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Review Article

Well-Posedness of the Cauchy Problem for Hyperbolic Equations with Non-Lipschitz Coefficients

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We consider hyperbolic equations with anisotropic elliptic part and some non-Lipschitz coefficients. We prove well-posedness of the corresponding Cauchy problem in some functional spaces. These functional spaces have finite smoothness with respect to variables corresponding to regular coefficients and infinite smoothness with respect to variables corresponding to singular coefficients.

1. Introduction

Let us consider the Cauchy problem for a second-order hyperbolic equation:

$$\ddot{u} - \sum_{i,j=1}^{n} a_{ij}(t) u_{x_i x_j} + \sum_{j=1}^{n} b_j(t) u_{x_j} + c(t)u = 0, \quad (t,x) \in [0,T] \times R^n, \tag{1.1}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in R^n, \tag{1.2}$$

where the matrix $(a_{ij}(t))$ is real and symmetric for all $t \in (0,T]$, $\ddot{u} = u_{tt}$.

Suppose that (1.1) is strictly hyperbolic, that is, there exists $\lambda_0 > 0$ such that

$$a(t,\xi) \equiv \sum_{i,j=1}^{n} a_{ij}(t) \frac{\xi_i \xi_j}{|\xi|^2} \geq \lambda_0 > 0, \tag{1.3}$$

for all $(t,\xi) \in (0,T] \times R^n \setminus \{0\}$.

It is known that if $a(t,\xi)$ satisfies the Lipschitz condition and $b_j(t), c(t) \in L_\infty(0,T)$, $j = 1,2,\dots,n$, then for any $u_0 \in H^s(R^n)$, $u_1 \in H^{s-1}(R^n)$ the problem (1.1), (1.2) has a unique solution

$$u(\cdot) \in C([0,T]; H^s(R^n)) \cap C^1\Big([0,T], H^{s-1}(R^n)\Big), \tag{1.4}$$

where $s \geq 1$ (see [1, Chapter 5] and [2, Chapter 3]) .

If we reject the Lipschitz condition, this result, generally speaking, stops to be valid (see [3]).

In the paper [4] it is proved that if $a(t,\xi) \in LL_\omega(0,T)$, that is, if $a(t,\xi)$ satisfies the logarithmic Lipschitz condition:

$$|a(t+\tau,\xi) - a(t,\xi)| \leq c|\tau| \cdot \big|\log|\tau|\big| \cdot \omega(|\tau|), \tag{1.5}$$

where $\omega(|\tau|)$ monotonically decreasing tends to zero, and $\log|\tau| \cdot \omega(|\tau|)$ tends to infinity, then there exists $\delta > 0$ such that, for all $u_0 \in H^s(R^n)$, $u_1 \in H^{s-1}(R^n)$ the problem (1.1), (1.2) has a unique solution $u \in C([0,T]; H^{s-\delta}(R^n)) \cap C^1([0,T], H^{s-1-\delta}(R^n))$ (this behavior goes under the name of loss of derivatives).

In the paper [5] it is considered the case when $a_{i,j}(t) = 0$, $i \neq j$, a part of coefficients belongs to the class $LL_\omega(0,T)$, and another part of coefficients satisfies the Lipschitz condition. It is proved that the loss of derivatives occurs in those variables x_k for which appropriate coefficient $a_{kk}(t)$ belongs to the class $LL_\omega(0,T)$.

It is interesting to investigate the Cauchy problem for (1.1), with singular coefficients. Many interesting results have been obtained in this direction. For example, in the paper [6] it is supposed that for each $\xi \in R^n \setminus \{0\} a(t,\xi) \in C^1(0,T]$ and

$$t^q|\dot{a}(t,\xi)| \leq c, \quad (t,\xi) \in (0,T] \times R^n \setminus \{0\}, \tag{1.6}$$

where $q \geq 1$, $c > 0$. It is proved that if $q = 1$, the problem (1.1), (1.2) is well-posed in $C^\infty(R^n)$. If $q > 1$ and

$$t^p|a(t,\xi)| \leq c, \quad (t,\xi) \in (0,T] \times R^n \setminus \{0\}, \tag{1.7}$$

where $p \in [0,1) \cap [0,q-1)$, then the problem (1.1), (1.2) is well-posed in the Geverey class $\gamma^{(s)}(R^n)$, $s < (q-p)/(q-1)$ (see [6]). If the coefficients $a_{ij}(t)$ satisfy only Holder conditions of order $\alpha < 1$ then in [3] it is established that the problem (1.1), (1.2) is $\gamma^{(s)}$ well-posed for all $s < 1/(1-\alpha)$. In this direction see also the results obtained in the papers [6–13].

In this paper we consider the Cauchy problem for a higher-order hyperbolic equation with anisotropic elliptic part:

$$\begin{gathered} \ddot{u} + \sum_{k=1}^{n} (-1)^{l_k} a_k(t) D_{x_k}^{2l_k} u + \sum_{|\alpha:l|\leq 1} b_\alpha(t) D_x^\alpha u = 0, \quad (t,x) \in [0,T] \times R^n, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in R^n, \end{gathered} \tag{1.8}$$

where $l_k \in N, \{1,2,\dots,\}$, $\alpha_k \in N \cup \{0\}$, $k = 1,2,\dots,n$, $|\alpha : l| = \alpha_1/l_1 + \cdots + \alpha_n/l_n$.

Here the coefficients $a_k(t)$ satisfy different conditions of type (1.6) and (1.7), so that q_k and p_k corresponding to different k are different. The smoothness of the solution depending on smoothness on initial data with respect to each variable x_k depends not only on l_k but also on q_k and p_k.

