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Research Article

The Use of Cerami Sequences in Critical Point Theory

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The concept of linking was developed to produce Palais-Smale (PS) sequences $G(u_k) \to a$, $G'(u_k) \to 0$ for C^1 functionals G that separate linking sets. These sequences produce critical points if they have convergent subsequences (i.e., if G satisfies the PS condition). In the past, we have shown that PS sequences can be obtained even when linking does not exist. We now show that such situations produce more useful sequences. They not only produce PS sequences, but also Cerami sequences satisfying $G(u_k) \to a$, $(1+||u_k||)G'(u_k) \to 0$ as well. A Cerami sequence can produce a critical point even when a PS sequence does not. In this situation, it is no longer necessary to show that G satisfies the PS condition, but only that it satisfies the easier Cerami condition (i.e., that Cerami sequences have convergent subsequences). We provide examples and applications. We also give generalizations to situations when the separating criterion is violated.

1. Introduction

An important approach to critical point theory involves the concept of linking. Ideally, one would like to know that if A, B are subsets of a Banach space E and G is a C^1-functional on E such that

$$a_0 := \sup_A G \le b_0 := \inf_B G, \tag{1.1}$$

then G has a critical point, that is, a point $u \in E$ such that $G'(u) = 0$. Clearly, this cannot be true for arbitrary subsets A, B. However, there are pairs of subsets such that (1.1) produces a *Palais-Smale* PS sequence, a sequence of the form

$$G(u_k) \longrightarrow a, \qquad G'(u_k) \longrightarrow 0, \tag{1.2}$$

where $a \geq b_0$. Such a sequence may not produce a critical point, but if it has a convergent subsequence, then it does. If every PS sequence for G has a convergent subsequence, then we say that G satisfies the PS condition. If A, B are such that (1.1) always produces a Palais-Smale sequence, we say that A links B. Consequently, if A links B and G is a C^1-functional on E which satisfies (1.1) and the PS condition, then G has a critical point.

Sufficient conditions for A to link B are found in the literature (cf., e.g., [1–10] and the references quoted there in). The most comprehensive criteria are given in [7, 10].

There are situations in which a Palais-Smale sequence does not lead to a critical point, but a sequence of the form

$$G(u_k) \longrightarrow a, \qquad (1+||u_k||)G'(u_k) \longrightarrow 0 \tag{1.3}$$

does. Such a sequence was first introduced by Cerami [11]. In the first part of the present paper, we show that (1.1) always produces a Cerami sequence whenever A links B in the sense of [10]. Thus, it is not necessary to check if G satisfies the PS condition, but only that it satisfies the Cerami condition, that is, that every sequence of the form (1.3) has a convergent subsequence.

In the second part of the paper, we show that Cerami-type sequences can be produced even when the sets do not link or (1.1) is violated. Finally, we present some applications in which a Cerami sequence produces a critical point, while a PS sequence does not.

2. Linking

Let E be a Banach space, and let Φ be the set of all continuous maps $\Gamma = \Gamma(t)$ from $E \times [0,1]$ to E such that

(1) $\Gamma(0) = I$, the identity map;
(2) for each $t \in [0,1)$, $\Gamma(t)$ is a homeomorphism of E onto E and $\Gamma^{-1}(t) \in C(E \times [0,1), E)$;
(3) $\Gamma(1)E$ is a single point in E and $\Gamma(t)A$ converges uniformly to $\Gamma(1)E$ as $t \to 1$ for each bounded set $A \subset E$;
(4) for each $t_0 \in [0,1)$ and each bounded set $A \subset E$,

$$\sup_{0 \leq t \leq t_0,\, u \in A} \{||\Gamma(t)u|| + ||\Gamma^{-1}(t)u||\} < \infty. \tag{2.1}$$

We make the following.

Definition 2.1. For $A, B \subset E$, say that A links B if

(a) $A \cap B = \phi$,
(b) for each $\Gamma \in \Phi$, there is a $t \in (0,1]$ such that

$$\Gamma(t)A \cap B \neq \phi. \tag{2.2}$$

Roughly speaking, this says that A links B if it cannot be slipped away from B by one of the mappings $\Gamma \in \Phi$. The importance of this concept begins to appear in the following.

Theorem 2.2. *Let G be a C^1-functional on E, and let A, B be subsets of E such that A links B and*

$$a_0 := \sup_A G \le b_0 := \inf_B G. \tag{2.3}$$

Assume that

$$a := \inf_{\Gamma\in\Phi} \sup_{0\le s\le 1,\, u\in A} G(\Gamma(s)u) \tag{2.4}$$

is finite. Let $\psi(t)$ be a positive, nonincreasing, locally Lipschitz continuous function on $[0,\infty)$ such that

$$\int_0^\infty \psi(r)dr = \infty. \tag{2.5}$$

Then there is a sequence $\{u_k\} \subset E$ such that

$$G(u_k) \longrightarrow a, \qquad \frac{G'(u_k)}{\psi(\|u_k\|)} \longrightarrow 0. \tag{2.6}$$

Corollary 2.3. *Under the hypotheses of Theorem 2.2, there is a sequence $\{u_k\} \subset E$ such that*

$$G(u_k) \longrightarrow a, \qquad (1+\|u_k\|)G'(u_k) \longrightarrow 0. \tag{2.7}$$

Proof. We merely take

$$\psi(r) = \frac{1}{1+r} \tag{2.8}$$

in Theorem 2.2. □

Remark 2.4. A sequence satisfying (2.7) is said to be a Cerami sequence. If a functional G has the property that every Cerami sequence for it has a convergent subsequence, it is said to satisfy the Cerami condition. Thus, if a functional satisfies the hypotheses of Theorem 2.2 and the Cerami condition, then it has a critical point satisfying

$$G(u) = a, \qquad G'(u) = 0. \tag{2.9}$$

It is easier to verify the Cerami condition than the PS condition.

Theorem 2.2 was proved in [12] for the case when the set A is bounded. Here, that hypothesis is removed.

We now give some consequences of Theorem 2.2.

Theorem 2.5. *Let G be a C^1-functional on E and let A be a subset of E such that the quantity a given by (2.4) is finite. Assume that for each $\Gamma \in \Phi$, the set*

$$g_\Gamma := \{v = \Gamma(s)u : s \in (0,1],\ u \in A,\ v \notin A,\ G(v) \ge a_0\} \tag{2.10}$$

is not empty. Then there is a sequence satisfying (2.6).

Proof. Let

$$B = \bigcup_{\Gamma\in\Phi} g_\Gamma. \tag{2.11}$$

Then $A \cap B = \phi$, and for each $\Gamma \in \Phi$, there are a $v \in B$, an $s \in (0,1]$, and a $u \in A$ such that $v = \Gamma(s)u$. Thus $\Gamma(s)A \cap B \neq \phi$. This means that A links B. Since $a_0 \leq G(v)$ for all $v \in B$, we have $a_0 \leq b_0$. We can now apply Theorem 2.2 to conclude that a sequence satisfying (2.6) exists. □

Corollary 2.6. *If $a < \infty$ and $a_0 \neq a$, then a sequence satisfying (2.6) exists.*

Proof. If $a_0 < a$, then for each $\Gamma \in \Phi$, there are a $u \in A$, $s \in (0,1]$ such that $G(\Gamma(s)u) > a_0$. Clearly $v = \Gamma(s)u \notin A$. Thus the set g_Γ given by (2.10) is not empty. We can now apply Theorem 2.5. □

Theorem 2.7. *There is a $B \subset E$ such that A links B and $a_0 \leq b_0$ if and only if the set g_Γ defined by (2.10) is not empty for each $\Gamma \in \Phi$.*

Proof. If the sets g_Γ are not empty, then B given by (2.11) has the required properties, as was shown in the proof of Theorem 2.5. On the other hand, if $g_\Gamma = \phi$ for some $\Gamma \in \Phi$, then for every set B such that $A \cap B = \phi$ and $a_0 \leq b_0$, we must have $\Gamma(s)A \cap B = \phi$ for all $s \in [0,1]$. Thus A cannot link B. □

3. Weaker conditions

We now turn to the question as to what happens if some of the hypotheses of Theorem 2.2 do not hold. We are particularly interested in what happens when (2.3) is violated. In this case, we let

$$B' := \{v \in B : G(v) < a_0\}. \tag{3.1}$$

Note that

$$B' = \phi \quad \text{iff } a_0 \leq b_0. \tag{3.2}$$

Let $\psi(t)$ be a positive nonincreasing function on $[0,\infty)$ satisfying the hypotheses of Theorem 2.2 and such that

$$a_0 - b_0 < \int_\alpha^{R+\alpha} \psi(t)dt \tag{3.3}$$

for some finite $R \leq d' := d(B',A)$, where $\alpha = d(0,A)$. If $B' = \phi$, we take $d' = \infty$. We assume that $d' > 0$. We have the following.

