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Research Article

Legendre's Differential Equation and Its Hyers-Ulam Stability

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We solve the nonhomogeneous Legendre's differential equation and apply this result to obtaining a partial solution to the Hyers-Ulam stability problem for the Legendre's equation.

1. Introduction

In 1940, S. M. Ulam gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems [1]. Among those was the question concerning the stability of homomorphisms. Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given any $\delta > 0$, does there exist an $\varepsilon > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \varepsilon$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \delta$ for all $x \in G_1$?

In the following year, Hyers [2] partially solved the Ulam's problem for the case where G_1 and G_2 are Banach spaces. Furthermore, the result of Hyers has been generalized by Rassias [3]. Since then, the stability problems of various functional equations have been investigated by many authors (see [4–7]).

We will now consider the Hyers-Ulam stability problem for the differential equations. Assume that X is a normed space over a scalar field $\mathbb{K}$ and that I is an open interval, where $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. Let $a_0, a_1, \ldots, a_n : I \to \mathbb{K}$ be given continuous functions, let $g : I \to X$ be a given continuous function, and let $y : I \to X$ be an n times continuously differentiable function satisfying the inequality

$$\left\|a_n(t)y^{(n)}(t)+a_{n-1}(t)y^{(n-1)}(t)+\cdots+a_1(t)y'(t)+a_0(t)y(t)+g(t)\right\|\le\varepsilon \tag{1.1}$$

for all $t\in I$ and for a given $\varepsilon>0$. If there exists an n times continuously differentiable function $y_0 : I\to X$ satisfying

$$a_n(t)y_0^{(n)}(t)+a_{n-1}(t)y_0^{(n-1)}(t)+\cdots+a_1(t)y_0'(t)+a_0(t)y_0(t)+g(t)=0 \tag{1.2}$$

and $\|y(t)-y_0(t)\|\le K(\varepsilon)$ for any $t\in I$, where $K(\varepsilon)$ is an expression of ε with $\lim_{\varepsilon\to 0}K(\varepsilon)=0$, then we say that the above differential equation has the Hyers-Ulam stability. For more detailed definitions of the Hyers-Ulam stability, we refer the reader to [4–6].

Alsina and Ger were the first authors who investigated the Hyers-Ulam stability of differential equations. They proved in [8] that if a differentiable function $f : I\to\mathbb{R}$ is a solution of the differential inequality $|y'(t)-y(t)|\le\varepsilon$, where I is an open subinterval of $\mathbb{R}$, then there exists a solution $f_0 : I\to\mathbb{R}$ of the differential equation $y'(t)=y(t)$ such that $|f(t)-f_0(t)|\le 3\varepsilon$ for any $t\in I$.

This result of Alsina and Ger has been generalized by Takahasi et al. They proved in [9] that the Hyers-Ulam stability holds true for the Banach space valued differential equation $y'(t)=\lambda y(t)$ (see also [10, 11]).

Moreover, Miura et al. [12] investigated the Hyers-Ulam stability of the nth order linear differential equation with complex coefficients. They [13] also proved the Hyers-Ulam stability of linear differential equations of first order, $y'(t)+g(t)y(t)=0$, where $g(t)$ is a continuous function. Indeed, they dealt with the differential inequality $\|y'(t)+g(t)y(t)\|\le\varepsilon$ for some $\varepsilon>0$. Recently, the author proved the Hyers-Ulam stability of various linear differential equations of the first order (see [14–17]).

In Section 2 of this paper, we will investigate the general solution of the nonhomogeneous Legendre's differential equation of the form

$$(1-x^2)y''(x)-2xy'(x)+p(p+1)y(x)=\sum_{m=0}^{\infty}a_m x^m, \tag{1.3}$$

where the parameter p is a given real number and the coefficients a_m's of the power series are given such that the radius of convergence is positive.

In Section 3, we will give a partial solution to the Hyers-Ulam stability problem for the Legendre's differential equation (2.1) in the class of analytic functions.

2. Nonhomogeneous Legendre's equation

A function is called a Legendre function if it satisfies the Legendre's differential equation

$$(1-x^2)y''(x)-2xy'(x)+p(p+1)y(x)=0. \tag{2.1}$$

The Legendre's equation plays a great role in physics and engineering. In particular, this equation is most useful for treating the boundary value problems exhibiting spherical symmetry.

