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Research Article

Oscillatority Conditions for Nonlinear Systems with Delay

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Sufficient conditions for oscillatority in the sense of Yakubovich for a class of time delay nonlinear systems are proposed. Under proposed conditions, upper and lower bounds for oscillation amplitude are given. Examples illustrating analytical results by computer simulation are presented.

1. Introduction

Most works on analysis or synthesis of nonlinear systems are devoted to studying stability-like behavior. Their typical results show that the motions of a system are close to a certain limit motion (limit mode) either existing in the system or created by a controller. Evaluation of the system trajectory deflection from a limit mode provides quantitative information about system behavior [1, 2].

During recent years, an interest in studying more complex behavior of the systems related to oscillatory and chaotic modes has grown significantly [3–7]. An important and useful concept for studying irregular oscillations is that of “oscillatority” introduced by Yakubovich [8]. Frequency domain conditions for oscillatority were obtained for Lurie systems, composed of linear and nonlinear parts [6, 8, 9]. However, when studying physical systems in many cases, it is more natural to decompose the system description into two nonlinear parts. Oscillation analysis and design methods for nonlinear systems without delays were proposed in [3]. The results of [3] are formulated in terms of two Lyapunov functions.

In this paper, the conditions of oscillatority proposed in [3] are extended to nonlinear systems with time delay. The sufficient conditions of periodic solutions existence for time delay systems can be found in [10, 11]. Section 2 contains some useful auxiliary statements and definitions. Main definitions and oscillations existence conditions are presented in Section 3. Section 4 deals with examples of analytical calculations and computer simulations for the proposed solutions.

2. Preliminaries

As usual, continuous function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ ($\mathbb{R}_+ = \{\tau \in \mathbb{R} : \tau \geq 0\}$) is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\sigma(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if it is also radially unbounded.

We will denote by $C^n[a,b]$, $0 \leq a < b \leq +\infty$, the Banach space of continuous functions $\varphi : [a,b] \to \mathbb{R}^n$ with the norm $\|\varphi\| = \sup_{a\leq\varsigma\leq b} |\varphi(\varsigma)|$, where $|\cdot|$ is the standard Euclidean norm. It is said that property $P(\varphi)$, $\varphi \in \mathcal{A} \subset C^n[-\tau,0]$, holds for almost all $\varphi \in \mathcal{A}$ if it holds for all $\varphi \in \mathcal{A}$ with the exception of some countable subset of $\mathcal{A}$. The set of all Lebesgue measurable functions $\mathbf{u} : \mathbb{R}_+ \to \mathbb{R}^m$ with property $\|\mathbf{u}\| < +\infty$, where

$$\|\mathbf{u}\| = \|\mathbf{u}\|_{[0+\infty)}, \qquad \|\mathbf{u}\|_{[t_0,T)} = \operatorname{ess\,sup}\{\,|\mathbf{u}(t)|,\ t \in [t_0,T)\}, \tag{2.1}$$

is denoted by $\mathcal{M}_{\mathbb{R}^m}$.

Let the model of a system be described by functional differential equation

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}(t,\mathbf{x}_\tau(t),\mathbf{u}(t)), \quad t \geq 0, \tag{2.2}$$

where $\mathbf{x} \in \mathbb{R}^n$ is state vector, $\mathbf{x}_\tau(t) = \mathbf{x}(t+s)$, $-\tau \leq s \leq 0 \in C^n[-\tau,0]$ is extended (lifted) state vector; $\mathbf{u} \in \mathcal{M}_{\mathbb{R}^m}$ is input vector function; $\mathbf{f} : \mathbb{R}_+ \times C^n[-\tau,0] \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous with respect to the first argument and locally Lipschitz continuous function with respect to the rest ones, $\mathbf{f}(\cdot,0,0) = 0$. We will assume that all solutions of the system satisfy initial conditions

