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Research Article

Approximation Technics for an Unsteady Dynamic Koiter Shell

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We propose a mixed formulation in dynamical elasticity of shells which allows a locking-free finite element approximation in particular cases of Koiter shells.

1. Introduction

Finite element solutions of shell models suffer from lack of stability when the shell thickness goes to zero. Indeed, most often, a large error discretization appears and compromises the method [1, 2]. This lack of robustness, known as locking, is considered as an actual challenge to approximate thin shells.

Numerous approaches for overcoming locking [1–6] make an essential use of a stable element for a mixed formulation of the initial problem in which the new unknowns play a crucial role in the stability analysis. In the pioneering paper [3], Arnold and Brezzi treat the Naghdi shell model as an abstract saddle point problem and consider a mixed finite element method to approximate it. Their method is robust in particular cases since they provide a uniform error estimate under some geometrical restrictions, namely, the geometric coefficients are constant locally on each element. Bramble and Sun [5] have used the Arnold and Brezzi approach to provide a weaker stability condition when geometric coefficients are smooth enough. They establish an optimal error estimation as long as $h^2\varepsilon^{-1}$ is bounded.

In the present paper, we introduce a mixed formulation for a bending-dominated dynamic Koiter shell. The approach of Arnold and Brezzi [3] is used with significant modifications but with the same geometric restrictions. Our formulation is valid for Koiter shells. It includes dynamic effects and is valid for shells whose midsurface can have charts with discontinuous second derivatives.

The paper is organized as follows. The shell Koiter dynamic model is introduced in Section 2. In Section 3, we present our mixed formulation and prove an existence and uniqueness solution. In Section 4, we focus on the space discretization. A uniform convergence with respect to the thickness is obtained under the Arnold and Brezzi assumption [3]. This restrictive constraint satisfied in [3] for cylindrical shell is satisfied in our approach for C^1 junction of cylindrical shells. In Sections 5 and 6, we study the fully discrete problem and prove time and uniform space convergence.

2. The shell model

Greek indices take their values in the set $\{1,2\}$ and the Latin indices take their values in $\{1,2,3\}$. Products containing repeated indices are summed.

Let ω be a domain of $\mathbb{R}^2$. We consider a shell whose midsurface is given by $S = \vec{\varphi}(\overline{\omega})$, where $\vec{\varphi} \in W^{2,\infty}(\omega,\mathbb{R}^3)$ is an injective mapping. Let $\vec{a}_\alpha = \vec{\varphi}_\alpha$, $\alpha = 1,2$; $\vec{a}_3 = \vec{a}_1 \wedge \vec{a}_2 / \|\vec{a}_1 \wedge \vec{a}_2\|$ be the covariant basis vectors and let $\vec{a}^\alpha$ defined by $\vec{a}^\alpha \cdot \vec{a}_\beta = \delta^\alpha_\beta$; $\vec{a}_3 = \vec{a}^3$ be the contravariant basis vectors. Let ε be the shell thickness. The first and second fundamental forms of the midsurface are defined componentwise by

$$a_{\alpha\beta} = \vec{a}_\alpha \cdot \vec{a}_\beta, \qquad b_{\alpha\beta} = \vec{a}_3 \cdot \vec{a}_{\alpha,\beta} = -\vec{a}_\alpha \cdot \vec{a}_{3,\beta}. \tag{2.1}$$

Let $a = \|\vec{a}_1 \wedge \vec{a}_2\|^2$ be the determinant of $(a_{\alpha\beta})_{\alpha\beta}$. We note $a^{\alpha\beta} := \vec{a}^\alpha \cdot \vec{a}^\beta$ the first fundamental form contravariant components and $b^\alpha_\gamma := a^{\alpha\beta} b_{\beta\gamma}$ the mixed components of the second fundamental form. For a displacement field $\vec{u}$, we define the linearized change of curvature tensor $\underline{\underline{\Upsilon}} = (\Upsilon_{\alpha\beta})_{\alpha,\beta}$ and the linearized membrane strain tensor $\underline{\underline{\Lambda}} = (\Lambda_{\alpha\beta})_{\alpha,\beta}$ [7–9] by

$$\begin{aligned} \Upsilon_{\alpha\beta}(\vec{u}) &= \left(\vec{u}_{\alpha\beta} - \Gamma^\rho_{\alpha\beta}\vec{u}_\rho\right) \cdot a_3, \\ \Lambda_{\alpha\beta}(\vec{u}) &= \frac{\vec{u}_\alpha \cdot \vec{a}_\beta + \vec{u}_\beta \cdot \vec{a}_\alpha}{2}. \end{aligned} \tag{2.2}$$

Set $\underline{\underline{E}} = (E^{\alpha\beta\lambda\mu})_{\alpha\beta\lambda\mu}$ the elasticity tensor, assumed to be elliptic, given by $E^{\alpha\beta\lambda\mu} = (\epsilon/2(1-\nu^2))(a^{\alpha\lambda}a^{\beta\mu} + a^{\alpha\mu}a^{\beta\lambda} + 2\nu a^{\alpha\beta}a^{\lambda\mu})$, where $\epsilon > 0$ and $\nu \in (0,1/2)$ denote the Young's module and Poisson ratio of the material.

We suppose the shell clamped on a nonempty part Γ of its boundary and set

$$\begin{aligned} H^1_\Gamma(\omega) &= \{u \in H^1(\omega),\ u = 0 \text{ on } \Gamma\}, \\ H^2_\Gamma(\omega) &= \left\{u \in H^2(\omega),\ u = \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma\right\}, \\ V &= \{\vec{v} = v_i a^i,\ v_\alpha \in H^1_\Gamma(\omega),\ v_3 \in H^2_\Gamma(\omega)\}. \end{aligned} \tag{2.3}$$

Note that V is a Hilbert space when endowed with the norm

$$\|v\|_V = \left(\sum_\alpha ||v_\alpha||^2_{H^1} + ||v_3||^2_{H^2}\right)^{1/2}. \tag{2.4}$$

Consider the dynamic bending-dominated Koiter shell problem

$$\begin{gathered} \text{find } \vec{u} \in L^2(0,T;V) \\ \tilde{m}(\ddot{\vec{u}};\vec{v}) + \tilde{A}(\vec{u};\vec{v}) = \tilde{L}(\vec{v}) \quad \forall \vec{v} \in V \text{ a.e. in time,} \end{gathered} \tag{2.5}$$

where the double superscript $\ddot{v}$ indicates double differentiation in time of the field v, $\tilde{m}$ is the inertia term, $\tilde{L}$ is a linear form corresponding to external forces, and $\tilde{A}$ is a bilinear form corresponding to internal energy given by

$$\begin{aligned} \tilde{A}(\vec{u};\vec{\hat{v}}) &= \varepsilon^3 \int_\omega \frac{1}{12} E^{\alpha\sigma\lambda\mu} \Upsilon_{\alpha\sigma}(\vec{u}) \Upsilon_{\lambda\mu}(\vec{\hat{v}}) \sqrt{a}\, dx \\ &= \varepsilon \int_\omega E^{\alpha\sigma\lambda\mu} \Lambda_{\alpha\sigma}(\vec{u}) \Lambda_{\lambda\mu}(\vec{\hat{v}}) \sqrt{a}\, dx. \end{aligned} \tag{2.6}$$

Note that $\tilde{A}$ is continuous and coercive on V [8]. The following assumptions, to check that the shell is in a bending-dominated state [1, 3, 5, 10, 11], are made about the scaling of external forces and inertia term:

$$\begin{aligned} \tilde{L}(\vec{\hat{v}}) &= \varepsilon^3 \int_\omega \vec{f} \cdot \vec{\hat{v}} \sqrt{a}\, dx, \\ \tilde{m}(\ddot{\vec{u}};\vec{\hat{v}}) &= \varepsilon^3 \int_\omega \rho \ddot{\vec{u}} \cdot \vec{\hat{v}} \sqrt{a}\, dx, \end{aligned} \tag{2.7}$$

where ρ denotes the surface mass density of the shell.