Theorem 3.1. *Let G be a C^1-functional on E and let $A,B \subset E$ be such that A links B and*

$$-\infty < b_0, \qquad a < \infty. \tag{3.4}$$

Under the hypotheses given above, for each $\delta > 0$, there is a $u \in E$ such that

$$b_0 - \delta \leq G(u) \leq a + \delta, \qquad ||G'(u)|| < \psi(d(u,A)). \tag{3.5}$$

We can also consider a slightly different version of Theorem 3.1. We consider the set

$$A'' := \{u \in A : G(u) > b_0\}, \tag{3.6}$$

and we note that $A'' = \phi$ if and only if $a_0 \le b_0$. We assume that ψ satisfies the hypotheses of Theorem 2.2 and

$$a_0 - b_0 < \int_{\beta}^{R+\beta} \psi(t)dt \tag{3.7}$$

holds for some finite $R \le d'' := d(A'', B)$, where $\beta = d(0, B)$. We have the following.

Theorem 3.2. *If A links B and*

$$-\infty < b_0, \qquad a < \infty \tag{3.8}$$

holds then for each $\delta > 0$, there is a $u \in E$ such that

$$b_0 - \delta \le G(u) \le a + \delta, \qquad ||G'(u)|| < \psi(d(u,B)). \tag{3.9}$$

4. Some consequences

We now discuss some methods which follow from Theorems 3.1 and 3.2. Let $\{A_k, B_k\}$ be a sequence of pairs of subsets of E such that A_k links B_k for each k. For $G \in C^1(E, \mathbb{R})$, let

$$\begin{aligned} a_{k0} &= \sup_{A_k} G, \qquad b_{k0} = \inf_{B_k} G, \\ a_k &= \inf_{\Gamma \in \Phi} \sup_{0 \le s \le 1, u \in A_k} G(\Gamma(s)u). \end{aligned} \tag{4.1}$$

We assume that $a_k < \infty$ for each k. We define

$$\begin{aligned} B'_k &:= \{v \in B_k : G(v) < a_{k0}\}, \\ A''_k &:= \{u \in A_k : G(u) > b_{k0}\}, \\ d'_k := d(A_k, B'_k), &\qquad d''_k := d(A''_k, B_k). \end{aligned} \tag{4.2}$$

We have the following.

Theorem 4.1. *Assume that*

$$d'_k \longrightarrow \infty \quad \text{as } k \longrightarrow \infty, \tag{4.3}$$

and for each k there is a positive nonincreasing function $\psi_k(t)$ on $[0, \infty)$ satisfting the hypotheses of Theorem 2.2 and such that

$$(a_{k0} - b_{k0}) < \int_{\alpha_k}^{R_k + \alpha_k} \psi_k(t)dt, \tag{4.4}$$

where $\alpha_k = d(0, A_k)$ *and* $R_k \leq d'_k$. *Then there is a sequence* $\{u_k\} \subset E$ *such that*

$$b_{k0} - \left(\frac{1}{k}\right) \leq G(u_k) \leq a_k + \left(\frac{1}{k}\right),$$
$$\|G'(u_k)\| \leq \psi_k(d(u_k, A_k)). \tag{4.5}$$

Theorem 4.2. *Assume that*

$$d''_k \longrightarrow \infty \quad \text{as } k \longrightarrow \infty \tag{4.6}$$

and that for each k *there is a positive nonincreasing function* $\psi_k(t)$ *on* $[0, \infty)$ *satisfying the hypotheses of Theorem 2.2 and such that*

$$(a_{k0} - b_{k0}) < \int_{\beta_k}^{R_k+\beta_k} \psi_k(t)\,dt, \tag{4.7}$$

where $\beta_k = d(0, B_k)$ *and* $R_k \leq d''_k$. *Then there is a sequence* $\{u_k\} \subset E$ *such that*

$$b_{k0} - \left(\frac{1}{k}\right) \leq G(u_k) \leq a_k + \left(\frac{1}{k}\right),$$
$$\|G'(u_k)\| \leq \psi_k(d(u_k, B_k)). \tag{4.8}$$

We combine the proofs of Theorems 4.1 and 4.2.

Proof. For each k, take R_k equal to d'_k or d''_k, as the case may be. We may assume that $b_{k0} < a_{k0}$ for each k. Otherwise, the conclusions of the theorems follow from Corollary 2.6. We can now apply Theorems 3.1 and 3.2 for each k to conclude that there is a $u_k \in E$ such that

$$b_{k0} - \left(\frac{1}{k}\right) \leq G(u_k) \leq a_k + \left(\frac{1}{k}\right), \tag{4.9}$$

and either

$$\|G'(u_k)\| < \psi_k(d(u_k, A_k)) \tag{4.10}$$

or

$$\|G'(u_k)\| < \psi_k(d(u_k, B_k)) \tag{4.11}$$

as the case may be. □

Corollary 4.3. *In Theorem 4.1, assume that*

$$b_{k0} \geq m_0 > -\infty, \qquad a_{k0} \leq m_1 < \infty \tag{4.12}$$

in place of (4.4). Then there is a sequence $\{u_k\} \subset E$ *such that*

$$G(u_k) \longrightarrow c, \quad m_0 \leq c \leq m_1, \quad G'(u_k) \leq \psi_k(d(u_k, A_k)). \tag{4.13}$$

Proof. If there is a k such that $a_k \neq a_{k0}$, then we can apply Corollary 2.6 to find a sequence $\{u_j\} \subset E$ such that

$$G(u_j) \longrightarrow a_k, \qquad G'(u_j) \leq \psi_k(d(u_k, A_k)). \tag{4.14}$$

Since $b_{k0} \leq a_k$, this provides the desired sequence. If no such k exists, then $a_k = a_{k0}$ for each k. Then by Theorem 4.1, there is a sequence satisfying

$$m_0 - \left(\frac{1}{k}\right) \leq G(u_k) \leq m_1 + \left(\frac{1}{k}\right), \qquad G'(u_k) \leq \psi_k(d(u_k, A_k)), \tag{4.15}$$

from which we obtain (4.13). □

Corollary 4.4. *In Theorem 4.2, assume (4.12) in place of (4.4). Then there is a sequence satisfying*

$$G(u_k) \longrightarrow c, \quad m_0 \leq c \leq m_1, \quad G'(u_k) \leq \psi_k(d(u_k, B_k)). \tag{4.16}$$

Proof. We apply the same reasoning as in the proof of Corollary 4.3. We obtain a sequence satisfying

$$m_0 - \left(\frac{1}{k}\right) \leq G(u_k) \leq m_1 + \left(\frac{1}{k}\right), \qquad G'(u_k) \leq \psi_k(d(u_k, B_k)), \tag{4.17}$$

and this produces a sequence satisfying (4.16). □

5. Various geometries

We now apply the theorems of the preceding sections to various geometries in Banach space. As before, we assume that $G \in C^1(E, \mathbb{R})$ and that ψ satisfies the hypotheses of Theorem 2.2.

