Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2008, Article ID 753518, 29 pages
doi:10.1155/2008/753518

Research Article

Numerical Blow-Up Time for a Semilinear Parabolic Equation with Nonlinear Boundary Conditions

Louis A. Assalé,[1] Théodore K. Boni,[1] and Diabate Nabongo[2]

[1] *Institut National Polytechnique Houphouët-Boigny de Yamoussoukro, BP 1093, Yamoussoukro, Cote D'Ivoire*
[2] *Département de Mathématiques et Informatiques, Université d'Abobo-Adjamé, UFR-SFA, 16 BP 372 Abidjan 16, Cote D'Ivoire*

Correspondence should be addressed to Diabate Nabongo, nabongo_diabate@yahoo.fr

Received 29 April 2008; Revised 15 December 2008; Accepted 29 December 2008

Recommended by Jacek Rokicki

We obtain some conditions under which the positive solution for semidiscretizations of the semilinear equation $u_t = u_{xx} - a(x,t)f(u)$, $0 < x < 1$, $t \in (0,T)$, with boundary conditions $u_x(0,t) = 0$, $u_x(1,t) = b(t)g(u(1,t))$, blows up in a finite time and estimate its semidiscrete blow-up time. We also establish the convergence of the semidiscrete blow-up time and obtain some results about numerical blow-up rate and set. Finally, we get an analogous result taking a discrete form of the above problem and give some computational results to illustrate some points of our analysis.

1. Introduction

In this paper, we consider the following boundary value problem:

$$\begin{gathered} u_t - u_{xx} = -a(x,t)f(u), \quad 0 < x < 1, \; t \in (0,T), \\ u_x(0,t) = 0, \quad u_x(1,t) = b(t)g\big(u(1,t)\big), \quad t \in (0,T), \\ u(x,0) = u_0(x) \ge 0, \quad 0 \le x \le 1, \end{gathered} \tag{1.1}$$

where $f : [0,\infty) \to [0,\infty)$ is a C^1 function, $f(0) = 0$, $g : [0,\infty) \to [0,\infty)$ is a C^1 convex function, $g(0) = 0$, $a \in C^0([0,1] \times \mathbb{R}_+)$, $a(x,t) \ge 0$ in $[0,1] \times \mathbb{R}_+$, $a_t(x,t) \le 0$ in $[0,1] \times \mathbb{R}_+$, $b \in C^1(\mathbb{R}_+)$, $b(t) > 0$ in $\mathbb{R}_+$, $b'(t) \ge 0$ in $\mathbb{R}_+$. The initial data $u_0 \in C^2([0,1])$, $u_0'(0) = 0$, $u_0'(1) = b(1)g(u_0(1))$.

Here $(0, T)$ is the maximal time interval on which the solution u of (1.1) exists. The time T may be finite or infinite. Where T is infinite, we say that the solution u exists globally. When T is finite, the solution u develops a singularity in a finite time, namely

$$\lim_{t \to T} \|u(\cdot, t)\|_{\infty} = +\infty, \tag{1.2}$$

where $\|u(\cdot, t)\|_{\infty} = \max_{0 \leq x \leq 1} |u(x, t)|$.

In this last case, we say that the solution u blows up in a finite time and the time T is called the blow-up time of the solution u.

In good number of physical devices, the boundary conditions play a primordial role in the progress of the studied processes. It is the case of the problem described in (1.1) which can be viewed as a heat conduction problem where u stands for the temperature, and the heat sources are prescribed on the boundaries. At the boundary $x = 0$, the heat source has a constant flux whereas at the boundary $x = 1$, the heat source has a nonlinear radition haw. Intensification of the heat source at the boundary $x = 1$ is provided by the function b. The function g also gives a dominant strength of the heat source at the boundary $x = 1$.

The theoretical study of blow-up of solutions for semilinear parabolic equations with nonlinear boundary conditions has been the subject of investigations of many authors (see [1–7], and the references cited therein).

The authors have proved that under some assumptions, the solution of (1.1) blows up in a finite time and the blow-up time is estimated. It is also proved that under some conditions, the blow-up occurs at the point 1. In this paper, we are interested in the numerical study. We give some assumptions under which the solution of a semidiscrete form of (1.1) blows up in a finite time and estimate its semidiscrete blow-up time. We also show that the semidiscrete blow-up time converges to the theoretical one when the mesh size goes to zero. An analogous study has been also done for a discrete scheme. For the semidiscrete scheme, some results about numerical blow-up rate and set have been also given. A similar study has been undertaken in [8, 9] where the authors have considered semilinear heat equations with Dirichlet boundary conditions. In the same way in [10] the numerical extinction has been studied using some discrete and semidiscrete schemes (a solution u extincts in a finite time if it reaches the value zero in a finite time). Concerning the numerical study with nonlinear boundary conditions, some particular cases of the above problem have been treated by several authors (see [11–15]). Generally, the authors have considered the problem (1.1) in the case where $a(x, t) = 0$ and $b(t) = 1$. For instance in [15], the above problem has been considered in the case where $a(x, t) = 0$ and $b(t) = 1$. In [16], the authors have considered the problem (1.1) in the case where $a(x, t) = \lambda > 0$, $b(t) = 1$, $f(u) = u^p$, $g(u) = u^q$. They have shown that the solution of a semidiscrete form of (1.1) blows up in a finite time and they have localized the blow-up set. One may also find in [17–22] similar studies concerning other parabolic problems.

The paper is organized as follows. In the next section, we present a semidiscrete scheme of (1.1). In Section 3, we give some properties concerning our semidiscrete scheme. In Section 4, under some conditions, we prove that the solution of the semidiscrete form of (1.1) blows up in a finite time and estimate its semidiscrete blow-up time. In Section 5, we study the convergence of the semidiscrete blow-up time. In Section 6, we give some results on the numerical blow-up rate and Section 7 is consecrated to the study of the numerical blow-up set. In Section 8, we study a particular discrete form of (1.1). Finally, in the last section, taking some discrete forms of (1.1), we give some numerical experiments.

2. The semidiscrete problem

Let I be a positive integer and define the grid $x_i = ih$, $0 \le i \le I$, where $h = 1/I$. We approximate the solution u of (1.1) by the solution $U_h(t) = (U_0(t), U_1(t), \ldots, U_I(t))^T$ of the following semidiscrete equations

$$\frac{dU_i(t)}{dt} - \delta^2 U_i(t) = -a_i(t) f(U_i(t)), \quad 0 \le i \le I-1,\ t \in (0, T_b^h), \tag{2.1}$$

$$\frac{dU_I(t)}{dt} - \delta^2 U_I(t) = \frac{2}{h} b(t) g(U_I(t)) - a_I(t) f(U_I(t)), \quad t \in (0, T_b^h), \tag{2.2}$$

$$U_i(0) = \varphi_i \ge 0, \quad 0 \le i \le I, \tag{2.3}$$

where $\varphi_{i+1} \ge \varphi_i$, $0 \le i \le I-1$,

$$\begin{gathered} \delta^2 U_0(t) = \frac{2U_1(t) - 2U_0(t)}{h^2}, \qquad \delta^2 U_I(t) = \frac{2U_{I-1}(t) - 2U_I(t)}{h^2}, \\ \delta^2 U_i(t) = \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{h^2}. \end{gathered} \tag{2.4}$$

Here $(0, T_b^h)$ is the maximal time interval on which $\|U_h(t)\|_\infty$ is finite where $\|U_h(t)\|_\infty = \max_{0 \le i \le I} U_i(t)$. When T_b^h is finite, we say that the solution $U_h(t)$ blows up in a finite time and the time T_b^h is called the blow-up time of the solution $U_h(t)$.

3. Properties of the semidiscrete scheme

In this section, we give some lemmas which will be used later.

The following lemma is a semidiscrete form of the maximum principle.

Lemma 3.1. *Let $a_h(t) \in C^0([0,T), \mathbb{R}^{I+1})$ and let $V_h(t) \in C^1([0,T), \mathbb{R}^{I+1})$ such that*

$$\begin{gathered} \frac{dV_i(t)}{dt} - \delta^2 V_i(t) + a_i(t) V_i(t) \ge 0, \quad 0 \le i \le I,\ t \in (0,T), \\ V_i(0) \ge 0, \quad 0 \le i \le I. \end{gathered} \tag{3.1}$$

Then we have $V_i(t) \ge 0$, $0 \le i \le I$, $t \in (0,T)$.

Proof. Let $T_0 < T$ and define the vector $Z_h(t) = e^{\lambda t} V_h(t)$ where λ is large enough that $a_i(t) - \lambda > 0$ for $t \in [0, T_0]$, $0 \le i \le I$. Let $m = \min_{0 \le i \le I, 0 \le t \le T_0} Z_i(t)$. Since for $i \in \{0, \ldots, I\}$, $Z_i(t)$ is a continuous function, there exists $t_0 \in [0, T_0]$ such that $m = Z_{i_0}(t_0)$ for a certain $i_0 \in \{0, \ldots, I\}$.

It is not hard to see that

$$\begin{aligned}
\frac{dZ_{i_0}(t_0)}{dt} &= \lim_{k\to 0}\frac{Z_{i_0}(t_0)-Z_{i_0}(t_0-k)}{k} \le 0,\\
\delta^2 Z_{i_0}(t_0) &= \frac{Z_{i_0+1}(t_0)-2Z_{i_0}(t_0)+Z_{i_0-1}(t_0)}{h^2} \ge 0 \quad \text{if } 1\le i_0\le I-1,\\
\delta^2 Z_{i_0}(t_0) &= \frac{2Z_1(t_0)-2Z_0(t_0)}{h^2} \ge 0 \quad \text{if } i_0=0,\\
\delta^2 Z_{i_0}(t_0) &= \frac{2Z_{I-1}(t_0)-2Z_I(t_0)}{h^2} \ge 0 \quad \text{if } i_0=I.
\end{aligned} \tag{3.2}$$

A straightforward computation reveals that

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + (a_{i_0}(t_0)-\lambda)Z_{i_0}(t_0) \ge 0. \tag{3.3}$$

We observe from (3.2) that $(a_{i_0}(t_0)-\lambda)Z_{i_0}(t_0) \ge 0$ which implies that $Z_{i_0}(t_0)\ge 0$ because $a_{i_0}(t_0)-\lambda > 0$. We deduce that $V_h(t)\ge 0$ for $t\in[0,T_0]$ and the proof is complete. □

Another form of the maximum principle for semidiscrete equations is the following comparison lemma.

Lemma 3.2. *Let* $V_h(t), U_h(t) \in C^1([0,T),\mathbb{R}^{I+1})$ *and* $f\in C^0(\mathbb{R}\times\mathbb{R},\mathbb{R})$ *such that for* $t\in(0,T)$

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) + f(V_i(t),t) < \frac{dU_i(t)}{dt} - \delta^2 U_i(t) + f(U_i(t),t), \quad 0\le i\le I, \tag{3.4}$$

$$V_i(0) < U_i(0), \quad 0\le i\le I. \tag{3.5}$$

Then we have $V_i(t) < U_i(t)$, $0\le i\le I$, $t\in(0,T)$.

