^N)}-\varepsilon^{-N/p}\left|\left(g\chi_{B(0,2\rho_0)}\right)^{1/p^*}u_\varepsilon\right|_{p^*}^{p^*}\\&\le\int_{\mathbb{R}^N\setminus B(0,\delta_0)}\frac{g(0)}{\left(\varepsilon+|x|^{p/(p-1)}\right)^N}dx+\int_{B(0,\delta_0)}\frac{o\left(|x|^\beta\right)}{\left(\varepsilon+|x|^{p/(p-1)}\right)^N}dx\\&\le\int_{\mathbb{R}^N\setminus B(0,\delta_0)}\frac{g(0)}{|x|^{Np/(p-1)}}dx+\int_{B(0,\delta_0)}\frac{o\left(|x|^\beta\right)}{|x|^{Np/(p-1)}}dx\\&=N\omega_N\int_{\delta_0}^\infty\frac{r^{N-1}g(0)}{r^{pN/(p-1)}}dr+\int_0^{\delta_0}\frac{o(r^\beta)r^{N-1}}{r^{pN/(p-1)}}dr\\&=(p-1)\omega_N\delta_0^{-N/(p-1)}g(0)+\frac{o(1)\delta_0^{\beta-(N/(p-1))}}{\beta-(N/(p-1))}\le C_1=\text{Const.},\end{aligned}\tag{4.24}$$

which leads to

$$0 \leq 1 - \left|g^+\right|_\infty^{-1} \left|\left(g\chi_{B(0,2\rho_0)}\right)^{1/p^*} u_\varepsilon\right|_{p^*}^{p^*} |U|_{L^{p^*}(\mathbb{R}^N)}^{-p^*} \leq C_1 \left|g^+\right|_\infty^{-1} |U|_{L^{p^*}(\mathbb{R}^N)}^{-p^*} \varepsilon^{N/p}, \tag{4.25}$$

that is,

$$1 - C_1 \left|g^+\right|_\infty^{-1} |U|_{L^{p^*}(\mathbb{R}^N)}^{-p^*} \varepsilon^{N/p} \leq \left|g^+\right|_\infty^{-1} \left|\left(g\chi_{B(0,2\rho_0)}\right)^{1/p^*} u_\varepsilon\right|_{p^*}^{p^*} |U|_{L^{p^*}(\mathbb{R}^N)}^{-p^*} \leq 1. \tag{4.26}$$

Now, let ε be small enough such that $C_1 |g^+|_\infty^{-1} |U|_{p^*}^{-p^*} \varepsilon^{N/p} < 1$, then from (4.26) we can deduce that

$$\begin{aligned} 1 - C_1 \left|g^+\right|_\infty^{-1} |U|_{L^{p^*}(\mathbb{R}^N)}^{-p^*} \varepsilon^{N/p} &\leq \left(1 - C_1 \left|g^+\right|_\infty^{-1} |U|_{L^{p^*}(\mathbb{R}^N)}^{-p^*} \varepsilon^{N/p}\right)^{p/p^*} \\ &\leq \left|g^+\right|_\infty^{-(N-p)/N} \left|\left(g\chi_{B(0,2\rho_0)}\right)^{1/p^*} u_\varepsilon\right|_{p^*}^{p} |U|_{L^{p^*}(\mathbb{R}^N)}^{-p} \leq 1, \end{aligned} \tag{4.27}$$

which yields that

$$\left|g^+\right|_\infty^{(N-p)/N} |U|_{L^{p^*}(\mathbb{R}^N)}^{p} - C_1 \left|g^+\right|_\infty^{-p/N} |U|_{L^{p^*}(\mathbb{R}^N)}^{p-p^*} \varepsilon^{N/p} \leq \left|\left(g\chi_{B(0,2\rho_0)}\right)^{1/p^*} u_\varepsilon\right|_{p^*}^{p} \leq \left|g^+\right|_\infty^{(N-p)/N} |U|_{L^{p^*}(\mathbb{R}^N)}^{p}, \tag{4.28}$$

equivalently, equality (4.18) is valid.