Remark 2.1. As its thickness goes to zero, the asymptotic behavior of a shell is governed either by membrane or flexural two-dimensional equation [10]. This distinction rests on whether the space $V_1 = \{v \in V \Lambda_{\alpha\beta}(v) = 0,\ \alpha,\beta = 1,2\}$ of linearized inextensional displacement skipping invariant at first-order midsurface metric is reduced or not to $\{0\}$. The scaling of external forces plays an important role in this classification. By supposing $V_1 \neq \{0\}$ and the resultant of the applied forces of the form $\varepsilon^3 \vec{f}$, $\vec{f} \notin V_1^0$ the polar set of V_1, we suppose that the shell is in the bending-dominated state.

Remark 2.2. It has been proved [12] by asymptotic analysis that the dynamic equations of shells lead to the dynamic equations of flexural shells when the external forces and inertia term are multiplied by thickness on power 3.

3. Mixed formulation

We introduce a new unknown $\underline{\underline{\lambda}}$ which represents the membrane stress aside a multiplicator factor. We set, for a real c_0 such that $0 < c_0 < \varepsilon^{-2}$,

$$\underline{\underline{\lambda}} = (\lambda_{\alpha\gamma})_{\alpha\gamma}, \quad \lambda_{\alpha\gamma} = \left(\frac{1}{\varepsilon^2} - c_0\right) E^{\alpha\gamma\sigma\mu} \Lambda_{\sigma\mu}(\vec{u}), \tag{3.1}$$

and seek $(\vec{u},\underline{\underline{\lambda}}) \in L^2(0,T;V)\times L^\infty(0,T;W)$ such that we have a.e. in time

$$\begin{aligned} &m(\vec{\ddot{u}}(t);\vec{\hat{v}}) + A(\vec{u}(t);\vec{\hat{v}}) + B(\vec{\hat{v}};\underline{\underline{\lambda}}(t)) = L(\vec{\hat{v}}) \quad \forall \vec{\hat{v}} \in V, \\ &B(\vec{u}(t);\underline{\underline{\hat{\lambda}}}) - \frac{\varepsilon^2}{1-c_o\varepsilon^2} C(\underline{\underline{\lambda}}(t);\underline{\underline{\hat{\lambda}}}) = 0 \quad \forall \underline{\underline{\hat{\lambda}}} \in W := \{\underline{\underline{\varphi}}/\varphi^{\alpha\beta} \in L^2(\omega)\}, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} A(\vec{u};\vec{\hat{v}}) &:= \int_\omega \frac{1}{12} E^{\alpha\sigma\lambda\mu} \Upsilon_{\alpha\sigma}(\vec{u}) \Upsilon_{\lambda\mu}(\vec{\hat{v}}) \sqrt{a}\, dx \\ &\quad + c_0 \int_\omega E^{\alpha\sigma\lambda\mu} \Lambda_{\alpha\sigma}(\vec{u}) \Lambda_{\lambda\mu}(\vec{\hat{v}}) \sqrt{a}\, dx, \\ B(\vec{v};\underline{\underline{\xi}}) &:= \int_\omega \Lambda_{\alpha\sigma}(\vec{v}) \xi^{\alpha\sigma} \sqrt{a}\, dx, \\ C(\underline{\underline{\lambda}};\underline{\underline{\xi}}) &:= \int_\omega (E^{-1})_{\alpha\sigma\delta\mu} \lambda^{\delta\mu} \xi^{\alpha\sigma} \sqrt{a}\, dx, \\ m(\vec{\ddot{u}};\vec{\hat{v}}) &= \int_\omega \rho \vec{\ddot{u}} \cdot \vec{\hat{v}} \sqrt{a}\, dx, \qquad \varepsilon^3 L(\vec{\hat{v}}) = \tilde{L}(\vec{\hat{v}}). \end{aligned} \tag{3.3}$$

We endow W by the standard L^2 product norm and by the seminorm

$$|||\underline{\underline{\lambda}}||| = \sup_{\vec{v}\in V} \frac{B(\vec{v};\underline{\underline{\lambda}})}{\|\vec{v}\|}. \tag{3.4}$$

Note that the bilinear forms A, B, and C are continuous, respectively, on $V\times V$, $V\times W$, and $W\times W$, such that A is V-elliptic and C is W-elliptic. The form $\tilde{\tilde{A}}$ defined on $V\times W\times V\times W$ by

$$\tilde{\tilde{A}}(\vec{u},\underline{\underline{\lambda}};\vec{\hat{v}},\underline{\underline{\hat{\lambda}}}) = A(\vec{u};\vec{\hat{v}}) + B(\vec{\hat{v}},\underline{\underline{\lambda}}) - B(\vec{u},\underline{\underline{\hat{\lambda}}}) + \frac{\varepsilon^2}{1-c_o\varepsilon^2} C(\underline{\underline{\lambda}},\underline{\underline{\hat{\lambda}}}) \tag{3.5}$$

is then elliptic which allows, using Galerkin approximation [13], to establish the existence result proved in Theorem 3.2.

We introduce the Hilbert basis $(\underline{\underline{\xi}}_b^j)_{j=1,\infty}$ of W and $(\vec{v}_b^j)_{j=1,\infty}$ of V made of the normed eigenvectors solution of the elliptic eigenproblem

$$(\vec{v}_b^j;\vec{\hat{v}})_V = m(\vec{v}_b^j;\vec{\hat{v}}) \quad \forall \vec{\hat{v}} \in V. \tag{3.6}$$

We introduce the subspaces V^N of V and W^N of W by

$$\begin{aligned} V^N &= \left\{ \vec{v} \in V,\ \vec{v} = \sum_{j=1}^N g^j \vec{v}_b^j \right\}, \\ W^N &= \left\{ \underline{\underline{\xi}} \in W,\ \underline{\underline{\xi}} = \sum_{j=1}^N k^j \underline{\underline{\xi}}_b^j \right\}. \end{aligned} \tag{3.7}$$

We then define the well-posed finite-dimensional problem.

Find $\vec{u}^N(t) = \sum_{j=1}^N g^j(t)\vec{v}_b^j \in V^N$; $\underline{\underline{\lambda}}^N(t) = \sum_{j=1}^N k^j(t)\underline{\underline{\xi}}_b^j \in W^N$ such that we have a.e. in time

$$m(\ddot{\vec{u}}^N;\vec{v}_b^j) + A(\vec{u}^N;\vec{v}_b^j) + B(\vec{v}_b^j;\underline{\underline{\lambda}}^N) = L(\vec{v}_b^j) \quad \forall j = 1,N, \tag{3.8}$$

$$\begin{gathered} B\left(\vec{u}^N;\underline{\underline{\xi}}_b^j\right) = \frac{\varepsilon^2}{1-c_o\varepsilon^2}C\left(\underline{\underline{\lambda}}^N;\underline{\underline{\xi}}_b^j\right) \quad \forall j = 1,N, \\ \lim_{N\to\infty}\vec{u}^N(0) = \vec{u}(0) \quad \text{in } V, \qquad \lim_{N\to\infty}\dot{\vec{u}}^N(0) = \vec{v}(0) \quad \text{in } L^2(\omega,\mathbb{R}^3). \end{gathered} \tag{3.9}$$

In this framework, we can prove the following lemma. C_o will denote a positive constant independent of solution and of space and time discretization steps. It can vary from one equality to another.