$$\mathbf{x}_\tau(0) = \mathbf{x}_0 \in C^n[-\tau,0]. \tag{2.3}$$

It is known from the theory of functional differential equations [10–12] that under above assumptions the system (2.2) has a unique solution $\mathbf{x}(\mathbf{x}_0,\mathbf{u},t)$ satisfying initial condition $\mathbf{x}_0$, which is defined on some finite interval $[0,T)$. If $T = +\infty$ for every initial state $\mathbf{x}_0$ and all $\mathbf{u} \in \mathcal{M}_{\mathbb{R}^m}$, then system is called *forward complete*. It is said that set $\mathcal{R} \subset C^n[-\tau,0]$ is a zero input *repulsion* set for the system (2.2) if for almost all initial conditions $\mathbf{x}_0 \in \mathbb{R}$ trajectories $\mathbf{x}(\mathbf{x}_0,0,t)$ of the system are well defined for all $t \geq 0$ and there exists $T_{\mathbf{x}_0} > 0$ such that $\mathbf{x}_\tau(t) \notin \mathcal{R}$ for $t \geq T_{\mathbf{x}_0}$. Finally, let $V : \mathbb{R}_+ \times C^n[-\tau,0] \to \mathbb{R}_+$ be a locally Lipschitz continuous functional, then the time derivative of the functional V along solutions of the system (2.2) is defined as follows ($V(t) = V(t,\mathbf{x}(\mathbf{x}_0,\mathbf{u},t))$):

$$\dot{V}(t) = \limsup_{\Delta t \to 0^+} \frac{1}{\Delta t}[V(t+\Delta t) - V(t)]. \tag{2.4}$$

LEMMA 2.1 [12]. *Let there exist locally Lipschitz continuous functional* $V : \mathbb{R}_+ \times C^n[-\tau, 0] \to \mathbb{R}_+$ *and functions* $\alpha_i \in \mathcal{K}_\infty$, $i = \overline{1,4}$, *such that*

$$\alpha_1(|\mathbf{x}(t)|) \le V(t, \mathbf{x}_\tau(t)) \le \alpha_2(|\mathbf{x}(t)|) + \alpha_3\left(\int_{t-\tau}^{t} \alpha_4(|\mathbf{x}(\varsigma)|)\, d\varsigma\right),$$
$$\dot{V}(t) \le -\alpha_4(|\mathbf{x}(t)|) + M \tag{2.5}$$

for all $t \ge 0$ *and* $\mathbf{x}_0 \in C^n[-\tau, 0]$, $M > 0$. *Then all solutions* $\mathbf{x}(\mathbf{x}_0, 0, t)$ *of system (2.2) are uniformly bounded, and*

$$|\mathbf{x}(\mathbf{x}_0, 0, t)| \le \alpha_1^{-1}(\alpha_2(B) + \alpha_3(\tau\alpha_4(B))), \quad t \ge 0, \tag{2.6}$$

where $B = \max\{|\mathbf{x}_0|, \alpha_4^{-1}(M)\}$. *Besides, they are uniformly asymptotically bounded*

$$\lim_{t \to +\infty} |\mathbf{x}(\mathbf{x}_0, 0, t)| \le R, \tag{2.7}$$

where $R = \alpha_1^{-1}(\alpha_2(\alpha_4^{-1}(M)) + \alpha_3(\tau M))$.

LEMMA 2.2 [19]. *Let* $\mathbf{f}(t, \mathbf{x}_\tau, 0) = \mathbf{F}(t, \mathbf{x}, \mathbf{x}(t-\tau))$ *and there exist continuously differentiable Lyapunov function* $V : \mathbb{R}^n \to \mathbb{R}_+$, *continuous function* $\boldsymbol{\varepsilon} : \mathbb{R}^{2n} \to (0,1)$, *functions* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\chi \in \mathcal{K}$, *and* $X \in \mathbb{R}_+$ *such that the following inequalities hold:*

$$\alpha_1(|\mathbf{x}|) \le V(\mathbf{x}) \le \alpha_2(|\mathbf{x}|),$$
$$\frac{\partial V(\mathbf{x})\mathbf{F}(t, \mathbf{x}, \mathbf{x}(t-\tau))}{\partial(\mathbf{x})} \le -\chi(V(\mathbf{x})) + \boldsymbol{\varepsilon}(\mathbf{x}, \mathbf{x}(t-\tau))\chi[V(\mathbf{x}(t-\tau))] \tag{2.8}$$

for all $t \ge 0$, $\mathbf{x} \in \mathbb{R}^n$, $|\mathbf{x}| \ge X$, *and* $\mathbf{x}(t-\tau) \in \mathbb{R}^n$, $|\mathbf{x}(t-\tau)| \ge X$. *Then all solutions* $\mathbf{x}(t, \mathbf{x}_0, 0)$ *of the system (2.2) with* $\mathbf{f}(t, \mathbf{x}_\tau, 0) = \mathbf{F}(t, \mathbf{x}, \mathbf{x}(t-\tau))$ *are uniformly bounded:*

$$|\mathbf{x}(t, \mathbf{x}_0, 0)| \le R, \quad R = \alpha_1^{-1} \circ \alpha_2(\max\{||\mathbf{x}_0||, X\}), \ t \ge 0. \tag{2.9}$$