Combining (4.18) and (4.19), we obtain that

$$\begin{aligned} Q(u_\varepsilon) &= \frac{|\nabla U|_{L^p(\mathbb{R}^N)}^p + O\left(\varepsilon^{(N-p)/p}\right)}{\left|g^+\right|_\infty^{(N-p)/N} |U|_{L^{p^*}(\mathbb{R}^N)}^p + O\left(\varepsilon^{N/p}\right)} \\ &= \left|g^+\right|_\infty^{-(N-p)/N} \frac{|\nabla U|_{L^p(\mathbb{R}^N)}^p + O\left(\varepsilon^{(N-p)/p}\right)}{|U|_{L^{p^*}(\mathbb{R}^N)}^p + O\left(\varepsilon^{N/p}\right)}. \end{aligned} \tag{4.29}$$

Hence

$$\begin{aligned} Q(u_\varepsilon) - \left|g^+\right|_\infty^{-(N-p)/N} S &= \left|g^+\right|_\infty^{-(N-p)/N} \left[\frac{|\nabla U|_{L^p(\mathbb{R}^N)}^p + O\left(\varepsilon^{(N-p)/p}\right)}{|U|_{L^{p^*}(\mathbb{R}^N)}^p + O\left(\varepsilon^{N/p}\right)} - \frac{|\nabla U|_{L^p(\mathbb{R}^N)}^p}{|U|_{L^{p^*}(\mathbb{R}^N)}^p}\right] \\ &= \left|g^+\right|_\infty^{-(N-p)/N} \left[\frac{|U|_{L^{p^*}(\mathbb{R}^N)}^p O\left(\varepsilon^{(N-p)/p}\right) - |\nabla U|_{L^p(\mathbb{R}^N)}^p O\left(\varepsilon^{N/p}\right)}{\left(|U|_{L^{p^*}(\mathbb{R}^N)}^p + O\left(\varepsilon^{N/p}\right)\right) |U|_{L^{p^*}(\mathbb{R}^N)}^p}\right] \\ &= O\left(\varepsilon^{(N-p)/p}\right). \end{aligned} \tag{4.30}$$

Using the fact that

$$\max_{t\geq 0}\left(\frac{t^p}{p}a-\frac{t^{p*}}{p^*}b\right)=\frac{1}{N}\left(\frac{a}{b^{p/p^*}}\right)^{N/p}\quad \text{for any } a,b>0, \tag{4.31}$$

we can deduce that

$$\sup_{t\geq 0} I(tu_\varepsilon)=\frac{1}{N}(Q(u_\varepsilon))^{N/p}. \tag{4.32}$$

From (4.30), we conclude that $\sup_{t\geq 0} I(tu_\varepsilon)\leq (1/N)|g^+|_\infty^{-(N-p)/p}S^{N/p}+O(\varepsilon^{(N-p)/p})$.

Step 2. We claim that for any $\lambda>0$ there exists a constant $\varepsilon_\lambda>0$ such that $\sup_{t\geq 0} J_\lambda(tu_{\varepsilon_\lambda})<(1/N)|g^+|_\infty^{-(N-p)/p}S^{N/p}$.

Using the definitions of J_λ, u_ε and by $(f2)$, $(g3)$, we get

$$J_\lambda(tu_\varepsilon)\leq \frac{t^p}{p}|\nabla u_\varepsilon|_p^p,\quad \forall t\geq 0,\ \forall\lambda>0. \tag{4.33}$$

Combining this with (4.19), let $\varepsilon\in(0,1)$, then there exists $t_0\in(0,1)$ independent of ε such that

$$\sup_{0\leq t\leq t_0} J_\lambda(tu_\varepsilon)<\frac{1}{N}\left|g^+\right|_\infty^{-(N-p)/p}S^{N/p},\quad \forall\lambda>0,\ \forall\varepsilon\in(0,1). \tag{4.34}$$

Using the definitions of J_λ, u_ε, and by the results in Step 1 and $(f2)$, we have

$$\begin{aligned}\sup_{t\geq t_0} J_\lambda(tu_\varepsilon)&=\sup_{t\geq t_0}\left(I(tu_\varepsilon)-\frac{t^q}{q}\lambda\int f(x)|u_\varepsilon|^q dx\right)\\&\leq \frac{1}{N}\left|g^+\right|_\infty^{-(N-p)/p}S^{N/p}+O\left(\varepsilon^{(N-p)/p}\right)-\frac{t_0^q}{q}\beta_0\lambda\int_{B(0,\delta_0)}|u_\varepsilon|^q dx.\end{aligned} \tag{4.35}$$