Proof. Define the vector $Z_h(t)=U_h(t)-V_h(t)$. Let t_0 be the first $t\in(0,T)$ such that $Z_i(t)>0$ for $t\in[0,t_0)$, $0\le i\le I$, but $Z_{i_0}(t_0)=0$ for a certain $i_0\in\{0,\dots,I\}$. We observe that

$$\begin{aligned}
\frac{dZ_{i_0}(t_0)}{dt} &= \lim_{k\to 0}\frac{Z_{i_0}(t_0)-Z_{i_0}(t_0-k)}{k} \le 0,\\
\delta^2 Z_{i_0}(t_0) &= \frac{Z_{i_0+1}(t_0)-2Z_{i_0}(t_0)+Z_{i_0-1}(t_0)}{h^2} \ge 0 \quad \text{if } 1\le i_0\le I-1,\\
\delta^2 Z_{i_0}(t_0) &= \frac{2Z_1(t_0)-2Z_0(t_0)}{h^2} \ge 0 \quad \text{if } i_0=0,\\
\delta^2 Z_{i_0}(t_0) &= \frac{2Z_{I-1}(t_0)-2Z_I(t_0)}{h^2} \ge 0 \quad \text{if } i_0=I,
\end{aligned} \tag{3.6}$$

which implies that

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + f(U_{i_0}(t_0), t_0) - f(V_{i_0}(t_0), t_0) \leq 0. \tag{3.7}$$

But this inequality contradicts (3.4) and the proof is complete. □

4. Semidiscrete blow-up solutions

In this section under some assumptions, we show that the solution U_h of (2.1)–(2.3) blows up in a finite time and estimate its semidiscrete blow-up time.

Before starting, we need the following two lemmas. The first lemma gives a property of the operator δ^2 and the second one reveals a property of the semidiscrete solution.

Lemma 4.1. *Let $U_h \in \mathbb{R}^{I+1}$ be such that $U_h \geq 0$. Then we have*

$$\delta^2 g(U_i) \geq g'(U_i)\delta^2 U_i \quad \textit{for } 0 \leq i \leq I. \tag{4.1}$$

Proof. Apply Taylor's expansion to obtain

$$\begin{aligned}
g(U_1) &= g(U_0) + (U_1 - U_0)g'(U_0) + \frac{(U_1 - U_0)^2}{2} g''(\eta_0), \\
g(U_{i+1}) &= g(U_i) + (U_{i+1} - U_i)g'(U_i) + \frac{(U_{i+1} - U_i)^2}{2} g''(\theta_i), \quad 1 \leq i \leq I-1, \\
g(U_{i-1}) &= g(U_i) + (U_{i-1} - U_i)g'(U_i) + \frac{(U_{i-1} - U_i)^2}{2} g''(\eta_i), \quad 1 \leq i \leq I-1, \\
g(U_{I-1}) &= g(U_I) + (U_{I-1} - U_I)g'(U_I) + \frac{(U_{I-1} - U_I)^2}{2} g''(\eta_I),
\end{aligned} \tag{4.2}$$

where θ_i is an intermediate between U_i and U_{i+1} and η_i the one between U_{i-1} and U_i. The first and last equalities imply that

$$\begin{aligned}
\delta^2 g(U_0) &= g'(U_0)\delta^2 U_0 + \frac{(U_1 - U_0)^2}{h^2} g''(\eta_0), \\
\delta^2 g(U_I) &= g'(U_I)\delta^2 U_I + \frac{(U_{I-1} - U_I)^2}{h^2} g''(\eta_I).
\end{aligned} \tag{4.3}$$

Combining the second and third equalities, we see that

$$\delta^2 g(U_i) = g'(U_i)\delta^2 U_i + \frac{(U_{i+1} - U_i)^2}{2h^2} g''(\theta_i) + \frac{(U_{i-1} - U_i)^2}{2h^2} g''(\eta_i), \quad 1 \leq i \leq I-1. \tag{4.4}$$

Use the fact that $g''(s) \geq 0$ for $s \geq 0$ and $U_h \geq 0$ to complete the rest of the proof. □

Lemma 4.2. *Let U_h be the solution of* (2.1)–(2.3)*. Then we have*

$$U_{i+1}(t) > U_i(t), \quad 0 \le i \le I-1,\ t \in \left(0, T_b^h\right). \tag{4.5}$$

Proof. Let t_0 be the first $t > 0$ such that $U_{i+1}(t) > U_i(t)$ for $0 \le i \le I-1$ but $U_{i_0+1}(t_0) = U_{i_0}(t_0)$ for a certain $i_0 \in \{0, \dots, I-1\}$. Without loss of generality, we may suppose that i_0 is the smallest integer which satisfies the equality. Introduce the functions $Z_i(t) = U_{i+1}(t) - U_i(t)$ for $0 \le i \le I-1$. We get

$$\begin{gathered}
\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \le 0, \\
\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} > 0 \quad \text{if } 1 \le i_0 \le I-2, \\
\delta^2 Z_{i_0}(t_0) = \delta^2 Z_0(t_0) = \frac{Z_1(t_0) - 3Z_0(t_0)}{h^2} > 0 \quad \text{if } i_0 = 0, \\
\delta^2 Z_{i_0}(t_0) = \delta^2 Z_{I-1}(t_0) = \frac{Z_{I-2}(t_0) - 3Z_{I-1}(t_0)}{h^2} > 0 \quad \text{if } i_0 = I-1,
\end{gathered} \tag{4.6}$$

which implies that

$$\begin{aligned}
&\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) - a_{i_0+1}(t_0) f\left(U_{i_0+1}(t_0)\right) \\
&\quad + a_{i_0}(t_0) f\left(U_{i_0}(t_0)\right) < 0 \quad \text{if } 0 \le i_0 \le I-2, \\
&\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + \frac{2}{h} b(t_0) g_{i_0+1}(t_0) - a_{i_0+1}(t_0) f\left(U_{i_0+1}(t_0)\right) \\
&\quad + a_{i_0}(t_0) f\left(U_{i_0}(t_0)\right) < 0 \quad \text{if } i_0 = I-1.
\end{aligned} \tag{4.7}$$

But this contradicts (2.1)-(2.2) and we have the desired result. □

The above lemma says that the semidiscrete solution is increasing in space. This property will be used later to show that the semidiscrete solution attains its minimum at the last node x_I.

Now, we are in a position to state the main result of this section.

Theorem 4.3. *Let U_h be the solution of* (2.1)–(2.3)*. Suppose that there exists a positive integer A such that*

$$\begin{gathered}
\delta^2 \varphi_i - a_i(0) f(\varphi_i) \ge 0, \quad 1 \le i \le I-1, \\
\delta^2 \varphi_I - a_I(0) f(\varphi_I) + b(0) g_I(\varphi_I) \ge A g(\varphi_I).
\end{gathered} \tag{4.8}$$

Assume that

$$f(s) g'(s) - f'(s) g(s) \ge 0 \quad \textit{for } s \ge 0. \tag{4.9}$$

Then the solution U_h blows up in a finite time T_b^h and we have the following estimate

$$T_b^h \leq \frac{1}{A}\int_{\|\varphi_h\|_\infty}^{+\infty} \frac{ds}{g(s)}. \tag{4.10}$$

Proof. Since $(0, T_b^h)$ is the maximal time interval on which $\|U_h(t)\|_\infty < \infty$, our aim is to show that T_b^h is finite and satisfies the above inequality. Introduce the vector J_h such that

$$J_i(t) = \frac{dU_i(t)}{dt}, \quad 0 \leq i \leq I-1, \qquad J_I(t) = \frac{dU_I(t)}{dt} - Ag(U_I(t)). \tag{4.11}$$

A straightforward calculation gives

$$\begin{aligned} &\frac{dJ_i}{dt} - \delta^2 J_i = \frac{d}{dt}\left(\frac{dU_i}{dt} - \delta^2 U_i\right), \quad 0 \leq i \leq I-1, \\ &\frac{dJ_I}{dt} - \delta^2 J_I = \frac{d}{dt}\left(\frac{dU_I}{dt} - \delta^2 U_I\right) - Ag'(U_I)\frac{dU_I}{dt} + A\delta^2 g(U_I). \end{aligned} \tag{4.12}$$

From Lemma 4.1, we have $\delta^2 g(U_I) \geq g'(U_I)\delta^2 U_I$ which implies that

$$\frac{dJ_I}{dt} - \delta^2 J_I \geq \frac{d}{dt}\left(\frac{dU_I}{dt} - \delta^2 U_I\right) - Ag'(U_I)\left(\frac{dU_I}{dt} - \delta^2 U_I\right). \tag{4.13}$$

Using (2.1), we get

$$\begin{aligned} \frac{dJ_i}{dt} - \delta^2 J_i &\geq -a_i'(t)f(U_i) - a_i(t)f'(U_i)\frac{dU_i}{dt}, \quad 0 \leq i \leq I-1, \\ \frac{dJ_I}{dt} - \delta^2 J_I &\geq -a_I'(t)f(U_I) - a_I(t)f'(U_I)\frac{dU_I}{dt} + \frac{2}{h}b'(t)g(U_I) \\ &\quad + \frac{2}{h}b(t)g'(U_I)\frac{dU_I}{dt} - Ag'(U_I)\left(-a_I(t)f(U_I) + \frac{2}{h}b(t)g(U_I)\right). \end{aligned} \tag{4.14}$$

It follows from the fact that $a_i'(t) \leq 0$, $b'(t) \geq 0$ and $dU_i/dt = J_i + Ag(U_i)$ that

$$\frac{dJ_I}{dt} - \delta^2 J_I \geq \left(-a_I(t)f'(U_I) + \frac{2}{h}b(t)g'(U_I)\right)J_I + Aa_I(t)\left(g'(U_I)f(U_I) - f'(U_I)g(U_I)\right). \tag{4.15}$$

We deduce from (4.9) that

$$\begin{aligned} &\frac{dJ_i}{dt} - \delta^2 J_i \geq -a_i(t)f'(U_i)J_i, \quad 0 \leq i \leq I-1, \\ &\frac{dJ_I}{dt} - \delta^2 J_I \geq \left(-a_I(t)f'(U_I) + \frac{2}{h}b(t)g'(U_I)\right)J_I. \end{aligned} \tag{4.16}$$

From (4.8), we observe that

$$J_i(0) = \delta^2\varphi_i - a_i(0)f(\varphi_i) \geq 0, \quad 0 \leq i \leq I-1,$$
$$J_I(0) = \delta^2\varphi_I - a_I(0)f(\varphi_I) + b(0)g_I(\varphi_I) - Ag(\varphi_I) \geq 0. \tag{4.17}$$

We deduce from Lemma 3.1 that $J_i(t) \geq 0$, $0 \leq i \leq I$, which implies that $dU_I/dt \geq g(U_I)$, $0 \leq i \leq I$. Obviously we have

$$\frac{dU_I}{g(U_I)} \geq A\,dt. \tag{4.18}$$

Integrating this inequality over (t, T_b^h), we arrive at

$$T_b^h - t \leq \frac{1}{A}\int_{U_I(t)}^{+\infty} \frac{ds}{g(s)}, \tag{4.19}$$

which implies that

$$T_b^h \leq \frac{1}{A}\int_{\|U_h(0)\|_\infty}^{+\infty} \frac{ds}{g(s)}. \tag{4.20}$$

Since the quantity on the right hand side of the above inequality is finite, we deduce that the solution U_h blows up in a finite time. Use the fact that $\|U_h(0)\|_\infty = \|\varphi_h\|_\infty$ to complete the rest of the proof. □

Remark 4.4. The inequality (4.19) implies that

$$T_b^h - t_0 \leq \frac{1}{A}\int_{\|U_h(t_0)\|_\infty}^{+\infty} \frac{ds}{g(s)} \quad \text{if } 0 < t_0 < T_b^h,$$
$$U_i(t) \leq H\left(A\left(T_b^h - t\right)\right), \quad 0 \leq i \leq I, \tag{4.21}$$

where $H(s)$ is the inverse of $G(s) = \int_s^{+\infty}(dz/g(z))$.