In contrast to Lemma 2.1, Lemma 2.2 provides delay-independent conditions for boundedness of trajectories of the system (2.2).

3. Oscillatority conditions

At first, extending the result of [3] we give a precise definition of the term "oscillatority" for time delay systems.

Definition 3.1. Solution $\mathbf{x}(\mathbf{x}_0, 0, t)$ with $\mathbf{x}_0 \in C^n[-\tau, 0]$ of system (2.2) is called $[\pi^-, \pi^+]$-oscillation with respect to output $\psi = \eta(\mathbf{x})$ (where $\eta : \mathbb{R}^n \to \mathbb{R}$ is a continuous monotonous function with respect to all arguments) if the solution is well defined for all $t \ge 0$ and

$$\underline{\lim}_{t \to +\infty} \psi(t) = \pi^-, \quad \overline{\lim}_{t \to +\infty} \psi(t) = \pi^+, \quad -\infty < \pi^- < \pi^+ < +\infty. \tag{3.1}$$

Solution $\mathbf{x}(\mathbf{x}_0,0,t)$ is called *oscillating* if there exist some output ψ and constants π^-, π^+ such that $\mathbf{x}(\mathbf{x}_0,0,t)$ is $[\pi^-,\pi^+]$-oscillation with respect to the output ψ. System (2.2) with $\mathbf{u}(t)\equiv 0$, $t\geq 0$, is called *oscillatory* if for almost all $\mathbf{x}_0\in C^n[-\tau,0]$ the solutions of the system $\mathbf{x}(\mathbf{x}_0,0,t)$ are oscillating. Oscillatory system (2.2) is called *uniformly oscillatory* if for almost all $\mathbf{x}_0\in C^n[-\tau,0]$ for corresponding solutions $\mathbf{x}(\mathbf{x}_0,0,t)$ there exist common output ψ and constants π^-, π^+ not depending on initial conditions.

Note that term "almost all solutions" is used to emphasize that generally system (2.2) has a nonempty set of equilibrium points, thus, there exists a set of initial conditions, that corresponding solutions are not oscillating.

The oscillatority property introduced in Definition 3.1 is defined for zero input and any initial conditions of system (2.2). The following property is a closely related characterization of the system behavior, extending the proposed-above property to the case of nonzero input and specific initial conditions [3, 5].

Definition 3.2. Let $\mathbf{u}\in\mathcal{M}_{\mathbb{R}^m}$ and $\mathbf{x}_0\in C^n[-\tau,0]$ be given such that $\mathbf{x}(\mathbf{x}_0,\mathbf{u},t)$ is well defined for all $t\geq 0$. The functions $\chi^-_{\psi,\mathbf{x}_0}(\gamma)$, $\chi^+_{\psi,\mathbf{x}_0}(\gamma)$ defined for $0\leq\gamma<+\infty$ are called lower and upper excitability indices of system (2.2) at $\mathbf{x}_0$ with respect to output $\psi=\eta(\mathbf{x})$ (where $\eta:\mathbb{R}^n\to\mathbb{R}$ is a continuous monotonous function with respect to all arguments), if

$$\left(\chi^-_{\psi,\mathbf{x}_0}(\gamma),\chi^+_{\psi,\mathbf{x}_0}(\gamma)\right)=\underset{(a,b)\in\mathscr{E}(\gamma)}{\arg\sup}\,\{b-a\},$$
$$\mathscr{E}(\gamma)=\left\{(a,b):\begin{pmatrix}a=\underline{\lim}_{t\to+\infty}\,\eta(\mathbf{x}(\mathbf{x}_0,\mathbf{u},t)),\\ b=\overline{\lim}_{t\to+\infty}\,\eta(\mathbf{x}(\mathbf{x}_0,\mathbf{u},t))\end{pmatrix}\right\}_{\|\mathbf{u}\|\leq\gamma}. \tag{3.2}$$

