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Research Article

Asymptotic Behavior of a Competition-Diffusion System with Variable Coefficients and Time Delays

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A class of time-delay reaction-diffusion systems with variable coefficients which arise from the model of two competing ecological species is discussed. An asymptotic global attractor is established in terms of the variable coefficients, independent of the time delays and the effect of diffusion by the upper-lower solutions and iteration method.

1. Introduction and main result

The Lotka-Volterra competition model with diffusion and time delays has been the object of analysis by numerous authors under different approaches. For the case of two species, Ruan and Zhao [1] considered uniform persistence and global extinction; Lu [2] studied global attractivity, and Gourley and Ruan [3] analyzed stability and traveling fronts. The periodic case has also been considered, Feng and Wang [4] studied asymptotic stability and Zhou et al. [5] investigated the Hopf bifurcation. The cases of three and N-species have also been analyzed in [6–8].

In this paper, we consider the asymptotic behavior of solutions for the competition-diffusion system with time delays of the following two species:

$$\frac{\partial u_1}{\partial t}=A\Delta u_1(t,x)+u_1(t,x)\left[a_1(t,x)-b_1(t,x)u_1(t,x)-c_1(t,x)\int_0^\infty u_1(t-\tau,x)\,d\mu_1(\tau)-d_1u_2(t-r_2,x)\right],$$
$$t>0,\quad x\in\Omega,$$
$$\frac{\partial u_2}{\partial t}=A\Delta u_2(t,x)+u_2(t,x)\left[a_2(t,x)-b_2(t,x)u_2(t,x)-c_2(t,x)\int_0^\infty u_2(t-\tau,x)\,d\mu_2(\tau)-d_2u_1(t-r_1,x)\right],$$
$$t>0,\quad x\in\Omega,$$

$$\frac{\partial u_1}{\partial \eta} = \frac{\partial u_2}{\partial \eta} = 0, \quad t > 0,\ x \in \partial\Omega,$$
$$u_i(t,x) = \phi_i(t,x) \quad (i = 1,2)\ t \le 0,\ x \in \overline{\Omega}, \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$ $(N \ge 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $\partial/\partial\eta$ denotes the differentiation outward normal on $\partial\Omega$, $A \ge 0$, $d_i > 0$, $r_i > 0$, $0 < a_i \le a_i(t,x) \le A_i$, $0 < b_i \le b_i(t,x) \le B_i$, $0 \le c_i \le c_i(t,x) \le C_i$, $\mu_i(\cdot)$ is of bounded variation with $\mu_i(0) = 0$, $\phi_i \in C^1((-\infty,0]) \times \overline{\Omega}$ is bounded and nonnegative and $0 \not\equiv \phi_i(0,\cdot) \in C^1(\overline{\Omega})$ $(i = 1,2)$, and u_1 and u_2 are the density functions of two species competing for a shared limited resource.

The functions $a_1(x,t)$ and $a_2(x,t)$ denote the intrinsic growth rate of the species, $b_1(x,t)$ and $b_2(x,t)$ represent self-limitation rates, and $c_1(x,t)$ and $c_2(x,t)$ represent the coefficients of the infinite continuous delay. The constants d_1, d_2 represent the competition rates. The distributed time delay should be viewed as the effects of past history.

Let $M_i(t)$ denote the total variation of $\mu_i(\cdot)$ on $[0,t]$ and let $M_i^{\pm}(t) = (M_i(t) \pm \mu_i(t))/2$, for all $t \in \mathbb{R}^+$. Then, $M_i(t)$ and $M_i^{\pm}(t)$ are nonnegative and nondecreasing on $\mathbb{R}^+$. It easy to see that

$$M_i^+(t) + M_i^-(t) = M_i(t), \qquad M_i^+(t) - M_i^-(t) = \mu_i(t). \tag{1.2}$$

Denote $M_{0i} = \lim_{t\to\infty} M_i(t)$, $M_{0i}^{\pm} = \lim_{t\to\infty} M_i^{\pm}(t)$, and $\mu_{0i} = \lim_{t\to\infty}\mu_i(t)$. Then,

$$M_{0i}^+ + M_{0i}^- = M_{0i}, \qquad M_{0i}^+ - M_{0i}^- = \mu_{0i}. \tag{1.3}$$

Our result can be stated as follows.

Theorem 1.1. *Assume that* $b_i > c_i M_{0i}^-$, $a_i - (C_i M_{0i}^+ A_i)/(b_i - c_i M_{0i}^-) > 0$, *and*

$$\frac{d_2 A_2 (b_2 - c_2 M_{02}^-)}{(b_1 - c_1 M_{01}^-)\left[a_2(b_2 - c_2 M_{02}^-) - C_2 M_{02}^+ A_2\right]} < \frac{A_2}{A_1} < \frac{(b_2 - c_2 M_{02}^-)\left[a_1(b_1 - c_1 M_{01}^-) - C_1 M_{01}^+ A_1\right]}{d_1 A_1 (b_1 - c_1 M_{01}^-)}. \tag{1.4}$$

Then, for any $\phi_i \in C^1((-\infty,0]) \times \overline{\Omega}$ *with* $\phi(0,x) \not\equiv 0$, *the solution of* (1.1) *satisfies*

$$0 < \alpha_1 \le \liminf_{t\to\infty} \min_{x\in\overline{\Omega}} u_1(t,x) \le \limsup_{t\to\infty} \max_{x\in\overline{\Omega}} u_1(t,x) \le \beta_1,$$
$$0 < \alpha_2 \le \liminf_{t\to\infty} \min_{x\in\overline{\Omega}} u_2(t,x) \le \limsup_{t\to\infty} \max_{x\in\overline{\Omega}} u_2(t,x) \le \beta_2, \tag{1.5}$$

where α_1, α_2, β_1, β_2 *are constants given by the linear system*

$$a_1 - C_1 M_{01}^+ \beta_1 - d_1 \beta_2 - \alpha_1 (B_1 - c_1 M_{01}^-) = 0,$$
$$a_2 - C_2 M_{02}^+ \beta_2 - d_2 \beta_1 - \alpha_2 (B_2 - c_2 M_{02}^-) = 0,$$
$$A_1 - c_1 M_{01}^+ \alpha_1 - d_1 \alpha_2 - \beta_1 (b_1 - c_1 M_{01}^-) = 0,$$
$$A_2 - c_2 M_{02}^+ \alpha_2 - d_2 \alpha_1 - \beta_2 (b_2 - c_2 M_{02}^-) = 0. \tag{1.6}$$

Remark 1.2. If $\mu_i(\tau) = \int_0^\tau f_i(t)dt$ with $f_i \in C(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, $A = 1$, and $0 < a_i = a_i(t,x) = A_i$, $0 < b_i = b_i(t,x) = B_i$, $c_i = c_i(t,x) = C_i = 1$, then by Theorem 1.1, we get

$$\begin{aligned} 0 < \alpha_1 \le \liminf_{t\to\infty} \min_{x\in\overline{\Omega}} u_1(t,x) \le \limsup_{t\to\infty} \max_{x\in\overline{\Omega}} u_1(t,x) \le \beta_1, \\ 0 < \alpha_2 \le \liminf_{t\to\infty} \min_{x\in\overline{\Omega}} u_2(t,x) \le \limsup_{t\to\infty} \max_{x\in\overline{\Omega}} u_2(t,x) \le \beta_2, \end{aligned} \tag{1.7}$$

where α_1, α_2, β_1, β_2 are constants given by the linear system

$$\begin{aligned} a_1 - M_{01}^+\beta_1 - d_1\beta_2 - \alpha_1(b_1 - M_{01}^-) = 0, \\ a_2 - M_{02}^+\beta_2 - d_2\beta_1 - \alpha_2(b_2 - M_{02}^-) = 0, \\ a_1 - M_{01}^+\alpha_1 - d_1\alpha_2 - \beta_1(b_1 - M_{01}^-) = 0, \\ a_2 - M_{02}^+\alpha_2 - d_2\alpha_1 - \beta_2(b_2 - M_{02}^-) = 0. \end{aligned} \tag{1.8}$$

Solving this system, we have the following result:

$$\begin{aligned} \alpha_1 = \beta_1 = \frac{a_1(b_2 + M_{02}^+ - M_{02}^-) - a_2 d_1}{(b_1 + M_{01}^+ - M_{01}^-)(b_2 + M_{02}^+ - M_{02}^-) - d_1 d_2}, \\ \alpha_2 = \beta_2 = \frac{a_2(b_1 + M_{01}^+ - M_{01}^-) - a_1 d_2}{(b_1 + M_{01}^+ - M_{01}^-)(b_2 + M_{02}^+ - M_{02}^-) - d_1 d_2}, \end{aligned} \tag{1.9}$$

which coincides with the result of [9], where the authors considered a system like (1.1) with constant coefficients.