2. Statement of the Problem and Results

We considered the Cauchy problem (1.8). Suppose that $a_k(t)$ and $b_\alpha(t)$ satisfy the following conditions:

$$a_k(t) \geq a > 0, \quad t \in [0,T], \ k = 1,2,\ldots,n, \tag{2.1}$$

$$t^{q_k} |\dot{a}_k(t)| \leq c, \quad t \in (0,T], \ k = 1,2,\ldots,n, \tag{2.2}$$

$$b_\alpha(t) \in L_\infty(0,T), \quad |\alpha : l| \leq 1. \tag{2.3}$$

In order to formulate the basic results we introduce some denotation. Let H be some Hilbert space. By $W_2^{\lambda,L}(R^m, H)$ we will denote a functional space with the norm

$$\|u\|_{W_2^{\lambda,L}(R^m,H)} = \left[\int_{R^m} \left(1 + \sum_{k=1}^{m} \eta_k^{2L_k}\right)^{\lambda} \|\widehat{u}(\eta)\|_H^2 d\eta\right]^{1/2}, \tag{2.4}$$

where $L = (L_1,\ldots,L_m)$, $L_j \in N$, $j = 1,2,\ldots,m$, $\lambda \geq 0$, and $\widehat{u}(\eta) = F_x[u](\eta)$; F_x is a Fourier transformation with respect to variable $x \in R^n$.

For $s \geq 1$ by $\gamma_\beta^{s,L}(R^m, H)$ we will denote a functional space with the norm

$$\|u\|_{\gamma_\beta^{s,L}(R^m,H)} = \left[\int \exp\left\{\beta \left|\sum_{k=1}^{m} \eta_k^2\right|^{1/s}\right\} \|\widehat{u}(\eta)\|_H^2 d\eta\right]^{1/2}. \tag{2.5}$$

Denote $W_2^{\lambda,L}(R^m, R) = W_2^{\lambda,L}(R^m)$, $\gamma_\beta^{s,L}(R^m, R) = \gamma_\beta^{s,L}(R^m)$,

$$C^\infty(R^m; H) = \bigcap_{\lambda \geq 0} W_2^{\lambda,L}(R^m; H), \qquad \gamma^{(s)}(R^m; H) = \bigcap_{\beta \geq 0} \gamma_\beta^{(s)}(R^m; H). \tag{2.6}$$

If $L = (1,\ldots,1)$ then $W_2^{\lambda,L}(R^m, H) = H^\lambda(R^m; H)$, $\gamma_\beta^{s,L}(R^m, H) = \gamma_\beta^{s}(R^m, H)$, and $\gamma_\beta^{s,L}(R^m, R) = \gamma_\beta^{(s)}$, where $\gamma_\beta^{(s)}$ is the Geverey space of order s (see [12, 13]). If $\lambda \in H$ then $W_2^{\lambda,L}(R^m, H)$ is Hilbert-valued anisotropic Sobolev space $W_2^{(\lambda L_1,\ldots,\lambda L_m)}(R^m; H)$. For the read valued functions the anisotropic Sobolev spaces are stated in [14]. The basic results led in [14] are also valid for abstract-valued functions.

We introduce also the following denotation:

$$\begin{gathered} x' = (x_1, \dots, x_{n_1}), \qquad x'' = (x_{n_1+1}, \dots, x_n), \\ \xi' = (\xi_1, \dots, \xi_{n_1}), \qquad \xi'' = (\xi_{n_1+1}, \dots, \xi_n), \\ l' = (l_1, \dots, l_{n_1}), \qquad l'' = (l_{n_1+1}, \dots, l_n), \\ |\xi| = \sum_{k=1}^{n} \xi_k^{2l_k}, \quad |\xi'|_{l'} = \sum_{k=1}^{n_1} \xi_k^{2l_k}, \quad |\xi''|_{l''} = \sum_{k=n_1+1}^{n} \xi_k^{2l_k}, \quad n_2 = n - n_1. \end{gathered} \tag{2.7}$$

The main results are the following theorems.

Theorem 2.1. *Let the conditions* (2.1)–(2.3) *be satisfied, where*

$$q_k \in [0,1), \quad \text{for } k = 1, 2, \dots, n_1, \tag{2.8}$$

$$q_k = 1, \quad \text{for } k = n_1 + 1, \dots, n. \tag{2.9}$$

Then for any $\lambda' \geq 0$, $\lambda'' \geq 0$ the energy estimates

$$E(t, \lambda', \lambda'') \leq M E(0, \lambda', \lambda'' + \lambda_0), \tag{2.10}$$

hold, where M and λ_0 are some constants indepent of $t \in [0, T]$,

$$\begin{gathered} E(t, \lambda', \lambda'' + \lambda) = \int_{R^n} (1 + |\xi'|_{l'})^{\lambda'} (1 + |\xi''|_{l''})^{\lambda'' + \lambda} \left[|\dot{v}(t, \xi)|^2 + (1 + |\xi|_l) |v(t, \xi)|^2 \right] d\xi, \\ \lambda \geq 0, \quad \dot{v}(t, \xi) = \frac{\partial v(t, \xi)}{\partial t}. \end{gathered} \tag{2.11}$$

Theorem 2.2. *Let the conditions* (2.1)–(2.3) *be satisfied, where*

$$q_k \in [0,1), \quad \text{for } k = 1, 2, \dots, n_1, \tag{2.12}$$

$$q_k = q > 1, \quad \text{for } k = n_1 + 1, \dots, n. \tag{2.13}$$

Additionally, let the conditions

$$t^p |a_k(t)| \leq c, \quad t \in [0, T], \quad \text{for } k = n_1 + 1, \dots, n. \tag{2.14}$$

be satisfied, where $p \in [0,1) \cap [0, q-1)$. Then for any $\beta > 0$, $\lambda' \geq 0$, and $1 \leq s < (q-p)/(q-1)$ the energy estimates,

$$\mathcal{E}(t, \beta, s, \lambda') \leq M \mathcal{E}(0, \beta + \delta, s, \lambda'), \tag{2.15}$$

hold, where M and δ are some constants independent of $t \in [0,T]$,

$$\mathcal{E}(t,\beta,s,\lambda') = \int_{R^n} \exp\left\{\beta\left|\xi''\right|_{l''}^{1/s}\right\}\left(1+\left|\xi'\right|_{l'}\right)^{\lambda'}\left[|\dot{v}(t,\xi)|^2 + (1+|\xi|_l)|v(t,\xi)|^2\right]d\xi. \tag{2.16}$$