Lower and upper excitation indices of forward complete system (2.2) with respect to output ψ are

$$\chi^-_{\psi}(\gamma)=\inf_{\mathbf{x}_0\in C^n[-\tau,0]}\chi^-_{\psi,\mathbf{x}_0}(\gamma),\qquad \chi^+_{\psi}(\gamma)=\sup_{\mathbf{x}_0\in C^n[-\tau,0]}\chi^+_{\psi,\mathbf{x}_0}(\gamma). \tag{3.3}$$

Excitation indices characterize ability of the system (2.2) to perform forced or controllable oscillations caused by bounded inputs. It is clear that properties $\pi^-=\chi^-_{\psi}(0)$ and $\pi^+=\chi^+_{\psi}(0)$ are satisfied. For nonzero inputs the indices characterize maximum (over specified set of inputs $\|\mathbf{u}\|\leq\gamma$) asymptotic amplitudes $\chi^+_{\psi}(\gamma)-\chi^-_{\psi}(\gamma)$ of the output ψ.

Sufficient conditions for oscillatority of system (2.2) are formulated in the following theorem.

Theorem 3.3. *Let system (2.2) have Lyapunov functional $V:\mathbb{R}_+\times C^n[-\tau,0]\to\mathbb{R}_+$ satisfying conditions of Lemma 2.1 or Lyapunov function $V:\mathbb{R}^n\to\mathbb{R}_+$ satisfying conditions of Lemma 2.2. Let the origin of the system be locally unstable with repulsion set $\mathscr{R}=\{\varphi\in C^n[-\tau,0]:0<|\varphi(0)|\leq r\}$, $0<r<R$. Then system (2.2) is oscillatory, provided that set $\Omega=\{\mathbf{x}:r\leq|\mathbf{x}|\leq R\}\subset\mathbb{R}^n$ does not contain equilibrium points of system (2.2) for $\mathbf{u}(t)\equiv 0$, $t\geq 0$.*

Proof. By Lemmas 2.1 or 2.2 all solutions of system (2.2) with $\mathbf{u}(t) \equiv 0$, $t \geq 0$, are bounded and asymptotically converge to region where $|\mathbf{x}| \leq R$. Since there exists repulsive set $\mathcal{R}$ containing the origin, the set Ω is the global attractor for the system (2.2).

As it was supposed, Ω does not contain equilibrium points of the system. Then for almost all $\mathbf{x}_0 \in C^n[-\tau,0]$ there exists a number i, $1 \leq i \leq n$, such that the solution is $[\pi^-,\pi^+]$-oscillation with respect to output $|x_i|$ with $0 \leq \pi^- < \pi^+ < R$. Suppose that there is no such an output. It means that for all $1 \leq i \leq n$ and for every output $|x_i|$ the equality $\pi^- = \pi^+$ holds. However, the latter can be true only at equilibrium points, which are excluded from the set Ω by the theorem conditions. Therefore, for almost all initial conditions there exist such oscillating outputs and system (2.2) is oscillatory by Definition 3.1. Note that for different $\mathbf{x}_0 \in C^n[-\tau,0]$ oscillations of the outputs $|x_i|$ are possible for different i, $1 \leq i \leq n$. □

Conditions of the above theorem define a class of systems, which oscillatory behavior can be investigated by the proposed approach. The systems should have attracting compact set in state space, which contains oscillatory movements of the systems. For such systems, Theorem 3.3 provides a useful tool for testing their oscillating behavior and obtaining estimates of oscillations amplitude.

The Poincaré-Bendixson theorem [13] provides another method to detect more stronger oscillating behavior in the system, like limit cycles presence. But as a price, Poincaré-Bendixson theorem imposes additional restrictions on structure properties of system (2.2) and does not allow to investigate behavior of chaotic systems. Another conditions on periodic solutions existence can be found in [10, 11].

Remark 3.4. Note that set Ω determines lower bound for value of π^- and upper bound for values of π^+.