Reaction-diffusion systems with delay have been treated by many authors. There are two ways to approach them. The first one is in the framework of semigroup theory of dynamical systems [1, 10]. The second one is a method of upper and lower solutions, using associated monotone iterations; several authors have studied their dynamic properties [2, 9, 11]. Sometimes the birth and death rates depend on both space and time, so when we consider instantaneous and delayed interference within the species and the diffusive effects of the species, system (1.1) will be the appropriate model.

The way we organize the paper is as follows: we first introduce several results which play an important role in the proof of Theorem 1.1 which we will prove in Section 2. We will provide some numerical simulations in Section 3 in order to illustrate our theory.

The following results are developed in [11]. They considered the Volterra reaction-diffusion equations with variable coefficients:

$$\begin{aligned} \frac{\partial u}{\partial t} = A\Delta u(t,x) + u(t,x)\left[a(t,x) - b(t,x)u(t,x) - c(t,x)\int_0^\infty u(t-\tau,x)d\mu(\tau)\right], \quad t > 0,\ x \in \Omega, \\ \frac{\partial u}{\partial \eta} = 0, \quad t > 0,\ x \in \partial\Omega, \\ u(t,x) = \phi(t,x), \quad t \le 0,\ x \in \overline{\Omega}, \end{aligned} \tag{1.10}$$

where $0 < a_1 \le a(t,x) \le A_1$, $0 < b_1 \le b(t,x) \le B_1$, $0 \le c_1 \le c(t,x) \le C_1$, and $\phi \in C^1((-\infty,0]) \times \overline{\Omega}$ is bounded and nonnegative and $0 \not\equiv \phi(0,\cdot) \in C^1(\overline{\Omega})$. Let $M(t)$ denote the total variation of $\mu(\cdot)$ and define $M^\pm$, $M_0^\pm$ of the same form as that of $M_i^\pm$, $M_{0i}^\pm$, respectively.

Lemma 1.3. *Assume that* $b_1 > c_1 M_0^-$ *and* $a_1 - (C_1 M_0^+ A_1)/(b_1 - c_1 M_0^-) > 0$, *then the solution of* (1.10) *satisfies*

$$\alpha \leq \liminf_{t\to\infty} \min_{x\in\overline{\Omega}} u(t,x) \leq \limsup_{t\to\infty} \max_{x\in\overline{\Omega}} u(t,x) \leq \beta, \tag{1.11}$$

where

$$\begin{aligned}\alpha &= \frac{(b_1 - c_1 M_0^-)a_1 - C_1 M_0^+ A_1}{(b_1 - c_1 M_0^-)(B_1 - c_1 M_0^-) - C_1^2 (M_0^+)^2},\\ \beta &= \frac{(B_1 - c_1 M_0^-)A_1 - C_1 M_0^+ a_1}{(b_1 - c_1 M_0^-)(B_1 - c_1 M_0^-) - C_1^2 (M_0^+)^2}.\end{aligned} \tag{1.12}$$

If $M^+(t) = 0$, *then* $\mu(t) = -M^-(t)$. *Thus,*

$$\begin{gathered}\frac{\partial u}{\partial t} = A\Delta u(t,x) + u(t,x)\left[a(t,x) - b(t,x)u(t,x) + c(t,x)\int_0^\infty u(t-\tau,x)\,dM^-(\tau)\right], \quad t>0,\ x\in\Omega,\\ \frac{\partial u}{\partial \eta} = 0, \quad t>0,\ x\in\partial\Omega,\\ u(t,x) = \phi(t,x), \quad t\leq 0,\ x\in\overline{\Omega}.\end{gathered} \tag{1.13}$$

Lemma 1.4. *Assume that* $b_1 > c_1 M_0^-$ *and* $a_1 > 0$, *then the solution of* (1.13) *satisfies*

$$0 < \alpha \leq \liminf_{t\to\infty} \min_{x\in\overline{\Omega}} u(t,x) \leq \limsup_{t\to\infty} \max_{x\in\overline{\Omega}} u(t,x) \leq \beta, \tag{1.14}$$

where

$$\alpha = \frac{a_1}{B_1 - c_1 M_0^-}, \qquad \beta = \frac{A_1}{b_1 - c_1 M_0^-}. \tag{1.15}$$

Lemma 1.5. *If* $u(t,x)$ *is the solution of* (1.10), *then* $0 < u(t,x)$.

Now, we introduce the existence-comparison result for the competition-diffusion system (1.1) which is a particular case of Theorem 2.2 in[12].

Definition 1.6. A pair of smooth functions $\tilde{u} = (\tilde{u}_1, \tilde{u}_2)$ and $\hat{u} = (\hat{u}_1, \hat{u}_2)$ are called upper-lower solutions of (1.1) if $\tilde{u}_i \geq \hat{u}_i$ $(i = 1,2)$ in $\mathbb{R}\times\overline{\Omega}$ and the following differential inequalities hold:

$$\begin{gathered}\frac{\partial \tilde{u}_i}{\partial t} - A\Delta\tilde{u}_i \geq \tilde{u}_i(t,x)\left[a_i(t,x) - b_i(t,x)\tilde{u}_i(t,x) - c_i(t,x)\int_0^\infty \tilde{u}_i(t-\tau,x)\,d\mu_i(\tau) - d_i\hat{u}_j(t-r_j,x)\right],\\ j\neq i,\ t>0,\ x\in\Omega,\\ \frac{\partial \hat{u}_i}{\partial t} - A\Delta\hat{u}_i \leq \hat{u}_i(t,x)\left[a_i(t,x) - b_i(t,x)\hat{u}_i(t,x) - c_i(t,x)\int_0^\infty \hat{u}_i(t-\tau,x)\,d\mu_i(\tau) - d_i\tilde{u}_j(t-r_j,x)\right],\\ j\neq i,\ t>0,\ x\in\Omega,\\ \frac{\partial \hat{u}_i}{\partial \eta} \leq 0 \leq \frac{\partial \tilde{u}_i}{\partial \eta}, \quad t>0,\ x\in\partial\Omega,\\ \hat{u}_i(t,x) \leq \phi_i(t,x) \leq \tilde{u}_i(t,x), \quad t\leq 0,\ x\in\overline{\Omega}\ (i=1,2).\end{gathered} \tag{1.16}$$

Lemma 1.7. *If there exists a pair of upper-lower solutions* $\tilde{u} = (\tilde{u}_1, \tilde{u}_2)$ *and* $\hat{u} = (\hat{u}_1, \hat{u}_2)$ *of* (1.1), *then the problem* (1.1) *has a unique solution* $u^* = (u_1^*, u_2^*)$ *and* $\hat{u}_i \le u_i^* \le \tilde{u}_i$, $i = 1, 2$.