Remark 2.3. It is clear by our notation that

$$\begin{aligned} E(t,\lambda',\lambda'') &\leq \|\dot{u}(t,\cdot)\|_{W_2^{\lambda'',l''}\left(R^{n_2}_{x''};W_2^{\lambda'+1,l'}(R^{n_1}_{x'})\right)} + \|u(t,\cdot)\|_{W_2^{\lambda'',l''}\left(R^{n_2}_{x''};W_2^{\lambda'+1,l'}(R^{n_1}_{x'})\right)} \\ &\quad + \|u(t,\cdot)\|_{W_2^{\lambda''+1,l''}\left(R^{n_2}_{x''};W_2^{\lambda',l'}(R^{n_1}_{x'})\right)} \\ &\leq 2E(\lambda',\lambda'',t), \end{aligned} \tag{2.17}$$

and we can write

$$\mathcal{E}(t,\beta,s,\lambda') = \|u(t,\cdot)\|_{\gamma_\beta^{s,l''}\left(R^{n_2}_{x''};W_2^{\lambda',l'}(R^{n_1}_{x'})\right)}. \tag{2.18}$$

Remark 2.4. It is possible to replace the conditions $a_1(t),\ldots,a_{n_1}(t) \in C^1(0,T]$ and (2.8) or (2.12) by Lipschitz conditions.

The following theorems are obtained from Theorems 2.1 and 2.2.

Theorem 2.5. *Let condition* (2.1)–(2.9) *be satisfied. Then for any* $s \geq 0$, $u_0 \in C^\infty(R^{n_2}_{x''};W_2^{s+1,l'}(R^{n_1}_{x'}))$, $u_1 \in C^\infty(R^{n_2}_{x''};W_2^{s,l'}(R^{n_1}_{x'}))$ *the problem* (1.1), (1.2) *admits a unique solution*

$$u \in C\left([0,T];C^\infty\left(R^{n_2}_{x''};W_2^{s+1,l'}(R^{n_1}_{x'})\right)\right) \cap C^1\left([0,T];C^\infty\left(R^{n_2}_{x''};W_2^{s,l'}(R^{n_1}_{x'})\right)\right). \tag{2.19}$$

Theorem 2.6. *Let conditions* (2.1)–(2.3) *and* (2.12)–(2.14) *be satisfied. Then for any* $s' \geq 0$, $1 \leq s'' < (q-p)/(q-1)$, $u_0 \in \gamma^{s''}(R^{n_2}_{x''};W_2^{s'+1,l'}(R^{n_1}_{x'}))$, $u_1 \in \gamma^{s''}(R^{n_2}_{x''};W_2^{s',l'}(R^{n_1}_{x'}))$ *the problem* (1.1), (1.2) *admits a unique solution*

$$u \in C\left([0,T]\ ;\gamma^{s''}\left(R^{n_2}_{x''};W_2^{s'+1,l'}(R^{n_1}_{x'})\right)\right) \cap C^1\left([0,T]\ ;\gamma^{s''}\left(R^{n_2}_{x''};W_2^{s',l'}(R^{n_1}_{x'})\right)\right). \tag{2.20}$$

In particular it follows from Theorem 2.1 that if the conditions (2.1)–(2.3) are satisfied, then the problem (1.1), (1.2) is well-posed in $C^\infty(R^n)$, and if the conditions (2.1)–(2.3) and (2.12)–(2.14) are satisfied then the problem (1.1), (1.2) is well-posed in the Geverey class $\gamma^{(s)}$.

3. Proof of Theorems

At first we reduce some auxiliary statements.

We denote $v(t,\xi) = F_x[u](t,\xi)$ and define the weighted energetic function in the following way:

$$\Phi(t) = \Phi(t,\xi,\lambda',\lambda'',\beta,r) = \left[|\dot{v}(t,\xi)|^2 + \left(1+\left|\xi'\right|_{l'} + d(t,\xi'')\right)|v(t,\xi)|^2\right]\cdot H(t,\xi), \tag{3.1}$$

where

$$
\begin{aligned}
H(t,\xi) &= H(t,\xi,\lambda',\lambda'',\beta,r) \\
&= \left(1+|\xi'|_{l'}\right)^{\lambda'}\left(1+|\xi''|_{l''}\right)^{\lambda''} \\
&\quad\times \exp\left[-\int_0^t \alpha(\tau,\xi'')d\tau + \beta|\xi''|_{l''}^{(q-1)/r}\right], \quad \lambda'\geq 0,\ \lambda''\geq 0,\ \beta>0, \\
r &= \begin{cases} s(q-1), & \text{for } q>1, \\ 1, & \text{for } q=1, \end{cases}
\end{aligned}
$$

$$
d(t,\xi'')\begin{cases} \sum_{k=n_1+1}^{n} a_k(T)\xi_k^{2l_k}, & \text{for } T^r|\xi''|_{l''}\leq 1, \\ \sum_{k=n_1+1}^{n} a_k\left(|\xi''|_{l''}^{-1/r}\right)\xi_k^{2l_k}, & \text{for } T^r|\xi''|_{l''}>1,\ t^r|\xi''|_{l''}\leq 1, \\ \sum_{k=n_1+1}^{n} a_k(t)\xi_k^{2l_k}, & \text{for } t^r|\xi''|_{l''}>1, \end{cases} \tag{3.2}
$$

$$
\alpha(t,\xi'')\begin{cases} \left|d(t,\xi'') - \sum_{k=n_1+1}^{n} a_k(t)\xi_k^{2l_k}\right|, & \text{for } t^r|\xi''|_{l''}\leq 1, \\ \dfrac{\left|\sum_{k=n_1+1}^{n} \dot{a}_k(t)\xi_k^{2l_k}\right|}{\sum_{k=n_1+1}^{n} a_k(t)\xi_k^{2l_k}}, & \text{for } t^r|\xi''|_{l''}>1. \end{cases}
$$