Theorem 5.1. *Assume that there is a $\delta > 0$ such that*

$$G(0) \leq \alpha \leq G(u), \quad u \in \partial B_\delta, \tag{5.1}$$

and that there is a $\varphi_0 \in \partial B_1$ such that

$$G(R\varphi_0) \leq \gamma, \quad R > R_0. \tag{5.2}$$

Then there is a sequence $\{u_k\} \subset E$ such that

$$G(u_k) \longrightarrow c, \quad \alpha \leq c \leq \gamma, \quad \frac{G'(u_k)}{\psi(\|u_k\|)} \longrightarrow 0. \tag{5.3}$$

Proof. We take $A = \{0, R\varphi_0\}$, $B = \partial B_\delta$. Then $A'' = \{R\varphi_0\}$. Note that a given by (2.4) is finite for each R since

$$a_R \leq \max_{0 \leq r \leq R} G(r\varphi_0). \tag{5.4}$$

We apply Theorem 4.2. We note that in each case,

$$a_R \leq \gamma, \quad R > R_0. \tag{5.5}$$

In each case, the mapping

$$\Gamma(s)u = su \tag{5.6}$$

(which is in Φ) satisfies

$$G(\Gamma(s)u) \leq \gamma, \quad 0 \leq s \leq 1,\ u \in A. \tag{5.7}$$

This implies (5.5). We replace $\psi(t)$ with $\tilde{\psi}(t) = \psi(t+\delta)$, which also satisfies the hypotheses of Theorem 2.2. By Theorem 4.2, we can find a sequence satisfying

$$\alpha - \left(\frac{1}{k}\right) \leq G(u_k) \leq \gamma + \left(\frac{1}{k}\right), \qquad \frac{G'(u_k)}{\tilde{\psi}(d(u_k,B))} \longrightarrow 0. \tag{5.8}$$

This implies (5.3) since

$$\|u\| \leq d(u,B) + \delta. \tag{5.9}$$

□

Theorem 5.2. *Let M, N be closed subspaces of E such that*

$$E = M \oplus N, \quad M \neq E,\ N \neq E \tag{5.10}$$

with

$$\dim M < \infty \quad \text{or} \quad \dim N < \infty. \tag{5.11}$$

Let $G \in C^1(E,\mathbb{R})$ be such that

$$\begin{gathered} G(v) \leq \gamma, \quad v \in \partial B_R \cap N,\ R > R_0, \\ G(w) \geq \alpha, \quad w \in M. \end{gathered} \tag{5.12}$$

Then there is a sequence $\{u_k\} \subset E$ such that

$$G(u_k) \longrightarrow c, \quad \alpha \leq c \leq \gamma, \quad \frac{G'(u_k)}{\psi(d(u_k,M))} \longrightarrow 0. \tag{5.13}$$

Proof. This time, we take A and B as in [7, Section 2.6, Example 2]. Thus A links B. Again a_R given by (2.4) is finite for each R since

$$a_R \leq \max_{u \in \overline{B}_R \cap N} G(u). \tag{5.14}$$

Again we see that we can apply Theorem 4.2 to conclude that the desired sequence exists.

□

Theorem 5.3. *Let M, N be as in Theorem 5.2, and let* $G \in C^1(E,\mathbb{R})$ *satisfy*

$$\begin{aligned} G(v) &\le \alpha, \quad v \in N, \\ G(w) &\ge \alpha, \quad w \in \partial B_\delta \cap M, \\ G(sw_0+v) &\le \gamma, \quad s \ge 0,\ v \in N,\ \|sw_0+v\| = R > R_0, \end{aligned} \tag{5.15}$$

for some $w_0 \in \partial B_1 \cap M$, *where* $0 < \delta < R_0$. *Then there is a sequence* $\{u_k\} \subset E$ *such that (5.3) holds.*

Proof. Here we take A, B as in [7, Section 2.6, Example 3]. Thus A and B link each other. Here

$$A'' = \{sw_0 + v : s \ge 0,\ v \in N,\ \|sw_0+v\| = R\}. \tag{5.16}$$

Again for each R, the quantity a given by (2.4) is finite since

$$a_R \le \max_Q G, \tag{5.17}$$

where

$$Q = \{sw_0 + v : s \ge 0,\ v \in N,\ \|sw_0+v\| \le R\}. \tag{5.18}$$

We now apply Theorem 4.2 to conclude that the desired sequence exists. □

Theorem 5.4. *Let M, N be as in Theorem 5.2, and let* $v_0 \in \partial B_1 \cap N$. *Take* $N = \{v_0\} \oplus N'$. *Let* $G \in C^1(E,\mathbb{R})$ *be such that*

$$\begin{aligned} G(v) &\le \gamma, \quad v \in \partial B_R \cap N, \\ G(w) &\ge \alpha, \quad w \in M,\ \|w\| \ge \delta, \\ G(sv_0+w) &\ge \alpha, \quad s \ge 0,\ w \in M,\ \|sv_0+w\| = \delta, \end{aligned} \tag{5.19}$$

where $0 < \delta < R$. *Then there is a sequence satisfying (5.3).*

Proof. We take A, B as in [7, Section 2.6, Example 5]. Thus A links B. As before, we note that $a_R < \infty$ for each R. Hence (5.3) holds for some sequence by Theorem 4.2. □

6. Some applications

Many elliptic semilinear problems can be described in the following way. Let Ω be a domain in $\mathbb{R}^n$, and let A be a selfadjoint operator on $L^2(\Omega)$. We assume that $A \ge \lambda_0 > 0$ and that

$$C_0^\infty(\Omega) \subset D := D(A^{1/2}) \subset H^{m,2}(\Omega) \tag{6.1}$$

for some $m > 0$, where $C_0^\infty(\Omega)$ denotes the set of test functions in Ω (i.e., infinitely differentiable functions with compact supports in Ω), and $H^{m,2}(\Omega)$ denotes the Sobolev space.

If m is an integer, the norm in $H^{m,2}(\Omega)$ is given by

$$\|u\|_{m,2} := \Big(\sum_{|\mu|\le m} \|D^{\mu}u\|^2 \Big)^{1/2}. \tag{6.2}$$

Here D^{μ} represents the generic derivative of order $|\mu|$ and the norm on the right-hand side of (6.2) is that of $L^2(\Omega)$. We will not assume that m is an integer.

Let q be any number satisfying

$$2 \le q \begin{cases} \le \dfrac{2n}{n-2m}, & 2m < n, \\ < \infty, & n \le 2m, \end{cases} \tag{6.3}$$

and let $f(x,t)$ be a Carathéodory function on $\Omega \times \mathbb{R}$. This means that $f(x,t)$ is continuous in t for a.e. $x \in \Omega$ and measurable in x for every $t \in \mathbb{R}$. We make the following assumptions.

(A) The function $f(x,t)$ satisfies

$$\begin{gathered} |f(x,t)| \le V_0(x)^q |t|^{q-1} + V_0(x)W_0(x), \\ \frac{f(x,t)}{V_0(x)^q} = o\big(|t|^{q-1}\big) \quad \text{as } |t| \longrightarrow \infty, \end{gathered} \tag{6.4}$$

where $V_0(x) > 0$ is a function in $L^q(\Omega)$ such that

$$\|V_0 u\|_q \le C\|u\|_D, \quad u \in D, \tag{6.5}$$

and W_0 is a function in $L^{q'}(\Omega)$. Here

$$\|u\|_q := \Big(\int_{\Omega} |u(x)|^q dx \Big)^{1/q}, \tag{6.6}$$

$$\|u\|_D := \|A^{1/2}u\|, \tag{6.7}$$

and $q' = q/(q-1)$. If Ω and $V_0(x)$ are bounded, then (6.5) will hold automatically by the Sobolev inequality. However, there are functions $V_0(x)$ which are unbounded and such that (6.5) holds even on unbounded regions Ω. With the norm (6.7), D becomes a Hilbert space. Define

$$\begin{gathered} F(x,t) := \int_0^t f(x,s)ds, \\ G(u) := \|u\|_D^2 - 2\int_{\Omega} F(x,u)dx. \end{gathered} \tag{6.8}$$

It follows that G is a continuously differentiable functional on the whole of D (cf., e.g., [7]).

We assume further that

$$H(x,t) = 2F(x,t) - tf(x,t) \ge -W_1(x) \in L^1(\Omega), \quad x \in \Omega,\ t \in \mathbb{R}, \tag{6.9}$$

$$H(x,t) \longrightarrow \infty \quad \textit{a.e. as } |t| \longrightarrow \infty. \tag{6.10}$$

Moreover, we assume that there are functions $V(x), W(x) \in L^2(\Omega)$ such that multiplication by $V(x)$ is a compact operator from D to $L^2(\Omega)$ and

$$F(x,t) \leq C\big(V(x)^2|t|^2 + V(x)W(x)|t|\big). \tag{6.11}$$

We wish to obtain a solution of

$$Au = f(x,u), \quad u \in D. \tag{6.12}$$

By a solution of (6.12), we will mean a function $u \in D$ such that

$$(u,v)_D = \big(f(\cdot,u),v\big), \quad v \in D. \tag{6.13}$$

If $f(x,u)$ is in $L^2(\Omega)$, then a solution of (6.13) is in $D(A)$ and solves (6.12) in the classical sense. Otherwise, we call it a weak (or semistrong) solution. We have the following.