2. Proof of the main result

The method of proof is via successive improvements of upper-lower solutions of suitable systems.

For given $\phi = (\phi_1, \phi_2)$ as initial conditions for the system (1.1), let K_1, K_2 be constants such that

$$K_1 \ge \max\left\{ ||\phi_1||, \frac{A_1}{b_1 - c_1 M_{01}^-} \right\}, \qquad K_2 \ge \max\left\{ ||\phi_2||, \frac{A_2}{b_2 - c_2 M_{02}^-} \right\}, \tag{2.1}$$

where $||\phi_i|| = \sup\{|\phi_i(t,x)| : (t,x) \in (-\infty, 0] \times \overline{\Omega}\}$, $i = 1, 2$.

Then, $(0,0)$ and (K_1, K_2) are a pair of lower-upper solutions of (1.1). By Lemma 1.7, there exists a unique global nonnegative solution (u_1, u_2) of (1.1) and it satisfies $0 \le u_1(t,x) \le K_1$, $0 \le u_2(t,x) \le K_2$.

Define $\overline{u}_1^{(1)}(x,t)$ and $\overline{u}_2^{(1)}(x,t)$ by

$$\begin{gathered}
\frac{\partial \overline{u}_1^{(1)}}{\partial t} = A\Delta \overline{u}_1^{(1)} + \overline{u}_1^{(1)}\left[a_1(t,x) - b_1(t,x)\overline{u}_1^{(1)} + c_1(t,x)\int_0^\infty \overline{u}_1^{(1)}(t-\tau, x)\, dM_1^-(\tau)\right] \quad t > 0,\ x \in \Omega, \\
\frac{\partial \overline{u}_2^{(1)}}{\partial t} = A\Delta \overline{u}_2^{(1)} + \overline{u}_2^{(1)}\left[a_2(t,x) - b_2(t,x)\overline{u}_2^{(1)} + c_2(t,x)\int_0^\infty \overline{u}_2^{(1)}(t-\tau, x)\, dM_2^-(\tau)\right] \quad t > 0,\ x \in \Omega, \\
\frac{\partial \overline{u}_1^{(1)}}{\partial \eta} = \frac{\partial \overline{u}_2^{(1)}}{\partial \eta} = 0, \quad t > 0,\ x \in \partial\Omega, \\
\overline{u}_i^{(1)}(t,x) = K_i, \quad t \le 0,\ x \in \overline{\Omega}\ (i = 1, 2).
\end{gathered} \tag{2.2}$$

Then, $(0,0)$ and $(\overline{u}_1^{(1)}, \overline{u}_2^{(1)})$ are lower and upper solutions, and by Lemma 1.7,

$$0 \le u_1 \le \overline{u}_1^{(1)}, \qquad 0 \le u_2 \le \overline{u}_i^{(1)}. \tag{2.3}$$

By Lemma 1.4, we can get

$$0 < \limsup_{t\to\infty} \max_{x\in\overline{\Omega}} \overline{u}_i^{(1)}(t,x) \le \beta_i^{(0)}, \tag{2.4}$$

where

$$\beta_i^{(0)} = \frac{A_i}{b_i - c_i M_{0i}^-}, \quad i = 1, 2. \tag{2.5}$$

Then from (2.3) and (2.4), we get

$$0 \le \limsup_{t\to\infty} \max_{x\in\overline{\Omega}} u_i(t,x) \le \beta_i^{(0)}. \tag{2.6}$$

From (2.6) and the definition of $M_{0i}^{+} = \lim_{t\to\infty} M_i^{+}(t)$, we have that for any sufficiently small $\varepsilon > 0$, there exist $t_1' > 0$ and $t_1 > t_1'$ such that

$$\begin{aligned}
&\max_{x\in\overline{\Omega}} \overline{u}_i^{(1)}(t,x) < \beta_i^{(0)} + \varepsilon, \quad \text{for } t \geq t_1', \\
&\overline{u}_i^{(1)}(t - r_i, x) < \beta_i^{(0)} + \varepsilon, \quad \text{for } t \geq t_1, \\
&C_i\left(M_{0i}^{+} - M_i^{+}(t - t_1')\right) < \varepsilon, \quad \text{for } t \geq t_1.
\end{aligned} \tag{2.7}$$

Define $\underline{u}_1^{(1)}(t,x)$ and $\underline{u}_2^{(1)}(t,x)$ by

$$\begin{aligned}
\frac{\partial \underline{u}_1^{(1)}}{\partial t} &= A\Delta \underline{u}_1^{(1)} + \underline{u}_1^{(1)}\Big[a_1(t,x) - b_1(t,x)\underline{u}_1^{(1)} + c_1(t,x)\int_0^\infty \underline{u}_1^{(1)}(t-\tau,x)dM_1^{-}(\tau) \\
&\quad - c_1(t,x)\int_0^\infty \overline{u}_1^{(1)}(t-\tau,x)dM_1^{+}(\tau) - d_1\overline{u}_2^{(1)}(t - r_2, x)\Big], \quad t > t_1,\ x \in \Omega, \\
\frac{\partial \underline{u}_2^{(1)}}{\partial t} &= A\Delta \underline{u}_2^{(1)} + \underline{u}_2^{(1)}\Big[a_2(t,x) - b_2(t,x)\underline{u}_2^{(1)} + c_2(t,x)\int_0^\infty \underline{u}_2^{(1)}(t-\tau,x)dM_2^{-}(\tau) \\
&\quad - c_2(t,x)\int_0^\infty \overline{u}_2^{(1)}(t-\tau,x)dM_2^{+}(\tau) - d_2\overline{u}_1^{(1)}(t - r_1, x)\Big], \quad t > t_1,\ x \in \Omega, \\
&\frac{\partial \underline{u}_1^{(1)}}{\partial \eta} = \frac{\partial \underline{u}_2^{(1)}}{\partial \eta} = 0, \quad t > t_1,\ x \in \partial\Omega, \\
&\underline{u}_i^{(1)}(t,x) = \frac{1}{2}u_i(t,x), \quad (t,x) \in \left(-\infty, t_1\right] \times \overline{\Omega}.
\end{aligned} \tag{2.8}$$

Then, $(\underline{u}_1^{(1)}, \underline{u}_2^{(1)})$ and $(\overline{u}_1^{(1)}, \overline{u}_2^{(1)})$ are a pair of lower-upper solutions of (1.1), and by Lemma 1.7,

$$\underline{u}_1^{(1)} \leq u_1 \leq \overline{u}_1^{(1)}, \qquad \underline{u}_2^{(1)} \leq u_2 \leq \overline{u}_2^{(1)}. \tag{2.9}$$

From $\overline{u}_i^{(1)}(t,x) \leq K_i$, for all $t \in \mathbb{R}$, and (2.8), for $t \geq t_1$, we get

$$\begin{aligned}
&c_i(t,x)\int_0^\infty \overline{u}_i^{(1)}(t-\tau,x)dM_i^{+}(\tau) + d_i\overline{u}_j^{(1)}(t - r_j, x) \\
&\quad = c_i(t,x)\int_0^{t-t_1'} \overline{u}_i^{(1)}(t-\tau,x)dM_i^{+}(\tau) + c_i(t,x)\int_{t-t_1'}^\infty \overline{u}_i^{(1)}(t-\tau,x)dM_i^{+}(\tau) + d_i\overline{u}_j^{(1)}(t - r_j, x) \\
&\quad \leq c_i(t,x)\int_0^{t-t_1'} \left(\beta_i^{(0)} + \varepsilon\right)dM_i^{+}(\tau) + c_i(t,x)\int_{t-t_1'}^\infty K_i dM_i^{+}(\tau) + d_i\left(\beta_j^{(0)} + \varepsilon\right) \\
&\quad \leq C_i\left(M_i^{+}(t - t_1) - M(0)\right)\left(\beta_i^{(0)} + \varepsilon\right) + K_iC_i\left(M_0i^{+} - M_i^{+}(t - t_1)\right) + d_i\left(\beta_j^{(0)} + \varepsilon\right) \\
&\quad \leq C_iM_{0i}^{+}\left(\beta_i^{(0)} + \varepsilon\right) + d_i\left(\beta_j^{(0)} + \varepsilon\right) + K_i\varepsilon.
\end{aligned} \tag{2.10}$$