Remark 3.5. Like in [9] one can use linearization near the origin of system (2.2) to prove local instability of the system solutions. Instead of existence of Lyapunov functional V one can require just boundedness of the system solution $\mathbf{x}(t)$ with known upper bound obtained using another approach not dealing with time derivative of Lyapunov functional analysis.

Let us show a link between oscillatority and excitation indices.

Corollary 3.6. *Let for almost all initial conditions $\mathbf{x}_0 \in C^n[-\tau,0]$ there exist solutions $\mathbf{x}(\mathbf{x}_0,\mathbf{k}(\mathbf{x}),t)$ of the system (2.2) with control $\mathbf{u} = \mathbf{k}(\mathbf{x})$, $\mathbf{k}(0) = 0$, which are $[\pi^-,\pi^+]$ -oscillations with respect to the output $\psi = \eta(\mathbf{x})$:*

$$\kappa_1(|\mathbf{x}|) \leq \eta(\mathbf{x}) \leq \kappa_2(|\mathbf{x}|), \quad \mathbf{x} \in \mathbb{R}^n,\ \kappa_1,\kappa_2 \in \mathcal{K}_\infty. \tag{3.4}$$

Then excitation indices of system (2.2) satisfy inequality

$$\pi^+ - \pi^- \leq \chi_\psi^+(\gamma) - \chi_\psi^-(\gamma) \tag{3.5}$$

for $\gamma \geq \gamma^*$, where $\gamma^* = \sup_{\mathbf{x}\in\tilde{\Omega}} |\mathbf{k}(\mathbf{x})|$,

$$\tilde{\Omega} = \{\mathbf{x} : \kappa_2^{-1}(\pi^-) \leq |\mathbf{x}| \leq \kappa_1^{-1}(\pi^+)\}. \tag{3.6}$$

Proof. From oscillatority property with respect to output ψ and according to the properties of the output, the solutions of the system (2.2) with $\mathbf{u} = \mathbf{k}(\mathbf{x})$ are bounded for all $\mathbf{x}_0 \in C^n[-\tau, 0]$ (almost all solutions are oscillating, while others are equilibriums):

$$|\mathbf{x}(t)| \leq P, \quad P > 0,\ t \geq 0. \tag{3.7}$$

Therefore, input $\mathbf{u} = \mathbf{k}(\mathbf{x})$ is upper bounded by

$$\gamma = \sup_{|\mathbf{x}| \leq P} |\mathbf{k}(\mathbf{x})| \tag{3.8}$$

and $\pi^+ - \pi^- \leq \chi_\psi^+(\gamma) - \chi_\psi^-(\gamma)$. Here P is some positive constant calculated along solutions of the closed loop system. Also solutions asymptotically converge to set $\widetilde{\Omega}$ (i.e., assumed to be nonempty), where the norm of control $\mathbf{k}$ is upper bounded by γ^*. Therefore, the statement follows from Definitions 3.1 and 3.2 (excitation indices are nondecreasing functions of γ). □

Hence, to compute estimates of excitation indices it is sufficient to find some control law $\mathbf{k}$ for system (2.2), which ensures oscillations existence in the closed loop system.

4. Applications

4.1. Delayed model of testosterone dynamics. Let us consider the following model of testosterone dynamics [14]:

$$\dot{R} = f(T(t-\tau_1)) - b_1 R, \qquad \dot{L} = g_1 R - b_2 L, \qquad \dot{T} = g_2 L(t-\tau_2) - b_3 T, \tag{4.1}$$

where L is luteinising hormone concentration; R is luteinising hormone releasing hormone concentration, T is concentration of testosterone in the blood, b_1, b_2, b_3, g_1, and g_2 are from $\mathbb{R}_+$, $f : \mathbb{R}_+ \to \mathbb{R}_+$ is differentiable, bounded from above and monotone decreasing (during computer simulation we will use $f(T) = A/(K + T^2)$), τ_1 and τ_2 are time delays. It is assumed that the presence of R in the blood induces the secretion of L, which induces testosterone to be secreted in the testes. The testosterone in turn causes a negative feedback effect on the secretion of R. As it was proposed in [14] the presence of delay in this stable model leads to oscillations arising, see also [15] for additional results in this field. Let us apply proposed approach to the above system.