In this section, we define

$$c_m = \frac{1}{m!}\sum_{i=1}^{[m/2]}(m-2i)!a_{m-2i}\prod_{j=1}^{i-1}(m-2j-p)(m-2j+p+1) \tag{2.2}$$

for each $m \in \{2,3,\dots\}$, where $[m/2]$ denotes the largest integer not exceeding $m/2$ and we refer to (1.3) for the a_m's. By some manipulations, we get

$$c_{m+2} = \frac{1}{(m+2)(m+1)}a_m + \frac{(m-p)(m+p+1)}{(m+2)(m+1)}c_m \tag{2.3}$$

for any $m \in \{2,3,\dots\}$.

Using these definitions and relations above, we will solve the nonhomogeneous Legendre's equation (1.3).

Theorem 2.1. *Assume that p is a given real number and the radius of convergence of the power series $\sum_{m=0}^{\infty} a_m x^m$ is $\rho_0 > 0$. Moreover, suppose that there exist real numbers σ_1 and σ_2 with*

$$\sigma_1 = \begin{cases} \lim\limits_{k\to\infty}\left|\dfrac{1}{(2k+2)(2k+1)}\dfrac{a_{2k}}{c_{2k}}\right| & \text{if the limit exists} \\ -1 & \text{if } c_{2k} = 0 \text{ for all sufficiently large } k, \end{cases}$$
$$\sigma_2 = \begin{cases} \lim\limits_{k\to\infty}\left|\dfrac{1}{(2k+3)(2k+2)}\dfrac{a_{2k+1}}{c_{2k+1}}\right| & \text{if the limit exists} \\ -1 & \text{if } c_{2k+1} = 0 \text{ for all sufficiently large } k. \end{cases} \tag{2.4}$$

A positive number ρ is defined by

$$\rho = \min\left\{\frac{1}{\sqrt{1+\sigma_1}}, \frac{1}{\sqrt{1+\sigma_2}}, \rho_0, 1\right\} \tag{2.5}$$

with the convention $1/0 = \infty$. Then, every solution $y : (-\rho,\rho) \to \mathbb{C}$ of the differential equation (1.3) can be expressed by

$$y(x) = y_h(x) + \sum_{m=2}^{\infty} c_m x^m, \tag{2.6}$$

where $y_h(x)$ is a Legendre function.

Remark 2.2. If $c_{2k} = 0$ for all sufficiently large k, then $\sum_{k=1}^{\infty} c_{2k}x^{2k}$ is indeed a polynomial which can obviously be defined on the whole real numbers and this fact is not contrary to our definition $\sigma_1 = -1$, since in this case we have

$$\begin{aligned}\rho &= \min\left\{\frac{1}{\sqrt{1+\sigma_1}}, \frac{1}{\sqrt{1+\sigma_2}}, \rho_0, 1\right\} \\ &= \min\left\{\frac{1}{\sqrt{1+\sigma_2}}, \rho_0, 1\right\}.\end{aligned} \tag{2.7}$$

A similar argument is applicable to σ_2.

Proof. Since each coefficient of (1.3) is analytic at $x = 0$, every solution of (1.3) can be expressed as a power series of the form

$$y(x) = \sum_{m=0}^{\infty} b_m x^m. \tag{2.8}$$

(0 is an ordinary point of (1.3) and ± 1 are the nearest singular points of the equation. So, the radius of convergence of the above power series is at least 1. This fact is consistent with the domain of y).

Substituting (2.8) into (1.3) and collecting like powers together, we have

$$\begin{aligned}&(1-x^2)y''(x) - 2xy'(x) + p(p+1)y(x) \\ &\quad= \sum_{m=0}^{\infty}\{(m+2)(m+1)b_{m+2} - (m-p)(m+p+1)b_m\}x^m = \sum_{m=0}^{\infty} a_m x^m\end{aligned} \tag{2.9}$$

for all $x \in (-\rho, \rho)$. Comparing the coefficients of like powers of two power series, we get

$$b_{m+2} = \frac{1}{(m+2)(m+1)}a_m + \frac{(m-p)(m+p+1)}{(m+2)(m+1)}b_m \tag{2.10}$$

for any $m \in \{0, 1, 2, \dots\}$.

We now assert that

$$b_m = c_m + \frac{b_{m-2[m/2]}}{m!}\prod_{j=1}^{[m/2]}(m-2j-p)(m-2j+p+1) \tag{2.11}$$

for any $m \in \{2, 3, \dots\}$.