It follows from (2.9) that, for $t > t_1$, $x \in \Omega$,

$$\begin{aligned}
\frac{\partial \underline{u}_1^{(1)}}{\partial t} &\geq A\Delta \underline{u}_1^{(1)} + \underline{u}_1^{(1)}\Big[a_1(t,x) - b_1(t,x)\underline{u}_1^{(1)} + c_1(t,x)\int_0^\infty \underline{u}_1^{(1)}(t-\tau,x)dM_1^-(\tau) \\
&\qquad - C_1M_{01}^+\Big(\beta_1^{(0)}+\varepsilon\Big) - d_1\Big(\beta_2^{(0)}+\varepsilon\Big) - K_1\varepsilon\Big], \\
\frac{\partial \underline{u}_2^{(1)}}{\partial t} &\geq A\Delta \underline{u}_2^{(1)} + \underline{u}_2^{(1)}\Big[a_2(t,x) - b_2(t,x)\underline{u}_2^{(1)} + c_2(t,x)\int_0^\infty \underline{u}_2^{(1)}(t-\tau,x)dM_2^-(\tau) \\
&\qquad - C_2M_{02}^+\Big(\beta_2^{(0)}+\varepsilon\Big) - d_2\Big(\beta_1^{(0)}+\varepsilon\Big) - K_2\varepsilon\Big].
\end{aligned} \tag{2.11}$$

By the comparison principle, we get, for $t > t_1$, $x \in \Omega$,

$$\underline{u}_1^{(1)} \geq v_1^{(1)}, \qquad \underline{u}_2^{(1)} \geq v_2^{(1)}, \tag{2.12}$$

where $v_1^{(1)}$ and $v_2^{(1)}$ are the solutions of the following problem, respectively:

$$\begin{aligned}
\frac{\partial v_1^{(1)}}{\partial t} &= A\Delta v_1^{(1)} + v_1^{(1)}\Big[a_1(t,x) - b_1(t,x)v_1^{(1)} + c_1(t,x)\int_0^\infty v_1^{(1)}(t-\tau,x)dM_1^-(\tau) \\
&\qquad - C_1M_{01}^+\Big(\beta_1^{(0)}+\varepsilon\Big) - d_1\Big(\beta_2^{(0)}+\varepsilon\Big) - K_1\varepsilon\Big], \\
&\frac{\partial v_1^{(1)}}{\partial \eta} = 0, \quad t > t_1,\ x \in \partial\Omega, \\
&v_1^{(1)}(t,x) = \frac{1}{2}u_1(t,x), \quad (t,x) \in (-\infty, t_1] \times \overline{\Omega}, \\
\frac{\partial v_2^{(1)}}{\partial t} &= A\Delta v_2^{(1)} + v_2^{(1)}\Big[a_2(t,x) - b_2(t,x)v_2^{(1)} + c_2(t,x)\int_0^\infty v_2^{(1)}(t-\tau,x)dM_2^-(\tau) \\
&\qquad - C_2M_{02}^+\Big(\beta_2^{(0)}+\varepsilon\Big) - d_2\Big(\beta_1^{(0)}+\varepsilon\Big) - K_2\varepsilon\Big], \\
&\frac{\partial v_2^{(1)}}{\partial \eta} = 0, \quad t > t_1,\ x \in \partial\Omega, \\
&v_2^{(1)}(t,x) = \frac{1}{2}u_2(t,x), \quad (t,x) \in (-\infty, t_1] \times \overline{\Omega}.
\end{aligned} \tag{2.13}$$

Using the three initial conditions and ε sufficiently small, we have

$$\begin{aligned}
a_1 - C_1M_{01}^+\Big(\beta_1^{(0)}+\varepsilon\Big) - d_1\Big(\beta_2^{(0)}+\varepsilon\Big) - K_1\varepsilon > 0, \\
a_2 - C_2M_{02}^+\Big(\beta_2^{(0)}+\varepsilon\Big) - d_2\Big(\beta_1^{(0)}+\varepsilon\Big) - K_2\varepsilon > 0.
\end{aligned} \tag{2.14}$$

By Lemma 1.4, we get

$$
\begin{aligned}
0 &< \frac{a_1 - C_1 M_{01}^{+}\beta_1^{(0)} - d_1\beta_2^{(0)}}{B_1 - c_1 M_{01}^{-}} - \varepsilon\frac{C_1 M_{01}^{+} + d_1 + K_1}{B_1 - c_1 M_{01}^{-}} \le \liminf_{t\to\infty}\ \min_{x\in\overline{\Omega}} v_1^{(1)}(t,x),\\
0 &< \frac{a_2 - C_2 M_{02}^{+}\beta_2^{(0)} - d_2\beta_1^{(0)}}{B_2 - c_2 M_{02}^{-}} - \varepsilon\frac{C_2 M_{02}^{+} + d_2 + K_2}{B_2 - c_2 M_{02}^{-}} \le \liminf_{t\to\infty}\ \min_{x\in\overline{\Omega}} v_2^{(1)}(t,x).
\end{aligned}
\tag{2.15}
$$

Then from (2.12), (2.14), and ε sufficiently small, we can conclude that

$$
\begin{aligned}
0 < \alpha_1^{(0)} &\le \liminf_{t\to\infty}\ \min_{x\in\overline{\Omega}} u_1(t,x) \le \limsup_{t\to\infty}\ \max_{x\in\overline{\Omega}} u_1(t,x) \le \beta_1^{(0)},\\
0 < \alpha_2^{(0)} &\le \liminf_{t\to\infty}\ \min_{x\in\overline{\Omega}} u_2(t,x) \le \limsup_{t\to\infty}\ \max_{x\in\overline{\Omega}} u_2(t,x) \le \beta_2^{(0)},
\end{aligned}
\tag{2.16}
$$

where

$$
\alpha_1^{(0)} = \frac{a_1 - C_1 M_{01}^{+}\beta_1^{(0)} - d_1\beta_2^{(0)}}{B_1 - c_1 M_{01}^{-}}, \qquad \alpha_2^{(0)} = \frac{a_2 - C_2 M_{02}^{+}\beta_2^{(0)} - d_2\beta_1^{(0)}}{B_2 - c_2 M_{02}^{-}}.
\tag{2.17}
$$

For any sufficiently small $\varepsilon > 0$, there exist $t_2' > t_1$ and $t_2 > t_2'$ such that

$$
\begin{aligned}
&\min_{x\in\overline{\Omega}} \underline{u}_i^{(1)}(t,x) > \alpha_i^{(0)} - \varepsilon, \quad \text{for } t \ge t_2',\\
&\underline{u}_i^{(1)}(t - r_i, x) > \alpha_i^{(0)} + \varepsilon, \quad \text{for } t \ge t_2,\\
&M_{0i}^{+} - \varepsilon < M_i^{+}(t - t_2'), \quad \text{for } t \ge t_2,
\end{aligned}
\tag{2.18}
$$