The following auxiliary lemmas are proved similar to the paper [6]. The proofs of the lemmas are in appendix.

Lemma 3.1. *If $q_k=1$, $k=n_1+1,\ldots,n$, then there exits such $c_1>0$, $c_2>0$, that*

$$
a|\xi''|_{l''} \leq d(t,\xi'') \leq \left[c_1 + c_2\ln\left(1+|\xi''|_{l''}\right)\right]|\xi''|_{l''}. \tag{3.3}
$$

If $q_k>1$, $k=n_1+1,\ldots,n$, then there exits such $c_1>0$, $c_2>0$, that

$$
a|\xi''|_{l''} \leq d(t,\xi'') \leq \left[c_1 + c_2|\xi''|_{l''}^{p/r}\right]|\xi''|_{l''}. \tag{3.4}
$$

Lemma 3.2. *If $q_k=1$, $k=1,2,\ldots,n_1$, then there exits such constant $c_3>0$, $\gamma>0$, that $\int_0^t\alpha(\tau,\xi)d\tau \leq c_3 + c_4\ln(1+|\xi''|_{l''})$.*

If $q_k>1$, $k=1,2,\ldots,n_1$ then there exits such $c_3>0$, $c_4>0$, that

$$
\int_0^t \alpha(\tau,\xi)d\xi \leq c_3 + c_4|\xi''|_{l''}^{(q-1)/r}. \tag{3.5}
$$

By the definition of $\Phi(t) = \Phi(t, \xi, \lambda', \lambda'', \beta, r)$ we have

$$\frac{d\Phi(t)}{dt} = 2\mathrm{Re}\Big[\ddot{v}(t,\xi)\overline{\dot{v}(t,\xi)} + \big(1 + |\xi'|_{\ell'} + d\big(t,\xi''\big)\big)v(t,\xi)\overline{\dot{v}(t,\xi)}\Big]H(t,\xi) + \dot{d}\big(t,\xi''\big)|v(t,\xi)|^2 H(t,\xi) - \alpha(t,\xi)\Phi(t). \tag{3.6}$$

On the other hand from (1.8) we have

$$\ddot{v}(t,\xi) + \sum_{k=1}^{n} a_k(t)\ \xi_k^{2\ell_k} v(t,\xi) + \sum_{|\alpha:\ell|\le 1} b_\alpha(t)(i\xi)^\alpha v(t,\xi) = 0, \tag{3.7}$$

$$v(0,\xi) = v_0(\xi), \qquad \dot{v}(0,\xi) = v_1(\xi), \tag{3.8}$$

where $v_0(\xi) = F[u_0](\xi)$, $v_1(\xi) = F[u_1](\xi)$, $\ddot{v}(t,\xi) = \partial^2 v(t,\xi)/\partial t^2$.

From (3.6) and (3.7) we obtain

$$\frac{d\Phi(t)}{dt} = 2\mathrm{Re}\left[-\sum_{k=1}^{n_1} a_k(t)\ \xi_k^{2\ell_k} + \big(1 + |\xi'|_{\ell'}\big) + \left(d\big(t,\xi''\big) - \sum_{k=n_1+1}^{n} a_k(t)\ \xi_k^{2\ell_k}\right)\right] \times v(t,\xi)\overline{\dot{v}(t,\xi)}H(t,\xi) - 2\mathrm{Re}\sum_{|\alpha:\ell|\le 1} b_\alpha(t)(i\xi)^\alpha v(t,\xi)\overline{\dot{v}(t,\xi)}H(t,\xi) + \dot{d}\big(t,\xi''\big)|v(t,\xi)|^2 H(t,\xi) - \alpha(t,\xi)\Phi(t). \tag{3.9}$$

If $t^r|\xi''| < 1$, then by definition of $d(t,\xi)$ and $\alpha(t,\xi'')$ we have

$$\frac{d\Phi(t)}{dt} = 2\mathrm{Re}\left[-\sum_{k=1}^{n_1} a_k(t)\ \xi_k^{2\ell_k} + \big(1 + |\xi'|_{\ell'}\big) + \alpha(t,\xi)\right]v(t,\xi)\overline{\dot{v}(t,\xi)}H(t,\xi) - 2\mathrm{Re}\sum_{|\alpha:\ell|\le 1} b_\alpha(t)(i\xi)^\alpha v(t,\xi)\overline{\dot{v}(t,\xi)}H(t,\xi) - \alpha(t,\xi)\Phi(t). \tag{3.10}$$

By our supposition $q_k < 1$ for $k = 1, 2, \ldots, n_1$. Therefore we can easily see that

$$a \le a_k(t) \le a_T, \quad k = 1, 2, \ldots, n_1 \tag{3.11}$$

with some constant $a_T > a$.