Remark 4.5. If $g(s) = s^q$, then $G(s) = s^{1-q}/(q-1)$ and $H(s) = ((q-1)s)^{1/(1-q)}$.

5. Convergence of the semidiscrete blow-up time

In this section, we show the convergence of the semidiscrete blow-up time. Now we will show that for each fixed time interval $[0, T]$ where u is defined, the solution $U_h(t)$ of (2.1)–(2.3) approximates u, when the mesh parameter h goes to zero.

Theorem 5.1. *Assume that* (1.1) *has a solution* $u \in C^{4,1}([0,1] \times [0,T])$ *and the initial condition at* (2.3) *satisfies*

$$\left\|\varphi_h - u_h(0)\right\|_\infty = o(1) \quad \text{as } h \longrightarrow 0, \tag{5.1}$$

where $u_h(t) = (u(x_0,t),\ldots,u(x_I,t))^T$. *Then, for* h *sufficiently small, the problem* (2.1)–(2.3) *has a unique solution* $U_h \in C^1([0,T],\mathbb{R}^{I+1})$ *such that*

$$\max_{0\le t\le T}\|U_h(t) - u_h(t)\|_\infty = O\left(\|\varphi_h - u_h(0)\|_\infty + h^2\right) \quad \text{as } h \longrightarrow 0. \tag{5.2}$$

Proof. Let $\alpha > 0$ be such that

$$\|u(\cdot,t)\|_\infty \le \alpha \quad \text{for } t \in [0,T]. \tag{5.3}$$

The problem (2.1)–(2.3) has for each h, a unique solution $U_h \in C^1([0,T_b^h),\mathbb{R}^{I+1})$. Let $t(h) \le \min\{T,T_b^h\}$ the greatest value of $t > 0$ such that

$$\|U_h(t) - u_h(t)\|_\infty < 1 \quad \text{for } t \in (0,t(h)). \tag{5.4}$$

The relation (5.1) implies that $t(h) > 0$ for h sufficiently small. By the triangle inequality, we obtain

$$\|U_h(t)\|_\infty \le \|u(\cdot,t)\|_\infty + \|U_h(t) - u_h(t)\|_\infty \quad \text{for } t \in (0,t(h)), \tag{5.5}$$

which implies that

$$\|U_h(t)\|_\infty \le 1 + \alpha \quad \text{for } t \in (0,t(h)). \tag{5.6}$$

Let $e_h(t) = U_h(t) - u_h(t)$ be the error of discretization. Using Taylor's expansion, we have for $t \in (0,t(h))$,

$$\begin{aligned}
&\frac{de_i(t)}{dt} - \delta^2 e_i(t) = -a_i(t)f'(\xi_i(t))e_i(t) + o(h^2), \quad 0 \le i \le I-1,\\
&\frac{de_I(t)}{dt} - \delta^2 e_I(t) = -a_I(t)f'(\xi_I(t))e_I(t) + \frac{2}{h}b(t)g'(U_I(t))e_I(t) + o(h^2),
\end{aligned} \tag{5.7}$$

where $\theta_I(t)$ is an intermediate value between $U_I(t)$ and $u(x_I,t)$ and $\xi_i(t)$ the one between $U_i(t)$ and $u(x_i,t)$. Using (5.3) and (5.6), there exist two positive constants K and L such that

$$\begin{aligned}
&\frac{de_i(t)}{dt} - \delta^2 e_i(t) \le L|e_i(t)| + Kh^2, \quad 0 \le i \le I-1,\\
&\frac{de_I(t)}{dt} - \frac{(2e_{I-1}(t) - 2e_I(t))}{h^2} \le \frac{L|e_I(t)|}{h} + L|e_I(t)| + Kh^2.
\end{aligned} \tag{5.8}$$

Consider the function $z(x,t) = e^{((M+1)t+Cx^2)}(\|\varphi_h - u_h(0)\|_\infty + Qh^2)$ where M, C, Q are constants which will be determined later. We get

$$\begin{gathered} z_t(x,t) - z_{xx}(x,t) = (M+1-2C-4C^2x^2)z(x,t), \\ z_x(0,t) = 0, \qquad z_x(1,t) = 2Cz(1,t), \\ z(x,0) = e^{Cx^2}(\|\varphi_h - u_h(0)\|_\infty + Qh^2). \end{gathered} \tag{5.9}$$

By a semidiscretization of the above problem, we may choose M, C, Q large enough that

$$\begin{gathered} \frac{d}{dt}z(x_i,t) > \delta^2 z(x_i,t) + L|z(x_i,t)| + Kh^2, \quad 0 \le i \le I-1, \\ \frac{d}{dt}z(x_I,t) > \delta^2 z(x_I,t) + \frac{L}{h}|z(x_I,t)| + L|z(x_I,t)| + Kh^2, \\ z(x_i,0) > e_i(0), \quad 0 \le i \le I. \end{gathered} \tag{5.10}$$

It follows from Lemma 3.2 that

$$z(x_i,t) > e_i(t) \quad \text{for } t \in (0,t(h)),\ 0 \le i \le I. \tag{5.11}$$

By the same way, we also prove that

$$z(x_i,t) > -e_i(t) \quad \text{for } t \in (0,t(h)),\ 0 \le i \le I, \tag{5.12}$$

which implies that

$$z(x_i,t) > |e_i(t)| \quad \text{for } t \in (0,t(h)),\ 0 \le i \le I. \tag{5.13}$$

We deduce that

$$\|U_h(t) - u_h(t)\|_\infty \le e^{(Mt+C)}(\|\varphi_h - u_h(0)\|_\infty + Qh^2), \quad t \in (0,t(h)). \tag{5.14}$$

Let us show that $t(h) = T$. Suppose that $T > t(h)$. From (5.4), we obtain

$$1 = \|U_h(t(h)) - u_h(t(h))\|_\infty \le e^{(MT+C)}(\|\varphi_h - u_h(0)\|_\infty + Qh^2). \tag{5.15}$$

Since the term on the right hand side of the above inequality goes to zero as h tends to zero, we deduce that $1 \le 0$, which is impossible. Consequently $t(h) = T$, and the proof is complete. □

Now, we are in a position to prove the main result of this section.

Theorem 5.2. *Suppose that the problem* (1.1) *has a solution u which blows up in a finite time T_b such that $u \in C^{4,1}([0,1] \times [0,T_b))$ and the initial condition at* (2.3) *satisfies*

$$\|\varphi_h - u_h(0)\|_\infty = o(1) \quad \text{as } h \longrightarrow 0. \tag{5.16}$$

Under the assumptions of Theorem 4.3, the problem (2.1)–(2.3) *admits a unique solution* U_h *which blows up in a finite time* T_b^h *and we have the following relation*

$$\lim_{h\to 0} T_b^h = T_b. \tag{5.17}$$

Proof. Let $\varepsilon > 0$. There exists a positive constant N such that

$$\frac{1}{A}\int_x^{+\infty}\frac{ds}{g(s)} \le \frac{\varepsilon}{2} \quad \text{for } x \in [N, +\infty). \tag{5.18}$$

Since the solution u blows up at the time T_b, then there exists $T_1 \in (T_b - \varepsilon/2, T_b)$ such that $\|u(\cdot,t)\|_\infty \ge 2N$ for $t \in [T_1, T_b)$. Setting $T_2 = (T_1 + T_b)/2$, then we have $\sup_{t\in[0,T_2]}|u(x,t)| < \infty$. It follows from Theorem 5.1 that

$$\sup_{t\in[0,T_2]}\left|U_h(t) - u_h(t)\right|_\infty \le N. \tag{5.19}$$

Applying the triangle inequality, we get

$$\left\|U_h(T_2)\right\|_\infty \ge \left\|u_h(T_2)\right\|_\infty - \left\|U_h(T_2) - u_h(T_2)\right\|_\infty, \tag{5.20}$$

which leads to $\|U_h(T_2)\|_\infty \ge N$. From Theorem 4.3, $U_h(t)$ blows up at the time T_b^h. We deduce from Remark 4.4 and (5.18) that

$$\left|T_b - T_b^h\right| \le \left|T_b - T_2\right| + \left|T_b^h - T_2\right| \le \frac{\varepsilon}{2} + \frac{1}{A}\int_{\|U_h(T_2)\|_\infty}^{+\infty}\frac{ds}{g(s)} \le \varepsilon, \tag{5.21}$$

and the proof is complete. □

6. Numerical blow-up rate

In this section, we determine the blow-up rate of the solution U_h of (2.1)–(2.3) in the case where $b(t) = 1$. Our result is the following.

Theorem 6.1. *Let* $U_h(t)$ *be the solution of* (2.1)–(2.3). *Under the assumptions of Theorem 4.3,* $U_h(t)$ *blows up in a finite time* T_b^h *and there exist two positive constants* C_1, C_2 *such that*

$$H\left(C_1\left(T_b^h - t\right)\right) \le U_I(t) \le H\left(C_2\left(T_b^h - t\right)\right), \quad \textit{for } t \in \left(0, T_b^h\right), \tag{6.1}$$

where $H(s)$ *is the inverse of the function* $G(s) = \int_s^{+\infty}(d\sigma/g(\sigma))$.

Proof. From Theorem 4.3 and Remark 4.4, $U_h(t)$ blows up in a finite time T_b^h and there exists a constant $C_2 > 0$ such that

$$U_I(t) \le H\left(C_2\left(T_b^h - t\right)\right) \quad \text{for } t \in \left(0, T_b^h\right). \tag{6.2}$$

From Lemma 4.2, $U_{I-1} < U_I$. Then using (2.2), we deduce that $dU_I/dt \le (2/h)b(t)g(U_I) - a_I(t)f(U_I)$, which implies that $dU_I/dt \le (2b(t)/h)g(U_I)$. Integration this inequality over (t, T_b^h), there exists a positive constant C_1 such that

$$U_I(t) \ge H\left(C_1\left(T_b^h - t\right)\right) \quad \text{for } t \in \left(0, T_b^h\right), \tag{6.3}$$

which leads us to the result. □

7. Numerical blow-up set

In this section, we determine the numerical blow-up set of the semidiscrete solution. This is stated in the theorem below.

Theorem 7.1. *Suppose that there exists a positive constant C_0 such that $sF'(s) \le C_0$ and*

$$\frac{d}{dt}U_i - \delta^2 U_i \le 0, \quad 0 \le i \le I-1. \tag{7.1}$$

Assume that there exists a positive constant C such

$$U_i \le H\left(C(T-t)\right), \quad 0 \le i \le I. \tag{7.2}$$

Then the numerical blow-up set is $B = \{1\}$.