This model for monotone decreasing positive f has one unique equilibrium (R_0, L_0, T_0) being the solution of equations

$$f(T_0) = \frac{b_1 b_2 b_3}{g_1 g_2} T_0, \qquad R_0 = \frac{b_2 b_3}{g_1 g_2} T_0, \qquad L_0 = \frac{b_3}{g_2} T_0. \tag{4.2}$$

The instability property can be established based on linearization of testosterone model near the equilibrium:

$$\delta\dot{R} = f'(T_0)\delta T(t-\tau_1) - b_1 \delta R, \qquad \delta\dot{L} = g_1 \delta R - b_2 \delta L, \qquad \delta\dot{T} = g_2 \delta L(t-\tau_2) - b_3 \delta T, \tag{4.3}$$

where δR, δL and δT are deviations of R, L, and T from the equilibrium, respectively, f' derivative of f. The characteristic polynomial has the form

$$(s+b_1)(s+b_2)(s+b_3) - g_1 g_2 f'(T_0) e^{-(\tau_1+\tau_2)s} = 0. \tag{4.4}$$

For chosen during simulation parameters

$$A = 10, \qquad K = 2, \qquad b_1 = b_2 = b_3 = 1, \qquad g_1 = 10, \qquad g_2 = 50, \tag{4.5}$$

the computation shows that for

$$\tau_1 + \tau_2 > 1.556 \tag{4.6}$$

characteristic polynomial has roots with positive real parts. Thus, to apply Theorem 3.3 for testosterone model we should find a Lyapunov functional satisfying Lemma 2.1. One possible solution is as follows:

$$\begin{aligned} V = {} & 0.5 b_1^{-1}\left(a_R + b_2^{-2}(a_L + b_3^{-1} g_2^2) g_1^2\right) R^2 + 0.5 T^2 \\ & + 0.5 b_2^{-1}\left(a_L + b_3^{-1} g_2^2\right) L^2 + 0.5 b_3^{-1} g_2^2 \int_{t-\tau_2}^{t} L(s)^2 ds, \end{aligned} \tag{4.7}$$

where $a_R, a_L > 0$. Its time derivative for testosterone model admits the upper estimate

$$\dot{V} \le -0.5 a_R R^2 - 0.5 b_3 T^2 - 0.5 a_L L^2 + M, \tag{4.8}$$

where $M = 0.5 b_1^{-2}(a_R + b_2^{-2}(a_L + b_3^{-1} g_2^2) g_1^2) f_{\max}^2$ and $f_{\max} \ge f(T)$, $T \in \mathbb{R}_+$ (in our example $f_{\max} = A/K$). Therefore, the system is uniformly oscillatory. It is worth to stress that testosterone model with the two time delays does not satisfy conditions of Poincaré-Bendixson theorem [13] as well as conditions of book [10]. The value of $\sqrt{M}$ serves as estimate on upper bounds of oscillation amplitude for state vector $\mathbf{x} = [RLT]^T$ (for $a_R = a_L = 1$, $\sqrt{M} \approx 1768$). A corresponding trajectory for $\tau_1 = \tau_2 = 1$ is presented in Figure 4.1.

4.2. Circadian oscillations model. Let us consider circadian model from [16, 17] with time delays:

$$\begin{aligned} \dot{M} &= \frac{v_s K_n}{K_n + P_N(t-\tau_1)^n} - \frac{v_m M}{k_m + M}, \\ \dot{P}_0 &= k_s M(t-\tau_2) - \frac{V_1 P_0}{K_1 + P_0} + \frac{V_2 P_1}{K_2 + P_1}, \\ \dot{P}_1 &= \frac{V_1 P_0}{K_1 + P_0} - \frac{V_2 P_1}{K_2 + P_1} - \frac{V_3 P_1}{K_3 + P_1} + \frac{V_4 P_2}{K_4 + P_2}, \\ \dot{P}_2 &= \frac{V_3 P_1}{K_3 + P_1} - \frac{V_4 P_2}{K_4 + P_2} - k_1 P_2 + k_2 P_N - \frac{v_d P_2}{k_d + P_2}, \\ \dot{P}_N &= k_1 P_2 - k_2 P_N, \end{aligned} \tag{4.9}$$