Lemma 3.1. *The sequence of solutions* $(\vec{u}^N;\underline{\underline{\lambda}}^N)$ *satisfies*

$$\vec{u}^N \text{ is bounded in } L^\infty(0,T,V), \tag{3.10}$$

$$\dot{\vec{u}}^N \text{ is bounded in } L^\infty(0,T,L^2(\omega;\mathbb{R}^3)), \tag{3.11}$$

$$\ddot{\vec{u}}^N \text{ is bounded in } L^2(0,T,V'), \tag{3.12}$$

$$\underline{\underline{\lambda}}^N \text{ is bounded in } L^\infty(0,T,W). \tag{3.13}$$

Proof. By multiplying (3.8) by $\dot{g}^j(t)$, summing in j, integrating in time from 0 to t, and using (3.9) after multiplying by $\dot{k}^j(t)$ and summing in j, we observe that the solution satisfies the fundamental energy estimation

$$\begin{aligned} &\frac{1}{2}m(\dot{\vec{u}}^N(t);\dot{\vec{u}}^N(t)) + \frac{1}{2}A(\vec{u}^N(t);\vec{u}^N(t)) + \frac{\varepsilon^2}{2(1-c_o\varepsilon^2)}C\left(\underline{\underline{\lambda}}^N(t);\underline{\underline{\lambda}}^N(t)\right) \\ &\quad = \int_0^t L(\dot{\vec{u}}^N)(\tau)d\tau + \frac{1}{2}m(\vec{v}(0);\vec{v}(0)) + \frac{1}{2}A(\vec{u}(0);\vec{u}(0)) + \frac{\varepsilon^2}{2(1-c_o\varepsilon^2)}C\left(\underline{\underline{\lambda}}(0);\underline{\underline{\lambda}}(0)\right). \end{aligned} \tag{3.14}$$

Using the positivity of m and the coercivity of A on V and C on W, we get

$$C_o\left\{\|\vec{u}^N(t)\|_V^2 + \frac{\varepsilon^2}{2(1-c_o\varepsilon^2)}\|\underline{\underline{\lambda}}^N(t)\|_W^2\right\} \le \int_0^t \|L\|_{V'}\|\dot{\vec{u}}^N(\tau)\|_V d\tau + C_o. \tag{3.15}$$

Applying the Gronwall's lemma and using the positivity of $\|\underline{\underline{\lambda}}^N(t)\|_W^2$, we first deduce that $\|\vec{u}^N(t)\|_V^2$ is uniformly bounded in time, which implies in turn that $\|\underline{\underline{\lambda}}^N(t)\|_W$ is uniformly bounded in time and proves (3.10) and (3.13). Using the positivity of A and C, we get

$$\frac{1}{2}m(\vec{\dot{u}}^N(t);\vec{\dot{u}}^N(t)) \leq \int_0^t \|L\|_{V'}\|\vec{\dot{u}}^N(\tau)\|_V d\tau + C_0 \tag{3.16}$$

which proves (3.11). P_N and Q_N assign the projection operators defined, respectively, from V in V_N and W in W_N by

$$\begin{aligned} P_N(\vec{v}) &:= \sum_{j=0}^{N}\left(\vec{v};\vec{v}_b^{\,j}\right)_V \vec{v}_b^{\,j} \quad \forall \vec{v} \in V, \\ Q_N(\underline{\underline{\lambda}}) &= \sum_{j=0}^{N}\left(\underline{\underline{\lambda}};\vec{\xi}_b^{\,j}\right)_W \vec{\xi}_b^{\,j} \quad \forall \underline{\underline{\lambda}} \in W. \end{aligned} \tag{3.17}$$

By construction, we have $\|P_N\|_{L(V,V)} \leq 1$, $\|Q_N\|_{L(W,W)} \leq 1$, and

$$m(\vec{v}_b^{\,l};\vec{v} - P_N(\vec{v})) = \sum_{j=N+1}^{\infty}(\vec{v};\vec{v}_b^{\,j})_{H^1} m(\vec{v}_b^{\,l};\vec{v}_b^{\,j}) = 0 \quad \forall l = 1,N. \tag{3.18}$$

We can then write, from (3.8), for each $\vec{\hat{v}} \in L^2(0,T;V)$,

$$\begin{aligned} &\int_0^T m(\vec{\ddot{u}}^N;\vec{\hat{v}}) \\ &\quad = \int_0^T m(\vec{\ddot{u}}^N;P_N(\vec{\hat{v}})) = -\int_0^T A(\vec{u}^N;P_N(\vec{\hat{v}})) - \int_0^T B(P_N(\vec{\hat{v}});\underline{\underline{\lambda}}^N) + L(P_N(\vec{\hat{v}})). \end{aligned} \tag{3.19}$$

Using (3.10) and (3.13), we obtain for each $\vec{\hat{v}} \in L^2(0,T;V)$,

$$\left|\int_0^T m(\vec{\ddot{u}}^N;\vec{\hat{v}})\right| \leq C_o\|\vec{\hat{v}}\|_{L^2(0,T,V)} \tag{3.20}$$

which is (3.12), and the lemma is proved. □

The above bounds can now be used to construct a solution for the problem (3.2) by compactness arguments and we get the following theorem.

Theorem 3.2. *The mixed problem (3.2) has a unique solution* $(\vec{u};\underline{\underline{\lambda}})$. *The primal variable* $\vec{u}$ *is the solution of the Koiter shell problem (2.5). The auxiliary variable* $\underline{\underline{\lambda}}$ *verifies* $\lambda_{\alpha\gamma}(t) = (1/\varepsilon^2 - c_o)E^{\alpha\gamma\sigma\mu}\Lambda_{\sigma\mu}(\vec{u}(t))$ *a.e in time.*

Proof. For any solution $(\vec{u};\underline{\underline{\lambda}})$ of (3.2), it is clear that $\vec{u}$ is a solution of (2.5) and $\underline{\underline{\lambda}}(t)$ is given by (3.1) a.e. in time. To construct a solution, we deduce from (3.10)–(3.13) that there exist subsequences $(\vec{u}^N)_N$, $(\vec{\dot{u}}^N)_N$, $(\vec{\ddot{u}}^N)_N$, and $(\underline{\underline{\lambda}}^N)_N$ weakly converging towards

$\vec{\tilde{u}}$, $\vec{\tilde{v}}$, $\vec{\tilde{\gamma}}$, and $\underline{\underline{\tilde{\lambda}}}$ in $L^2(0,T,V)$, $L^2(0,T,L^2(\omega;\mathbb{R}^3\times\mathbb{R}^3))$, $L^2(0,T,V')$, and $L^2(0,T,W)$, respectively. Consequently, we have

$$\begin{aligned} \vec{\tilde{u}}(t) &= \vec{u}(0)+\int_0^t \vec{\tilde{v}}(\tau)d\tau, \quad \text{a.e. in time,} \\ \vec{\tilde{v}}(t) &= \vec{v}(0)+\int_0^t \vec{\tilde{\gamma}}(\tau)d\tau, \quad \text{a.e. in time} \end{aligned} \tag{3.21}$$

and then $(\partial/\partial t)\vec{\tilde{u}}(t)=\vec{\tilde{v}}(t)$ and $(\partial/\partial t)\vec{\tilde{v}}(t)=\vec{\tilde{\gamma}}(t)$, a.e. in time.