Proof. Let $v(x) = 1 - x^2$ and define

$$W(x,t) = H\left(\delta v(x) + \delta B(T-t)\right) \quad \text{for } 0 \le x \le 1,\ t \ge t_0, \tag{7.3}$$

where δ is small enough. We have

$$W_x(0,t) = 0, \qquad W(1,t) = H\left(\delta B(T-t)\right) \ge u(1,t), \tag{7.4}$$

and for $t \ge t_0$, we get

$$\begin{aligned} W(x,t_0) &= H\left(\delta v(x) + \delta\right) \ge H(2\delta) = H\left(2\delta B(T-t_0)\right) \\ &\ge H\left(C(T-t_0)\right) \ge u(x,t_0). \end{aligned} \tag{7.5}$$

A straightforward computation yields

$$\begin{aligned} W_t(x,t) - W_{xx}(x,t) &= \delta F\left(H(\tau)\right)\left(B - 2 - 4xF'\left(H(\tau)\right)\right) \\ &\ge \delta F\left(H(\tau)\right)\left(B - 2 - 4\delta C_0\right). \end{aligned} \tag{7.6}$$

This implies that there exists $\alpha > 0$ such that

$$W_t(x,t) - W_{xx}(x,t) \ge \alpha F\left(H(\delta + \delta BT)\right). \tag{7.7}$$

Using Taylor's expansion, there exists a constant $K > 0$ such that

$$\frac{d}{dt}W(x_i,t) - \delta^2 W(x_i,t) \geq \alpha F\big(H(\delta + \delta BT)\big) - Kh^2, \quad 0 \leq i \leq I, \tag{7.8}$$

which implies that

$$\frac{dW(x_i,t)}{dt} - \delta^2 W(x_i,t) \geq 0. \tag{7.9}$$

The maximum principle implies that

$$U_i(t) \leq H\big(\delta v(x) + \delta B(T - t_0)\big) \quad \text{for } t \geq t_0,\ 0 \leq i \leq I. \tag{7.10}$$

Hence, we get

$$U_i(t) \leq H\big(\delta v(x)\big), \quad 0 \leq i \leq I. \tag{7.11}$$

Therefore $U_i(T) < +\infty,\ 0 \leq i \leq I-1$, and we have the desired result. □

8. Full discretization

In this section, we consider the problem (1.1) in the case where $a(x,t) = 1$, $b(t) = 1$, $f(u) = u^p$, $g(u) = u^p$ with $p = \text{const} > 1$. Thus our problem is equivalent to

$$\begin{gathered} u_t(x,t) = u_{xx}(x,t) - u^p(x,t), \quad 0 < x < 1,\ t \in (0,T), \\ u_x(0,t) = 0, \quad u_x(1,t) = u^p(1,t), \quad t \in (0,T), \\ u(x,0) = u_0(x) > 0, \quad 0 \leq x \leq 1, \end{gathered} \tag{8.1}$$

where $p > 1$, $u_0 \in C^1([0,1])$, $u_0'(0) = 0$ and $u_0'(1) = u_0^p(1)$.

We start this section by the construction of an adaptive scheme as follows. Let I be a positive integer and let $h = 1/I$. Define the grid $x_i = ih$, $0 \leq i \leq I$ and approximate the solution $u(x,t)$ of the problem (8.1) by the solution $U_h^{(n)} = (U_0^{(n)}, U_1^{(n)}, \ldots, U_I^{(n)})^T$ of the following discrete equations

$$\delta_t U_i^{(n)} = \delta^2 U_i^{(n)} - \big(U_i^{(n)}\big)^p, \quad 0 \leq i \leq I-1, \tag{8.2}$$

$$\delta_t U_I^{(n)} = \delta^2 U_I^{(n)} - \big(U_I^{(n)}\big)^p + \frac{2}{h}\big(U_I^{(n)}\big)^p, \tag{8.3}$$

$$U_i^{(0)} = \varphi_i, \quad 0 \leq i \leq I, \tag{8.4}$$

where $n \geq 0$, $\varphi_{i+1} \geq \varphi_i$, $0 \leq i \leq I-1$,

$$\begin{gathered}
\delta_t U_i^{(n)} = \frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n}, \\
\delta^2 U_i^{(n)} = \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2}, \quad 1 \leq i \leq I-1, \\
\delta^2 U_0^{(n)} = \frac{2U_1^{(n)} - 2U_0^{(n)}}{h^2}, \qquad \delta^2 U_I^{(n)} = \frac{2U_{I-1}^{(n)} - 2U_I^{(n)}}{h^2}.
\end{gathered} \tag{8.5}$$

In order to permit the discrete solution to reproduce the property of the continuous one when the time t approaches the blow-up time, we need to adapt the size of the time step so that we take $\Delta t_n = \min\{(1-p\tau)h^2/3, \tau/\|U_h^{(n)}\|_\infty^{p-1}\}$, $0 < \tau < 1/p$.

Let us notice that the restriction on the time step ensures the nonnegativity of the discrete solution. The lemma below shows that the discrete solution is increasing in space.

Lemma 8.1. *Let $U_h^{(n)}$ be the solution of* (8.2)–(8.4). *Then we have*

$$U_{i+1}^{(n)} \geq U_i^{(n)}, \quad 0 \leq i \leq I-1. \tag{8.6}$$

Proof. Let $Z_i^{(n)} = U_{i+1}^{(n)} - U_i^{(n)}$, $0 \leq i \leq I-1$. We observe that

$$\begin{gathered}
\frac{Z_0^{(n+1)} - Z_0^{(n)}}{\Delta t_n} = \frac{Z_1^{(n)} - 3Z_0^{(n)}}{h^2} - \left((U_1^{(n)})^p - (U_0^{(n)})^p\right), \\
\frac{Z_i^{(n+1)} - Z_i^{(n)}}{\Delta t_n} = \frac{Z_{i+1}^{(n)} - 2Z_i^{(n)} + Z_{i-1}^{(n)}}{h^2} - \left((U_{i+1}^{(n)})^p - (U_i^{(n)})^p\right), \quad 1 \leq i \leq I-2, \\
\frac{Z_{I-1}^{(n+1)} - Z_{I-1}^{(n)}}{\Delta t_n} = \frac{Z_{I-2}^{(n)} - 3Z_{I-1}^{(n)}}{h^2} - \left((U_I^{(n)})^p - (U_{I-1}^{(n)})^p\right) + \frac{2}{h}(U_I^{(n)})^p.
\end{gathered} \tag{8.7}$$

Using the Taylor's expansion, we find that

$$\begin{aligned}
Z_0^{(n+1)} &= \frac{\Delta t_n}{h^2} Z_1^{(n)} + \left(1 - 3\frac{\Delta t_n}{h^2}\right) Z_0^{(n)} - \Delta t_n p \left(\xi_0^{(n)}\right)^{p-1} Z_0^{(n)}, \\
Z_i^{(n+1)} &= \frac{\Delta t_n}{h^2} Z_{i+1}^{(n)} + \left(1 - 2\frac{\Delta t_n}{h^2}\right) Z_i^{(n)} + \frac{\Delta t_n}{h^2} Z_{i-1}^{(n)} \\
&\quad - \Delta t_n p \left(\xi_i^{(n)}\right)^{p-1} Z_i^{(n)}, \quad 1 \leq i \leq I-2, \\
Z_{I-1}^{(n+1)} &\geq \frac{\Delta t_n}{h^2} Z_{I-2}^{(n)} + \left(1 - 3\frac{\Delta t_n}{h^2}\right) Z_{I-1}^{(n)} - \Delta t_n p \left(\xi_{I-1}^{(n)}\right)^{p-1} Z_{I-1}^{(n)},
\end{aligned} \tag{8.8}$$

where $\xi_i^{(n)}$ is an intermediate value between $U_i^{(n)}$ and $U_{i+1}^{(n)}$. If $Z_i^{(n)} \leq 0$, $0 \leq i \leq I-1$, we deduce that

$$\begin{aligned}
Z_0^{(n+1)} &\geq \frac{\Delta t_n}{h^2} Z_1^{(n)} + \left(1 - 3\frac{\Delta t_n}{h^2} - \Delta t_n p \|U_h^{(n)}\|_\infty^{p-1}\right) Z_0^{(n)}, \\
Z_i^{(n+1)} &\geq \frac{\Delta t_n}{h^2} Z_{i+1}^{(n)} + \left(1 - 2\frac{\Delta t_n}{h^2} - \Delta t_n p \|U_h^{(n)}\|_\infty^{p-1}\right) Z_i^{(n)} \\
&\quad + \frac{\Delta t_n}{h^2} Z_{i-1}^{(n)}, \quad 1 \leq i \leq I-2, \\
Z_{I-1}^{(n+1)} &\geq \frac{\Delta t_n}{h^2} Z_{I-2}^{(n)} + \left(1 - 3\frac{\Delta t_n}{h^2} - \Delta t_n p \|U_h^{(n)}\|_\infty^{p-1}\right) Z_{I-1}^{(n)}.
\end{aligned} \tag{8.9}$$

Using the restriction $\Delta t_n \leq \tau / \|U_h^{(n)}\|_\infty^{p-1}$, we find that

$$\begin{aligned}
Z_0^{(n+1)} &\geq \frac{\Delta t_n}{h^2} Z_1^{(n)} + \left(1 - 3\frac{\Delta t_n}{h^2} - p\tau\right) Z_0^{(n)}, \\
Z_i^{(n+1)} &\geq \frac{\Delta t_n}{h^2} Z_{i+1}^{(n)} + \left(1 - 2\frac{\Delta t_n}{h^2} - p\tau\right) Z_i^{(n)} \\
&\quad + \frac{\Delta t_n}{h^2} Z_{i-1}^{(n)}, \quad 1 \leq i \leq I-2, \\
Z_{I-1}^{(n+1)} &\geq \frac{\Delta t_n}{h^2} Z_{I-2}^{(n)} + \left(1 - 3\frac{\Delta t_n}{h^2} - p\tau\right) Z_{I-1}^{(n)}.
\end{aligned} \tag{8.10}$$

We observe that $1 - 3(\Delta t_n / h^2) - p\tau$ is nonnegative and by induction, we deduce that $Z_i^{(n)} \leq 0$, $0 \leq i \leq I-1$. This ends the proof. □

The following lemma is a discrete form of the maximum principle.

Lemma 8.2. *Let $a_h^{(n)}$ be a bounded vector and let $V_h^{(n)}$ a sequence such that*

$$\delta_t V_i^{(n)} - \delta^2 V_i^{(n)} + a_i^{(n)} V_i^{(n)} \geq 0, \quad 0 \leq i \leq I, \ n \geq 0, \tag{8.11}$$

$$V_i^{(0)} \geq 0, \quad 0 \leq i \leq I. \tag{8.12}$$

Then $V_i^{(n)} \geq 0$ for $n \geq 0$, $0 \leq i \leq I$ if $\Delta t_n \leq h^2 / (2 + \|a_h^{(n)}\|_\infty h^2)$.