By the mathematical induction on m, we will prove the formula (2.11) for all even integers m. If we put $m = 2$ in (2.11) and recall the definition (2.2), then we obtain

$$b_2 = c_2 - \frac{p(p+1)}{2!} b_0 = \frac{1}{2!} a_0 - \frac{p(p+1)}{2!} b_0 \tag{2.12}$$

which is identical with the formula induced from (2.10) for $m = 0$. Assume now that formula (2.11) is true for some even m. It then follows from (2.10), (2.11), and (2.2) that

$$\begin{aligned}
b_{m+2} &= \frac{m!}{(m+2)!} a_m \\
&\quad + \frac{1}{(m+2)!} \sum_{i=1}^{[m/2]} (m-2i)! a_{m-2i} \prod_{j=0}^{i-1} (m-2j-p)(m-2j+p+1) \\
&\quad + \frac{b_0}{(m+2)!} \prod_{j=0}^{[m/2]} (m-2j-p)(m-2j+p+1) \\
&= \frac{1}{(m+2)!} \sum_{i=0}^{[m/2]} (m-2i)! a_{m-2i} \prod_{j=0}^{i-1} (m-2j-p)(m-2j+p+1) \\
&\quad + \frac{b_0}{(m+2)!} \prod_{j=0}^{[m/2]} (m-2j-p)(m-2j+p+1) \\
&= \frac{1}{(m+2)!} \sum_{i=1}^{[m/2]+1} (m+2-2i)! a_{m+2-2i} \\
&\quad \cdot \prod_{j=1}^{i-1} (m+2-2j-p)(m+2-2j+p+1) \\
&\quad + \frac{b_0}{(m+2)!} \prod_{j=1}^{[m/2]+1} (m+2-2j-p)(m+2-2j+p+1) \\
&= c_{m+2} + \frac{b_0}{(m+2)!} \prod_{j=1}^{[m/2]+1} (m+2-2j-p)(m+2-2j+p+1),
\end{aligned} \tag{2.13}$$

which is identical with formula (2.11) when m is replaced by $m+2$. (We assume that $\prod_{j=1}^{i-1}(\cdots)(\cdots) = 1$ for $i \leq 1$.) Hence, (2.11) is valid for any even m. Similarly, we can verify that (2.11) is true for all odd m.

Consequently, it follows from (2.8) and (2.11) that

$$
\begin{aligned}
y(x) &= b_0 + b_1 x + \sum_{k=1}^{\infty} b_{2k} x^{2k} + \sum_{k=1}^{\infty} b_{2k+1} x^{2k+1} \\
&= \sum_{k=1}^{\infty} c_{2k} x^{2k} + \sum_{k=1}^{\infty} c_{2k+1} x^{2k+1} \\
&\quad + b_0 \left[1 + \sum_{k=1}^{\infty} \frac{x^{2k}}{(2k)!} \prod_{j=1}^{k} (2k-2j-p)(2k-2j+p+1) \right] \\
&\quad + b_1 \left[x + \sum_{k=1}^{\infty} \frac{x^{2k+1}}{(2k+1)!} \prod_{j=1}^{k} (2k-2j-p+1)(2k-2j+p+2) \right] \\
&= y_h(x) + \sum_{m=2}^{\infty} c_m x^m,
\end{aligned}
\tag{2.14}
$$

where y_h stands for the last two power series, that is,

$$
y_h(x) = b_0 \left[1 + \sum_{k=1}^{\infty} \cdots \right] + b_1 \left[x + \sum_{k=1}^{\infty} \cdots \right]. \tag{2.15}
$$

Using the ratio test, we can easily show that the power series in the brackets converge for each $x \in (-1,1)$. For any real numbers b_0 and b_1, $y_h(x)$ is a Legendre function, that is, it is a solution of the Legendre's equation (2.1) (see [18]).

