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Research Article

Generalized Stability of C^*-Ternary Quadratic Mappings

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We prove the generalized stability of C^*-ternary quadratic mappings in C^*-ternary rings for the quadratic functional equation $f(x+y)+f(x-y)=2f(x)+2f(y)$.

1. Introduction and preliminaries

A C^*-ternary ring is a complex Banach space A, equipped with a ternary product $(x,y,z) \mapsto [x,y,z]$ of A^3 into A, which is $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $[x,y,[z,w,v]] = [x,[w,z,y],v] = [[x,y,z],w,v]$, and satisfies $\|[x,y,z]\| \le \|x\| \cdot \|y\| \cdot \|z\|$ and $\|[x,x,x]\| = \|x\|^3$ (see [1]).

If a C^*-ternary ring $(A,[\cdot,\cdot,\cdot])$ has an identity, that is, an element $e \in A$ such that $x = [x,e,e] = [e,e,x]$ for all $x \in A$, then it is routine to verify that A, endowed with $x \circ y := [x,e,y]$ and $x^* := [e,x,e]$, is a unital C^*-algebra. Conversely, if $(A,\circ)$ is a unital C^*-algebra, then $[x,y,z] := x \circ y^* \circ z$ makes A into a C^*-ternary ring (see [2]).

Ulam [3] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems, containing the stability problem of homomorphisms. Hyers [4] proved the stability problem of additive mappings in Banach spaces. Rassias [5] provided a generalization of Hyers' theorem which allows the *Cauchy difference to be unbounded*: let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon\left(\|x\|^p + \|y\|^p\right) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Inequality (1.1) provided a lot of influence in the development of a generalization of the Hyers-Ulam stability concept. Găvruţa [6] provided a further generalization of Hyers-Ulam theorem (see [7, 8]).

A square norm on an inner product space satisfies the important parallelogram equality

$$\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2. \tag{1.2}$$

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.3}$$

is called the *quadratic functional equation* whose solution is said to be a *quadratic mapping*. A generalized stability problem for the quadratic functional equation was proved by Skof [9] for mappings $f:E_1\to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [10] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. Czerwik [11] proved the generalized stability of the quadratic functional equation, and Park [12] proved the generalized stability of the quadratic functional equation in Banach modules over a C^*-algebra. Jun and Lee [13] proved the further generalized stability of a Pexiderized quadratic functional equation

$$f(x+y)+g(x-y)=2h(x)+2k(y). \tag{1.4}$$

Recently, a fixed point approach to the stability of Pexiderized quadratic equation was established by Mirzavaziri and Moslehian [14].

Throughout this paper, assume that A is a C^*-ternary ring with norm $\|\cdot\|_A$ and that B is a C^*-ternary ring with norm $\|\cdot\|_B$.

A quadratic mapping $Q:A\to B$ is called a *C^*-ternary quadratic mapping* if

$$Q([x,y,z])=[Q(x),Q(y),Q(z)] \tag{1.5}$$

for all $x,y,z\in A$.

Example 1.1. Let $(A,[\cdot,\cdot,\cdot])$ be a C^*-ternary ring derived from a unital commutative C^*-algebra A, and let $Q:A\to A$ satisfy $Q(x)=x^2$ for all $x\in A$. It is easy to show that the mapping $Q:A\to A$ is a C^*-ternary quadratic mapping.

In this paper, we prove the further generalized stability of C^*-ternary quadratic mappings in C^*-ternary rings.

2. Stability of C^*-ternary quadratic mappings

We prove the further generalized stability of C^*-ternary quadratic mappings in C^*-ternary rings for the quadratic functional equation

$$Q(x+y)+Q(x-y)=2Q(x)+2Q(y). \tag{2.1}$$

Theorem 2.1. *Let $f : A \to B$ be a mapping for which there exists a function $\varphi : A^3 \to [0,\infty)$ such that*

$$\sum_{j=0}^{\infty} 4^{3j}\varphi\left(\frac{x}{2^j},\frac{y}{2^j},\frac{z}{2^j}\right) < \infty, \tag{2.2}$$

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\|_B \le \varphi(x,y,0), \tag{2.3}$$

$$\|f([x,y,z])-[f(x),f(y),f(z)]\|_B \le \varphi(x,y,z) \tag{2.4}$$

for all $x,y,z \in A$. Then there exists a unique C^*-ternary quadratic mapping $Q : A \to B$ such that

$$\|f(x)-Q(x)\|_B \le \widetilde{\varphi}\left(\frac{x}{2},\frac{x}{2},0\right) \tag{2.5}$$

for all $x \in A$. Here,

$$\widetilde{\varphi}(x,y,z) := \sum_{j=0}^{\infty} 4^{j}\varphi\left(\frac{x}{2^j},\frac{y}{2^j},\frac{z}{2^j}\right) \tag{2.6}$$

for all $x,y,z \in A$.