On the other, hand we have for each $\vec{\hat{v}}\in L^2(0,T,V)$,

$$\begin{aligned} & m(\vec{\tilde{\gamma}};\vec{\hat{v}})+A(\vec{\tilde{u}};\vec{\hat{v}})+B(\vec{\hat{v}};\underline{\underline{\tilde{\lambda}}})-L(\vec{\hat{v}}) \\ &= \lim_{M\to\infty}\Big\{\lim_{N\to\infty}\{m(\vec{\ddot{u}}^N;P_M(\vec{\hat{v}}))+A(\vec{u}^N;P_M(\vec{\hat{v}}))+B(P_M(\vec{\hat{v}});\underline{\underline{\lambda}}^N)-L(P_M(\vec{\hat{v}})) \\ &\qquad +m(\vec{\tilde{\gamma}}-\vec{\ddot{u}}^N;P_M(\vec{\hat{v}}))+A(\vec{\tilde{u}}-\vec{u}^N;P_M(\vec{\hat{v}}))+B(P_M(\vec{\hat{v}});\underline{\underline{\tilde{\lambda}}}-\underline{\underline{\lambda}}^N)\} \\ &\qquad -m(\vec{\tilde{\gamma}};P_M(\vec{\hat{v}})-\vec{\hat{v}})-A(\tilde{u};P_M(\vec{\hat{v}})-\vec{\hat{v}})-B(P_M(\vec{\hat{v}})-\vec{\hat{v}};\tilde{\lambda})+(P_M(\vec{\hat{v}})-\vec{\hat{v}})\Big\}. \end{aligned} \tag{3.22}$$

Using at first, with M fixed, the weak convergence of $\vec{u}^N$, $\vec{\ddot{u}}^N$, and $\underline{\underline{\lambda}}^N$ towards $\vec{\tilde{u}}$, $\vec{\tilde{\gamma}}$, and $\underline{\underline{\tilde{\lambda}}}$ afterwards the strong convergence of $P_M(\vec{\hat{v}})$ to $\vec{\hat{v}}$ in $L^2(0,T,V)$, we get

$$m(\vec{\tilde{\gamma}};\vec{\hat{v}})+A(\vec{\tilde{u}};\vec{\hat{v}})+B(\vec{\hat{v}};\underline{\underline{\tilde{\lambda}}})-L(\vec{\hat{v}})=0 \quad \forall\vec{\hat{v}}\in L^2(0,T,V). \tag{3.23}$$

In the same way, we have for every $\underline{\underline{\hat{\lambda}}}\in L^2(0,T,W)$,

$$\begin{aligned} & B\Big(\vec{\tilde{u}};\underline{\underline{\hat{\lambda}}}\Big)-\frac{\varepsilon^2}{1-c_o\varepsilon^2}C\Big(\underline{\underline{\tilde{\lambda}}};\underline{\underline{\hat{\lambda}}}\Big) \\ &= \lim_{M\to\infty}\Big\{\lim_{N\to\infty}\Big\{B\Big(\vec{u}^N;Q_M\Big(\underline{\underline{\hat{\lambda}}}\Big)\Big)-\frac{\varepsilon^2}{1-c_o\varepsilon^2}C\Big(\underline{\underline{\lambda}}^N;Q_M\Big(\underline{\underline{\hat{\lambda}}}\Big)\Big) \\ &\qquad +B\Big(\vec{\tilde{u}}-\vec{u}^N;Q_M\Big(\underline{\underline{\hat{\lambda}}}\Big)\Big)-\frac{\varepsilon^2}{1-c_o\varepsilon^2}C\Big(\underline{\underline{\tilde{\lambda}}}-\underline{\underline{\lambda}}^N;Q_M\Big(\underline{\underline{\hat{\lambda}}}\Big)\Big)\Big\} \\ &\qquad -B\Big(\vec{\tilde{u}};Q_M\Big(\underline{\underline{\hat{\lambda}}}\Big)-\underline{\underline{\hat{\lambda}}}\Big)+\frac{\varepsilon^2}{1-c_o\varepsilon^2}C\Big(\underline{\underline{\tilde{\lambda}}};Q_M\Big(\underline{\underline{\hat{\lambda}}}\Big)-\underline{\underline{\hat{\lambda}}}\Big)\Big\}. \end{aligned} \tag{3.24}$$

Using the weak convergence of $\vec{u}^N$, $\underline{\underline{\lambda}}^N$ towards $\vec{\tilde{u}}$, $\underline{\underline{\tilde{\lambda}}}$ and the strong convergence of $Q_M(\underline{\underline{\hat{\lambda}}})$ to $\underline{\underline{\hat{\lambda}}}$ in $L^2(0,T,W)$, we get

$$B\Big(\vec{\tilde{u}};\underline{\underline{\hat{\lambda}}}\Big)-\frac{\varepsilon^2}{1-c_o\varepsilon^2}C\Big(\underline{\underline{\tilde{\lambda}}};\underline{\underline{\hat{\lambda}}}\Big)=0 \quad \forall\underline{\underline{\hat{\lambda}}}\in L^2(0,T,W). \tag{3.25}$$

Combining (3.23) and (3.25), we deduce that $(\vec{\tilde{u}};\underline{\underline{\tilde{\lambda}}})$ is the solution of (3.2). □

4. Space discretization

Henceforth we assume that the domain ω is a polygon which is triangulated by a regular triangulation τ^h, $\omega = \bigcup_{T\in\tau^h} T$. The set Γ where Dirichlet conditions are imposed is assumed to be a union of edges of triangles in τ^h. The set $P_k(T)$ denotes the space of functions on T which are the restrictions of polynomial of degree $\leq k$. We approximate H^1_Γ using P_3 Lagrange finite elements augmented by bubble functions and introduce the spaces

$$\begin{aligned}
L_h^1 &= \{v \in H_\Gamma^1(\omega),\ v_{/T} \in P_3(T)\ \forall T \in \tau^h\},\\
B_h^1 &= \{v \in H^1(\omega),\ v_{/T} = \lambda_1\lambda_2\lambda_3 p,\ p \in P_1(T)\ \forall T \in \tau^h\},\\
H_h^1 &= \{v = v_1 + v_2 \text{ such that } v_1 \in L_h^1,\ v_2 \in B_h^1\} = L_h^1 \oplus B_h^1,\\
L_h^2 &= \{v \in H_\Gamma^2(\omega),\ v_{/T} \in P_5(T)\ \forall T \in \tau^h\},\\
B_h^2 &= \{v \in H^2 \cap H_0^1(\omega),\ v_{/T} = \lambda_1^2\lambda_2^2\lambda_3^3 p,\ p \in P_1(T)\ \forall T \in \tau^h\},\\
H_h^2 &= \{v = v_1 + v_2 \text{ such that } v_1 \in L_h^2,\ v_2 \in B_h^2\} = L_h^2 \oplus B_h^2.
\end{aligned} \tag{4.1}$$