Theorem 6.1. *Let A be a selfadjoint operator in $L^2(\Omega)$ such that $A \geq \lambda_0 > 0$ and (6.1) holds for some $m > 0$. Assume that λ_0 is an eigenvalue of A with eigenfunction φ_0. Assume also that*

$$2F(x,t) \leq \lambda_0 t^2, \quad |t| \leq \delta \text{ for some } \delta > 0, \tag{6.14}$$

$$2F(x,t) \geq \lambda_0 t^2 - W_0(x), \quad t > 0,\ x \in \Omega, \tag{6.15}$$

where $W_0 \in L^1(\Omega)$. Assume that $f(x,t)$ is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying (6.4). Then (6.12) has a solution $u \neq 0$.

Proof. Under the hypotheses of the theorem, it was shown in [7, Theorem 3.2.1] that the following alternative holds.

Either

(a) there are an infinite number of $y(x) \in D(A) \setminus \{0\}$ such that

$$Ay = f(x,y) = \lambda_0 y, \tag{6.16}$$

or

(b) for each $\rho > 0$ sufficiently small, there is an $\varepsilon > 0$ such that

$$G(u) \geq \varepsilon, \qquad \|u\|_D = \rho. \tag{6.17}$$

We may assume that option (b) holds, for otherwise we are done. By (6.15), we have

$$G(R\varphi_0) \leq R^2\Big(||\varphi_0||_D^2 - \lambda_0||\varphi_0||^2\Big) + \int_\Omega W_0(x)dx = \int_\Omega W_0(x)dx. \tag{6.18}$$

Thus (5.2) holds. By Theorem 5.1, there is a sequence satisfying (5.3). Taking $\psi(r) = 1/(r+1)$, we conclude that there is a sequence $\{u_k\} \subset D$ such that

$$G(u_k) \longrightarrow c, \quad m_0 \leq c \leq m_1, \quad (1+||u_k||_D)G'(u_k) \longrightarrow 0. \tag{6.19}$$

In particular, we have

$$||u_k||_D^2 - 2\int_\Omega F(x,u_k)\,dx \longrightarrow c, \tag{6.20}$$

$$||u_k||_D^2 - (f(\cdot,x_k),u_k) \longrightarrow 0. \tag{6.21}$$

Consequently,

$$\int_\Omega H(x,u_k)\,dx \longrightarrow -c. \tag{6.22}$$

These imply

$$\int_\Omega H(x,u_k)\,dx \le K. \tag{6.23}$$

If $\rho_k = \|u_k\|_D \to \infty$, let $\tilde{u}_k = u_k/\rho_k$. Then $\|\tilde{u}_k\|_D = 1$. Consequently there is a renamed subsequence such that $\tilde{u}_k \to \tilde{u}$ weakly in D, strongly in $L^2(\Omega)$, and a.e. in Ω. We have by (6.11) that

$$1 \le \frac{m_1+\delta}{\rho_k^2} + 2C\int_\Omega \{V(x)^2\tilde{u}_k^2 + V(x)W(x)\,|\,\tilde{u}_k\,|\,\rho_k^{-1}\}\,dx. \tag{6.24}$$

Consequently,

$$1 \le 2C\int_\Omega V(x)^2\tilde{u}^2\,dx. \tag{6.25}$$

This shows that $\tilde{u} \not\equiv 0$. Let Ω_0 be the subset of Ω on which $\tilde{u} \neq 0$. Then

$$|\,u_k(x)\,| = \rho_k\,|\,\tilde{u}_k(x)\,| \longrightarrow \infty, \quad x \in \Omega_0. \tag{6.26}$$

If $\Omega_1 = \Omega \setminus \Omega_0$, then we have

$$\int_\Omega H(x,u_k)\,dx = \int_{\Omega_0} + \int_{\Omega_1} \ge \int_{\Omega_0} H(x,u_k)\,dx - \int_{\Omega_1} W_1(x)\,dx \longrightarrow \infty. \tag{6.27}$$

This contradicts (6.23), and we see that $\rho_k = \|u_k\|_D$ is bounded. Once we know that the ρ_k are bounded, we can apply [7, Theorem 3.4.1] to obtain the desired conclusion. □

Remark 6.2. It should be noted that the crucial element in the proof of Theorem 6.1 was (6.21). If we had been dealing with an ordinary Palais-Smale sequence, we could only conclude that

$$||u_k||_D^2 - (f(\cdot,u_k),u_k) = o(\rho_k), \tag{6.28}$$

which would imply only that

$$\int_\Omega H(x,u_k)\,dx = o(\rho_k). \tag{6.29}$$

This would not contradict (6.27), and the argument would not go through.

Theorem 6.3. *Assume that the spectrum of A consists of isolated eigenvalues of finite multiplicity*

$$0 < \lambda_0 < \lambda_1 < \cdots < \lambda_k < \cdots, \tag{6.30}$$

and let ℓ be a nonnegative integer. Take N to be the subspace of D spanned by the eigenspaces of A corresponding to the eigenvalues $\lambda_0, \lambda_1, \dots, \lambda_\ell$. Take $M = N^\perp \cap D$. Assume that there are numbers a_1, a_2 such that $\alpha_\ell < a_1 \leq a_2$ and

$$\begin{aligned} a_1(t^-)^2 + \gamma_\ell(a_1)(t^+)^2 - W_1(x) &\leq 2F(x,t) \\ &\leq a_2(t^-)^2 + \Gamma_\ell(a_2)(t^+)^2 + W_2(x), \quad x \in \Omega,\ t \in \mathbb{R}, \end{aligned} \tag{6.31}$$

where

$$\alpha_\ell := \max\{(Av,v) : v \in N,\ v \geq 0,\ \|v\| = 1\}, \tag{6.32}$$

the W_j are in $L^1(\Omega)$, and the functions $\gamma_\ell(a)$, $\Gamma_\ell(a)$ are defined by

$$\gamma_\ell(a) := \max\{(Av,v) - a\|v^-\|^2 : v \in N,\ \|v^+\| = 1\}, \tag{6.33}$$

$$\Gamma_\ell(a) := \inf\{(Aw,w) - a\|w^-\|^2 : w \in M,\ \|w^+\| = 1\}, \tag{6.34}$$

where $u^\pm = \max\{\pm u, 0\}$. Assume that (6.9) and (6.10) hold. Then (6.12) has at least one solution.

Proof. First, we note that

$$\sup_N G \leq B_1, \qquad \inf_M G \geq -B_2, \qquad B_j = \int_\Omega W_j(x)dx. \tag{6.35}$$

To see this, note that by (6.33), we have

$$\|v\|_D^2 \leq a_1\|v^-\|^2 + \gamma_\ell(a_1)\|v^+\|^2, \quad v \in N. \tag{6.36}$$

By (6.34) we have

$$a_2\|w^-\|^2 + \Gamma_\ell(a_2)\|w^+\|^2 \leq \|w\|_D^2, \quad w \in M, \tag{6.37}$$

Hence

$$\begin{aligned} G(v) &\leq B_1, \quad v \in N, \\ G(w) &\geq -B_2, \quad w \in M \end{aligned} \tag{6.38}$$

by (6.31).

Moreover, (6.10) implies that

$$G(v) \longrightarrow -\infty \quad \text{as } \|v\| \longrightarrow \infty, \quad v \in N. \tag{6.39}$$

To see this, we fix $x \in \Omega$, $K \in \mathbb{R}$ and take T so large that

$$H(x,t) \geq K, \quad |t| \geq T. \tag{6.40}$$

Since

$$\frac{\partial(t^{-2}F(x,t))}{\partial t} = -t^{-3}H(x,t), \tag{6.41}$$

we have for $T < t_1 < t_2$ that

$$t_2^{-2}F(x,t_2) - t_1^{-2}F(x,t_1) \leq \frac{K(t_2^{-2} - t_1^{-2})}{2}. \tag{6.42}$$

Consequently,

$$t_2^{-2}[2F(x,t_2) - K] \leq t_1^{-2}[2F(x,t_1) - K]. \tag{6.43}$$

Thus,

$$\frac{[2F(x,t) - K]}{t^2} \tag{6.44}$$

is a monotone nonincreasing function in t for $t > T$. By (6.31), it is bounded below by

$$\gamma_\ell(a_1) - \frac{[W_1(x) + K]}{t^2} \longrightarrow \gamma_\ell(a_1). \tag{6.45}$$

Thus,

$$\frac{[2F(x,t) - K]}{t^2} \longrightarrow h(x) \geq \gamma_\ell(a_1) \quad \text{a.e. as } t \longrightarrow \infty. \tag{6.46}$$

This implies that

$$K \leq 2F(x,t) - \gamma_\ell(a_1)t^2. \tag{6.47}$$

Since K was arbitrary, we have

$$2F(x,t) - \gamma_\ell(a_1)t^2 \longrightarrow \infty \quad \textit{a.e. as } t \longrightarrow \infty. \tag{6.48}$$