Define $\overline{u}_1^{(2)}$ and $\overline{u}_2^{(2)}$ by

$$
\begin{aligned}
\frac{\partial \overline{u}_1^{(2)}}{\partial t} &= A\Delta\overline{u}_1^{(2)} + \overline{u}_1^{(2)}\Big[a_1(t,x) - b_1(t,x)\overline{u}_1^{(2)} + c_1(t,x)\int_0^\infty \overline{u}_1^{(2)}(t-\tau,x)\,dM_1^{-}(\tau)\\
&\quad - c_1(t,x)\int_0^\infty \underline{u}_1^{(1)}(t-\tau,x)\,dM_1^{+}(\tau) - d_1\underline{u}_2^{(1)}(t-r_2,x)\Big], \quad t > t_2,\ x\in\Omega,\\
\frac{\partial \overline{u}_2^{(2)}}{\partial t} &= A\Delta\overline{u}_2^{(2)} + \overline{u}_2^{(2)}\Big[a_2(t,x) - b_2(t,x)\overline{u}_2^{(2)} + c_2(t,x)\int_0^\infty \overline{u}_2^{(2)}(t-\tau,x)\,dM_2^{-}(\tau)\\
&\quad - c_2(t,x)\int_0^\infty \underline{u}_2^{(1)}(t-\tau,x)\,dM_2^{+}(\tau) - d_2\underline{u}_1^{(1)}(t-r_1,x)\Big], \quad t > t_2,\ x\in\Omega,\\
&\frac{\partial \overline{u}_1^{(2)}}{\partial \eta} = \frac{\partial \overline{u}_2^{(2)}}{\partial \eta} = 0, \quad t > t_2,\ x\in\partial\Omega,\\
&\overline{u}_i^{(2)}(t,x) = K_i, \quad (t,x)\in(-\infty,t_2]\times\overline{\Omega}\ (i=1,2).
\end{aligned}
\tag{2.19}
$$

Then $(\underline{u}_1^{(1)}, \underline{u}_2^{(1)})$ and $(\overline{u}_1^{(2)}, \overline{u}_2^{(2)})$ are a pair of lower-upper solutions of (1.1). By Lemma 1.7,

$$\underline{u}_1^{(1)} \le u_1 \le \overline{u}_1^{(2)}, \qquad \underline{u}_2^{(1)} \le u_2 \le \overline{u}_2^{(2)}. \tag{2.20}$$

From (2.19) and (2.20), for $t > t_2$, $x \in \Omega$, we get

$$\begin{aligned}\frac{\partial \overline{u}_1^{(2)}}{\partial t} &\le A\Delta \overline{u}_1^{(2)} + \overline{u}_1^{(2)}\Big[a_1(t,x) - b_1(t,x)\overline{u}_1^{(2)} + c_1(t,x)\int_0^\infty \overline{u}_1^{(2)}(t-\tau,x)dM_1^-(\tau) \\ &\qquad - c_1 M_{01}^+\Big(\alpha_1^{(0)} - \varepsilon\Big) - d_1\Big(\alpha_2^{(0)} - \varepsilon\Big) + \varepsilon c_1 \alpha_1^{(0)}\Big], \\ \frac{\partial \overline{u}_2^{(2)}}{\partial t} &\le A\Delta \overline{u}_2^{(2)} + \overline{u}_2^{(2)}\Big[a_2(t,x) - b_2(t,x)\overline{u}_2^{(2)} + c_2(t,x)\int_0^\infty \overline{u}_2^{(2)}(t-\tau,x)dM_2^-(\tau) \\ &\qquad - c_2 M_{02}^+\Big(\alpha_2^{(0)} - \varepsilon\Big) - d_2\Big(\alpha_1^{(0)} - \varepsilon\Big) + \varepsilon c_2 \alpha_2^{(0)}\Big].\end{aligned} \tag{2.21}$$

By the comparison principle, for $t > t_2,\ x \in \Omega$, we get

$$\overline{u}_1^{(2)} \le w_1^{(1)}, \qquad \overline{u}_2^{(2)} \le w_2^{(1)}, \tag{2.22}$$

where $w_1^{(1)}$ and $w_2^{(1)}$ are the solutions of the following problem, respectively:

$$\begin{gathered}\frac{\partial w_1^{(1)}}{\partial t} = A\Delta w_1^{(1)} + w_1^{(1)}\Big[a_1(t,x) - b_1(t,x)w_1^{(1)} + c_1(t,x)\int_0^\infty w_1^{(1)}(t-\tau,x)dM_1^-(\tau) \\ - c_1 M_{01}^+\Big(\alpha_1^{(0)} - \varepsilon\Big) - d_1\Big(\alpha_2^{(0)} - \varepsilon\Big) + \varepsilon c_1 \alpha_1^{(0)}\Big], \quad t > t_2,\ x \in \Omega, \\ \frac{\partial w_1^{(1)}}{\partial \eta} = 0, \quad t > t_2,\ x \in \partial\Omega, \\ w_1^{(1)}(t,x) = K_1, \quad (t,x) \in \big(-\infty, t_2\big] \times \overline{\Omega}, \\ \frac{\partial w_2^{(1)}}{\partial t} = A\Delta w_2^{(1)} + w_2^{(1)}\Big[a_2(t,x) - b_2(t,x)w_2^{(1)} + c_2(t,x)\int_0^\infty w_2^{(1)}(t-\tau,x)dM_2^-(\tau) \\ - c_2 M_{02}^+\Big(\alpha_2^{(0)} - \varepsilon\Big) - d_2\Big(\alpha_1^{(0)} - \varepsilon\Big) + \varepsilon c_2 \alpha_2^{(0)}\Big], \quad t > t_2,\ x \in \Omega, \\ \frac{\partial w_2^{(1)}}{\partial \eta} = 0, \quad t > t_2,\ x \in \partial\Omega, \\ w_2^{(1)}(t,x) = K_2, \quad (t,x) \in \big(-\infty, t_2\big] \times \overline{\Omega}.\end{gathered} \tag{2.23}$$

From (2.16), (2.18), and sufficiently small ε, we get

$$\begin{gathered}a_1 - c_1 M_{01}^+\Big(\alpha_1^{(0)} - \varepsilon\Big) - d_1\Big(\alpha_2^{(0)} - \varepsilon\Big) > 0, \\ a_2 - c_2 M_{02}^+\Big(\alpha_2^{(0)} - \varepsilon) - d_2\Big(\alpha_1^{(0)} - \varepsilon\Big) > 0.\end{gathered} \tag{2.24}$$

By Lemma 1.4, we get

$$0<\limsup_{t\to\infty}\max_{x\in\overline{\Omega}} w_1^{(1)}(t,x)\le \frac{A_1-c_1M_{01}^{+}\alpha_1^{(0)}-d_1\alpha_2^{(0)}}{b_1-c_1M_{01}^{-}}-\varepsilon\frac{c_1M_{01}^{+}+d_1-c_1\alpha_1^{(0)}}{b_1-c_1M_{01}^{-}},$$
$$0<\limsup_{t\to\infty}\max_{x\in\overline{\Omega}} w_2^{(1)}(t,x)\le \frac{A_2-c_2M_{02}^{+}\alpha_2^{(0)}-d_2\alpha_1^{(0)}}{b_2-c_2M_{02}^{-}}-\varepsilon\frac{c_2M_{02}^{+}+d_2-c_2\alpha_2^{(0)}}{b_2-c_2M_{02}^{-}}. \tag{2.25}$$

Then from (2.22), (2.24), and sufficiently small ε,

$$0<\alpha_1^{(0)}\le\liminf_{t\to\infty}\min_{x\in\overline{\Omega}} u_1(t,x)\le\limsup_{t\to\infty}\max_{x\in\overline{\Omega}} u_1(t,x)\le\beta_1^{(1)},$$
$$0<\alpha_2^{(0)}\le\liminf_{t\to\infty}\min_{x\in\overline{\Omega}} u_2(t,x)\le\limsup_{t\to\infty}\max_{x\in\overline{\Omega}} u_2(t,x)\le\beta_2^{(1)}, \tag{2.26}$$

where

$$\beta_1^{(1)}=\frac{A_1-c_1M_{01}^{+}\alpha_1^{(0)}-d_1\alpha_2^{(0)}}{b_1-c_1M_{01}^{-}},\qquad \beta_2^{(1)}=\frac{A_2-c_2M_{02}^{+}\alpha_2^{(0)}-d_2\alpha_1^{(0)}}{b_2-c_2M_{02}^{-}}. \tag{2.27}$$