Above, λ_1, λ_2, and λ_3 denote the barycentric coordinates for each triangle T. We note that the space L_h^2 consists of the Argyris element and B_h^α are bubble function spaces that will be used for the local adjustment to achieve discrete stability. We introduce the discrete displacement and stress spaces by

$$\begin{aligned}
V_h &= \{\vec{v}_h \in V, \vec{v}_h \cdot a_\alpha \in H_h^1, \vec{v}_h \cdot \vec{a}_3 \in H_h^2\},\\
W_h &= \left\{\underline{\underline{\lambda}}, \lambda_{\alpha\beta/T} \in P_1(T)\ \forall T\right\}
\end{aligned} \tag{4.2}$$

and consider the discrete static problem

$$\begin{aligned}
&\text{Find } \vec{u}_h \in V_h \text{ and } \underline{\underline{\lambda}}_h \in W_h \text{ such that}\\
&A\left(\vec{u}_h;\vec{\hat{v}}\right) + B\left(\vec{\hat{v}};\underline{\underline{\lambda}}_h\right) = L\left(\vec{\hat{v}}\right) \quad \forall \vec{\hat{v}} \in V_h,\\
&B\left(\vec{u}_h;\underline{\underline{\hat{\lambda}}}\right) - \frac{\varepsilon^2}{1-c_0\varepsilon^2} C\left(\underline{\underline{\lambda}}_h;\underline{\underline{\hat{\lambda}}}\right) = 0 \quad \forall \underline{\underline{\hat{\lambda}}} \in W_h.
\end{aligned} \tag{4.3}$$

To prove uniform convergence with respect to the shell thickness, we need the inf-sup stability hypothesis where we assume that there exists a constant $\tilde{C} > 0$ for which we have

$$\inf_{0\neq\lambda\in W_h} \sup_{0\neq v\in V_h} \frac{B(\vec{v};\underline{\underline{\lambda}})}{\|\vec{v}\|_V |||\underline{\underline{\lambda}}|||} \geq \tilde{C}. \tag{4.4}$$

In this framework, we use the following theorem proved in [3] in an abstract framework which proves a uniform convergence with respect to the thickness.

Theorem 4.1. *Let* $(\vec{u}_h;\underline{\underline{\lambda}}_h) \in V_h \times W_h$ *be the solution of the static discrete problem (4.3) and let* $(\vec{u};\underline{\underline{\lambda}}) \in V \times W$ *be the static solution associated to (3.2). If (4.4) is satisfied, there* *exists a constant* $C > 0$ *such that*

$$\begin{aligned} &\|\vec{u} - \vec{u}_h\|_V + |||\underline{\underline{\lambda}} - \underline{\underline{\lambda}}_h||| + \frac{\varepsilon^2}{1 - c_o\varepsilon^2}\|\underline{\underline{\lambda}} - \underline{\underline{\lambda}}_h\|_W \\ &\quad \leq C \inf_{\hat{v}\in V_h, \hat{\lambda}\in W_h} \left\{\|\vec{u}_s - \vec{\hat{v}}\|_V + |||\underline{\underline{\lambda}} - \underline{\underline{\hat{\lambda}}}||| + \frac{\varepsilon^2}{1 - c_o\varepsilon^2}\|\underline{\underline{\lambda}} - \underline{\underline{\hat{\lambda}}}\|_W\right\}. \end{aligned} \tag{4.5}$$

It remains to prove that our discrete spaces verify (4.4) which is the purpose of the following lemma.

Lemma 4.2. *If the chart* φ *defining the shell midsurface is in* $W^{2,\infty}(\omega)^3$ *and the associated first and second fundamental forms are piecewise constant, then (4.4) is verified.*

Proof. The proof is based on the construction of an adequate projection operator $\pi : V \to V_h$ satisfying

(i) $B(\pi\vec{v};\underline{\underline{\lambda}}) = B(v;\underline{\underline{\lambda}})$ for all $\vec{v} \in V$, for all $\underline{\underline{\lambda}} \in W_h$,

(ii) $\|\pi\vec{v}\|_V \leq C_o\|v\|_V$ for all $\vec{v} \in V$.

In fact, we have $\sup_{v\in V}(B(\vec{v};\underline{\underline{\lambda}})/\|\vec{v}\|_V|||\lambda|||) = 1$, so given $\underline{\underline{\lambda}} \in W_h$, we can choose $\vec{v} \in V$ for which $B(\vec{v};\underline{\underline{\lambda}})/\|\vec{v}\| \geq 1/2|||\underline{\underline{\lambda}}|||$. The condition (4.4) is then verified since we have

$$\frac{B\left(\pi\vec{v};\underline{\underline{\lambda}}\right)}{\|\pi\vec{v}\|_V} = \frac{B\left(\vec{v};\underline{\underline{\lambda}}\right)}{\|\pi\vec{v}\|_V} \geq \frac{B\left(\vec{v};\underline{\underline{\lambda}}\right)}{C_o\|\vec{v}\|_V} \geq \frac{1}{2C_o}|||\underline{\underline{\lambda}}|||. \tag{4.6}$$

Let $\pi^1 : H^1 \to H_h^1$ be the projection constructed in [3] which satisfies

$$\begin{gathered} \|\pi^1 v\|_{H^1(T)} \leq C\|v\|_{H^1(\widetilde{T})} \quad \forall T \in \tau^h, \\ \int_e (v - \pi^1 v)p = 0 \quad \forall p \in P_1(T)\ \forall e \in \partial T,\ T \in \tau^h, \\ \int_T (v - \pi^1 v)p = 0 \quad \forall p \in P_1(T)\ \forall T \in \tau^h, \end{gathered} \tag{4.7}$$

where $\widetilde{T}$ is the union of triangles in τ^h which meet T.

We also have to construct a projection $\pi^2 : H^2 \to H_h^2$ satisfying

$$\|\pi^2 v\|_{H^2(T)} \leq C\|v\|_{H^2(\widetilde{T})}, \quad \int_T (v - \pi^1 v)p = 0 \quad \forall p \in P_1(T) \quad \forall T \in \tau^h. \tag{4.8}$$

A constructive way to define a map $\pi_0^2 : H^2 \to L_h^2$ satisfying for any $T \in \tau^h$,

$$\|v - \pi_0^2 v\|_{0,T} + h\|v - \pi_0^2 v\|_{1,T} + h^2\|v - \pi_0^2 v\|_{2,T} \leq Ch^2\|v\|_{2,\widetilde{T}} \quad \forall v \in H^2 \tag{4.9}$$

can be found in [14]. We define $\pi_1^2 : H_{\partial\omega}^2 \to B_h^2$ by the conditions

$$\int_T (v - \pi_1^2 v)p = 0 \quad \forall p \in P_1(T),\ \forall T \in \tau^h \tag{4.10}$$

and obtain by scaling argument

$$\|v - \pi_1^2 v\|_{0,T} + h\|v - \pi_1^2 v\|_{1,T} + h^2\|v - \pi_1^2 v\|_{2,T} \leq C\|v\|_{0,T}. \tag{4.11}$$

Finally, we set $\pi^2 v = \pi_0^2 v + \pi_1^2(v - \pi_0^2 v)$ and obtain an operator π^2 which verifies (4.8) and an operator $\pi = (\pi^1, \pi^1, \pi^2)$ which verifies (i)-(ii). □

We hence obtain a finite element which is locking-free in particular cases, cylindrical shells for example. But let us note that the combination of the elements is highly unbalanced from the point of view of approximation.