On the other hand, if $t_1 < t_2 < -T$, then

$$t_2^{-2}F(x,t_2) - t_1^{-2}F(x,t_1) \geq \frac{K(t_2^{-2} - t_1^{-2})}{2}. \tag{6.49}$$

Consequently, function (6.44) is monotone nonincreasing in t for $t < -T$. In view of (6.31), this implies that

$$2F(x,t) - a_1t^2 \longrightarrow \infty \quad \text{a.e. as } t \longrightarrow -\infty. \tag{6.50}$$

Combining (6.48) and (6.50), we have

$$2F(x,t) - a_1(t^-)^2 - \gamma_\ell(a_1)(t^+)^2 \longrightarrow \infty \quad \text{a.e. as } |t| \longrightarrow \infty. \tag{6.51}$$

Now

$$G(v) = \|v\|_D^2 - a_1\|v^-\|^2 - \gamma_\ell(a_1)\|v^+\|^2 - \int_\Omega L(x,v)dx, \tag{6.52}$$

where $L(x,t)$ is the left-hand side of (6.51). In view of (6.33), we have

$$G(v) \le -\int_\Omega L(x,v)dx, \quad v \in N. \tag{6.53}$$

Let $\{v_k\} \subset N$ be such that $\rho_k = \|v_k\|_D \to \infty$. Take $\tilde{v}_k = v_k/\rho_k$. Then $\|\tilde{v}_k\|_D = 1$, and consequently there is a renamed subsequence such that $\tilde{v}_k \to \tilde{v}$ strongly in N. Thus $\|\tilde{v}\|_D = 1$ showing that $\tilde{v} \not\equiv 0$. Let Ω_1 be the set on which $\tilde{v} \neq 0$ and let $\Omega_2 = \Omega \setminus \Omega_1$. Then

$$G(v_k) \le -\int_{\Omega_1} L(x,v_k)dx - \int_{\Omega_2} W_1(x)dx \longrightarrow -\infty \tag{6.54}$$

since

$$-W_1(x) \le L(x,t) \longrightarrow \infty \quad \text{a.e. as } |t| \longrightarrow \infty, \tag{6.55}$$

and $|v_k(x)| = \rho_k|\tilde{v}_k(x)| \to \infty$ for $x \in \Omega_1$. Since this is true for any such sequence, (6.39) follows.

Take R so large that

$$G(v) \le -B_2, \quad v \in N \cap \partial B_R. \tag{6.56}$$

Since $N \cap \partial B_R$ links M, we have by Corollary 2.3 that there are a constant $c \in \mathbb{R}$ and a sequence $\{u_k\} \subset E$ such that

$$G(u_k) \longrightarrow c, \quad -B_2 \le c \le B_1, \quad (1+\|u_k\|)G'(u_k) \longrightarrow 0. \tag{6.57}$$

We can now follow the proof of Theorem 6.1 to conclude that (6.20)–(6.27) hold to complete the proof. □

We also have the following.

Theorem 6.4. *The conclusion of Theorem 6.3 holds if in place of (6.9), (6.10), one assumes that*

$$H(x,t) \le W_1(x) \in L^1(\Omega), \quad x \in \Omega,\ t \in \mathbb{R}, \tag{6.58}$$

$$H(x,t) \longrightarrow -\infty \quad \text{a.e. as } |t| \longrightarrow \infty. \tag{6.59}$$

Proof. We use (6.58) and (6.59) to replace (6.39) with

$$G(w) \longrightarrow \infty \quad \text{as } \|w\| \longrightarrow \infty,\ w \in M. \tag{6.60}$$

We then proceed as before. □

Remark 6.5. We could have assumed

$$a_1 \le \liminf_{t\to-\infty} \frac{2F(x,t)}{t^2} \le \limsup_{t\to-\infty} \frac{2F(x,t)}{t^2} \le a_2,$$
$$\gamma_\ell(a_1) \le \liminf_{t\to\infty} \frac{2F(x,t)}{t^2} \le \limsup_{t\to\infty} \frac{2F(x,t)}{t^2} \le \Gamma_\ell(a_2) \tag{6.61}$$

in place of (6.31).

Remark 6.6. The above theorems apply to the equation

$$Au = -\Delta u + a(x)u = f(x,u), \quad x \in \mathbb{R}^n, \tag{6.62}$$

where $a(x) \ge c_0 > 0$ and A has compact resolvent. We do not need to restrict the sizes of $a(x)$ or $V(x)$. The limits (6.10) or (6.59) need only hold on a subset of Ω with positive measure.

7. Ordinary differential equations

In proving Theorem 2.2, we will make use of various extensions of Picard's theorem in a Banach space. Some are well known (cf., e.g., [13]).

Theorem 7.1. *Let X be a Banach space, and let*

$$B_0 = \{x \in X : \|x - x_0\| \le R_0\},$$
$$I_0 = \{t \in \mathbb{R} : |t - t_0| \le T_0\}. \tag{7.1}$$

Assume that $g(t,x)$ is a continuous map of $I_0 \times B_0$ into X such that

$$\|g(t,x) - g(t,y)\| \le K_0\|x - y\|, \quad x,y \in B_0,\ t \in I_0,$$
$$\|g(t,x)\| \le M_0, \quad x \in B_0,\ t \in I_0. \tag{7.2}$$

Let T_1 be such that

$$T_1 \le \min\left(T_0, \frac{R_0}{M_0}\right), \quad K_0T_1 < 1. \tag{7.3}$$

Then there is a unique solution $x(t)$ of

$$\frac{dx(t)}{dt} = g(t,x(t)), \quad |t - t_0| \le T_1,\ x(t_0) = x_0. \tag{7.4}$$

Lemma 7.2. *Let $\gamma(t)$ and $\rho(t)$ be continuous functions on $[0,\infty)$, with $\gamma(t)$ nonnegative and $\rho(t)$ positive. Assume that*

$$\int_{u_0}^{\infty} \frac{d\tau}{\rho(\tau)} > \int_{t_0}^{T} \gamma(s)ds, \tag{7.5}$$

where $t_0 < T$ and u_0 are given positive numbers. Then there is a unique solution of

$$u'(t) = \gamma(t)\rho(u(t)), \quad t \in [t_0, T),\ u(t_0) = u_0, \tag{7.6}$$

which is positive in $[t_0, T)$ and depends continuously on u_0.

Proof. One can separate variables to obtain

$$W(u) = \int_{u_0}^{u} \frac{d\tau}{\rho(\tau)} = \int_{t_0}^{t} \gamma(s)ds. \tag{7.7}$$

The function $W(u)$ is differentiable and increasing in $\mathbb{R}$, positive in $[u_0, \infty)$, depends continuously on u_0, and satisfies

$$W(u) \longrightarrow L = \int_{u_0}^{\infty} \frac{d\tau}{\rho(\tau)} > \int_{t_0}^{T} \gamma(s)ds, \quad \text{as } u \longrightarrow \infty. \tag{7.8}$$

Thus, for each $t \in [t_0, T)$, there is a unique $u \in [u_0, \infty)$ such that

$$u = W^{-1}\left(\int_{t_0}^{t} \gamma(s)ds\right) \tag{7.9}$$

is the unique solution of (7.6), and it depends continuously on u_0. □

Lemma 7.3. *Let $\gamma(t)$ and $\rho(t)$ be continuous functions on $[0, \infty)$, with $\gamma(t)$ nonnegative and $\rho(t)$ positive. Assume that*

$$\int_{m}^{u_0} \frac{d\tau}{\rho(\tau)} > \int_{t_0}^{T} \gamma(s)ds, \tag{7.10}$$

where $t_0 < T$ and $m < u_0$ are given positive numbers. Then there is a unique solution of

$$u'(t) = -\gamma(t)\rho(u(t)), \quad t \in [t_0, T),\ u(t_0) = u_0, \tag{7.11}$$

which is $\geq m$ in $[t_0, T)$ and depends continuously on u_0.