Using the Cauchy inequality, definition of $\alpha(t,\xi)$, $H(t,\xi)$, and $\varphi(t)$ we have

$$2\mathrm{Re}\,\alpha(t,\xi)v(t,\xi)\overline{\dot{v}(t,\xi)}H(t,\xi) - \alpha(t,\xi)\Phi(t) \le 0, \tag{3.12}$$

$$\begin{aligned} 2\mathrm{Re}&\sum_{|\alpha:\ell|\le 1} b_\alpha(t)(i\xi)^\alpha v(t,\xi)\overline{\dot{v}(t,\xi)}H(t,\xi) \\ &\le 2b_T\sum_{|\alpha:l|\le 1}|\xi^\alpha||v(t,\xi)|\cdot|\dot{v}(t,\xi)|\cdot H(t,\xi) \\ &\le 2b_T c_5\left[\left(1+\sum_{k=1}^n|\xi_k|^{2l_k}\right)|v(t,\xi)|^2+|\dot{v}(t,\xi)|^2\right]H(t,\xi), \end{aligned} \tag{3.13}$$

where $b_T = \sup_{|\alpha:l|\le 1}\|b_\alpha(t)\|_{L_\infty(0,T)}$, $c_5 = \sup_{\xi\in R^n}(\sum_{|\alpha:\ell|\le 1}|\xi^\alpha|)^2/(\sum_{k=1}^n|\xi_k|^{2l_k}+1)$.

From (3.10)–(3.13) we get that when $t^r|\xi''|_{l''} < 1$, then there exists such a constant $M_1 > 0$, that

$$\frac{d\Phi(t)}{dt} \le M_1\Phi(t). \tag{3.14}$$

If $t^r|\xi''|_{l''} \ge 1$ then by definition of $d(t,\xi)$ and $\alpha(t,\xi'')$ from (3.9) we have that

$$\begin{aligned} \frac{d\Phi(t)}{dt} &= 2\mathrm{Re}\left[-\sum_{k=1}^{n_1}a_k(t)\xi_k^{2\ell_k} - \sum_{|\alpha:\ell|\le 1}b_\alpha(t)(i\xi)^\alpha v(t,\xi)\overline{\dot{v}(t,\xi)}\right]H(t,\xi) \\ &\quad + \sum_{k=n_1+1}^n \dot{a}_k(t)\ \xi_k^{2\ell_k}|v(t,\xi)|^2H(t,\xi) - \frac{\left|\sum_{k=n_1+1}^n \dot{a}_k(t)\xi_k^{2\ell_k}\right|}{\sum_{k=n_1+1}^n a_k(t)\xi_k^{2\ell_k}}\Phi(t). \end{aligned} \tag{3.15}$$

On the other hand

$$\begin{aligned} &\sum_{k=n_1+1}^n \dot{a}_k(t)\ \xi_k^{2\ell_k}|v(t,\xi)|^2H(t,\xi) - \frac{\left|\sum_{k=n_1+1}^n \dot{a}_k(t)\xi_k^{2\ell_k}\right|}{\sum_{k=n_1+1}^n a_k(t)\xi_k^{2\ell_k}}\Phi(t) \\ &\quad= \sum_{k=n_1+1}^n \dot{a}_k(t)\xi_k^{2\ell_k}|v(t,\xi)|^2H(t,\xi) - \frac{\left|\sum_{k=n_1+1}^n \dot{a}_k(t)\xi_k^{2\ell_k}\right|}{\sum_{k=n_1+1}^n a_k(t)\xi_k^{2\ell_k}} \\ &\quad\quad\times\left[|\dot{v}(t,\xi)|^2+\left(1+\left|\xi'\right|_{l'}^2+\sum_{k=n_1+1}^n a_k(t)\xi_k^{2\ell_k}\right)|v(t,\xi)|^2\right]H(t,\xi) \le 0. \end{aligned} \tag{3.16}$$

From (3.13), (3.15), and (3.16) we again get inequality (3.14).

It follows from (3.14) that

$$\Phi(t) \le M\Phi(0), \quad t\in[0,T], \tag{3.17}$$

where $M = M_1e^T$.

Proof of Theorem 2.1. Let $q_k = q = 1$, $k = n_1 + 1, \dots, n$, then $r = 1$. From Lemmas 3.1 and 3.2 we have

$$\begin{aligned}\int_{R^n} \Phi(t,\xi,\lambda',\lambda'',0,1)d\xi &\le \int_{R^n} (1+|\xi'|_{\ell'})^{\lambda'} (1+|\xi''|_{\ell''})^{\lambda''} \\ &\quad \times \left[|\dot{v}(t,\xi)|^2 + \left(1+|\xi'|_{l'}^2 + |\xi''|_{l''}(c_1 + c_2 \ln(1+|\xi''|_{l''}))\right)\right]|v(t,\xi)|^2 \\ &\quad \times \exp(c_3 + c_4 \ln(1+|\xi''|_{l''}))d\xi \\ &\le c_6 \int_{R^n} (1+|\xi'|_{\ell'})^{\lambda'} (1+|\xi''|_{\ell''})^{\lambda''+c_2} \left[|\dot{v}(t,\xi)|^2 + (1+|\xi|_{\ell})|v(t,\xi)|^2\right] d\xi \\ &= c_6 E(t,\lambda',\lambda''+\lambda_0),\end{aligned} \tag{3.18}$$

where $\lambda_0 = c_4 + 1$, $c_6 = \max\{1, c_1 e^{c_3}, c_2 e^{c_3}\}$.