Then,

$$0<\alpha_1^{(0)}\le\beta_1^{(1)}\le\beta_1^{(0)},$$
$$0<\alpha_2^{(0)}\le\beta_2^{(1)}\le\beta_2^{(0)}. \tag{2.28}$$

For any sufficiently small $\varepsilon>0$, there exist $t_3'>t_2$ and $t_3>t_3'$ such that

$$\max_{x\in\overline{\Omega}}\overline{u}_i^{(2)}(t,x)<\beta_i^{(1)}+\varepsilon,\quad \text{for } t\ge t_3',$$
$$\overline{u}_i^{(1)}(t-r_i,x)<\beta_i^{(1)}+\varepsilon,\quad \text{for } t\ge t_3,$$
$$C_i\left(M_{0i}^{+}-M_i^{+}(t-t_3')\right)<\varepsilon,\quad \text{for } t\ge t_3. \tag{2.29}$$

Define $\underline{u}_1^{(2)}(t,x)$ and $\underline{u}_2^{(2)}(t,x)$ by

$$\frac{\partial\underline{u}_1^{(2)}}{\partial t}=A\Delta\underline{u}_1^{(2)}+\underline{u}_1^{(2)}\Big[a_1(t,x)-b_1(t,x)\underline{u}_1^{(2)}+c_1(t,x)\int_0^\infty\underline{u}_1^{(2)}(t-\tau,x)dM_1^{-}(\tau)$$
$$-c_1(t,x)\int_0^\infty\overline{u}_1^{(2)}(t-\tau,x)dM_1^{+}(\tau)-d_1\overline{u}_2^{(2)}(t-r_2,x)\Big],\quad t>t_3,\ x\in\Omega,$$
$$\frac{\partial\underline{u}_2^{(2)}}{\partial t}=A\Delta\underline{u}_2^{(2)}+\underline{u}_2^{(2)}\Big[a_2(t,x)-b_2(t,x)\underline{u}_2^{(2)}+c_2(t,x)\int_0^\infty\underline{u}_2^{(2)}(t-\tau,x)dM_2^{-}(\tau)$$
$$-c_2(t,x)\int_0^\infty\overline{u}_2^{(2)}(t-\tau,x)dM_2^{+}(\tau)-d_2\overline{u}_1^{(2)}(t-r_1,x)\Big],\quad t>t_3,\ x\in\Omega,$$
$$\frac{\partial\underline{u}_1^{(2)}}{\partial\eta}=\frac{\partial\underline{u}_2^{(2)}}{\partial\eta}=0,\quad t>t_3,\ x\in\partial\Omega,$$
$$\underline{u}_i^{(2)}(t,x)=\frac{1}{2}u_i(t,x),\quad (t,x)\in\left(-\infty,t_3\right]\times\overline{\Omega}. \tag{2.30}$$

Then, $(\underline{u}_1^{(2)}, \underline{u}_2^{(2)})$ and $(\overline{u}_1^{(2)}, \overline{u}_2^{(2)})$ are a pair of lower-upper solutions of (1.1). By Lemma 1.7,

$$\underline{u}_1^{(2)} \le u_1 \le \overline{u}_1^{(2)}, \qquad \underline{u}_2^{(2)} \le u_2 \le \overline{u}_2^{(2)}. \tag{2.31}$$

From (2.30) and (2.31), for $t > t_3$, $x \in \Omega$, we get

$$\begin{aligned}
\frac{\partial \underline{u}_1^{(2)}}{\partial t} \ge{}& A\Delta \underline{u}_1^{(2)} + \underline{u}_1^{(2)} \bigg[a_1(t,x) - b_1(t,x)\underline{u}_1^{(2)} + c_1(t,x)\int_0^\infty \underline{u}_1^{(2)}(t-\tau,x)\,dM_1^-(\tau) \\
&\qquad - C_1 M_{01}^+\Big(\beta_1^{(1)} + \varepsilon\Big) - d_1\Big(\beta_2^{(1)} + \varepsilon\Big) - K_1\varepsilon\bigg], \\
\frac{\partial \underline{u}_2^{(2)}}{\partial t} \ge{}& A\Delta \underline{u}_2^{(2)} + \underline{u}_2^{(2)} \bigg[a_2(t,x) - b_2(t,x)\underline{u}_2^{(2)} + c_2(t,x)\int_0^\infty \underline{u}_2^{(2)}(t-\tau,x)\,dM_2^-(\tau) \\
&\qquad - C_2 M_{02}^+\Big(\beta_2^{(1)} + \varepsilon\Big) - d_2\Big(\beta_1^{(1)} + \varepsilon\Big) - K_2\varepsilon\bigg].
\end{aligned} \tag{2.32}$$

By the comparison principle, for $t > t_3$, $x \in \Omega$, we get

$$\underline{u}_1^{(2)} \ge v_1^{(2)}, \qquad \underline{u}_2^{(2)} \ge v_2^{(2)}, \tag{2.33}$$

where $v_1^{(2)}$ and $v_2^{(2)}$ are the solutions of the following problem, respectively:

$$\begin{aligned}
\frac{\partial v_1^{(2)}}{\partial t} ={}& A\Delta v_1^{(2)} + v_1^{(2)} \bigg[a_1(t,x) - b_1(t,x)v_1^{(2)} + c_1(t,x)\int_0^\infty v_1^{(2)}(t-\tau,x)\,dM_1^-(\tau) \\
&\qquad - C_1 M_{01}^+\Big(\beta_1^{(1)} + \varepsilon\Big) - d_1\Big(\beta_2^{(1)} + \varepsilon\Big) - K_1\varepsilon\bigg], \quad t > t_3,\ x \in \Omega, \\
&\frac{\partial v_1^{(2)}}{\partial \eta} = 0, \quad t > t_3,\ x \in \partial\Omega, \\
&v_1^{(2)}(t,x) = \frac{1}{2}u_1(t,x), \quad (t,x) \in \big(-\infty, t_3\big] \times \overline{\Omega}, \\
\frac{\partial v_2^{(2)}}{\partial t} ={}& A\Delta v_2^{(2)} + v_2^{(2)} \bigg[a_2(t,x) - b_2(t,x)v_2^{(2)} + c_2(t,x)\int_0^\infty v_2^{(2)}(t-\tau,x)\,dM_2^-(\tau) \\
&\qquad - C_2 M_{02}^+\Big(\beta_2^{(1)} + \varepsilon\Big) - d_2\Big(\beta_1^{(1)} + \varepsilon\Big) - K_2\varepsilon\bigg], \quad t > t_3,\ x \in \Omega, \\
&\frac{\partial v_2^{(2)}}{\partial \eta} = 0, \quad t > t_3,\ x \in \partial\Omega, \\
&v_2^{(2)}(t,x) = \frac{1}{2}u_2(t,x), \quad (t,x) \in \big(-\infty, t_3\big] \times \overline{\Omega}.
\end{aligned} \tag{2.34}$$

From (2.29), (2.30), and ε sufficiently small, we get

$$a_1 - C_1 M_{01}^{+}\left(\beta_1^{(1)} + \varepsilon\right) - d_1\left(\beta_2^{(1)} + \varepsilon\right) - K_1\varepsilon > 0,$$
$$a_2 - C_2 M_{02}^{+}\left(\beta_2^{(1)} + \varepsilon\right) - d_2\left(\beta_1^{(1)} + \varepsilon\right) - K_2\varepsilon > 0. \tag{2.35}$$