Proof. One can separate variables to obtain

$$W(u) = \int_{u}^{u_0} \frac{d\tau}{\rho(\tau)} = \int_{t_0}^{t} \gamma(s)ds. \tag{7.12}$$

The function $W(u)$ is differentiable and decreasing in $\mathbb{R}$, positive in $[m, u_0]$, depends continuously on u_0, and satisfies

$$W(u) \longrightarrow L = \int_{m}^{u_0} \frac{d\tau}{\rho(\tau)} > \int_{t_0}^{T} \gamma(s)ds, \quad \text{as } u \longrightarrow m. \tag{7.13}$$

Thus, for each $t \in [t_0, T)$, there is a unique $u \in [m, u_0]$ such that

$$u = W^{-1}\left(\int_{t_0}^{t} \gamma(s)ds\right) \tag{7.14}$$

is the unique solution of (7.11), and it depends continuously on u_0. □

Theorem 7.4. *Assume, in addition to the hypotheses of Theorem 7.1, that*

$$||g(t,x)|| \leq \gamma(t)\rho(\|x\|), \quad x \in B_0,\ t \in I_0, \tag{7.15}$$

where $\gamma(t)$ and $\rho(t)$ satisfy the hypotheses of Lemma 7.2 with $T = t_0 + T_1$. Let $u(t)$ be the positive solution of

$$u'(t) = \gamma(t)\rho(u(t)), \quad t \in [t_0,T),\ u(t_0) = u_0 \geq ||x_0||, \tag{7.16}$$

provided by Lemma 7.2. Then the unique solution of (7.4) satisfies

$$||x(t)|| \leq u(t), \quad t \in [t_0,T). \tag{7.17}$$

Proof. Assume that there is a $t_1 \in [t_0,T)$ such that

$$u(t_1) < ||x(t_1)||. \tag{7.18}$$

For $\varepsilon > 0$, let $u_\varepsilon(t)$ be the solution of

$$u'(t) = [\gamma(t)+\varepsilon]\rho(u(t)), \quad t \in [t_0,T),\ u(t_0) = u_0. \tag{7.19}$$

By Lemma 7.2, a solution exists for $\varepsilon > 0$ sufficiently small. Moreover, $u_\varepsilon(t) \to u(t)$ uniformly on any compact subset of $[t_0,T)$. Let

$$w(t) = ||x(t)|| - u_\varepsilon(t). \tag{7.20}$$

Then, we may take ε sufficiently small so that

$$w(t_0) \leq 0, \qquad w(t_1) > 0. \tag{7.21}$$

Let t_2 be the largest number in $[t_0,t_1)$ such that $w(t_2) = 0$ and

$$w(t) > 0, \quad t \in (t_2,t_1]. \tag{7.22}$$

For $h > 0$ sufficiently small, we have

$$\frac{w(t_2+h) - w(t_2)}{h} > 0. \tag{7.23}$$

Consequently,

$$D^+w(t_2) \geq 0. \tag{7.24}$$

But

$$\begin{aligned} D^+w(t_2) &= D^+||x(t_2)|| - u_\varepsilon'(t_2) \leq ||x'(t_2)|| - u_\varepsilon'(t_2) \\ &= ||g(t_2,x(t_2))|| - [\gamma(t_2)+\varepsilon]\rho(u_\varepsilon(t_2)) \\ &\leq \gamma(t_2)\rho(||x(t_2)||) - [\gamma(t_2)+\varepsilon]\rho(u_\varepsilon(t_2)) = -\varepsilon\rho(u_\varepsilon(t_2)) < 0. \end{aligned} \tag{7.25}$$

This contradiction proves the theorem. □

Theorem 7.5. *Let $g(t,x)$ be a continuous map from $\mathbb{R}\times H$ to H, where H is a Banach space. Assume that for each point $(t_0,x_0)\in\mathbb{R}\times H$, there are constants $K,b>0$ such that*

$$||g(t,x)-g(t,y)||\le K\|x-y\|,\quad |t-t_0|<b,\ ||x-x_0||<b,\ ||y-x_0||<b. \tag{7.26}$$

Assume also that

$$||g(t,x)||\le\gamma(t)\rho(\|x\|),\quad x\in H,\ t\in[t_0,T_M), \tag{7.27}$$

where $T_M\le\infty$, and $\gamma(t)$, $\rho(t)$ satisfy the hypotheses of Lemma 7.2 with ρ nondecreasing. Then for each $x_0\in H$ and $t_0>0$, there is a unique solution $x(t)$ of the equation

$$\frac{dx(t)}{dt}=g(t,x(t)),\quad t\in[t_0,T_M),\ x(t_0)=x_0. \tag{7.28}$$

Moreover, $x(t)$ depends continuously on x_0 and satisfies

$$||x(t)||\le u(t),\quad t\in[t_0,T_M), \tag{7.29}$$

where $u(t)$ is the solution of (7.6) in that interval satisfying $u(t_0)=u_0\ge\|x_0\|$.

Before proving Theorem 7.5, we note that the following is an immediate consequence.

Corollary 7.6. *Let $V(y)$ be a locally Lipschitz continuous map from H to itself satisfying*

$$||V(y)||\le C(1+\|y\|),\quad y\in H. \tag{7.30}$$

Then for each $y_0\in H$, there is a unique solution of

$$y'(t)=V(y(t)),\quad t\in\mathbb{R}^+,\ y(0)=y_0. \tag{7.31}$$

We now give the proof of Theorem 7.5.

Proof. By Theorems 7.1 and 7.4, there is an interval $[t_0,t_0+m]$, $m>0$, in which a unique solution of

$$\frac{dx(t)}{dt}=g(t,x(t)),\quad t\in[t_0,t_0+m],\ x(t_0)=x_0, \tag{7.32}$$

exists and satisfies

$$||x(t)||\le u(t),\quad t\in[t_0,t_0+m], \tag{7.33}$$

where $u(t)$ is the unique solution of

$$u'(t)=\gamma(t)\rho(u(t)),\quad t\in[t_0,T_M),\ u(t_0)=u_0=||x_0||. \tag{7.34}$$

Let $T\le T_M$ be the supremum of all numbers t_0+m for which this holds. If $t_1<t_2<T$, then the solution in $[t_0,t_2]$ coincides with that in $[t_0,t_1]$, since such solutions are unique.

Thus a unique solution of (7.32) satisfying (7.33) exists for each $t_0 < t < T$. Moreover, we have

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} g(t,x(t))\,dt. \tag{7.35}$$

Consequently,

$$\begin{aligned} ||x(t_2) - x(t_1)|| &\le \int_{t_1}^{t_2} ||g(t,x(t))||dt \le \int_{t_1}^{t_2} \gamma(t)\rho(||x(t)||)\,dt \\ &\le \int_{t_1}^{t_2} \gamma(t)\rho(u(t))\,dt = u(t_2) - u(t_1). \end{aligned} \tag{7.36}$$

Assume that $T < T_M$. Let t_k be a sequence such that $t_0 < t_k < T$ and $t_k \to T$. Then

$$||x(t_k) - x(t_j)|| \le u(t_k) - u(t_j) \longrightarrow 0. \tag{7.37}$$

Thus $\{x(t_k)\}$ is a Cauchy sequence in H. Since H is complete, $x(t_k)$ converges to an element $x_1 \in H$. Since $\|x(t_k)\| \le u(t_k)$, we see that $\|x_1\| \le u(T)$. Moreover, we note that

$$x(t) \longrightarrow x_1 \quad \text{as } t \longrightarrow T. \tag{7.38}$$

To see this, let $\varepsilon > 0$ be given. Then there is a k such that

$$||x(t_k) - x_1|| < \varepsilon, \quad u(T) - u(t_k) < \varepsilon. \tag{7.39}$$

Then for $t_k \le t < T$,

$$\begin{aligned} ||x(t) - x_1|| &\le ||x(t) - x(t_k)|| + ||x(t_k) - x_1|| \\ &\le u(t) - u(t_k) + ||x(t_k) - x_1|| < 2\varepsilon. \end{aligned} \tag{7.40}$$

We define $x(T) = x_1$. Then, we have a solution of (7.32) satisfying (7.33) in $[0,T]$. By Theorem 7.1, there is a unique solution of

$$\frac{dy(t)}{dt} = g(t,y(t)), \quad y(T) = x_1, \tag{7.41}$$

satisfying $\|y(t)\| \le u(t)$ in some interval $|t - T| < \delta$. By uniqueness, the solution of (7.41) coincides with the solution of (7.32) in the interval $(T-\delta,T]$. Define

$$\begin{gathered} z(t) = x(t), \quad t_0 \le t < T, \\ z(T) = x_1, \\ z(t) = y(t), \quad T < t \le T + \delta. \end{gathered} \tag{7.42}$$

This gives a solution of (7.32) satisfying (7.33) in the interval $[t_0, T+\delta)$, contradicting the definition of T. Hence, $T = T_M$. □