5. Fully discretization

Because of its superior accuracy and lack of dissipation, the acceleration is usually approximated by a mid point rule

$$\left(\frac{\partial \vec{u}}{\partial t}\right)_n = \frac{\vec{u}^{\,n+1} - \vec{u}^{\,n}}{\Delta t} = \frac{\vec{v}^{\,n+1} + \vec{v}^{\,n}}{2}. \tag{5.1}$$

This leads to the fully discrete mixed formulation

$$(P_h^n)\begin{cases} \text{For each } n, \text{ find } \vec{u}_h^{\,n+1} \in V_h,\ \underline{\underline{\lambda}}_h^{n+1} \in W_h \text{ such that} \\ m\left(\dfrac{\vec{v}_h^{\,n+1} - \vec{v}_h^{\,n}}{\Delta t}; \hat{\vec{v}}_h\right) + A\left(\dfrac{\vec{u}_h^{\,n+1} + \vec{u}_h^{\,n}}{2}, \hat{\vec{v}}_h\right) \\ \qquad + B\left(\hat{\vec{v}}_h; \dfrac{\underline{\underline{\lambda}}_h^{n+1} + \underline{\underline{\lambda}}_h^{n}}{2}\right) = L^{n+1/2}(\hat{\vec{v}}_h) \quad \forall \hat{\vec{v}}_h \in V_h, \\ B(\vec{u}_h^{\,n+1}; \underline{\underline{\mu}}_h) - \dfrac{\varepsilon^2}{1 - c_o\varepsilon^2} C(\underline{\underline{\lambda}}_h^{n+1}; \underline{\underline{\xi}}_h) = 0 \quad \forall \underline{\underline{\xi}}_h \in W_h, \\ \text{with} \\ \dfrac{\vec{u}_h^{\,n+1} - \vec{u}_h^{\,n}}{\Delta t} = \dfrac{\vec{v}_h^{\,n+1} + \vec{v}_h^{\,n}}{2}, \qquad L^{n+1/2} = \dfrac{L^{n+1} + L^n}{2}. \end{cases} \tag{5.2}$$

The convergence analysis will be based on the study of an error equation between the discrete solution and an appropriate projection of the continuous one. Let $(\vec{u}, \underline{\underline{\lambda}})$ be the solution of (3.2); we construct displacement and multiplicator projections by solving at time t_n,

$$\begin{gathered} \left(\pi \vec{u}^{\,n}, \pi_m \underline{\underline{\lambda}}^n\right) \in V_h \times W_h, \\ A\left(\pi \vec{u}^{\,n}; \hat{\vec{v}}_h\right) + B\left(\hat{\vec{v}}_h; \pi_m \underline{\underline{\lambda}}^n\right) = A\left(\vec{u}(t_n); \hat{\vec{v}}_h\right) + B\left(\hat{\vec{v}}_h; \underline{\underline{\lambda}}(t_n)\right) \quad \forall \hat{\vec{v}}_h \in V_h \\ B\left(\pi \vec{u}^{\,n}; \hat{\underline{\underline{\lambda}}}_h\right) - \frac{\varepsilon^2}{1 - c_o\varepsilon^2} C\left(\pi_m \underline{\underline{\lambda}}^n; \hat{\underline{\underline{\lambda}}}_h\right) = 0 \quad \forall \hat{\underline{\underline{\lambda}}}_h \in W_h. \end{gathered} \tag{5.3}$$

We define a discrete approximation of velocity fields and errors fields by

$$\begin{gathered} \frac{\pi \vec{v}^{\,n+1} + \pi \vec{v}^{\,n}}{2} = \frac{\pi \vec{u}^{\,n+1} - \pi \vec{u}^{\,n}}{\Delta t}, \qquad \pi \vec{v}^{\,0} = \pi \vec{u}^{\,0}, \\ e\vec{u}^{\,n} = \vec{u}_h^{\,n} - \pi \vec{u}^{\,n}, \qquad e\vec{v}^{\,n} = \vec{v}_h^{\,n} - \pi \vec{v}^{\,n}, \qquad e\underline{\underline{\lambda}}^n = \underline{\underline{\lambda}}_h^n - \pi_m \underline{\underline{\lambda}}^n \end{gathered} \tag{5.4}$$

and have the following theorem.

Theorem 5.1. *The errors defined below are solutions of the problem:*

$$
\begin{aligned}
&m\left(\frac{e\vec{v}^{\,n+1}-e\vec{v}^{\,n}}{\Delta t};\hat{\vec{v}}_h\right)+A\left(\frac{e\vec{u}^{\,n+1}+e\vec{u}^{\,n}}{2},\hat{\vec{v}}_h\right)+B\left(\hat{\vec{v}}_h;\frac{e\underline{\underline{\lambda}}^{n+1}+e\underline{\underline{\lambda}}^{n}}{2}\right)\\
&\quad=m\left(\frac{\dot{\vec{v}}(t_{n+1})+\dot{\vec{v}}(t_n)}{2}-\frac{\pi\vec{v}^{\,n+1}-\pi\vec{v}^{\,n}}{\Delta t};\hat{\vec{v}}_h\right)\quad\forall\hat{\vec{v}}_h\in V_h,
\end{aligned}
\tag{5.5}
$$

$$
B\left(e\vec{u}^{\,n+1};\underline{\underline{\xi}}_h\right)-\frac{\varepsilon^2}{1-c_o\varepsilon^2}C\left(e\underline{\underline{\lambda}}^{n+1};\underline{\underline{\xi}}_h\right)=0\quad\forall\underline{\underline{\xi}}_h\in W_h.
\tag{5.6}
$$

Proof. Since $(\vec{u}_h^{\,n};\underline{\underline{\lambda}}_h^{n})$ is solution of (P_n^h) and $(\vec{u};\underline{\underline{\lambda}})$ is solution of (3.2), we have

$$
\begin{aligned}
&m\left(\frac{e\vec{v}^{\,n+1}-e\vec{v}^{\,n}}{\Delta t};\vec{v}_h\right)+A\left(\frac{e\vec{u}^{\,n+1}+e\vec{u}^{\,n}}{2},\vec{v}_h\right)+B\left(\vec{v}_h;\frac{e\underline{\underline{\lambda}}^{n+1}+e\underline{\underline{\lambda}}^{n}}{2}\right)\\
&\quad=L^{n+1/2}\left(\hat{\vec{v}}_h\right)-m\left(\frac{\pi\vec{v}^{\,n+1}-\pi\vec{v}^{\,n}}{\Delta t};\hat{\vec{v}}_h\right)-A\left(\frac{\pi\vec{u}^{\,n+1}+\pi\vec{u}^{\,n}}{2},\hat{\vec{v}}_h\right)\\
&\qquad-B\left(\hat{\vec{v}}_h;\frac{\pi_m\underline{\underline{\lambda}}^{n+1}+\pi_m\underline{\underline{\lambda}}^{n}}{2}\right)\\
&\quad=m\left(\frac{\dot{\vec{v}}(t_{n+1})+\dot{\vec{v}}(t_n)}{2};\hat{\vec{v}}_h\right)-m\left(\frac{\pi\vec{v}^{\,n+1}-\pi\vec{v}^{\,n}}{\Delta t};\hat{\vec{v}}_h\right)\\
&\qquad-A\left(\frac{\vec{u}(t_{n+1})+\vec{u}(t_n)}{2},\hat{\vec{v}}_h\right)-B\left(\hat{\vec{v}}_h;\frac{\underline{\underline{\lambda}}(t_{n+1})+\underline{\underline{\lambda}}(t_n)}{2}\right)\quad\forall\hat{\vec{v}}_h\in V_h,
\end{aligned}
\tag{5.7}
$$

which proves (5.5). Equation (5.6) is a direct consequence of the π_m definition. □