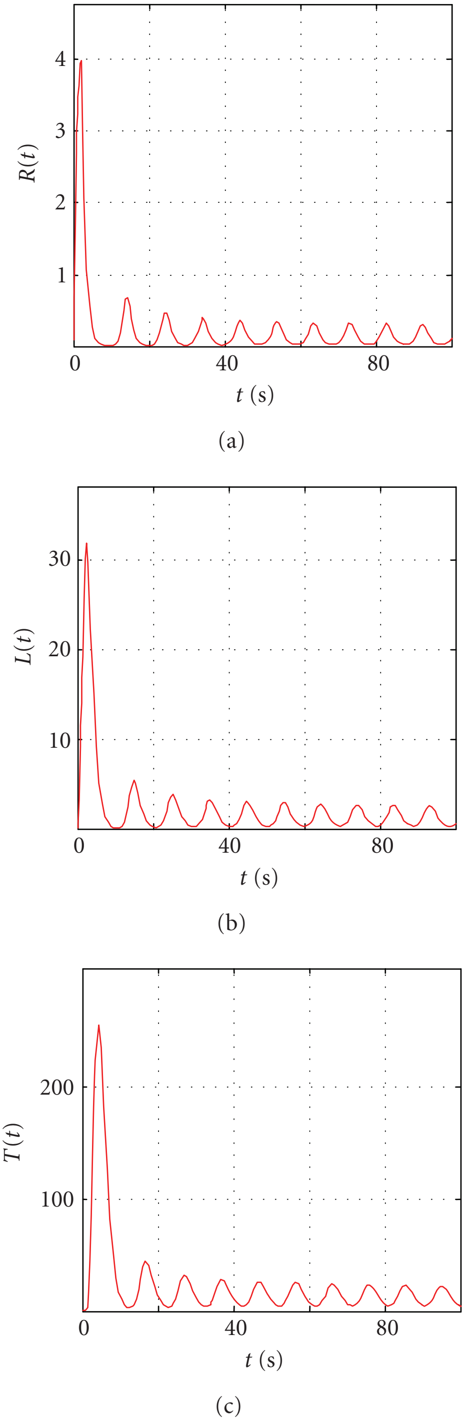

Figure 4.1. Trajectories of testosterone model.

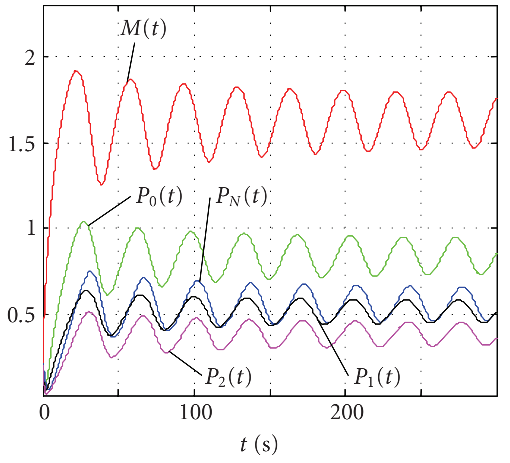

Figure 4.2. Circadian model oscillations.

where P_i, $i = \overline{0,2}$ are concentration degrees of phosphorylation of PER protein, P_N indicates concentration of PER in nucleus, M is concentration of *per* mRNA; τ_1 and τ_2 are time delays. The following values of all other parameters were chosen [16]:

$$\begin{gathered} V_1 = 3.2, \quad V_2 = 1.58, \quad V_3 = 5, \quad V_4 = 2.5, \quad k_1 = 1.9, \\ k_2 = 1.3, \quad k_m = 0.5, \quad k_d = 0.2, \quad k_s = 0.38, \quad v_s = 0.55, \\ v_d = 0.95, \quad v_m = 0.65, \quad n = 4, \quad K_I = 1, \\ K_1 = K_2 = K_3 = K_4 = 2. \end{gathered} \tag{4.10}$$

The description of functionality of the above model can be found in [16, 17]. In [16], it was mentioned without proof that for $v_s = 0.5$ and for bigger values system with delays exhibits oscillations. Also it was proven in [16] that this system has bounded solutions (even with time delays) and unique equilibrium under some mild restrictions on values of model parameters (like chosen for computer simulation)

$$M^0 = 1.758, \quad P_0^0 = 0.95, \quad P_1^0 = 0.595, \quad P_2^0 = 0.474, \quad P_N^0 = 0.693. \tag{4.11}$$