Lemma 7.7. *Let ρ, γ satisfy the hypotheses of Lemma 7.3, with ρ locally Lipschitz continuous. Let $u(t)$ be the solution of (7.11), and let $h(t)$ be a continuous function satisfying*

$$h(t) \geq h(s) - \int_s^t \gamma(r)\rho(h(r))dr, \quad t_0 \leq s < t < T,\ h(t_0) \geq u_0. \tag{7.43}$$

Then

$$u(t) \leq h(t), \quad t \in [t_0, T). \tag{7.44}$$

Proof. Assume that there is a point t_1 in the interval such that

$$h(t_1) < u(t_1). \tag{7.45}$$

Let

$$y(t) = u(t) - h(t), \quad t \in [t_0, T). \tag{7.46}$$

Then, $y(t_0) \leq 0$ and $y(t_1) > 0$. Let τ be the largest point $< t_1$ such that $y(\tau) = 0$. Then

$$y(t) > 0, \quad t \in (\tau, t_1]. \tag{7.47}$$

Moreover, by (7.11) and (7.43), we have

$$y(t) \leq -\int_\tau^t \gamma(s)[\rho(u(s)) - \rho(h(s))]ds \leq L\int_\tau^t y(s)ds, \tag{7.48}$$

where L is the Lipschitz constant for ρ at $u(\tau)$ times the maximum of γ in the interval. Let

$$w(t) = \int_\tau^t y(s)ds. \tag{7.49}$$

Then

$$[e^{-Lt}w(t)]' = e^{-Lt}[y(t) - Lw(t)] \leq 0, \quad t \in [\tau, t_1]. \tag{7.50}$$

Consequently,

$$e^{-Lt}w(t) \leq e^{-L\tau}w(\tau) = 0, \quad t \in [\tau, t_1]. \tag{7.51}$$

Hence,

$$y(t) \leq Lw(t) \leq 0, \quad t \in [\tau, t_1], \tag{7.52}$$

contradicting (7.47). This completes the proof. □

8. Cerami sequences

We are now ready for the proof of Theorem 2.2.

Proof. First we note that if the theorem were false, there would be a $\delta >$ and a ψ satisfying (2.5) such that

$$\|G'(u)\| \geq \psi(\|u\|) \tag{8.1}$$

when

$$u \in Q = \{u \in E : |G(u) - a| \leq 3\delta\}. \tag{8.2}$$

Assume first that $b_0 < a$, and reduce δ so that $3\delta < a - b_0$. Since $G \in C^1(E, \mathbb{R})$, there is a locally Lipschitz continuous mapping $Y(u)$ of $\widehat{E} = \{u \in E : G'(u) \neq 0\}$ into E such that

$$\|Y(u)\| \leq 1, \quad \theta\|G'(u)\| \leq (G'(u), Y(u)), \quad u \in \widehat{E} \tag{8.3}$$

holds for some $\theta > 0$ (cf., e.g., [7]). Let

$$\begin{aligned} Q_0 &= \{u \in E : |G(u) - a| \leq 2\delta\}, \\ Q_1 &= \{u \in E : |G(u) - a| \leq \delta\}, \\ Q_2 &= E \setminus Q_0, \\ \eta(u) &= \frac{d(u, Q_2)}{[d(u, Q_1) + d(u, Q_2)]}. \end{aligned} \tag{8.4}$$

It is easily checked that $\eta(u)$ is locally Lipschitz continuous on E and satisfies

$$\begin{aligned} \eta(u) &= 1, \quad u \in Q_1, \\ \eta(u) &= 0, \quad u \in \overline{Q}_2, \\ \eta(u) &\in (0,1) \quad \text{otherwise}. \end{aligned} \tag{8.5}$$

Let $\rho(t) = 1/\psi(t)$. Then ρ is a positive, nondecreasing, locally Lipschitz continuous function on $[0, \infty)$ such that

$$\int_0^\infty \frac{d\tau}{\rho(\tau)} = \infty \tag{8.6}$$

by (2.5). Let

$$W(u) = -\eta(u) Y(u) \rho(\|u\|). \tag{8.7}$$

Then

$$\|W(u)\| \leq \rho(\|u\|), \quad u \in E. \tag{8.8}$$

By Theorem 7.5, for each $u \in E$ there is a unique solution of

$$\sigma'(t) = W(\sigma(t)), \quad t \in \mathbb{R}^+, \ \sigma(0) = u. \tag{8.9}$$

We have

$$\frac{dG(\sigma(t)u)}{dt} = -\eta(\sigma(t)u)\left(G'(\sigma(t)u), Y(\sigma(t)u)\right)\rho(||\sigma(t)u||) \\ \leq -\theta\eta(\sigma)||G'(\sigma)||\rho(\|\sigma\|) \leq -\theta\eta(\sigma). \tag{8.10}$$

By the definition (2.4) of a, there is a $\Gamma \in \Phi$ such that

$$G(\Gamma(s)u) < a+\delta, \quad s \in [0,1],\ u \in A. \tag{8.11}$$

Let $v = \Gamma(s)u$, where $s \in [0,1]$ and $u \in A$. If there is a $t_1 \leq T$ such that $\sigma(t_1)v \notin Q_1$, then

$$G(\sigma(T)v) < a-\delta, \tag{8.12}$$

since

$$G(\sigma(T)v) \leq G(\sigma(t_1)v) \tag{8.13}$$

and the right-hand side cannot be greater than $a+\delta$ by (8.11). On the other hand, if $\sigma(t)v \in Q_1$ for all $t \in [0,T]$, then we have by (8.10) that

$$G(\sigma(T)v) \leq a+\delta-\theta\int_0^T dt < a-\delta \tag{8.14}$$

if we take $T \geq 3\delta/\theta$. Hence

$$G(\sigma(T)\Gamma(s)u) < a-\delta, \quad s \in [0,1],\ u \in A. \tag{8.15}$$

Let

$$\Gamma_1(s) = \begin{cases} \sigma(2sT), & 0 \leq s \leq \dfrac{1}{2}, \\ \sigma(T)\Gamma(2s-1), & \dfrac{1}{2} < s \leq 1. \end{cases} \tag{8.16}$$

Then $\Gamma_1 \in \Phi$. Since

$$G(\sigma(t)u) \leq a_0, \quad t \geq 0, \tag{8.17}$$

we see by (8.15) that

$$G(\Gamma_1(s)u) < a-\delta, \quad s \in [0,1],\ u \in A. \tag{8.18}$$

But this contradicts the definition (2.4) of a. Hence (8.1) cannot hold for u satisfying (8.2).

If $b_0 = a$, we proceed as before, but we cannot use (8.17) to imply (8.18). However, we note that (8.10) implies that

$$G(\sigma(t)u) \leq b_0 - \theta\int_0^t \eta(\sigma(\tau)u)\,d\tau \tag{8.19}$$

for $u \in A$. This shows that

$$\sigma(t)A \cap B = \phi, \quad t \geq 0. \tag{8.20}$$

To see this, note that the only way we can have $\sigma(t)u \in B$ is if

$$\eta(\sigma(\tau)u) \equiv 0, \quad 0 \leq \tau \leq t. \tag{8.21}$$

But this implies that $\sigma(\tau)u \in \overline{Q}_2$, and consequently that

$$G(\sigma(\tau)u) < a - \delta, \quad 0 \leq \tau \leq t, \tag{8.22}$$

which cannot happen if $\sigma(\tau)u \in B$. Thus (8.20) holds. Similarly, (8.15) shows that

$$\sigma(T)\Gamma(t)A \cap B = \phi, \quad 0 \leq t \leq 1. \tag{8.23}$$

Combining (8.20) and (8.23), we see that

$$\Gamma_1(s)A \cap B = \phi, \quad 0 \leq s \leq 1, \tag{8.24}$$

contradicting the fact that A links B. This completes the proof of the theorem. □

9. The remaining proofs

We can now prove Theorem 3.1.