Thus,

$$\int_{R^n} \Phi(t,\xi,\lambda',\lambda'',0,1)d\xi \le c_6 E(t,\lambda',\lambda''+\lambda_0). \tag{3.19}$$

On the other hand from the definition of Φ and E we have

$$\int_{R^n} \Phi(t,\xi,\lambda',\lambda'',0,1)d\xi \ge c_7 E(t,\lambda',\lambda''). \tag{3.20}$$

It follows from (3.17)–(3.20) that

$$E(t,\lambda',\lambda'') \le c_8 E(0,\lambda',\lambda''+d). \tag{3.21}$$

□

Proof of Theorem 2.2. Let $q_k = q > 1$, $k = n_1 + 1, \dots, n$, then $r = (q-1)s$. Taking into account Lemmas 3.1 and 3.2 and Theorem 2.5 we have

$$\begin{aligned}&\int_{R^n} \Phi(t,\xi,\lambda',0,\beta,r)d\xi \\ &\quad \le \int_{R^n} (1+|\xi'|_{\ell})^{\lambda'} \left[|v(t,\xi)|^2 + \left(1+|\xi'|_{l'} + c_1|\xi''|_{l'} + c_2|\xi''|^{p/2+1}\right)\right] \\ &\quad\quad \times |v(t,\xi)|^2 \exp\left(c_3 + (\beta+c_4)|\xi|^{(q-1)/2}\right)d\xi.\end{aligned} \tag{3.22}$$

Further using the inequality $\eta^{p/2+1} \le c_9 \exp(c\eta^{1/s})$ we obtain

$$\int_{R^n} \Phi(t,\xi,\lambda',0,\beta,r)d\xi \le c_{10} \mathcal{E}(t,\lambda',s,\beta+\delta), \tag{3.23}$$

where $\delta = c_4 + c$.

On the other hand from the definition of ϕ and $\mathcal{E}$ we have

$$\int_{R^n} \Phi(t,\xi,\lambda',0,\beta,r)\,d\xi \geq c_{11}\mathcal{E}(t,\lambda',s,\beta). \tag{3.24}$$

From inequalities (3.17), (3.23), and (3.24) it follows that

$$\mathcal{E}(t,\lambda',s,\beta) \leq c_{12}\mathcal{E}(0,\lambda',s,\beta+d). \tag{3.25}$$

□

Proof of Theorem 2.5. For any $\xi \in R^n$ the problem (3.7), (3.8) has a unique solution $v(t,\xi) \in C^1[0,T]$ (see [15, Chapter I]).

Let $u_0 \in C^\infty(R^{n_2}_{x''}; W_2^{\lambda'+1,l'}(R^{n_1}_{x'}))$, $u_1 \in C^\infty(R^{n_2}_{x''}; W_2^{\lambda',l'}(R^{n_1}_{x'}))$, then for any $s \geq 0$, $\lambda' \geq 0$,

$$E(0,\lambda',s+\lambda_0) \leq c_{s,\lambda'}, \tag{3.26}$$

where $c_{s,\lambda} > 0$ is some constant dependent on $s \geq 0$ and $\lambda' \geq 0$.

Taking into account Theorem 2.1 it follows from (3.20) that

$$E(t,\lambda',s) \leq Mc_{\lambda',s}, \quad t \in [0,T], \tag{3.27}$$

that is,

$$\begin{aligned} &\|\dot{u}(t,\cdot)\|_{W_2^{s,l''}\left(R^{n_2}_{x''};W_2^{\lambda',l'}(R^{n_1}_{x'})\right)} + \|u(t,\cdot)\|_{W_2^{s+1,l''}\left(R^{n_2}_{x''};W_2^{\lambda',l'}(R^{n_1}_{x'})\right)} \\ &\quad + \|u(t,\cdot)\|_{W_2^{s,l''}\left(R^{n_2}_{x''};W_2^{\lambda'+1,l'}(R^{n_1}_{x'})\right)} \leq Mc_{\lambda',s}, \quad t \in [0,T]. \end{aligned} \tag{3.28}$$

It follows from (3.28) that

$$\begin{aligned} &u \in C\Big([0,T]; C^\infty\Big(R^{n_2}_{x''}; W_2^{\lambda'+1,l'}(R^{n_1}_{x'})\Big)\Big), \\ &\dot{u} \in C\Big([0,T]; C^\infty\Big(R^{n_2}_{x''}; W_2^{\lambda',l'}(R^{n_1}_{x'})\Big)\Big). \end{aligned} \tag{3.29}$$

By the expression of $u(t,x)$ it follows that the function $u(t,x)$ is the solution of problem (1.8).

The uniqueness of the solution is proved by standard method. □

The proof of Theorem 2.6 is carried out in the similar way.

Appendices

A. Proof of Lemmas

Proof of Lemma 3.1. Let $q_k = 1$, $k = n_1 + 1, \ldots, n$. Then from (2.2) we have

$$\begin{aligned} a_k(t) &\le a_k(T) + |a_k(t) - a_k(T)| \\ &\le a_k(T) + \int_t^T |\dot{a}_k(s)| ds \\ &\le a_k(T) + c \ln \frac{T}{t} \\ &\le c_1 + c_2 \ln\left(1 + \frac{1}{t}\right). \end{aligned} \tag{A.1}$$

It follows from (2.1) and (2.14) that

$$a\left|\xi''\right|_{l''} \le d(t, \xi''). \tag{A.2}$$

By definition of $d(t, \xi'')$ for $T|\xi''|_{l''} \le 1$ we have

$$d(t, \xi'') = \sum_{k=n_1}^{n} a_k(T) \xi_k^{2l_k} \le c_1 \left|\xi''\right|_{l''}. \tag{A.3}$$

If $T|\xi''|_{l''} > 1$, and $t|\xi''|_{l''} < 1$, then from (A.1) we have

$$\begin{aligned} d(t, \xi'') &= \sum_{k=n_1}^{n} a_k\left(|\xi''|_{l''}^{-1}\right) \xi_k^{2l_k} \\ &\le \left[c_1 + c_2 \ln\left(1 + \frac{1}{|\xi''|_{l''}^{-1}}\right)\right] \sum_{k=n_1}^{n} \xi_k^{2l_k} \\ &= \left(c_1 + c_2 \ln\left(1 + \left|\xi''\right|_{l''}\right)\right) \left|\xi''\right|_{l''}. \end{aligned} \tag{A.4}$$