Proof. If $V_h^{(n)} \geq 0$ then a routine computation yields

$$\begin{aligned}
V_0^{(n+1)} &\geq \frac{2\Delta t_n}{h^2} V_1^{(n)} + \left(1 - 2\frac{\Delta t_n}{h^2} - \Delta t_n \|a_h^{(n)}\|_\infty\right) V_0^{(n)}, \\
V_i^{(n+1)} &\geq \frac{\Delta t_n}{h^2} V_{i+1}^{(n)} + \left(1 - 2\frac{\Delta t_n}{h^2} - \Delta t_n \|a_h^{(n)}\|_\infty\right) V_i^{(n)} \\
&\quad + \frac{\Delta t_n}{h^2} V_{i-1}^{(n)}, \quad 1 \leq i \leq I-1, \\
V_I^{(n+1)} &\geq \frac{2\Delta t_n}{h^2} V_{I-1}^{(n)} + \left(1 - 2\frac{\Delta t_n}{h^2} - \Delta t_n \|a_h^{(n)}\|_\infty\right) V_I^{(n)}.
\end{aligned} \tag{8.13}$$

Since $\Delta t_n \leq h^2/(2+\|a_h^{(n)}\|_\infty h^2)$, we see that $1-2(\Delta t_n/h^2)-\Delta t_n\|a_h^{(n)}\|_\infty$ is nonnegative. From (8.12), we deduce by induction that $V_h^{(n)} \geq 0$ which ends the proof. □

A direct consequence of the above result is the following comparison lemma. Its proof is straightforward.

Lemma 8.3. *Suppose that $a_h^{(n)}$ and $b_h^{(n)}$ two vectors such that $a_h^{(n)}$ is bounded. Let $V_h^{(n)}$ and $W_h^{(n)}$ two sequences such that*

$$\begin{gathered}\delta_t V_i^{(n)} - \delta^2 V_i^{(n)} + a_i^{(n)} V_i^{(n)} + b_i^{(n)} \leq \delta_t W_i^{(n)} - \delta^2 W_i^{(n)} + a_i^{(n)} W_i^{(n)} + b_i^{(n)}, \quad 0 \leq i \leq I,\ n \geq 0, \\ V_i^{(0)} \leq W_i^{(0)}, \quad 0 \leq i \leq I.\end{gathered} \tag{8.14}$$

Then $V_i^{(n)} \leq W_i^{(n)}$ for $n \geq 0$, $0 \leq i \leq I$ if $\Delta t_n \leq h^2/(2+\|a_h^{(n)}\|_\infty h^2)$.

Now, let us give a property of the operator δ_t.

Lemma 8.4. *Let $U^{(n)} \in \mathbb{R}$ be such that $U^{(n)} \geq 0$ for $n \geq 0$. Then we have*

$$\delta_t (U^{(n)})^p \geq p (U^{(n)})^{p-1} \delta_t U^{(n)}, \quad n \geq 0. \tag{8.15}$$

Proof. From Taylor's expansion, we find that

$$\delta_t (U^{(n)})^p = p (U^{(n)})^{p-1} \delta_t U^{(n)} + \frac{p(p-1)}{2} \Delta t_n (\delta_t U^{(n)})^2 (\theta^{(n)})^{p-2}, \tag{8.16}$$

where $\theta^{(n)}$ is an intermediate value between $U^{(n)}$ and $U^{(n+1)}$. Use the fact that $U^{(n)} \geq 0$ for $n \geq 0$ to complete the rest of the proof. □

To handle the phenomenon of blow-up for discrete equations, we need the following definition.

Definition 8.5. We say that the solution $U_h^{(n)}$ of (8.2)–(8.4) blows up in a finite time if

$$\lim_{n \to +\infty} \|U_h^{(n)}\|_\infty = +\infty, \qquad T_h^{\Delta t} = \lim_{n \to \infty} \sum_{i=0}^{n-1} \Delta t_i < +\infty. \tag{8.17}$$

The number $T_h^{\Delta t}$ is called the numerical blow-up time of $U_h^{(n)}$.

The following theorem reveals that the discrete solution $U_h^{(n)}$ of (8.2)–(8.4) blows up in a finite time under some hypotheses.

Theorem 8.6. *Let $U_h^{(n)}$ be the solution of* (8.2)–(8.4). *Suppose that there exists a constant $A \in (0,1]$ such that the initial data at* (8.4) *satisfies*

$$\begin{aligned} \delta^2\varphi_i - \varphi_i^p &\geq 0, \quad 0 \leq i \leq I-1. \\ \delta^2\varphi_I - \varphi_I^p + \frac{2}{h}\varphi_I^p &\geq A\varphi_I^p. \end{aligned} \tag{8.18}$$

Then $U_h^{(n)}$ blows up in a finite time $T_h^{\Delta t}$ which satisfies the following estimate

$$T_h^{\Delta t} \leq \frac{\tau(1+\tau')^{p-1}}{\left((1+\tau')^{p-1}-1\right)\|\varphi_h\|_\infty^{p-1}}, \tag{8.19}$$

where $\tau' = A\min\{(1-p\tau)h^2\|\varphi_h\|_{\inf}^{-p-1}/3, \tau\}$.

Proof. Introduce the vector $J_h^{(n)}$ defined as follows

$$\begin{aligned} J_i^{(n)} &= \delta_t U_i^{(n)}, \quad 0 \leq i \leq I-1, \ n \geq 0, \\ J_I^{(n)} &= \delta_t U_I^{(n)} - A\left(U_I^{(n)}\right)^{-p}, \quad n \geq 0. \end{aligned} \tag{8.20}$$

A straightforward computation yields

$$\begin{aligned} \delta_t J_i^{(n)} - \delta^2 J_i^{(n)} &= \delta_t\left(\delta_t U_i^{(n)} - \delta^2 U_i^{(n)}\right), \quad 0 \leq i \leq I-1, \\ \delta_t J_I^{(n)} - \delta^2 J_I^{(n)} &= \delta_t\left(\delta_t U_I^{(n)} - \delta^2 U_I^{(n)}\right) - A\delta_t\left(U_I^{(n)}\right)^p + A\delta^2\left(U_I^{(n)}\right)^p. \end{aligned} \tag{8.21}$$

Using (8.2), we arrive at

$$\begin{aligned} \delta_t J_i^{(n)} - \delta^2 J_i^{(n)} &= -\delta_t\left(U_i^{(n)}\right)^p, \quad 0 \leq i \leq I-1, \\ \delta_t J_I^{(n)} - \delta^2 J_I^{(n)} &= \left(\frac{2}{h} - 1 - A\right)\delta_t\left(U_I^{(n)}\right)^p + A\delta^2\left(U_I^{(n)}\right)^p. \end{aligned} \tag{8.22}$$

Due to the mean value theorem, we get

$$\delta_t\left(U_i^{(n)}\right)^p = p\left(\xi_i^{(n)}\right)^{p-1}\delta_t\left(U_i^{(n)}\right) = p\left(\xi_i^{(n)}\right)^{p-1}J_i^{(n)}, \tag{8.23}$$

where $\xi_i^{(n)}$ is an intermediate value between $U_i^{(n)}$ and $U_{i+1}^{(n)}$. On the other hand, from Lemmas 2.4 and 2.5, we deduce that

$$\begin{aligned} \delta_t J_i^{(n)} - \delta^2 J_i^{(n)} &= -p\left(\xi_i^{(n)}\right)^{p-1}J_i^{(n)}, \quad 0 \leq i \leq I-1, \\ \delta_t J_I^{(n)} - \delta^2 J_I^{(n)} &= \left(\frac{2}{h} - 1 - A\right)p\left(U_I^{(n)}\right)^{p-1}\delta_t U_I^{(n)} + Ap\delta_t\left(U_I^{(n)}\right)^{p-1}\delta^2 U_I^{(n)}. \end{aligned} \tag{8.24}$$

It follows from (8.3) that

$$\delta_t J_I^{(n)} - \delta^2 J_I^{(n)} = \left(\frac{2}{h} - 1\right) p \left(U_I^{(n)}\right)^{p-1} \delta_t U_I^{(n)} - Ap\delta_t \left(U_I^{(n)}\right)^{p-1} \left(\left(\frac{2}{h} - 1\right) \left(U_I^{(n)}\right)^p\right), \tag{8.25}$$

which implies that

$$\begin{gathered} \delta_t J_i^{(n)} - \delta^2 J_i^{(n)} = -p\left(\xi_i^{(n)}\right)^{p-1} J_i^{(n)}, \quad 0 \le i \le I-1, \\ \delta_t J_I^{(n)} - \delta^2 J_I^{(n)} = \left(\frac{2}{h} - 1\right) p \left(U_I^{(n)}\right)^{p-1} J_I^{(n)}. \end{gathered} \tag{8.26}$$

From (8.18), we observe that $J_h^{(0)} \ge 0$. It follows from Lemma 8.2 that $J_h^{(n)} \ge 0$ which implies that

$$U_I^{(n+1)} \ge U_I^{(n)} \left(1 + A\Delta t_n \left(U_I^{(n)}\right)^{p-1}\right). \tag{8.27}$$

From Lemma 8.1, we see that $U_I^{(n)} = \|U_h^{(n)}\|_\infty$ which implies that

$$\left\|U_h^{(n+1)}\right\|_\infty \ge \left\|U_h^{(n)}\right\|_\infty \left(1 + A\Delta t_n \left\|U_h^{(n)}\right\|_\infty^{p-1}\right). \tag{8.28}$$

It is not hard to see that

$$A\Delta t_n \left\|U_h^{(n)}\right\|_\infty^{p-1} = A \min\left\{\frac{(1-p\tau)h^2 \left\|U_h^{(n)}\right\|_\infty^{p-1}}{3}, \tau\right\}. \tag{8.29}$$

From (8.28), we get $\|U_h^{(n+1)}\|_\infty \ge \|U_h^{(n)}\|_\infty$. By induction, we arrive at $\|U_h^{(n+1)}\|_\infty \ge \|U_h^{(0)}\|_\infty = \|\varphi_h\|_\infty$, which implies that $\|U_h^{(n)}\|_\infty^{p-1} \ge \|\varphi_h\|_\infty^{p-1}$. Therefore, we find that

$$A\Delta t_n \left\|U_h^{(n)}\right\|_\infty^{p-1} \ge A \min\left\{\frac{(1-p\tau)h^2 \left\|\varphi_h\right\|_\infty^{p-1}}{3}, \tau\right\} = \tau'. \tag{8.30}$$

Consequently, we arrive at

$$\left\|U_h^{(n+1)}\right\|_\infty \ge \left\|U_h^{(n)}\right\|_\infty (1 + \tau') \tag{8.31}$$

and by induction, we get

$$\left\|U_h^{(n)}\right\|_\infty \ge \left\|U_h^{(0)}\right\|_\infty (1+\tau')^n = \left\|\varphi_h\right\|_\infty (1+\tau')^n, \quad n \ge 0. \tag{8.32}$$

Since the term on the right hand side of the above equality tends to infinity as n approaches infinity, we conclude that $\|U_h^{(n)}\|_\infty$ tends to infinity as n approaches infinity. Now, let us estimate the numerical blow-up time. Due to (8.32), the restriction on the time step ensures that

$$\Sigma_{n=0}^{+\infty} \Delta t_n \leq \Sigma_{n=0}^{+\infty} \frac{\tau}{\|U_h^{(n)}\|_\infty^{p-1}} \leq \frac{\tau}{\|\varphi_h\|_\infty^{p-1}} \Sigma_{n=0}^{+\infty} \left(\frac{1}{(1+\tau')^{p-1}}\right)^n. \tag{8.33}$$

Using the fact that the series on the right hand side of the above inequality converges towards $\tau(1+\tau')^{p-1}/((1+\tau')^{p-1}-1)$, we deduce that $\Sigma_{n=0}^{+\infty}\Delta t_n \leq \Sigma_{n=0}^{+\infty}(\tau(1+\tau')^{p-1}/((1+\tau')^{p-1}-1)\|\varphi_h\|_\infty^{p-1})$ and the proof is complete. □

Remark 8.7. Apply Taylor's expansion to obtain $(1+\tau')^{p-1} = 1-(p-1)\tau'+o(\tau')$, which implies that

$$\frac{\tau}{(1+\tau')^{p-1}-1} = \frac{\tau}{\tau'}\left(\frac{1}{p-1+o(1)}\right) \leq \frac{2\tau}{\tau'(p-1)}. \tag{8.34}$$

If we take $\tau = h^2$, we see that

$$\frac{\tau}{\tau'} = A \min\left\{\frac{(1-ph^2)}{3}\|\varphi_h\|_\infty^{p-1}, 1\right\} \geq A \min\left\{\frac{1}{4}\|\varphi_h\|_\infty^{p-1}, 1\right\}. \tag{8.35}$$

We deduce that τ/τ' is bounded from above. We conclude that $\tau/((1+\tau')^{p-1}-1)$ is bounded from above.