Let $0<\varepsilon\leq\delta_0^{p/(p-1)}$, we have

$$\begin{aligned}\int_{B(0,\delta_0)}|u_\varepsilon|^q dx&=\int_{B(0,\delta_0)}\frac{\varepsilon^{q(N-p)/p^2}}{\left(\varepsilon+|x|^{p/(p-1)}\right)^{((N-p)/p)q}}dx\\&\geq\int_{B(0,\delta_0)}\frac{\varepsilon^{q(N-p)/p^2}}{\left(2\delta_0^{p/(p-1)}\right)^{((N-p)/p)q}}dx\\&=C_2(N,p,q,\delta_0)\varepsilon^{(q(N-p))/p^2}.\end{aligned} \tag{4.36}$$

Combining (4.35) and (4.36), for all $\varepsilon \in (0, \delta_0^{p/(p-1)})$, we get

$$\sup_{t\geq t_0} J_\lambda(tu_\varepsilon) \leq \frac{1}{N}\left|g^+\right|_\infty^{-(N-p)/p} S^{N/p} + O\left(\varepsilon^{(N-p)/p}\right) - \frac{t_0^q}{q}\beta_0 C_2 \lambda \varepsilon^{q(N-p)/p^2}. \tag{4.37}$$

Hence, for any $\lambda > 0$, we can choose small positive constant $\varepsilon_\lambda < min\{1, \delta_0^{p/(p-1)}\}$ such that

$$O\left({\varepsilon_\lambda}^{(N-p)/p}\right) - \frac{t_0^q}{q}\beta_0 C_2 \lambda {\varepsilon_\lambda}^{q(N-p)/p^2} < 0. \tag{4.38}$$

From (4.34), (4.37), (4.38), we can deduce that for any $\lambda > 0$, there exists $\varepsilon_\lambda > 0$ such that

$$\sup_{t\geq 0} J_\lambda(tu_{\varepsilon_\lambda}) < \frac{1}{N}\left|g^+\right|_\infty^{-(N-p)/p} S^{N/p}. \tag{4.39}$$

Step 3. Prove that $\alpha_\lambda^- < (1/N)S^{N/p}$ for all $\lambda \in (0, \Lambda_1)$.

By $(f2)$, $(g2)$, and the definition of u_ε, we have

$$\int_\Omega f(x)|u_\varepsilon|^q dx > 0, \qquad \int_\Omega g(x)|u_\varepsilon|^{p^*} dx > 0. \tag{4.40}$$

Combining this with Lemma 2.7(ii), from the definition of α_λ^- and the results in Step 2, for any $\lambda \in (0, \Lambda_1)$, we obtain that there exists $t_{\varepsilon_\lambda} > 0$ such that $t_{\varepsilon_\lambda} u_{\varepsilon_\lambda} \in \mathcal{N}_\lambda^-$ and

$$\alpha_\lambda^- \leq J_\lambda(t_{\varepsilon_\lambda} u_{\varepsilon_\lambda}) \leq \sup_{t\geq 0} J_\lambda(tu_{\varepsilon_\lambda}) < \frac{1}{N}\left|g^+\right|_\infty^{-(N-p)/p} S^{N/p}. \tag{4.41}$$

This completes the proof. □

Now, we establish the existence of a local minimum of J_λ on $\mathcal{N}_\lambda^-$.

Theorem 4.4. *If $\lambda \in (0, (q/p)\Lambda_1)$, then J_λ satifies the $(PS)_{\alpha_\lambda^-}$ condition. Moreover, J_λ has a minimizer U_λ in $\mathcal{N}_\lambda^-$ and satisfies that*

(i) *$J_\lambda(U_\lambda) = \alpha_\lambda^-$;*

(ii) *U_λ is a positive solution of $(E_{\lambda f, g})$ in $C^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1)$,*

where Λ_1 is as in (1.5).