Furthermore, in view of (2.3) and (2.4), we can apply the ratio test and show that power series

$$
\sum_{k=1}^{\infty} c_{2k} x^{2k}, \quad \sum_{k=1}^{\infty} c_{2k+1} x^{2k+1} \quad \text{converge for all } x \in (-\rho, \rho). \tag{2.16}
$$

We will now show that each function $y : (-\rho,\rho) \to \mathbb{C}$ defined by

$$
y(x) = y_h(x) + \sum_{m=2}^{\infty} c_m x^m \tag{2.17}
$$

is a solution of the nonhomogeneous Legendre differential equation (1.3), where $y_h(x)$ is a Legendre funcion and c_m is given by (2.2). For this purpose, it only needs to show that

$$
y_p(x) = \sum_{m=2}^{\infty} c_m x^m \tag{2.18}
$$

satisfies (1.3). It is not difficult to see

$$\begin{aligned}(1-x^2)y_p''(x) &- 2xy_p'(x) + p(p+1)y_p(x)\\ &= 2c_2 + 6c_3x + \sum_{m=2}^{\infty}\{(m+2)(m+1)c_{m+2} - (m-p)(m+p+1)c_m\}x^m \\ &= a_0 + a_1x + \sum_{m=2}^{\infty} a_m x^m,\end{aligned} \tag{2.19}$$

since we obtain $a_0 = 2c_2$ and $a_1 = 6c_3$ by putting $m = 2$ and $m = 3$ in (2.2), respectively, and since it follows from (2.3) that

$$(m+2)(m+1)c_{m+2} - (m-p)(m+p+1)c_m = a_m \tag{2.20}$$

for all $m \in \{2,3,\dots\}$. □

Corollary 2.3. *Under the same notations and conditions of Theorem 2.1, it holds that*

$$\sum_{m=2}^{\infty} c_m x^m = \sum_{i=1}^{\infty} x^{2i} \sum_{m=0}^{\infty} \frac{a_m x^m}{(m+2i)(m+1)} \prod_{j=1}^{i-1}\left\{1 - \frac{p(p+1)}{(m+2i-2j+1)(m+2i-2j)}\right\} \tag{2.21}$$

for any $x \in (-\rho,\rho)$.

Proof. Since

$$\frac{(m-2i)!}{m!} = \frac{1}{m(m-2i+1)} \prod_{j=1}^{i-1} \frac{1}{(m-2j+1)(m-2j)}, \tag{2.22}$$

it follows from (2.2) that

$$\begin{aligned}\sum_{m=2}^{\infty} c_m x^m &= \sum_{m=2}^{\infty} \frac{x^m}{m!} \sum_{i=1}^{[m/2]} (m-2i)!a_{m-2i} \prod_{j=1}^{i-1}(m-2j-p)(m-2j+p+1)\\ &= \sum_{m=2}^{\infty}\sum_{i=1}^{[m/2]} x^m \frac{a_{m-2i}}{m(m-2i+1)} \prod_{j=1}^{i-1} \frac{(m-2j-p)(m-2j+p+1)}{(m-2j+1)(m-2j)}.\end{aligned} \tag{2.23}$$

Thus, we further obtain

$$\begin{aligned}\sum_{m=2}^{\infty} c_m x^m &= \sum_{m=2}^{\infty}\sum_{i=1}^{[m/2]} x^m \frac{a_{m-2i}}{m(m-2i+1)} \prod_{j=1}^{i-1}\left\{1 - \frac{p(p+1)}{(m-2j+1)(m-2j)}\right\}\\ &= \sum_{m=2}^{\infty}\sum_{i=1}^{[m/2]} \alpha_{mi} x^m,\end{aligned} \tag{2.24}$$

where we set

$$\alpha_{mi} = \frac{a_{m-2i}}{m(m-2i+1)} \prod_{j=1}^{i-1}\left\{1 - \frac{p(p+1)}{(m-2j+1)(m-2j)}\right\}. \tag{2.25}$$

As we already stated in (2.16), it follows from (2.3) and (2.4) that the power series $\sum_{m=2}^{\infty} c_m x^m$ is absolutely convergent for all $x \in (-\rho,\rho)$ (recall the Cauchy-Hadamard formula or the root test). Hence, we can rearrange the terms of the power series without changing its sum as follows:

$$
\begin{aligned}
\sum_{m=2}^{\infty}\sum_{i=1}^{[m/2]} \alpha_{mi}x^m = {} & \alpha_{21}x^2 + \alpha_{31}x^3 \\
& + \alpha_{41}x^4 + \alpha_{42}x^4 \\
& + \alpha_{51}x^5 + \alpha_{52}x^5 \\
& + \alpha_{61}x^6 + \alpha_{62}x^6 + \alpha_{63}x^6 \\
& + \alpha_{71}x^7 + \alpha_{72}x^7 + \alpha_{73}x^7 \\
& + \alpha_{81}x^8 + \alpha_{82}x^8 + \alpha_{83}x^8 + \alpha_{84}x^8 \\
& \quad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
= {} & \sum_{m=2}^{\infty} \alpha_{m1}x^m + \sum_{m=4}^{\infty} \alpha_{m2}x^m + \sum_{m=6}^{\infty} \alpha_{m3}x^m + \cdots \\
= {} & \sum_{i=1}^{\infty}\sum_{m=2i}^{\infty} \alpha_{mi}x^m.
\end{aligned}
\tag{2.26}
$$