Remark 8.8. From (8.31), we get

$$\|U_h^{(n)}\|_\infty \geq \|U_h^{(q)}\|_\infty (1+\tau')^{n-q} \quad \text{for } n \geq q \tag{8.36}$$

which implies that

$$\Sigma_{n=q}^{+\infty} \Delta t_n \leq \frac{\tau}{\|U_h^{(q)}\|_\infty^{p-1}} \Sigma_{n=q}^{+\infty} \left[\frac{1}{(1+\tau')^{p-1}}\right]^{n-q}. \tag{8.37}$$

We deduce that

$$T_h^{\Delta t} - t_q \leq \frac{\tau}{\|U_h^{(q)}\|_\infty^{p-1}} \frac{(1+\tau')^{p-1}}{(1+\tau')^{p-1}-1}. \tag{8.38}$$

In the sequel, we take $\tau = h^2$.

9. Convergence of the blow-up time

In this section, under some conditions, we show that the discrete solution blows up in a finite time and its numerical blow-up time goes to the real one when the mesh size goes to zero. To start, let us prove a result about the convergence of our scheme.

Theorem 9.1. *Suppose that the problem* (1.1) *has a solution $u \in C^{4,2}([0,1] \times [0,T])$. Assume that the initial data at* (8.4) *satisfies*

$$\|\varphi_h - u_h(0)\|_\infty = o(1) \quad \text{as } h \longrightarrow 0. \tag{9.1}$$

Then the problem (8.2)–(8.4) *has a solution $U_h^{(n)}$ for h sufficiently small, $0 \le n \le J$ and we have the following relation*

$$\max_{0\le n\le J}\|U_h^{(n)} - u_h(t_n)\|_\infty = O\left(\|\varphi_h - u_h(0)\|_\infty + h^2 + \Delta t_n\right) \quad \text{as } h \longrightarrow 0, \tag{9.2}$$

where J is such that $\sum_{n=0}^{J-1}\Delta t_n \le T$ and $t_n = \sum_{j=0}^{n-1}\Delta t_j$.

Proof. For each h, the problem (8.2)–(8.4) has a solution $U_h^{(n)}$. Let $N \le J$ be the greatest value of n such that

$$\|U_h^{(n)} - u_h(t_n)\|_\infty < 1 \quad \text{for } n < N. \tag{9.3}$$

We know that $N \ge 1$ because of (9.1). Due to the fact that $u \in C^{4,2}$, there exists a positive constant K such that $\|u\|_\infty \le K$. Applying the triangle inequality, we have

$$\|U_h^{(n)}\|_\infty \le \|u_h(t_n)\|_\infty + \|U_h^{(n)} - u_h(t_n)\|_\infty \le 1 + K \quad \text{for } n < N. \tag{9.4}$$

Since $u \in C^{4,2}$, using Taylor's expansion, we find that

$$\begin{aligned}
&\delta_t u(x_i,t_n) - \delta^2 u(x_i,t_n) = -u^p(x_i,t_n) + O(h^2) + O(\Delta t_n), \quad 0 \le i \le I-1,\\
&\delta_t u(x_I,t_n) - \delta^2 u(x_I,t_n) = -u^p(x_I,t_n) + \frac{2}{h}u^p(x_I,t_n) + O(h^2) + O(\Delta t_n).
\end{aligned} \tag{9.5}$$

Let $e_h^{(n)} = U_h^{(n)} - u_h(t_n)$ be the error of discretization. From the mean value theorem, we get

$$\begin{aligned}
&\delta_t e_i^{(n)} - \delta^2 e_i^{(n)} = -p\left(\xi_i^{(n)}\right)^{p-1} e_i^{(n)} + O(h^2) + O(\Delta t_n), \quad 0 \le i \le I-1,\\
&\delta_t e_I^{(n)} - \delta^2 e_I^{(n)} = p\left(\frac{2}{h} - 1\right)\left(\xi_I^{(n)}\right)^{p-1} e_I^{(n)} + O(h^2) + O(\Delta t_n),
\end{aligned} \tag{9.6}$$

where $\xi_i^{(n)}$ is an intermediate value between $u(x_i,t_n)$ and $U_i^{(n)}$. Hence, there exist positive constants L and K such that

$$\begin{aligned}
&\delta_t e_i^{(n)} - \delta^2 e_i^{(n)} \le -p\left(\xi_i^{(n)}\right)^{p-1} e_i^{(n)} + Lh^2 + L\Delta t_n, \quad 0 \le i \le I-1, \ n < N,\\
&\delta_t e_I^{(n)} - \delta^2 e_I^{(n)} \le p\left(\frac{2}{h} - 1\right)\left(\xi_I^{(n)}\right)^{p-1} e_I^{(n)} + Lh^2 + L\Delta t_n, \quad n < N.
\end{aligned} \tag{9.7}$$

Consider the function $Z(x,t) = e^{((M+1)t+Cx^2)}(\|\varphi_h - u_h(0)\|_\infty + Qh^2 + Q\Delta t_n)$ where M, C, Q are positive constants which will be determined later. We get

$$\begin{gathered} Z_t(x,t) - Z_{xx}(x,t) = \left(M+1-2C-4C^2x^2\right)Z(x,t), \\ Z_x(0,t) = 0, \qquad Z_x(1,t) = 2CZ(1,t), \\ Z(x,0) = e^{(Cx^2)}\left(\|\varphi_h - u_h(0)\|_\infty + Qh^2 + Q\Delta t_n\right). \end{gathered} \tag{9.8}$$

By a discretization of the above problem, we obtain

$$\begin{aligned} \delta_t Z(x_i,t_n) - \delta^2 Z(x_i,t_n) &= \left(M+1-2C-4C^2x_i^2\right)Z(x_i,t_n) + \frac{h^2}{12}Z_{xxxx}(\tilde{x}_i,t_n) \\ &\quad - \frac{\Delta t_n}{2}Z_{tt}(x_i,\tilde{t}_n), \\ \delta_t Z(x_I,t_n) - \delta^2 Z(x_I,t_n) &= \left(M+1-2C-4C^2x_I^2\right)Z(x_I,t_n) + \frac{4C}{h}Z(x_I,t_n) \\ &\quad + \frac{h^2}{12}Z_{xxxx}(\tilde{x}_I,t_n) - \frac{\Delta t_n}{2}Z_{tt}(x_I,\tilde{t}_n). \end{aligned} \tag{9.9}$$

We may choose M, C, Q large enough that

$$\begin{gathered} \delta_t Z(x_i,t_n) - \delta^2 Z(x_i,t_n) > -p\left(\xi_i^{(n)}\right)^{p-1}Z(x_i,t_n) + Lh^2 + L\Delta t_n, \quad 0 \le i \le I-1, \\ \delta_t Z(x_I,t_n) - \delta^2 Z(x_I,t_n) > p\left(\frac{2}{h}-1\right)\left(\xi_I^{(n)}\right)^{p-1}Z(x_I,t_n) + Lh^2 + L\Delta t_n, \\ Z_i^{(0)} > e_i^{(0)}, \quad 0 \le i \le I. \end{gathered} \tag{9.10}$$

It follows from Comparison Lemma 8.3 that

$$Z(x_i,t_n) > e_i^{(n)}, \quad 0 \le i \le I,\ n < N. \tag{9.11}$$

By the same way, we also prove that

$$Z(x_i,t_n) > -e_i^{(n)}, \quad 0 \le i \le I,\ n < N, \tag{9.12}$$

which implies that

$$\left\|U_h^{(n)} - u_h(t)\right\|_\infty \le e^{(Mt_n+C)}\left(\|\varphi_h - u_h(0)\|_\infty + Qh^2 + Q\Delta t_n\right), \quad n < N. \tag{9.13}$$

Let us show that $N = J$. Suppose that $N < J$. From (9.3), we obtain

$$1 \le \left\|U_h^{(N)} - u_h(t_N)\right\|_\infty \le e^{(MT+C)}\left(\|\varphi_h - u_h(0)\|_\infty + Qh^2 + Q\Delta t_n\right). \tag{9.14}$$

Since the term on the right hand side of the second inequality goes to zero as h goes to zero, we deduce that $1 \le 0$, which is a contradiction and the proof is complete. □

Now, we are in a position to state the main theorem of this section.