If $t|\xi''|_{l''} > 1$, then using (A.1) we get

$$\begin{aligned} d(t, \xi'') &= \sum_{k=n_1+1}^{n} a_k(t) \xi_k^{2l_k} \\ &\le \left[c_1 + c_2 \ln\left(1 + \frac{1}{t}\right)\right] \left|\xi''\right|_{l''} \\ &\le \left[c_1 + c_2 \ln\left(1 + \left|\xi''\right|_{l''}\right)\right] \left|\xi''\right|_{l''}. \end{aligned} \tag{A.5}$$

Consequently if $q = 1$, the statement of the lemma follows from (A.2)–(A.5).

Let $q_k > 1$, $k = n_1 + 1, \ldots, n$. By definition of $d(t, \xi'')$ for $T^r|\xi''|_{l''} < 1$ we have

$$d(t, \xi'') \leq c_1 |\xi''|_{l''}. \tag{A.6}$$

If $T^r|\xi''|_{l''} > 1$ and $t^r|\xi''| < 1$ then

$$\begin{aligned} d(t, \xi'') &= \sum_{k=n_1+1}^{n} a_k\left(|\xi''|_{l''}^{-1/r}\right)\xi_k^{2l_k} \\ &\leq \sum_{k=n_1+1}^{n} \frac{M}{\left(|\xi''|_{l''}^{-1/r}\right)^p}\xi_k^{2l_k} \\ &= M|\xi''|_{l''}^{1+p/r}. \end{aligned} \tag{A.7}$$

If $t^r|\xi''| > 1$ then

$$\begin{aligned} d(t, \xi'') &= \sum_{k=n_1+1}^{n} a_k(t)\xi_k^{2l_k} \\ &\leq \sum_{k=n_1+1}^{n} \frac{M}{t^p}\xi_k^{2l_k} \\ &= M|\xi''|_{l''} \cdot |\xi''|^{p/r} \\ &= M|\xi''|^{1+p/r}. \end{aligned} \tag{A.8}$$

Thus if $q_k > 1, k = n_1 + 1, \ldots, n$ then the statement of the lemma follows from (A.2), (A.6), and (A.8).

The lemma is proved. □

Proof of Lemma 3.2. At first we consider the case when $q_k = 1$, $k = n_1 + 1, \ldots, n$. If $T|\xi''|_{\ell''} \leq 1$, then

$$\begin{aligned} \int_0^t \alpha(\tau, \xi'')d\tau &\leq \int_0^T \alpha(\tau, \xi'')d\tau \\ &\leq \int_0^T \left| \sum_{k=n_1}^{n} a_k(T)\xi_k^{2\ell_k} - \sum_{k=n_1}^{n} a_k(\tau)\xi_k^{2\ell_k} \right| d\tau \\ &\leq \sum_{k=n_1}^{n} \xi_k^{2\ell_k} \int_0^T |a_k(T) - a_k(\tau)| d\tau \\ &\leq T \cdot \max_{k=n_1+1,\ldots,n} a_k(T)|\xi''|_{\ell''} + |\xi_k''|_{\ell''} \int_0^T a_k(\tau)d\tau \\ &\leq a_T, \end{aligned} \tag{A.9}$$

where $a_T = \max_{k=n_1+1,\ldots,n} a_k(\tau) + (1/T)\max_{k=n_1+1,\ldots,n} \int_0^T a_k(\tau)d\tau < +\infty$.

If $T|\xi''| > 1$, then

$$\begin{aligned}
\int_0^t \alpha(\tau,\xi'')ds &\le \int_0^{|\xi''|_{l''}^{-1}} \alpha(s,\xi'')d\tau + \int_{|\xi''|_{\ell''}^{-1}}^{T} \alpha(s,\xi'')ds \\
&\le \int_0^{|\xi''|_{l''}^{-1}} \left| d(\tau,\xi'') - \sum_{k=n_1}^{n} a_k(\tau)\xi_k^{2l_k} \right| d\tau + \int_{|\xi''|_{\ell''}^{-1}}^{T} \frac{\left|\sum_{k=n_1}^{n} \dot{a}_k(\tau)\xi_k^{2\ell_k}\right|}{\sum_{k=n_1}^{n} a_k(\tau)\xi_k^{2\ell_k}} d\tau \\
&\le \int_0^{|\xi''|_{l''}^{-1}} d(\tau,\xi'')d\tau + \sum_{k=n_1}^{n} \xi_k^{2l_k} \cdot \int_0^{|\xi''|_{l''}^{-1}} \alpha_k(\tau)d\tau + \frac{c}{a}\sum_{k=n_1}^{n} \xi_k^{2\ell_k} \int_{|\xi''|_{\ell''}^{-1}}^{T} \frac{d\tau}{\tau} \\
&\le \int_0^{|\xi''|_{\ell''}^{-1}} \left[c_1 + c_2 \ell n\left(1 + |\xi''|_{\ell''}\right)\right] |\xi''|_{\ell''} d\tau \\
&\quad + \sum_{k=n_1}^{n} \xi_k^{2\ell_k} \cdot c \int_0^{|\xi''|_{\ell''}^{-1}} \ell n \frac{T}{\tau} d\tau + \frac{c}{a}\sum_{k=n_1}^{n} \int_{|\xi''|_{\ell''}^{-1}}^{T} \frac{d\tau}{\tau} \\
&= c_1 + c_2 \ell n\left(1 + |\xi''|_{\ell''}^{-1}\right) + c|\xi''|_{\ell''} \cdot \int_0^{|\xi''|_{\ell''}^{-1}} \ell n \frac{T}{\tau} d\tau + \frac{c}{a} \int_{|\xi''|_{\ell''}^{-1}}^{T} \frac{d\tau}{\tau} \\
&\le c_3 + c_4 \ell n\left(1 + |\xi''|_{\ell''}\right).
\end{aligned} \tag{A.10}$$