So, we further obtain

$$
\begin{aligned}
\sum_{m=2}^{\infty} c_m x^m &= \sum_{i=1}^{\infty}\sum_{m=2i}^{\infty} \frac{a_{m-2i}x^m}{m(m-2i+1)} \prod_{j=1}^{i-1}\left\{1 - \frac{p(p+1)}{(m-2j+1)(m-2j)}\right\} \\
&= \sum_{i=1}^{\infty} x^{2i} \sum_{m=2i}^{\infty} \frac{a_{m-2i}x^{m-2i}}{m(m-2i+1)} \prod_{j=1}^{i-1}\left\{1 - \frac{p(p+1)}{(m-2j+1)(m-2j)}\right\}.
\end{aligned}
\tag{2.27}
$$

Finally, if we substitute m for $(m-2i)$ in the above equality, then we get the desired equality. □

3. Partial solution to Hyers-Ulam stability problem

In this section, we will investigate a property of the Legendre's differential equation (2.1) concerning the Hyers-Ulam stability problem. That is, we will try to answer the question, whether there exists a Legendre function near any approximate Legendre function.

If a function $y(x)$ can be expressed as a power series of the form (2.8), then we follow the first part of the proof of Theorem 2.1 to get

$$\begin{aligned}(1-x^2)y''(x)-2xy'(x)+p(p+1)y(x)\\ =\sum_{m=0}^{\infty}\{(m+2)(m+1)b_{m+2}-(m-p)(m+p+1)b_m\}x^m.\end{aligned}\tag{3.1}$$

Let us define

$$a_m=(m+2)(m+1)b_{m+2}-(m-p)(m+p+1)b_m \tag{3.2}$$

for all $m\in\{0,1,2,\dots\}$. By some tedious calculations, we can now express the c_m's defined in (2.2) in terms of the b_m's:

$$\begin{aligned}c_m&=\frac{1}{m!}\sum_{i=1}^{[m/2]}(m-2i)!a_{m-2i}\prod_{j=1}^{i-1}(m-2j-p)(m-2j+p+1)\\ &=b_m-\frac{b_{m-2[m/2]}}{m!}\prod_{j=1}^{[m/2]}(m-2j-p)(m-2j+p+1)\end{aligned}\tag{3.3}$$

for any $m\in\{2,3,\dots\}$ (cf. (2.11) in Section 2).

Theorem 3.1. *Assume that ρ and ρ_0 are positive constants with $\rho<\min\{1,\rho_0\}$. Let $y:(-\rho,\rho)\to\mathbb{C}$ be a function which can be represented by a power series of the form (2.8) whose radius of convergence is ρ_0. Assume moreover that the conditions in (2.4) are satisfied with a_m's and c_m's given in (3.2) and (3.3). If there exists a constant $\varepsilon>0$ such that*

$$|(1-x^2)y''(x)-2xy'(x)+p(p+1)y(x)|\le\varepsilon \tag{3.4}$$

for all $x\in(-\rho,\rho)$ and for some real number p, then there exists a Legendre function $y_h:(-\rho,\rho)\to\mathbb{C}$ and a constant $C>0$ such that

$$|y(x)-y_h(x)|\le C\frac{x^2}{1-x^2} \tag{3.5}$$

for all $x\in(-\rho,\rho)$.

Proof. We assumed that $y(x)$ can be represented by a power series (2.8) whose radius of convergence is $\rho_0>\rho$, so

$$(1-x^2)\sum_{m=2}^{\infty}m(m-1)b_mx^{m-2}-2x\sum_{m=1}^{\infty}mb_mx^{m-1}+p(p+1)\sum_{m=0}^{\infty}b_mx^m \tag{3.6}$$

is also a power series whose radius of convergence is ρ_0. More precisely, in view of (3.1) and (3.2), we have

$$(1-x^2)\sum_{m=2}^{\infty}m(m-1)b_mx^{m-2}-2x\sum_{m=1}^{\infty}mb_mx^{m-1}+p(p+1)\sum_{m=0}^{\infty}b_mx^m=\sum_{m=0}^{\infty}a_mx^m \tag{3.7}$$

for all $x \in (-\rho_0, \rho_0)$.