Theorem 9.2. *Suppose that the problem* (1.1) *has a solution u which blows up in a finite time T_0 and $u \in C^{4,2}([0,1] \times [0,T_0))$. Assume that the initial data at* (2.3) *satisfies*

$$\left\|\varphi_h - u_h(0)\right\|_\infty = o(1) \quad as\ h \longrightarrow 0. \tag{9.15}$$

Under the assumption of Theorem 8.6, the problem (8.2)–(8.4) *has a solution $U_h^{(n)}$ which blows up in a finite time $T_h^{\Delta t}$ and the following relation holds*

$$\lim_{h\to 0} T_h^{\Delta t} = T_0. \tag{9.16}$$

Proof. We know from Remark 8.7 that $\tau(1+\tau')/((1+\tau')^{p-1}-1)$ is bounded. Letting $\varepsilon > 0$, there exists a constant $R > 0$ such that

$$\frac{\tau(1+\tau')^{p-1}}{x^{p-1}\left((1+\tau')^{p-1}-1\right)} < \frac{\varepsilon}{2} \quad \text{for } x \in [R,\infty). \tag{9.17}$$

Since u blows up at the time T_0, there exists $T_1 \in (T_0 - \varepsilon/2, T_0)$ such that $\|u(\cdot,t)\|_\infty \geq 2R$ for $t \in [T_1, T_0)$. Let $T_2 = (T_1+T_0)/2$ and let q be a positive integer such that $t_q = \sum_{n=0}^{q-1}\Delta t_n \in [T_1, T_2]$ for h small enough. We have $0 < \|u_h(t_n)\|_\infty < \infty$ for $n \leq q$. It follows from Theorem 4.3 that the problem (2.1)–(2.3) has a solution $U_h^{(n)}$ which obeys $\|U_h^{(n)} - u_h(t_n)\|_\infty < R$ for $n \leq q$, which implies that

$$\left\|U_h^{(q)}\right\|_\infty \geq \left\|u_h(t_q)\right\|_\infty - \left\|U_h^{(q)} - u_h(t_q)\right\|_\infty \geq R. \tag{9.18}$$

From Theorem 8.6, $U_h^{(n)}$ blows up at the time $T_h^{\Delta t}$. It follows from Remark 8.8 and (9.17) that $|T_h^{\Delta t} - t_q| \leq \tau(1+\tau')^{p-1}\|U_h^{(q)}\|_\infty^{1-p}/((1+\tau')^{p-1}-1) < \varepsilon/2$ because $\|U_h^{(q)}\|_\infty \geq R$. We deduce that $|T_0 - T_h^{\Delta t}| \leq |T_0 - t_q| + |t_q - T_h^{\Delta t}| \leq \varepsilon/2 + \varepsilon/2 \leq \varepsilon$, which leads us to the result. □

10. Numerical experiments

In this section, we present some numerical approximations to the blow-up time of (1.1) in the case where $a(x,t) = \lambda > 0$, $f(u) = u^p$, $g(u) = u^q$, $b(t) = 1$ with $p = \text{const} > 1$, $q = \text{const} > 1$. We approximate the solution u of (1.1) by the solution $U_h^{(n)}$ of the following explicit scheme

$$\begin{gathered}
\delta_t U_i^{(n)} = \delta^2 U_i^{(n)} - \lambda\left(U_i^{(n)}\right)^{p-1} U_i^{(n+1)}, \quad 0 \leq i \leq I-1, \\
\delta_t U_I^{(n)} = \delta^2 U_I^{(n)} + \frac{2}{h}\left(U_I^{(n)}\right)^{q} - \lambda\left(U_I^{(n)}\right)^{p-1} U_I^{(n+1)}, \\
U_i^{(0)} = \varphi_i \geq 0, \quad 0 \leq i \leq I,
\end{gathered} \tag{10.1}$$

We also approximate the solution u of (1.1) by the solution $U_h^{(n)}$ of the implicit scheme below

$$\begin{aligned} \delta_t U_i^{(n)} &= \delta^2 U_i^{(n+1)} - \lambda \left(U_i^{(n)}\right)^{p-1} U_i^{(n+1)}, \quad 0 \leq i \leq I-1, \\ \delta_t U_I^{(n)} &= \delta^2 U_I^{(n+1)} + \frac{2}{h}\left(U_I^{(n)}\right)^q - \lambda \left(U_I^{(n)}\right)^{p-1} U_I^{(n+1)}, \\ U_i^{(0)} &= \varphi_i \geq 0, \quad 0 \leq i \leq I. \end{aligned} \tag{10.2}$$

For the time step, we take $n \geq 0$, $\Delta t_n = \min(h^2/2, \tau \|U_h^{(n)}\|_\infty^{1-p})$ for the explicit scheme and $\Delta t_n = \tau \|U_h^{(n)}\|_\infty^{1-p}$ for the implicit scheme.

The problem described in (10.1) may be rewritten as follows

$$\begin{aligned} U_0^{(n+1)} &= \frac{2(\Delta t_n/h^2) U_1^{(n)} + (1 - 2(\Delta t_n/h^2)) U_0^{(n)}}{1 + \lambda \Delta t_n \left(U_0^{(n)}\right)^{p-1}}, \\ U_i^{(n+1)} &= \frac{2(\Delta t_n/h^2) U_{i+1}^{(n)} + (1 - 2(\Delta t_n/h^2)) U_i^{(n)} + 2(\Delta t_n/h^2) U_{i-1}^{(n)}}{1 + \lambda \Delta t_n \left(U_i^{(n)}\right)^{p-1}}, \\ U_I^{(n+1)} &= \frac{2(\Delta t_n/h^2) U_{I-1}^{(n)} + (1 - 2(\Delta t_n/h^2)) U_I^{(n)} + 2(\Delta t_n/h^2) \left(U_I^{(n)}\right)^q}{1 + \lambda \Delta t_n \left(U_I^{(n)}\right)^{p-1}}. \end{aligned} \tag{10.3}$$

Let us notice that the restriction on the time step $\Delta t_n \leq h^2/2$ ensures the nonnegativity of the discrete solution.

The implicit scheme may be rewritten in the following form

$$A_h^n U_h^{(n+1)} = F^n, \tag{10.4}$$

where

$$A_h^{(n)} = \begin{pmatrix} a_0 & b_0 & 0 & \cdots & 0 \\ c_1 & a_1 & b_1 & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \\ \vdots & \ddots & \ddots & \ddots & b_{I-1} \\ 0 & \cdots & 0 & c_I & a_I \end{pmatrix},$$

$$\begin{aligned} a_i &= 1 + 2\frac{\Delta t_n}{h^2} + \lambda \Delta t_n \left(U_i^{(n)}\right)^{p-1}, \quad 0 \leq i \leq I, \\ b_i &= -2\frac{\Delta t_n}{h^2}, \quad i = 0, \ldots, I-1, \\ c_i &= -2\frac{\Delta t_n}{h^2}, \quad i = 1, \ldots, I, \\ (F^n)_i &= U_i^{(n)}, \quad i = 0, \ldots, I-1, \\ (F^n)_I &= U_I^{(n)} + \frac{2}{h} \Delta t_n \left(U_I^{(n)}\right)^q. \end{aligned} \tag{10.5}$$

Table 1: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method defined in (10.1).

I	T^n	n	CPU time	s
16	0.047927	451	—	—
32	0.044695	1260	0.5	—
64	0.043583	4075	5	1.54
128	0.043225	14555	60	1.64
256	0.043115	55061	1816	1.71

The matrix $A_h^{(n)}$ satisfies the following properties

$$\left(A_h^{(n)}\right)_{ii} > 0, \quad \left(A_h^{(n)}\right)_{ij} < 0, \quad \text{if } i \neq j,$$
$$\left|\left(A_h^{(n)}\right)_{ii}\right| > \sum_{j \neq i} \left|\left(A_h^{(n)}\right)_{ij}\right|. \tag{10.6}$$

It follows that $U_h^{(n)}$ exists for $n \geq 0$. In addition, since $U_h^{(0)}$ is nonnegative, $U_h^{(n)}$ is also nonnegative for $n \geq 0$. We need the following definition.

Definition 10.1. We say that the discrete solution $U_h^{(n)}$ of the explicit scheme or the implicit scheme blows up in a finite time if $\lim_{n\to+\infty}\|U_h^{(n)}\|_\infty = +\infty$ and the series $\sum_{n=0}^{+\infty}\Delta t_n$ converges. The quantity $\sum_{n=0}^{+\infty}\Delta t_n$ is called the numerical blow-up time of the solution $U_h^{(n)}$.

In Tables 1, 2, 3, 4, 5, 6, 7, and 8, in rows, we present the numerical blow-up times, values of n, the CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128, 256. For the numerical blow-up time we take $T^n = \sum_{j=0}^{n-1}\Delta t_j$ which is computed at the first time when

$$\Delta t_n = \left|T^{n+1} - T^n\right| \leq 10^{-16}. \tag{10.7}$$

The order (s) of the method is computed from

$$s = \frac{\log\left(\left(T_{4h} - T_{2h}\right)/\left(T_{2h} - T_h\right)\right)}{\log(2)}. \tag{10.8}$$

Case 1. $p = 0$, $q = 2$, $\varphi_i = 10 + 10 * \cos(\pi i h)$, $\lambda = 1$.

Case 2. $p = 2$, $q = 4$, $\varphi_i = 10 + 10 * \cos(\pi i h)$, $\lambda = 1$.

Case 3. $p = 2$, $q = 3$, $\varphi_i = 10 + 10 * \cos(\pi i h)$, $\lambda = 1$.

Case 4. $p = 2$, $q = 2$, $\varphi_i = 10 + 10 * \cos(\pi i h)$, $\lambda = 1$.

Remark 10.2. The different cases of our numerical results show that there is a relationship between the flow on the boundary and the absorption in the interior of the domain. Indeed,

Table 2: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method defined in (10.2).

I	T^n	n	CPU time	s
16	0.047631	423	—	—
32	0.044645	1234	1	—
64	0.043576	4050	5	1.49
128	0.043224	14533	99	1.61
256	0.043113	55035	2000	1.67

Table 3: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method defined in (10.1).

I	T^n	n	CPU time	s
16	0.018286	21750	3	—
32	0.017181	83838	17	—
64	0.016729	329960	108	1.30
128	0.016412	1298750	1570	0.51
256	0.016324	6447649	27049	1.85

Table 4: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method defined in (10.2).

I	T^n	n	CPU time	s
16	0.018283	21741	6	—
32	0.017181	83831	37	—
64	0.016729	3299953	347	1.30
128	0.016617	1208495	4640	2.01
256	0.016526	6348765	29957	0.30

Table 5: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method defined in (10.1).

I	T^n	n	CPU time	s
16	0.024197	1649	—	—
32	0.022570	6103	2	—
64	0.021950	23583	8	1.40
128	0.021734	92985	200	1.52
256	0.021712	369250	3243	3.30

Table 6: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method defined in (10.2).

I	T^n	n	CPU time	s
16	0.024169	1602	—	—
32	0.022566	6066	5	—
64	0.021950	23551	65	1.38
128	0.021734	92985	1140	1.52
256	0.021713	370240	6709	3.37

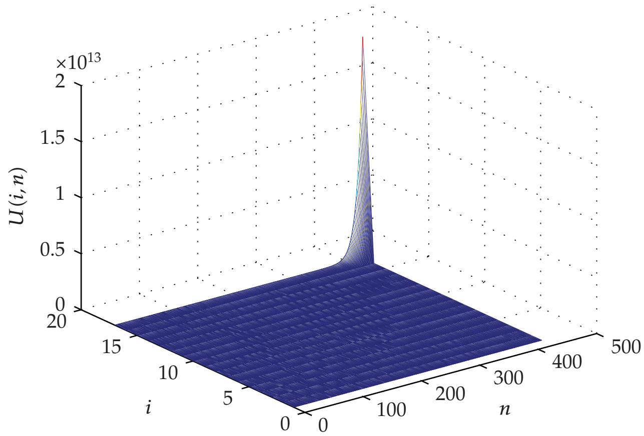

Figure 1: Evolution of the discrete solution, $q = 2$, $p = 2$ (explicit scheme).

Table 7: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method defined in (8.2)–(8.4).

I	T^n	n	CPU time	s
16	0.054342	422	—	—
32	0.050346	1130	—	—
64	0.049027	3539	4	1.60
128	0.048615	12020	28	1.68
256	0.048491	46439	937	1.74

Table 8: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method defined in (10.2).