6. Convergence

We set for $\vec{v}_h\in V_h$, $L_1^{n+1/2}(\vec{v}_h)=m((\dot{\vec{v}}(t_{n+1})+\dot{\vec{v}}(t_n))/2-(\pi\vec{v}^{\,n+1}-\pi\vec{v}^{\,n})/\Delta t,\vec{v}_h)$ and write (5.5) at time t_n, with the test function $\hat{\vec{v}}_h=\Delta t((e\vec{v}_h^{\,n}+e\vec{v}_h^{\,n-1})/2)\in V_h$. By adding the resulting equations from one to n and using (5.6), we obtain

$$
\begin{aligned}
&\frac{1}{2}m\left(e\vec{v}^{\,n};e\vec{v}^{\,n}\right)+\frac{1}{2}A\left(e\vec{u}^{\,n};e\vec{u}^{\,n}\right)+\frac{\varepsilon^2}{2(1-c_o\varepsilon^2)}C\left(e\underline{\underline{\lambda}}^{n};e\underline{\underline{\lambda}}^{n}\right)\\
&\quad=\sum_{i=1}^{n}L_1^{i-1/2}\left(e\vec{u}^{\,i}-e\vec{u}^{\,i-1}\right)+\frac{1}{2}m\left(e\vec{v}^{\,0},e\vec{v}^{\,0}\right)+\frac{1}{2}A\left(e\vec{u}^{\,0};e\vec{u}^{\,0}\right)+\frac{\varepsilon^2}{2(1-c_o\varepsilon^2)}C\left(e\underline{\underline{\lambda}}^{0};e\underline{\underline{\lambda}}^{0}\right).
\end{aligned}
\tag{6.1}
$$

Hence, the energy estimate on the error leads to an error bound if we control the truncation errors $L_1^{i-1/2}$. This is the purpose of the following lemma.

Lemma 6.1. *The inertia truncation error satisfies*

$$\sum_{i=1}^{n} L_1^{i-1/2}(e\vec{u}^{\,i} - e\vec{u}^{\,i-1}) \le \frac{\Delta t}{4}\sum_{i=1}^{n-1} m(e\vec{v}^{\,i}, e\vec{v}^{\,i}) + \frac{1}{4}m(e\vec{v}^{\,0}, e\vec{v}^{\,0}) + \frac{1}{4}m(e\vec{v}^{\,n}, e\vec{v}^{\,n}) + C_o\left\{\Delta t^2 \|\vec{u}\|_{W^{4,\infty}(0,T;L^2)} + \|(\pi - \mathrm{Id})(\vec{u})\|_{W^{2,\infty}(0,T;L^2)}\right\}^2. \tag{6.2}$$

Proof. We note for $n \ge 1$, $t\vec{v}_n = (\vec{v}(t_n) + \vec{v}(t_{n-1}))/2 - (\pi\vec{v}^{\,n} - \pi\vec{v}^{\,n-1})/\Delta t$. By definition of the error field, we have

$$\begin{aligned}\sum_{i=1}^{n} L_1^{i-1/2}(e\vec{u}^{\,i} - e\vec{u}^{\,i-1}) &= \Delta t\sum_{i=1}^{n} m\left(tv_i, \frac{e\vec{v}^{\,i} + e\vec{v}^{\,i-1}}{2}\right)\\ &= \frac{\Delta t}{2}\sum_{i=1}^{n-1} m(t\vec{v}_i + t\vec{v}_{i+1}, e\vec{v}^{\,i}) + \frac{\Delta t}{2}m(t\vec{v}_n, e\vec{v}^{\,n}) + \frac{\Delta t}{2}m(t\vec{v}_1, e\vec{v}^{\,0}).\end{aligned} \tag{6.3}$$

By Cauchy Schwarz, we deduce the estimate

$$\begin{aligned}\left|\sum_{i=1}^{n} L_1^{i-1/2}(e\vec{u}^{\,i} - e\vec{u}^{\,i-1})\right| &\le \frac{\Delta t}{4}\sum_{i=1}^{n-1} m(e\vec{v}^{\,i}, e\vec{v}^{\,i})\\ &\le C_o\left\{\frac{1}{4}m(e\vec{v}^{\,0}, e\vec{v}^{\,0}) + \frac{\Delta t^2}{4}\|t\vec{v}_1\|_{L^2}^2\right.\\ &\qquad \left. + \frac{\Delta t}{4}\sum_{i=1}^{n-1}\|t\vec{v}_i + t\vec{v}_{i+1}\|_{L^2}^2 + \frac{1}{4}m(e\vec{v}^{\,n}, e\vec{v}^{\,n}) + \frac{\Delta t^2}{4}\|t\vec{v}_n\|_{L^2}^2\right\}.\end{aligned} \tag{6.4}$$

By definition of projection operators, we have for each $i \ge 1$,

$$\begin{aligned}t\vec{v}_i + t\vec{v}_{i+1} &= \frac{\vec{v}(t_{i+1}) + 2\vec{v}(t_i) + \vec{v}(t_{i-1})}{2} - \frac{\pi\vec{v}^{\,i+1} - \pi\vec{v}^{\,i-1}}{\Delta t}\\ &= \left\{\frac{\vec{v}(t_{i+1}) + 2\vec{v}(t_i) + \vec{v}(t_{i-1})}{2} - 2\frac{\vec{u}^{\,i+1} - 2\vec{u}^{\,i} + \vec{u}^{\,i-1}}{\Delta t^2}\right\}\\ &\quad - 2(\pi - \mathrm{Id})\left(\frac{\vec{u}(t_{i+1}) - 2\vec{u}(t_i) + \vec{u}(t_{i-1})}{\Delta t^2}\right).\end{aligned} \tag{6.5}$$

By Taylor expansion, we get for $i = 1, n-1$,

$$\|t\vec{v}_i + t\vec{v}_{i+1}\|_{L^2} \le \frac{\Delta t^2}{2}\|\vec{u}\|_{W^{4,\infty}(0,T;L^2)} + C_o\|(\pi - \mathrm{Id})\vec{u}\|_{W^{2,\infty}(0,T;L^2)}. \tag{6.6}$$

For $i = n-1$, we obtain

$$\|t\vec{v}_n\|_{L^2} \le \|tv_{n-1}\|_{L^2} + \frac{\Delta t^2}{2}\|\vec{u}\|_{W^{4,\infty}(0,T;L^2)} + C_o\|(\pi - \mathrm{Id})(\vec{u})\|_{W^{2,\infty}(0,T;L^2)}. \tag{6.7}$$

By adding, from $i = 2$ until n, it remains

$$\|t\vec{v}_n\|_{L^2} \leq \|t\vec{v}_1\|_{L^2} + \frac{C_o}{\Delta t}\{\Delta t^2 \|\vec{u}\|_{W^{4,\infty}(0,T)} + \|(\pi - \mathrm{Id})\vec{u}\|_{W^{2,\infty}(0,T)}\}. \tag{6.8}$$

To estimate tv_1, we use its definition

$$\begin{aligned} t\vec{v}_1 = {} & \frac{\vec{v}(t_1) + \vec{v}(t_0)}{2} + 2\frac{\vec{v}(t_0)}{\Delta t} - 2\frac{\vec{u}(t_1) - \vec{u}(t_0)}{\Delta t^2} \\ & - 2(\pi - \mathrm{Id})\left(\frac{\vec{u}(t_1) - \vec{u}(t_0)}{\Delta t^2} - \frac{\vec{v}(t_0)}{\Delta t}\right). \end{aligned} \tag{6.9}$$

We deduce that

$$\Delta t\|t\vec{v}_1\|_{L^2} \leq C_o\frac{\Delta t^2}{2}\|\vec{u}\|_{W^{4,\infty}(0,T;L^2)} + C_o\|(\pi - \mathrm{Id})\vec{u}\|_{W^{2,\infty}(0,T;L^2)}. \tag{6.10}$$

The lemma is deduced by combining (6.4), (6.6), (6.8), and (6.10). □

We are thus able to prove the main convergence result.