Since

$$y(x) = \sum_{m=0}^{\infty} b_m x^m, \qquad y'(x) = \sum_{m=1}^{\infty} m b_m x^{m-1}, \qquad y''(x) = \sum_{m=2}^{\infty} m(m-1) b_m x^{m-2} \tag{3.8}$$

for any $x \in (-\rho, \rho)$, we get

$$(1-x^2)y''(x) - 2xy'(x) + p(p+1)y(x) = \sum_{m=0}^{\infty} a_m x^m \tag{3.9}$$

for all $x \in (-\rho, \rho)$, where the radius of convergence of $\sum_{m=0}^{\infty} a_m x^m$ is ρ_0. Thus, it follows from (3.4) that

$$\left| \sum_{m=0}^{\infty} a_m x^m \right| \le \varepsilon \tag{3.10}$$

for all $x \in (-\rho, \rho)$.

Since the power series $\sum_{m=0}^{\infty} a_m x^m$ is absolutely convergent on its interval of convergence, which includes the interval $[-\rho, \rho]$, and the power series $\sum_{m=0}^{\infty} |a_m x^m|$ is continuous on $[-\rho, \rho]$ (a power series is differentiable on its interval of convergence), there exists a constant $C_1 > 0$ with

$$\sum_{m=0}^{n} |a_m x^m| \le C_1 \tag{3.11}$$

for all integers $n \ge 0$ and for any $x \in (-\rho, \rho)$.

Moreover, we know that $\{1/(m+2i)(m+1)\}_{m=0,1,\dots}$ is a decreasing sequence of positive numbers. According to [19, Theorem 3.3], it holds that

$$\sum_{m=0}^{\infty} \frac{|a_m x^m|}{(m+2i)(m+1)} \le \frac{C_1}{2i} \tag{3.12}$$

for any $x \in (-\rho, \rho)$ and all $i \in \{1, 2, \dots\}$.

On the other hand, since

$$\begin{aligned} \sum_{k=1}^{\infty} \left| \frac{p(p+1)}{(m+2k+1)(m+2k)} \right| &= \frac{|p(p+1)|}{(m+3)(m+2)} + \frac{|p(p+1)|}{(m+5)(m+4)} + \cdots \\ &\le \frac{|p(p+1)|}{4} \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty \end{aligned} \tag{3.13}$$

for any integer $m \ge 0$, we may conclude that the infinite product

$$\prod_{k=1}^{\infty} \left\{ 1 - \frac{p(p+1)}{(m+2k+1)(m+2k)} \right\} \tag{3.14}$$

converges. (According to [20, Theorem 6.6.2], the above infinite product converges for $p(p+1) < 0$. The same argument can be applied for the case of $p(p+1) \geq 0$.) Hence, substituting $i-j$ for k in the above infinite product, there exists a constant $C_2 > 0$ with

$$\left| \prod_{j=1}^{i-1} \left\{ 1 - \frac{p(p+1)}{(m+2i-2j+1)(m+2i-2j)} \right\} \right| \leq C_2 \tag{3.15}$$

for all integers $i \geq 1$ and $m \geq 0$. Therefore, it follows from Corollary 2.3 that

$$\left| \sum_{m=2}^{\infty} c_m x^m \right| \leq C_2 \sum_{i=1}^{\infty} |x|^{2i} \sum_{m=0}^{\infty} \frac{|a_m x^m|}{(m+2i)(m+1)} \tag{3.16}$$

for every $x \in (-\rho, \rho)$.

By (3.12) and (3.16), we get

$$\left| \sum_{m=2}^{\infty} c_m x^m \right| \leq C_1 C_2 \sum_{i=1}^{\infty} \frac{|x|^{2i}}{2i} \leq \frac{C_1 C_2}{2} \frac{x^2}{1-x^2} \tag{3.17}$$

for all $x \in (-\rho, \rho)$. This completes the proof of our theorem. □

John M. Rassias' open problems. (1) It is an open problem whether Theorem 3.1 also holds for the function $y(x)$ which cannot be represented by a power series of the form (2.8).

(2) It seems to be interesting to investigate the stability problem for the case where the inequality (3.4) is controlled by a power of the absolute value of x.