I	T^n	n	CPU time	s
16	0.054158	364	—	—
32	0.050332	1077	0.6	—
64	0.049030	3491	7	1.56
128	0.048616	12358	79	1.66
256	0.048519	36919	1123	2.10

when there is not an absorption on the interior of the domain, we see that the blow-up time is slightly equal to 0.043 for $q = 2$ whereas if there is an absorption in the interior of the domain, we observe that the blow-up time is slightly equal to 0.048 for $q = 2$ and $p = 2$. We see that there is a diminution of the blow-up time. We also remark that if the power of flow on the boundary increases then the blow-up time diminishes. Thus the flow on the boundary make blow-up occurs whereas the absorption in the interior of domain prevents the blow-up. This phenomenon is well known in a theoretical point of view.

For other illustrations, in what follows, we give some plots to illustrate our analysis. In Figures 1, 2, 3, 4, 5, and, 6, we can appreciate that the discrete solution blows up in a finite time at the last node.

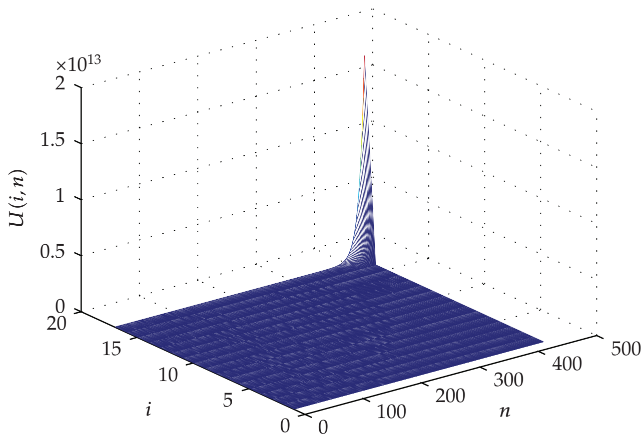

Figure 2: Evolution of the discrete solution, $q = 2$, $p = 2$ (implicit scheme).

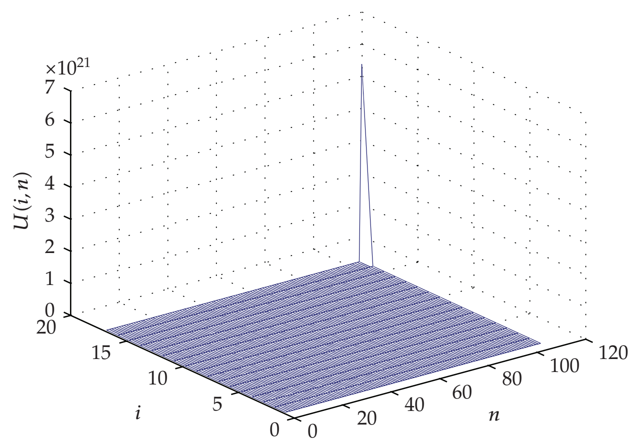

Figure 3: Evolution of the discrete solution, $q = 3$, $p = 2$ (explicit scheme).

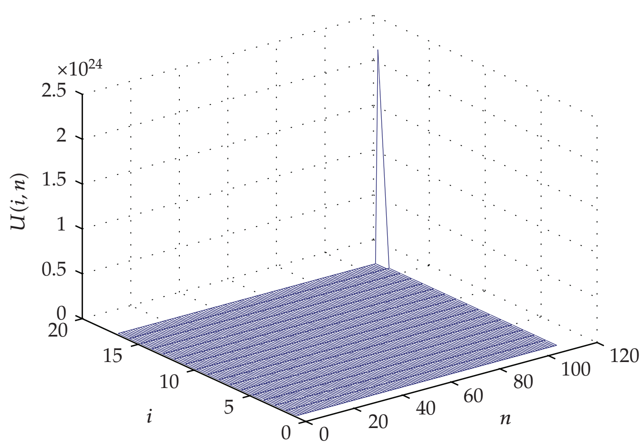

Figure 4: Evolution of the discrete solution, $q = 3$, $p = 2$ (implicit scheme).

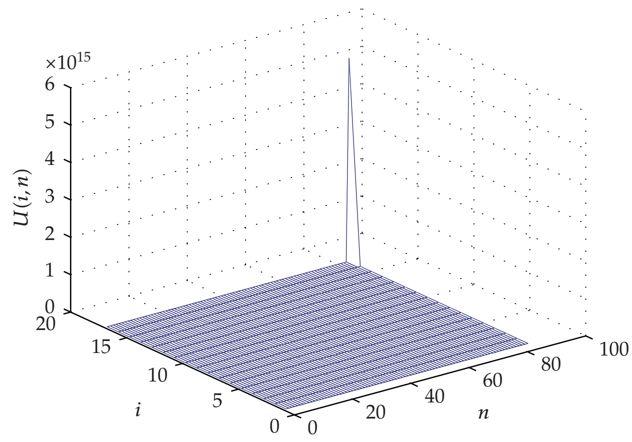

Figure 5: Evolution of the discrete solution, $q = 4$, $p = 2$ (explicit scheme).

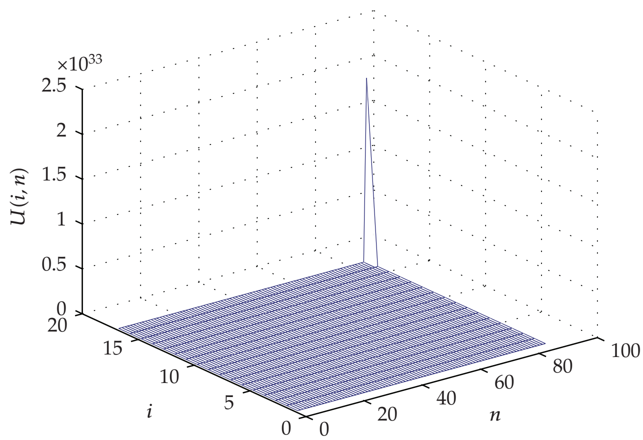

Figure 6: Evolution of the discrete solution, $q = 4$, $p = 2$ (implicit scheme).

Acknowledgments

We want to thank the anonymous referee for the throughout reading of the manuscript and several suggestions that help us to improve the presentation of the paper.

References

[1] T. K. Boni, "Sur l'explosion et le comportement asymptotique de la solution d'une équation parabolique semi-linéaire du second ordre," *Comptes Rendus de l'Académie des Sciences. Série I*, vol. 326, no. 3, pp. 317–322, 1998.

[2] T. K. Boni, "On blow-up and asymptotic behavior of solutions to a nonlinear parabolic equation of second order with nonlinear boundary conditions," *Commentationes Mathematicae Universitatis Carolinae*, vol. 40, no. 3, pp. 457–475, 1999.

[3] M. Chipot, M. Fila, and P. Quittner, "Stationary solutions, blow up and convergence to stationary solutions for semilinear parabolic equations with nonlinear boundary conditions," *Acta Mathematica Universitatis Comenianae*, vol. 60, no. 1, pp. 35–103, 1991.

[4] V. A. Galaktionov and J. L. Vázquez, "The problem of blow-up in nonlinear parabolic equations," *Discrete and Continuous Dynamical Systems. Series A*, vol. 8, no. 2, pp. 399–433, 2002.

[5] P. Quittner and P. Souplet, *Superlinear Parabolic Problems: Blow-up, Global Existence and Steady States*, Birkhäuser Advanced Texts. Basler Lehrbücher, Birkhäuser, Basel, Switzerland, 2007.
[6] A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov, *Blow-up in Problems for Quasilinear Parabolic Equations*, Nauka, Moscow, Russia, 1987.
[7] A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov, *Blow-up in Problems for Quasilinear Parabolic Equations*, Walter de Gruyter, Berlin, Germany, 1995.
[8] L. M. Abia, J. C. López-Marcos, and J. Martínez, "On the blow-up time convergence of semidiscretizations of reaction-diffusion equations," *Applied Numerical Mathematics*, vol. 26, no. 4, pp. 399–414, 1998.
[9] T. Nakagawa, "Blowing up of a finite difference solution to $u_t = u_{xx} + u^2$," *Applied Mathematics and Optimization*, vol. 2, no. 4, pp. 337–350, 1975.
[10] T. K. Boni, "Extinction for discretizations of some semilinear parabolic equations," *Comptes Rendus de l'Académie des Sciences. Série I*, vol. 333, no. 8, pp. 795–800, 2001.
[11] G. Acosta, J. Fernández Bonder, P. Groisman, and J. D. Rossi, "Simultaneous *vs.* non-simultaneous blow-up in numerical approximations of a parabolic system with non-linear boundary conditions," *M2AN*, vol. 36, no. 1, pp. 55–68, 2002.
[12] G. Acosta, J. Fernández Bonder, P. Groisman, and J. D. Rossi, "Numerical approximation of a parabolic problem with a nonlinear boundary condition in several space dimensions," *Discrete and Continuous Dynamical Systems. Series B*, vol. 2, no. 2, pp. 279–294, 2002.
[13] C. Brändle, P. Groisman, and J. D. Rossi, "Fully discrete adaptive methods for a blow-up problem," *Mathematical Models & Methods in Applied Sciences*, vol. 14, no. 10, pp. 1425–1450, 2004.
[14] C. Brändle, F. Quirós, and J. D. Rossi, "An adaptive numerical method to handle blow-up in a parabolic system," *Numerische Mathematik*, vol. 102, no. 1, pp. 39–59, 2005.
[15] R. G. Duran, J. I. Etcheverry, and J. D. Rossi, "Numerical approximation of a parabolic problem with a nonlinear boundary condition," *Discrete and Continuous Dynamical Systems*, vol. 4, no. 3, pp. 497–506, 1998.
[16] J. Fernández Bonder and J. D. Rossi, "Blow-up vs. spurious steady solutions," *Proceedings of the American Mathematical Society*, vol. 129, no. 1, pp. 139–144, 2001.
[17] A. de Pablo, M. Pérez-Llanos, and R. Ferreira, "Numerical blow-up for the p-Laplacian equation with a nonlinear source," in *Proceedings of the 11th International Conference on Differential Equations (Equadiff '05)*, pp. 363–367, Bratislava, Slovakia, July 2005.
[18] M. N. Le Roux, "Semidiscretization in time of nonlinear parabolic equations with blowup of the solution," *SIAM Journal on Numerical Analysis*, vol. 31, no. 1, pp. 170–195, 1994.
[19] M. N. Le Roux, "Semi-discretization in time of a fast diffusion equation," *Journal of Mathematical Analysis and Applications*, vol. 137, no. 2, pp. 354–370, 1989.
[20] D. Nabongo and T. K. Boni, "Numerical blow-up and asymptotic behavior for a semilinear parabolic equation with a nonlilnear boundary condition," *Albanian Journal of Mathematics*, vol. 2, no. 2, pp. 111–124, 2008.
[21] D. Nabongo and T. K. Boni, "Numerical blow-up solutions of localized semilinear parabolic equations," *Applied Mathematical Sciences*, vol. 2, no. 21–24, pp. 1145–1160, 2008.
[22] F. K. N'gohissé and T. K. Boni, "Numerical blow-up solution for some semilinear heat equation," *Electronic Transactions on Numerical Analysis*, vol. 30, pp. 247–257, 2008.