Proof. If $\lambda \in (0, (q/p)\Lambda_1)$, then by Theorem 2.6(ii), Proposition 3.3(ii), and Lemma 4.3, there exists a $(PS)_{\alpha_\lambda^-}$-sequence $\{u_n\} \subset \mathcal{N}_\lambda^-$ in W for J_λ with $\alpha_\lambda^- \in (0, (1/N)|g^+|_\infty^{-(N-p)/p} S^{N/p})$. From Lemma 4.2, there exists a subsequence still denoted by $\{u_n\}$ and nontrivial solution $U_\lambda \in W$ of $(E_{\lambda f, g})$ such that $u_n \rightharpoonup U_\lambda$ weakly in W. Now we prove that $u_n \to U_\lambda$ strongly in W and $J_\lambda(U_\lambda) = \alpha_\lambda^-$. By (3.29), if $u \in \mathcal{N}_\lambda$, then

$$J_\lambda(u) = \frac{p^* - p}{p^* p}\|u\|^p - \frac{p^* - q}{p^* q}\lambda \int_\Omega f|u|^q dx. \tag{4.42}$$

First, we prove that $U_\lambda \in \mathcal{N}_\lambda^-$. On the contrary, if $U_\lambda \in \mathcal{N}_\lambda^+$, then by $\mathcal{N}_\lambda^-$ closed in W, we have $\|U_\lambda\| < \liminf_{n\to\infty}\|u_n\|$. By Lemma 2.7, there exists a unique t_λ^- such that $t_\lambda^- U_\lambda \in \mathcal{N}_\lambda^-$. Since $u_n \in \mathcal{N}_\lambda^-$, $J_\lambda(u_n) \geq J_\lambda(tu_n)$ for all $t \geq 0$ and by (4.42), we have

$$\alpha_\lambda^- \leq J_\lambda(t_\lambda^- U_\lambda) < \lim_{n\to\infty} J_\lambda(t_\lambda^- u_n) \leq \lim_{n\to\infty} J_\lambda(u_n) = \alpha_\lambda^-, \tag{4.43}$$

and this is contradiction.

In order to prove that $J_\lambda(U_\lambda) = \alpha_\lambda^-$, it suffices to recall that $u_n, U_\lambda \in \mathcal{N}_\lambda^-$ for all n, by (4.42), and applying Fatou's lemma to get

$$\begin{aligned}\alpha_\lambda^- \leq J_\lambda(U_\lambda) &= \frac{p^*-p}{p^*p}\|U_\lambda\|^p - \frac{p^*-q}{p^*q}\lambda\int_\Omega f|U_\lambda|^q dx \\ &\leq \liminf_{n\to\infty}\left(\frac{p^*-p}{p^*p}\|u_n\|^p - \frac{p^*-q}{p^*q}\lambda\int_\Omega f|u_n|^q dx\right) \\ &\leq \liminf_{n\to\infty} J_\lambda(u_n) = \alpha_\lambda^-.\end{aligned} \tag{4.44}$$

This implies that $J_\lambda(U_\lambda) = \alpha_\lambda^-$ and $\lim_{n\to\infty}\|u_n\|^p = \|U_\lambda\|^p$. Let $v_n = u_n - U_\lambda$, then Brézis and Lieb lemma [15] implies that

$$\|v_n\|^p = \|u_n\|^p - \|U_\lambda\|^p + o_n(1). \tag{4.45}$$

Therefore, $u_n \to U_\lambda$ strongly in W.

Since $J_\lambda(U_\lambda) = J_\lambda(|U_\lambda|)$ and $|U_\lambda| \in \mathcal{N}_\lambda^-$, by Lemma 2.3 we may assume that U_λ is a nontrivial nonnegative solution of $(E_{\lambda f,g})$. Finally, by using the same arguments as in the proof of Theorem 3.4, for all $\lambda \in (0, (q/p)\Lambda_1)$, we have that U_λ is a positive solution of $(E_{\lambda f,g})$ in $C^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1)$. □

Now, we complete the proof of Theorem 1.5. By Theorems 3.4 and 4.4, if $\lambda \in (0, (q/p)\Lambda_1)$, then we obtain $(E_{\lambda f,g})$ that has two positive solutions u_λ and U_λ such that $u_\lambda \in \mathcal{N}_\lambda^+$, $U_\lambda \in \mathcal{N}_\lambda^-$, and $u_\lambda, U_\lambda \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1)$. Since $\mathcal{N}_\lambda^+ \cap \mathcal{N}_\lambda^- = \emptyset$, this implies that u_λ and U_λ are distinct.

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