4. Example

In this section, our task is to show that there certainly exist functions $y(x)$ which satisfy all the conditions given in Theorem 3.1.

Example 4.1. Let p be neither an odd number nor of the form, $-2k$, for some $k \in \mathbb{N}$, let ρ be a positive constant less than 1, and let q be given with

$$0 < q \leq \frac{\varepsilon}{p^2 + |p| + 3}. \tag{4.1}$$

We define a function $y:(-\rho,\rho)\to\mathbb{R}$ by

$$\begin{aligned}
y(x) &= \sum_{m=0}^{\infty} b_m x^m = y_h(x) + q\sin x \\
&= 1 + \sum_{k=1}^{\infty} \frac{x^{2k}}{(2k)!}\prod_{j=1}^{k}(2k-2j-p)(2k-2j+p+1) \\
&\quad + x + \sum_{k=1}^{\infty} \frac{x^{2k+1}}{(2k+1)!}\prod_{j=1}^{k}(2k-2j-p+1)(2k-2j+p+2) + q\sin x \\
&= 1 + \sum_{k=1}^{\infty} \frac{x^{2k}}{(2k)!}\prod_{j=1}^{k}(2k-2j-p)(2k-2j+p+1) + (1+q)x \\
&\quad + \sum_{k=1}^{\infty} \frac{x^{2k+1}}{(2k+1)!}\left\{(-1)^k q + \prod_{j=1}^{k}(2k-2j-p+1)(2k-2j+p+2)\right\},
\end{aligned} \tag{4.2}$$

which is a sum of a Legendre function and a sine function. Obviously, the radius of convergence of $y(x)$ is $\rho_0 = 1$ and we have

$$\begin{aligned}
b_0 &= 1, \qquad b_{2k} = \frac{1}{(2k)!}\prod_{j=1}^{k}(2k-2j-p)(2k-2j+p+1), \\
b_1 &= 1+q, \qquad b_{2k+1} = \frac{1}{(2k+1)!}\left\{(-1)^k q + \prod_{j=1}^{k}(2k-2j-p+1)(2k-2j+p+2)\right\}
\end{aligned} \tag{4.3}$$

for all $k\in\mathbb{N}$.

It follows from (3.3) and (3.2) that

$$c_{2k} = 0, \qquad a_{2k} = 0 \tag{4.4}$$

for any $k\in\mathbb{N}$. In this case, according to (2.4), we have $\sigma_1 = -1$. Similarly, using (3.3) and (3.2), we get

$$\begin{aligned}
c_{2k+1} &= \frac{q}{(2k+1)!}\left\{(-1)^k - \prod_{j=1}^{k}(2k-2j-p+1)(2k-2j+p+2)\right\}, \\
a_{2k+1} &= \frac{(-1)^{k+1}q}{(2k+1)!}\{1+(2k-p+1)(2k+p+2)\}
\end{aligned} \tag{4.5}$$

for any $k\in\mathbb{N}$. Thus, we get

$$\sigma_2 = \lim_{k\to\infty}\left|\frac{1}{(2k+3)(2k+2)}\frac{a_{2k+1}}{c_{2k+1}}\right| = 0. \tag{4.6}$$

Hence, both conditions in (2.4) are satisfied with $\sigma_1 = -1$ and $\sigma_2 = 0$.

Obviously, we get

$$\lim_{k\to\infty}\left|\frac{a_{2k+3}}{a_{2k+1}}\right| = 0 \tag{4.7}$$

and we can show, by applying the ratio test, that the power series

$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + \sum_{k=1}^{\infty} a_{2k+1} x^{2k+1} \tag{4.8}$$

converges for every real number x. (Notice that $a_{2k} = 0$ for all $k \in \mathbb{N}$.)

Since $y_h(x)$ is a Legendre function, we now have

$$\begin{aligned}
&\left|(1-x^2)y''(x) - 2xy'(x) + p(p+1)y(x)\right| \\
&\quad = \left| -(1-x^2)q\sin x - 2qx\cos x + p(p+1)q\sin x\right| \\
&\quad \le \left(\left|1-x^2\right| + 2|x| + \left|p(p+1)\right|\right)q \le \left(p^2 + |p| + 3\right)q \le \varepsilon
\end{aligned} \tag{4.9}$$

for all x with $|x| < \rho < 1$. Hence, $y(x)$ satisfies inequality (3.4).

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