THEOREM 6.2. *Under the hypothesis of Theorem 3.2, the errors on displacement field* $\vec{u}_h^n - \vec{u}(t_n)$, *on velocity field* $\vec{v}_h^n - \vec{v}(t_n)$ *and on membrane stress* $\underline{\underline{\lambda}}_h^n - \underline{\underline{\lambda}}(t_n)$ *satisfy*

$$\begin{aligned} & \Delta t\|\vec{v}(t_n) - \vec{v}_h^n\|_{L^2} + \|\vec{u}(t_n) - \vec{u}_h^n\|_V + \varepsilon\|\underline{\underline{\lambda}}(t_n) - \underline{\underline{\lambda}}_h^n\|_{L^2} + |||\underline{\underline{\lambda}}(t_n) - \underline{\underline{\lambda}}_h^n||| \\ & \quad \leq C_o\Big\{\Delta t\|e\vec{v}^0\|_{L^2} + \|e\vec{u}^0\|_{H^1} + \varepsilon\|e\underline{\underline{\lambda}}^0\|_{L^2} + \Delta t^2\|\vec{u}\|_{W^{4,\infty}(0,T;L^2)} \\ & \qquad + h^2\Big\{\|\vec{u}\|_{L^\infty(0,T;H^3)} + \|\underline{\underline{\lambda}}\|_{L^\infty(0,T;H^2)} + \|\vec{u}\cdot\vec{a}_3\|_{W^{2,\infty}(0,T;H^4)}\Big\}\Big\}. \end{aligned} \tag{6.11}$$

Proof. From (6.1) and (6.2), we have

$$\begin{aligned} & \frac{1}{4}m(e\vec{v}^n;e\vec{v}^n) + \frac{1}{2}A(e\vec{u}^n;e\vec{u}^n) + \frac{\varepsilon^2}{2(1-c_0\varepsilon^2)}C(e\underline{\underline{\lambda}}^n;e\underline{\underline{\lambda}}^n) \\ & \quad \leq \frac{3}{4}m(e\vec{v}^0,e\vec{v}^0) + \frac{1}{2}A(e\vec{u}^0;e\vec{u}^0) + \frac{\varepsilon^2}{2(1-c_0\varepsilon^2)}C(e\underline{\underline{\lambda}}^0;e\underline{\underline{\lambda}}^0) \\ & \qquad + \frac{\Delta t}{4}\sum_{i=1}^{n-1}m(e\vec{v}^i,e\vec{v}^i) + C_o\Big\{\Delta t^2\|\vec{u}\|_{W^{4,\infty}(0,T;L^2)} + \|(\pi - \mathrm{Id})(\vec{u})\|_{W^{2,\infty}(0,T;L^2)}\Big\}^2. \end{aligned} \tag{6.12}$$

Using the discrete Gronwall's lemma with $\theta = (1/4)\sum_{i=1}^{n-1}m(e\vec{v}^i,e\vec{v}^i)$ and

$$\begin{aligned} E = {} & \frac{3}{4}m(e\vec{v}^0,e\vec{v}^0) + \frac{1}{2}A(e\vec{u}^0;e\vec{u}^0) + \frac{\varepsilon^2}{2(1-c_0\varepsilon^2)}C\left(e\underline{\underline{\lambda}}^0;e\underline{\underline{\lambda}}^0\right) \\ & + C_o\{\Delta t^2\|\vec{u}\|_{W^{4,\infty}(0,T;L^2)} + \|(\pi - \mathrm{Id})(\vec{u})\|^2_{W^{2,\infty}(0,T;L^2)}\}^2, \end{aligned} \tag{6.13}$$

we obtain

$$\begin{aligned}&\frac{1}{4}m(e\vec{v}^{\,n};e\vec{v}^{\,n})+\frac{1}{2}A(e\vec{u}^{\,n};e\vec{u}^{\,n})+\frac{\varepsilon^2}{2(1-c_0\varepsilon^2)}C\left(e\underline{\underline{\lambda}}^{n};e\underline{\underline{\lambda}}^{n}\right)\\&\quad\le C_o\Big\{m(e\vec{v}^{\,0},e\vec{v}^{\,0})+A(e\vec{u}^{\,0};e\vec{u}^{\,0})+\frac{\varepsilon^2}{2(1-c_0\varepsilon^2)}C\left(e\underline{\underline{\lambda}}^{0};e\underline{\underline{\lambda}}^{0}\right)\\&\qquad+\left\{\Delta t^2\|\vec{u}\|_{W^{4,\infty}(0,T;L^2)}+\|(\pi-\mathrm{Id})(\vec{u})\|^2_{W^{2,\infty}(0,T;L^2)}\right\}^2\Big\}.\end{aligned}\tag{6.14}$$

To prove the required error estimation, we observe that we have

$$\begin{aligned}\vec{u}_h^{\,n}-\vec{u}(t_n)&=e\vec{u}^{\,n}-\vec{u}(t_n)-\pi\vec{u}^{\,n},\\\vec{v}_h^{\,n}-\vec{v}(t_n)&=e\vec{v}^{\,n}-\vec{v}(t_n)-\pi\vec{v}^{\,n},\\\underline{\underline{\lambda}}_h^{n}-\underline{\underline{\lambda}}(t_n)&=e\underline{\underline{\lambda}}^{n}-\underline{\underline{\lambda}}(t_n)-\pi_m^n\underline{\underline{\lambda}}.\end{aligned}\tag{6.15}$$

The terms $\vec{u}(t_n)-\pi\vec{u}^{\,n}$, $\underline{\underline{\lambda}}(t_n)-\pi_m^n\underline{\underline{\lambda}}$ can be bounded using Theorem 3.2 and we obtain for each n,

$$\begin{aligned}&\|\vec{u}(t_n)-\pi\vec{u}^{\,n}\|_V+\||\underline{\underline{\lambda}}(t_n)-\pi_m\underline{\underline{\lambda}}^{n}|\|+\varepsilon\|\underline{\underline{\lambda}}(t_n)-\pi_m\underline{\underline{\lambda}}^{n}\|_W\\&\quad\le C_oh^2\left\{\|\vec{u}(t_n)\|_{H^3}+\|\vec{u}(t_n)\cdot\vec{a}_3\|_{H^4}+\|\underline{\underline{\lambda}}(t_n)\|_{H^2}\right\}.\end{aligned}\tag{6.16}$$

Moreover, for the additional term associated to velocities, we write using projection velocities operators,

$$\begin{aligned}\pi\vec{v}^{\,n}-\vec{v}(t_n)&=2\sum_{i=1}^{n}(\pi-\mathrm{id})\left(\frac{\vec{u}(t_i)-\vec{u}(t_{i-1})}{\Delta t}\right)\\&\quad-\sum_{i=1}^{n}\left\{\vec{v}(t_i)-\vec{v}(t_{i-1})-2\frac{\vec{u}(t_i)-\vec{u}(t_{i-1})}{\Delta t}\right\}+(\pi-\mathrm{id})(\vec{u})(t_0).\end{aligned}\tag{6.17}$$

We deduce, by Taylor expansion, that

$$\|\pi\vec{v}^{\,n}-\vec{v}(t_n)\|_{L^2}\le C_o\left\{\Delta t\|\vec{u}\|_{W^{3,\infty}(0,T;L^2)}+\|(\pi-\mathrm{id})\vec{u}\|_{L^\infty(0,T;L^2)}\right\}.\tag{6.18}$$

The theorem follows by combining (6.14), (6.16), and (6.18). □

7. Conclusion

We have presented a well-posed dynamic mixed formulation for a shell problem and its approximation by finite elements. We also proved the time and uniform space convergence of our method. Our approach, which is valid for bending-dominated Koiter shells, avoids locking under a strong restriction assumption on the geometrical midsurface and a highly unbalanced combination of finite elements.

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