By Lemma 1.4 we get

$$0 < \frac{a_1 - C_1 M_{01}^{+}\beta_1^{(1)} - d_1\beta_2^{(1)}}{B_1 - c_1 M_{01}^{-}} - \varepsilon\frac{C_1 M_{01}^{+} + d_1 + K_1}{B_1 - c_1 M_{01}^{-}} \le \liminf_{t\to\infty} \min_{x\in\overline{\Omega}} v_1^{(1)}(t,x),$$
$$0 < \frac{a_2 - C_2 M_{02}^{+}\beta_2^{(1)} - d_2\beta_1^{(1)}}{B_2 - c_2 M_{02}^{-}} - \varepsilon\frac{C_2 M_{02}^{+} + d_2 + K_2}{B_2 - c_2 M_{02}^{-}} \le \liminf_{t\to\infty} \min_{x\in\overline{\Omega}} v_2^{(1)}(t,x). \tag{2.36}$$

From (2.33), (2.35), and for ε, we conclude that

$$0 < \alpha_1^{(1)} \le \liminf_{t\to\infty} \min_{x\in\overline{\Omega}} u_1(t,x) \le \limsup_{t\to\infty} \max_{x\in\overline{\Omega}} u_1(t,x) \le \beta_1^{(1)},$$
$$0 < \alpha_2^{(1)} \le \liminf_{t\to\infty} \min_{x\in\overline{\Omega}} u_2(t,x) \le \limsup_{t\to\infty} \max_{x\in\overline{\Omega}} u_2(t,x) \le \beta_2^{(1)}, \tag{2.37}$$

where

$$\alpha_1^{(1)} = \frac{a_1 - C_1 M_{01}^{+}\beta_1^{(1)} - d_1\beta_2^{(1)}}{B_1 - c_1 M_{01}^{-}}, \qquad \alpha_2^{(1)} = \frac{a_2 - C_2 M_{02}^{+}\beta_2^{(1)} - d_2\beta_1^{(1)}}{B_2 - c_2 M_{02}^{-}}. \tag{2.38}$$

Then,

$$0 < \alpha_1^{(0)} \le \alpha_1^{(1)} \le \beta_1^{(1)} \le \beta_1^{(0)},$$
$$0 < \alpha_2^{(0)} \le \alpha_2^{(1)} \le \beta_2^{(1)} \le \beta_2^{(0)}. \tag{2.39}$$

Define the sequences $\alpha_1^{(k)}$, $\alpha_2^{(k)}$, $\beta_1^{(k)}$, and $\beta_2^{(k)}$ as follows:

$$\alpha_1^{(k)} = \frac{a_1 - C_1 M_{01}^{+}\beta_1^{(k)} - d_1\beta_2^{(k)}}{B_1 - c_1 M_{01}^{-}}, \qquad \alpha_2^{(k)} = \frac{a_2 - C_2 M_{02}^{+}\beta_2^{(k)} - d_2\beta_1^{(k)}}{B_2 - c_2 M_{02}^{-}},$$
$$\beta_1^{(k+1)} = \frac{A_1 - c_1 M_{01}^{+}\alpha_1^{(k)} - d_1\alpha_2^{(k)}}{b_1 - c_1 M_{01}^{-}}, \qquad \beta_2^{(k+1)} = \frac{A_2 - c_2 M_{02}^{+}\alpha_2^{(k)} - d_2\alpha_1^{(k)}}{b_2 - c_2 M_{02}^{-}},$$
$$\beta_1^{(0)} = \frac{A_1}{b_1 - c_1 M_{01}^{-}}, \qquad \beta_2^{(0)} = \frac{A_2}{b_2 - c_2 M_{02}^{-}}. \tag{2.40}$$

Lemma 2.1. *For the above-defined sequences, one has*

$$\left[\alpha_1^{(k+1)},\beta_1^{(k+1)}\right] \subseteq \left[\alpha_1^{(k)},\beta_1^{(k)}\right], \quad \left[\alpha_2^{(k+1)},\beta_2^{(k+1)}\right] \subseteq \left[\alpha_2^{(k)},\beta_2^{(k)}\right], \quad k \geq 0. \tag{2.41}$$

Proof. For $k = 0$, it has been shown that $[\alpha_1^{(1)},\beta_1^{(1)}] \subseteq [\alpha_1^{(0)},\beta_1^{(0)}]$ and $[\alpha_2^{(1)},\beta_2^{(1)}] \subseteq [\alpha_2^{(0)},\beta_2^{(1)}]$. Using induction, we can complete the proof.

Lemma 2.1 implies that

$$\lim_{k\to\infty}\alpha_1^{(k)}, \quad \lim_{k\to\infty}\alpha_2^{(k)}, \quad \lim_{k\to\infty}\beta_1^{(k)}, \quad \lim_{k\to\infty}\beta_2^{(k)} \tag{2.42}$$

exist, denoted as α_1, α_2, β_1, and β_2, respectively. From (2.41), we have the following linear system by which we can obtain the numbers α_1, α_2, β_1, and β_2:

$$\begin{aligned}
\alpha_1 &= \frac{a_1 - C_1 M_{01}^{+}\beta_1 - d_1\beta_2}{B_1 - c_1 M_{01}^{-}}, \qquad & \alpha_2 &= \frac{a_2 - C_2 M_{02}^{+}\beta_2 - d_2\beta_1}{B_2 - c_2 M_{02}^{-}}, \\
\beta_1 &= \frac{A_1 - c_1 M_{01}^{+}\alpha_1 - d_1\alpha_2}{b_1 - c_1 M_{01}^{-}}, \qquad & \beta_2 &= \frac{A_2 - c_2 M_{02}^{+}\alpha_2 - d_2\alpha_1}{b_2 - c_2 M_{02}^{-}}.
\end{aligned} \tag{2.43}$$

□

Lemma 2.2. *For the solutions of* (1.1), *one has*

$$0 < \alpha_1^{(k)} \leq \liminf_{t\to\infty} \min_{x\in\overline{\Omega}} u_1(t,x) \leq \limsup_{t\to\infty} \max_{x\in\overline{\Omega}} u_1(t,x) \leq \beta_1^{(k)}, \quad k \geq 0, \tag{2.44}$$

$$0 < \alpha_2^{(k)} \leq \liminf_{t\to\infty} \min_{x\in\overline{\Omega}} u_2(t,x) \leq \limsup_{t\to\infty} \max_{x\in\overline{\Omega}} u_2(t,x) \leq \beta_2^{(k)}, \quad k \geq 0. \tag{2.45}$$

Proof. We have shown that (2.45) and (3.1) are valid for $k = 0, 1$. Using induction and repeating the above process, we can complete the proof. □

Combining the above lemmas, we can complete the proof of Theorem 1.1

Remark 2.3. Following the same kind of proof for Theorem 1.1, it can be shown that the same conclusions hold if instead of system (1.1) we work with solutions of

$$\begin{gathered}
\frac{\partial v_1}{\partial t} = A\Delta v_1(t,x) + v_1(t,x)\left[a_1(t,x) - b_1(t,x)v_1(t,x) - c_1(t,x)\int_0^\infty v_1(t-\tau,x)\,d\mu_1(\tau) - d_1 v_2(t-r_2,x)\right], \\
t > 0, \quad x \in \Omega, \\
\frac{\partial v_2}{\partial t} = B\Delta v_2(t,x) + v_2(t,x)\left[a_2(t,x) - b_2(t,x)v_2(t,x) - c_2(t,x)\int_0^\infty v_2(t-\tau,x)\,d\mu_2(\tau) - d_2 v_1(t-r_1,x)\right], \\
t > 0, \quad x \in \Omega, \\
\frac{\partial v_1}{\partial \eta} = \frac{\partial v_2}{\partial \eta} = 0, \quad t > 0,\ x \in \partial\Omega, \\
v_i(t,x) = \phi_i(t,x) \quad (i = 1,2)\ t \leq 0,\ x \in \overline{\Omega},
\end{gathered} \tag{2.46}$$

as expected since they do not include the coefficients of diffusion. We thank one of the referees for his comments regarding this matter.