As it was mentioned in Remark 3.5, to establish global boundedness of the system trajectories it is possible to use any other approaches not dealing with Lyapunov functions analysis. For example, [16, Proposition 3.1 and Theorem 1] help to establish global boundedness of the system trajectories here. So, to establish oscillatority property of the system we should investigate stability property of equilibrium. Linearizing circadian model near the equilibrium as in the previous example, we obtain that the equilibrium is unstable and oscillations exist for

$$\tau_1 + \tau_2 \geq 3.44. \tag{4.12}$$

For $\tau_1 = \tau_2 = 2$, the circadian system trajectory is shown in Figure 4.2.

4.3. Blood cells production. Let us consider the conditions of oscillations arising in the following simple model of blood cells production [18, 19]:

$$\dot{z}(t) = -\gamma z(t) + \beta_0 \frac{\phi^n}{\phi^n + z(t-\tau)^n}, \tag{4.13}$$

where $z \in \mathbb{R}_+$ is the density of blood cells, γ, β_0, ϕ, τ are strictly positive parameters, n is integer number. During simulation we will choose $\gamma = 10$, $\beta_0 = 5$, $\phi = 0.01$, $n = 2$, $\tau = 5$. Nonlinear function in right-hand side of the above model is bounded, separated from zero and strictly decreasing. Therefore, equation

$$\gamma z_c = \beta_0 \frac{\phi^n}{\phi^n + z_c^n} \tag{4.14}$$

has the single solution z_c for positive values of γ, β_0, ϕ. The value z_c corresponds to the single equilibrium point of the system. Rewriting the equation of the model in new coordinate $x = z - z_c$, we obtain

$$\begin{gathered} \dot{x}(t) = -\gamma x(t) + g\big(x(t-\tau)\big), \\ g(x) = \beta_0 \frac{\phi^n}{\phi^n + (x+z_c)^n} - \beta_0 \frac{\phi^n}{\phi^n + z_c^n}. \end{gathered} \tag{4.15}$$

Function g is bounded and $g(0) = 0$. To determine conditions of the system local instability let us consider the characteristic polynomial of linearized in the equilibrium $x = 0$ system:

$$\lambda + \gamma + \frac{n\phi^n \beta_0 z_c^{n-1}}{\left(\phi^n + z_c^n\right)^2} e^{-\tau\lambda} = 0, \tag{4.16}$$

which has roots with positive real parts for chosen values of parameters for all $\gamma \le 320$. To prove global boundedness of the system trajectories let us apply the result of Lemma 2.2. Due to boundedness of function g all conditions of the lemma are satisfied for

$$V = 0.5x^2, \qquad R = g_{\max}(\gamma - 0.5)^{-1}, \qquad g_{\max} = \frac{\beta_0 \phi^n}{\phi^n + z_c^n}. \tag{4.17}$$

A trajectory of the system is shown in Figure 4.3 together with estimate R on upper bound of amplitude of oscillations.

5. Conclusion

The paper presents definitions of oscillatority in the sense of Yakubovich and excitation indices for nonlinear dynamical systems with time delays (which models are described by functional differential equations). The sufficient conditions of oscillatority are given as extension of the results of [3]. The good potentiality of proposed approach for detecting of oscillations and evaluation of the amplitudes bounds is demonstrated through examples of analytical design and computer simulation.

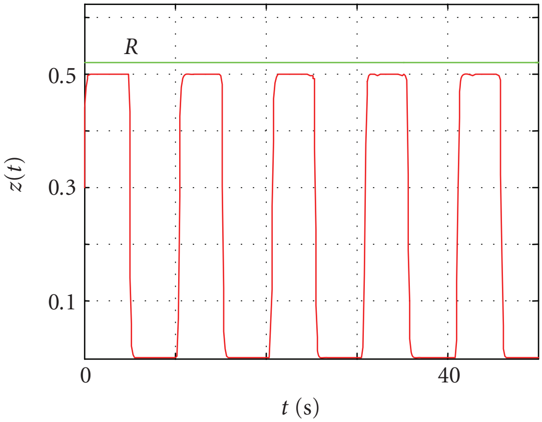

Figure 4.3. Trajectory of the blood cells production model.

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