Now let us consider the case $q_k > 1$, $k = n_1 + 1, \ldots, n$. In this case $r = (q-1)s$. If $T^r|\xi''|_{\ell''} \le 1$, then

$$\begin{aligned}
\int_0^t \alpha(\tau,\xi'')d\tau &\le \int_0^T \alpha(\tau,\xi'')d\tau \\
&\le \sum_{k=n_1+1}^{n} \int_0^T |a_k(T) - a_k(\tau)| \xi_k^{2\ell_k} d\tau \\
&\le \max_{k=n_1+1,\ldots,n} a_k(T) T |\xi''|_{\ell''} + \int_0^T c\tau^{-p} d\tau |\xi''|_{\ell''} \\
&\le a_T \cdot T^{1-r} + c \cdot \frac{1}{1-p} T^{1-p} |\xi''|_{\ell''} \le a_T T^{1-r} + \frac{c}{1-p} T^{1-p-r}.
\end{aligned} \tag{A.11}$$

If $T^r|\xi''|_{\ell''} > 1$, then

$$\begin{aligned}
\int_0^t \alpha(\tau,\xi'')d\tau &\le \int_0^{|\xi''|^{-1/r}} \alpha(\tau,\xi)d\tau + \int_{|\xi''|_{\ell''}^{-1/r}}^T \alpha(\tau,\xi'')d\tau \\
&\le \int_0^{|\xi''|_{\ell''}^{-1/r}} \left| d(\tau,\xi) - \sum_{k=n_1+1}^{n} a_k(\tau)\xi_k^{2l_k} \right| d\tau + \int_{|\xi''|_{\ell''}^{-1/r}}^T \alpha(\tau,\xi'')d\tau \\
&\le \sum_{k=n_1+1}^{n} a_k\left(|\xi''|_{\ell''}^{-1/r}\right)\xi_k^{2\ell_k} \int_0^{|\xi''|_{l''}^{-1/r}} d\tau + \sum_{k=n_1+1}^{n} \xi_k^{2\ell_k} \int_0^{|\xi''|_{\ell''}^{-1/r}} a_k(\tau)d\tau \\
&\quad + \int_{|\xi''|_{\ell''}^{-1/r}}^T \frac{\left|\sum_{k=n_1+1}^{n} \dot{a}_k(\tau)\xi_k^{2\ell_k}\right|}{\sum_{k=n_1+1}^{n} a_k(\tau)\xi_k^{2\ell_k}} d\tau \qquad (A.12)\\
&\le \frac{c}{\left(|\xi''|_{\ell''}^{-1/r}\right)^p} \cdot |\xi''|_{\ell''} \cdot \int_0^{|\xi''|_{\ell''}^{-1/r}} d\tau + c|\xi''|_{\ell''} \cdot \int_0^{|\xi''|^{-1/r}} \frac{d\tau}{\tau^p} d\tau + \frac{c}{a}\int_{|\xi''|^{-1/r}}^T \frac{d\tau}{\tau^q} \\
&\le c|\xi''|_{\ell''}^{p/r+1} \cdot |\xi''|^{-1/r} + c|\xi''|_{\ell''} \cdot \frac{1}{1-p}\left(|\xi''|^{-1/r}\right)^{1-p} \\
&\quad + \frac{c}{a}\frac{1}{1-q}\left(T^{1-q} - \left(|\xi''|^{-1/r}\right)^{1-q}\right) \\
&< c|\xi''|_{\ell''}^{1-((1-p)/r)} + \frac{c}{1-p}|\xi''|_{\ell''}^{1-((1-p)/r)} + \frac{c}{a(q-1)}|\xi''|^{(q-1)/r}.
\end{aligned}$$

As $r = (q-1)s$, and $s < (q-p)/(q-1)$, it follows that $1 - (1-p)/s < 1/s$ and $(q-1)/r = 1/s$. Then according to the Young inequality there exists such $\delta > 0$ that

$$|\xi''|^{1-((1-p)/r)} \le c_1\delta + \delta_1|\xi''|^{1/s}. \qquad (A.13)$$

Thus, by (A.9)–(A.13) the following inequality is valid:

$$\int_0^t \alpha(\tau,\xi'')d\tau \le \delta|\xi|^{1/s} + c_\delta, \qquad (A.14)$$

where $\delta = \delta_1 a(2+p)/(1-p) + (c/a(q-1))c_\delta = c_{1\delta}c(2+\delta)/(1-p)$. □

B. Example

Let us consider the Cauchy problem in $[0, T) \times R^2$:

$$u_{tt} - \left(1 + t^2\right) u_{xx} - \left(1 + \sqrt[3]{t^2}\right) u_{yy} = 0,$$
$$u(0, x, y) = \varphi_1(x)\psi_1(y), \tag{B.1}$$
$$u_t(o, x, y) = \varphi_2(x)\psi_2(y),$$

where $\varphi_1(x), \varphi_2(x) \in C^\infty(R) = \cap_{s\geq 0} W_2^s(R)$, $\psi_1(y) \in W_2^2(R)$, $\psi_2(y) \in W_2^1(R)$, $u = u(t, x, y)$.

It follows from Theorem 2.5 that the problem (B.1) has a unique solution

$$u \in C\left([0, T]; C^\infty\left(R; W_2^2(R)\right)\right) \cap C^1\left([0, T]; C^\infty\left(R; W_2^1(R)\right)\right). \tag{B.2}$$

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