3. Numerical simulations

In this section, we present some numerical results that agree with Theorem 1.1 proved above. We used the method of upper and lower solutions as developed by Pao [12, 13] discretizing the systems into finite difference systems. On both examples, the domain used is $\Omega = (0, 12)$.

Example 3.1. In this example, we work with coefficients that depend on t and initial values that depend on (t, x).

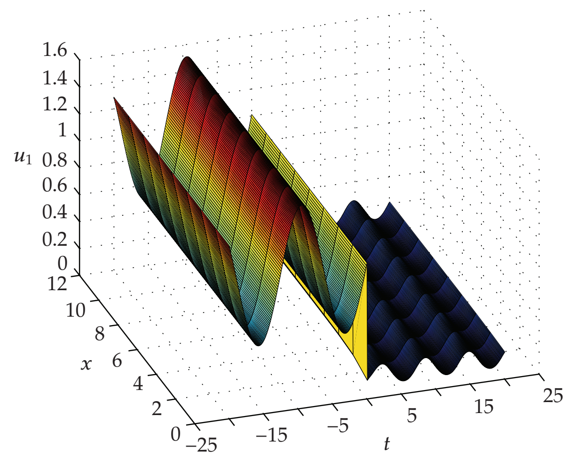
Consider the system

$$\begin{aligned}
\frac{\partial u_1}{\partial t} &= \frac{\partial^2 u_1}{\partial x^2} + u_1\Bigg[\left(\sin(t)+2\right) - \left(\frac{1}{1+x^2}+10\right)u_1 \\
&\quad - \left(\cos(t)+1.01\right)\int_0^\infty \frac{1}{1+\tau^2}u_1(t-\tau,x)\,d\tau - \frac{1}{10}u_2(t-2,x)\Bigg], \quad t\in[0,T],\ x\in[0,12], \\
\frac{\partial u_2}{\partial t} &= \frac{\partial^2 u_2}{\partial x^2} + u_2\Bigg[\left(\sqrt{\sin^2(t)}+1\right) - \left(\frac{x}{1+x^2}+7.5\right)u_2 \\
&\quad - \left(\sin\left(t^2\right)+1.01\right)\int_0^\infty \frac{1}{1+\tau^2}u_2(t-\tau,x)\,d\tau - \frac{1}{2}u_1(t-1.5,x)\Bigg], \quad t\in[0,T],\ x\in[0,12], \\
&\frac{\partial u_i}{\partial x}(0,t) = \frac{\partial u_i}{\partial x}(12,t) = 0 \quad (i=1,2),\ t>0, \\
u_1(t,x) &= \frac{1}{2}\sin\left(\frac{1}{2}t\right) + 1.01, \quad t\le 0,\ x\in[0,12], \\
u_2(t,x) &= \frac{1}{2}\left(\sin\left(\frac{1}{2}t\right) + \sin\left(\frac{1}{2}x\right)\right) + 1.01, \quad t\le 0,\ x\in[0,12].
\end{aligned} \tag{3.1}$$

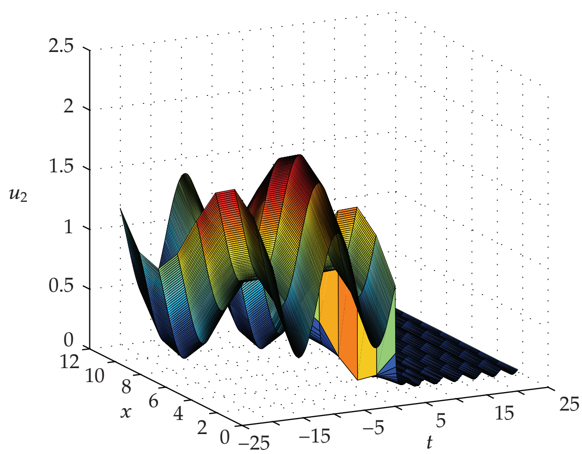
It is easy to see that this system satisfies the conditions of the main theorem (Theorem 1.1), therefore the global attractors are defined by the solutions of the linear system

$$\begin{aligned}
\alpha_1(11-0) &= 1 - 2.01\frac{\pi}{2}\beta_1 - \frac{1}{10}\beta_2, \\
\alpha_2(1.5-0) &= 1 - 2.01\frac{\pi}{2}\beta_2 - \beta_1, \\
\beta_1(10-0) &= 3 - 0.01\frac{\pi}{2}\alpha_1 - \frac{1}{10}\alpha_2, \\
\beta_2(0.5-0) &= 2 - 0.01\frac{\pi}{2}\alpha_2 - \alpha_1,
\end{aligned} \tag{3.2}$$

that is, the global attractors for u_1 and u_2 are given by $[0.002245, 0.299884]$ and $[0.011192, 0.285680]$, respectively. Figure 1 shows the numerical simulation of the solution of this system and on it we can notice the global attractors obtained before.



(a)

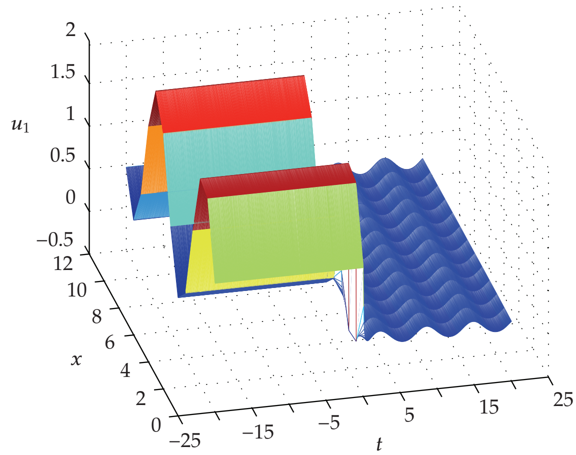


(b)

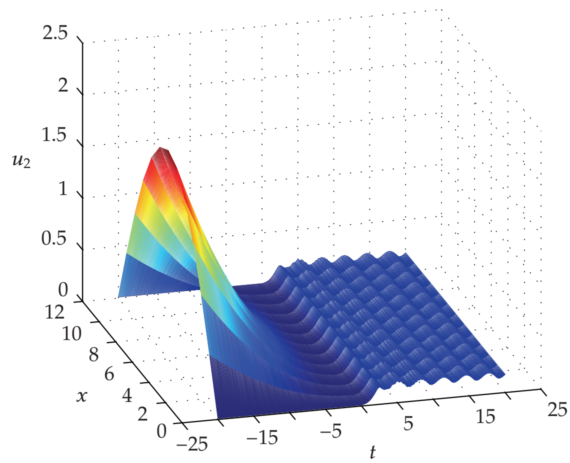
Figure 1: Numerical simulation of Example 3.1.

Example 3.2. According to the main theorem (Theorem 1.1), the global attractors depend only on the coefficients and not on the initial values. In this example, we only change the initial functions of the above example:

$$\begin{gathered} u_1(t,x) = \sin(x) + 1.01, \quad t \le 0,\ x \in [0,12], \\ u_2(t,x) = \frac{1}{200}\left(1+t^2\right)\sin\left(\frac{\pi x}{12}\right) + 0.01, \quad t \le 0,\ x \in [0,12], \end{gathered} \tag{3.3}$$



(a)



(b)

Figure 2: Numerical simulation of Example 3.2.

to obtain the same attractors as in Example 3.1. Figure 2 shows the numerical simulation of the solution of this system and we observe that they have the attractors expected.

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