Proof. We may assume that $a = a_0$. Otherwise by Corollary 2.6, a Cerami sequence (2.6) exists with ψ replaced by $\tilde{\psi}(t) = \psi(t+\alpha)$. Since $\tilde{\psi}$ satisfies the hypotheses of Theorem 2.2 and

$$d(u,A) \leq \|u\| + \alpha, \tag{9.1}$$

for each $\delta > 0$ we can find a $u \in E$ such that

$$a - \delta \leq G(u) \leq a + \delta, \qquad ||G'(u)|| < \tilde{\psi}(\|u\|) \leq \psi(d(u,A)), \tag{9.2}$$

which certainly implies (3.5). If the conclusion of the theorem was not true, there would be a $\delta > 0$ such that

$$\psi(d(u,A)) \leq ||G'(u)|| \tag{9.3}$$

would hold for all u in the set

$$Q = \{u \in E : b_0 - 3\delta \leq G(u) \leq a + 3\delta\}. \tag{9.4}$$

By reducing δ if necessary, we can find $\theta < 1$, $T > 0$ such that

$$a_0 - b_0 + \delta < \theta T, \quad T \leq \int_{\delta+\alpha}^{R+\alpha} \psi(s)ds. \tag{9.5}$$

Thus, by Lemma 7.3, if $u(t)$ is the solution of (7.11) with $\rho(t) = 1/\psi(t)$, $\gamma = 1$, $t_0 = 0$, and $u_0 = R$, then

$$u(t) \geq \delta, \quad t \in [0, T]. \tag{9.6}$$

Let

$$\begin{aligned} Q_0 &= \{u \in Q : b_0 - 2\delta \leq G(u) \leq a + 2\delta\} \\ Q_1 &= \{u \in Q : b_0 - \delta \leq G(u) \leq a + \delta\}, \\ Q_2 &= E \setminus Q_0, \qquad \eta(u) = \frac{d(u, Q_2)}{[d(u, Q_1) + d(u, Q_2)]}. \end{aligned} \tag{9.7}$$

As before, we note that η satisfies (8.5). There is a locally Lipschitz continuous map $Y(u)$ of $\widehat{E} = \{u \in E : G'(u) \neq 0\}$ into itself such that

$$||Y(u)|| \leq 1, \quad \theta||G'(u)|| \leq (G'(u), Y(u)), \quad u \in \widehat{E} \tag{9.8}$$

(cf., e.g., [7]). Let $\sigma(t)$ be the flow generated by

$$W(u) = \eta(u)Y(u)\rho(d(u, A)), \tag{9.9}$$

where $\rho(\tau) = 1/\psi(\tau)$. Since $\|W(u)\| \leq \rho(d(u, A)) \leq \widetilde{\rho}(\|u\|) = 1/\widetilde{\psi}(\|u\|)$ and is locally Lipschitz continuous, $\sigma(t)$ exists for all $t \in \mathbb{R}^+$ in view of Theorem 7.5. Since

$$\sigma(t)v - v = \int_0^t W(\sigma(\tau)v)\, d\tau, \tag{9.10}$$

we have

$$||\sigma(t)v - \sigma(s)v|| \leq \int_s^t \rho(d(\sigma(r)v, A))\, dr. \tag{9.11}$$

If $u \in A$, we have

$$h(s) = d(\sigma(s)v, A) \leq ||\sigma(s)v - u|| \leq ||\sigma(t)v - u|| + \int_s^t \rho(d(\sigma(r)v, A))\, dr. \tag{9.12}$$

This implies that

$$h(s) \leq h(t) + \int_s^t \rho(h(r))\, dr. \tag{9.13}$$

We also have

$$\begin{aligned} \frac{dG(\sigma(t)v)}{dt} &= (G'(\sigma), \sigma') = \eta(\sigma)(G'(\sigma), Y(\sigma))\rho(d(\sigma, A)) \\ &\geq \theta\eta(\sigma)||G'(\sigma)||\rho(d(\sigma, A)) \geq \theta\eta(\sigma)\psi(d(\sigma, A))\rho(d(\sigma, A)) = \theta\eta(\sigma) \end{aligned} \tag{9.14}$$

in view of (9.3) and (9.8). Now suppose $v \in B$ is such that there is a $t_1 \in [0,T]$ for which $\sigma(t_1)v \notin Q_1$. Then

$$G(\sigma(t_1)v) > a+\delta, \tag{9.15}$$

since we cannot have $G(\sigma(t_1)v) < b_0 - \delta$ for $v \in B$ by (9.14). But this implies that

$$G(\sigma(T)v) > a+\delta. \tag{9.16}$$

On the other hand, if $\sigma(t)v \in Q_1$ for all $t \in [0,T]$, then

$$G(\sigma(T)v) \geq G(v) + \theta \int_0^T dt \geq b_0 + \theta T > a + \delta \tag{9.17}$$

by (9.5). Thus, (9.16) holds for $v \in B$. the author claims that A links $B_1 = \sigma(T)B$. Assume this for the moment. By the definition (2.4) of a, there is a $\Gamma \in \Phi$ such that

$$G(\Gamma(s)u) < a + \frac{\delta}{2}, \quad 0 \leq s \leq 1,\ u \in A. \tag{9.18}$$

But if A links B_1, then there is a $t_1 \in [0,1]$ such that $\Gamma(t_1)A \cap B_1 \neq \phi$. This means that there is a $u_1 \in A$ such that $\Gamma(t_1)u_1 \in B_1$. In view of (9.16), this would imply that

$$G(\Gamma(t_1)u_1) > a+\delta, \tag{9.19}$$

contradicting (9.18). Thus it remains only to show that A links B_1. To this end, we note that $\sigma(t)v \notin A$ for $v \in B$ and $t \in [0,T]$. For $v \in B'$, this follows from (9.13) and the fact that

$$h(t) = d(\sigma(t)v, A) \geq u(t) \geq \delta, \quad t \in [0,T], \tag{9.20}$$

in view of Lemma 7.7. If $v \in B \setminus B'$, we have by (9.14) that

$$G(\sigma(t)v) \geq a + \theta \int_0^t \eta(\sigma(\tau)v)\, d\tau > a, \quad t > 0, \tag{9.21}$$

unless $\eta(v) = 0$. But this would mean that $v \in \overline{Q}_2$ in view of (8.5). But then we would have $G(v) \geq a + 2\delta$ since we cannot have $G(v) \leq b_0 - 2\delta$ for $v \in B$. Thus,

$$G(\sigma(t)v) > a, \quad t > 0,\ v \in B \setminus B'. \tag{9.22}$$

Hence

$$A \cap \sigma(t)B = \phi, \quad 0 \leq t \leq T. \tag{9.23}$$

Let Γ be any map in Φ. Define

$$\Gamma_1(s) = \begin{cases} \sigma(2sT)^{-1}, & 0 \leq s \leq \dfrac{1}{2}, \\ \sigma(T)^{-1}\Gamma(2s-1), & \dfrac{1}{2} < s \leq 1. \end{cases} \tag{9.24}$$

Clearly, $\Gamma_1 \in \Phi$. Since A links B, there is a $t_1 \in [0,1]$ such that $\Gamma_1(t_1)A \cap B \neq \phi$. If $0 \leq t_1 \leq 1/2$, this would mean that

$$\sigma(2t_1 T)^{-1} A \cap B \neq \phi \tag{9.25}$$

or, equivalently, that

$$A \cap \sigma(2t_1 T) B \neq \phi, \tag{9.26}$$

contradicting (9.23). Thus we must have $1/2 < t_1 \leq 1$. This says that

$$\sigma(T)^{-1}\Gamma(2t_1 - 1)A \cap B \neq \phi \tag{9.27}$$

or, equivalently,

$$\Gamma(2t_1 - 1)A \cap \sigma(T)B \neq \phi. \tag{9.28}$$

Hence A links B_1, and the proof is complete. □

We also give the proof of Theorem 3.2.

Proof. Again, we may assume that $a = a_0$. We interchange A and B and consider the functional $\widetilde{G}(u) = -G(u)$. Then

$$\begin{aligned} \widetilde{a}_0 &= \sup_B \widetilde{G} = -\inf_B G = -b_0 < \infty, \\ \widetilde{b}_0 &= \inf_A \widetilde{G} = -\sup_A G = -a_0 > -\infty. \end{aligned} \tag{9.29}$$

Moreover,

$$\widetilde{a}_0 - \widetilde{b}_0 = a_0 - b_0 < \int_\beta^{R+\beta} \psi(t)\,dt, \tag{9.30}$$

where

$$R \leq d'' = d(A'', B). \tag{9.31}$$

Since

$$A'' = \{u \in A : \widetilde{G}(u) < \widetilde{a}_0\}, \tag{9.32}$$

we can apply Theorem 3.1 to conclude that for each $\delta > 0$, there is a $u \in E$ such that

$$\widetilde{b}_0 - \delta \leq \widetilde{G}(u) \leq \widetilde{a}_0 + \delta, \qquad \|\widetilde{G}'(u)\| < \psi(d(u,B)). \tag{9.33}$$

This implies (3.9). □

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