Proof. If follows from (2.3) that $f(0)=0$. Letting $y=x$ in (2.3), we get

$$\|f(2x)-4f(x)\|_B \le \varphi(x,x,0) \tag{2.7}$$

for all $x \in A$. So

$$\left\|f(x)-4f\left(\frac{x}{2}\right)\right\|_B \le \varphi\left(\frac{x}{2},\frac{x}{2},0\right) \tag{2.8}$$

for all $x \in A$. Hence,

$$\left\|4^l f\left(\frac{x}{2^l}\right)-4^m f\left(\frac{x}{2^m}\right)\right\|_B \le \sum_{j=l}^{m-1}\left\|4^j f\left(\frac{x}{2^j}\right)-4^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\|_B \le \sum_{j=l}^{m-1} 4^j\varphi\left(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},0\right) \tag{2.9}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (2.9) that the sequence $\{4^n f(x/2^n)\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\{4^n f(x/2^n)\}$ converges. So one can define the mapping $Q : A \to B$ by

$$Q(x) := \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) \tag{2.10}$$

for all $x \in A$. Moreover, letting $l=0$ and passing the limit $m \to \infty$ in (2.9), we get (2.5).

It follows from (2.3) that

$$\begin{aligned} &\|Q(x+y)+Q(x-y)-2Q(x)-2Q(y)\|_B \\ &\quad = \lim_{n\to\infty} 4^n \left\| f\left(\frac{x+y}{2^n}\right)+f\left(\frac{x-y}{2^n}\right)-2f\left(\frac{x}{2^n}\right)-2f\left(\frac{y}{2^n}\right)\right\|_B \\ &\quad \le \lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n},\frac{y}{2^n},0\right)=0 \end{aligned} \tag{2.11}$$

for all $x, y \in A$. So

$$Q(x+y)+Q(x-y)=2Q(x)+2Q(z) \tag{2.12}$$

for all $x, y \in A$.

It follows from (2.4) and the continuity of the ternary product that

$$\begin{aligned} &\|Q([x,y,z])-[Q(x),Q(y),Q(z)]\|_B \\ &\quad = \lim_{n\to\infty} 4^{3n} \left\| f\left(\frac{[x,y,z]}{2^{3n}}\right)-\left[f\left(\frac{x}{2^n}\right),f\left(\frac{y}{2^n}\right),f\left(\frac{z}{2^n}\right)\right]\right\|_B \\ &\quad \le \lim_{n\to\infty} 4^{3n} \varphi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right)=0 \end{aligned} \tag{2.13}$$

for all $x, y, z \in A$. So

$$Q([x,y,z])=[Q(x),Q(y),Q(z)] \tag{2.14}$$

for all $x, y, z \in A$.

Now, let $T : A \to B$ be another quadratic mapping satisfying (2.5). Then we have

$$\begin{aligned} \|Q(x)-T(x)\|_B &= 4^n \left\| Q\left(\frac{x}{2^n}\right)-T\left(\frac{x}{2^n}\right)\right\|_B \\ &\le 4^n \left(\left\| Q\left(\frac{x}{2^n}\right)-f\left(\frac{x}{2^n}\right)\right\|_B + \left\| T\left(\frac{x}{2^n}\right)-f\left(\frac{x}{2^n}\right)\right\|_B \right) \\ &\le 2\cdot 4^n \varphi\left(\frac{x}{2^n},\frac{x}{2^n},0\right), \end{aligned} \tag{2.15}$$

which tends to zero as $n \to \infty$ for all $x \in A$. So we can conclude that $Q(x) = T(x)$ for all $x \in A$. This proves the uniqueness of Q. Thus, the mapping $Q : A \to B$ is a unique C^*-ternary quadratic mapping satisfying (2.5). □

THEOREM 2.2. *Let $f : A \to B$ be a mapping for which there exists a function $\varphi : A^3 \to [0,\infty)$ satisfying (2.3) and (2.4) such that*

$$\widetilde{\varphi}(x,y,z) := \sum_{j=0}^{\infty} \frac{1}{4^j}\varphi(2^j x, 2^j y, 2^j z) < \infty \tag{2.16}$$

for all $x, y, z \in A$. Then there exists a unique C^*-ternary quadratic mapping $Q : A \to B$ such that

$$\|f(x) - Q(x)\|_B \le \frac{1}{4}\widetilde{\varphi}(x,x,0) \tag{2.17}$$

for all $x \in A$.

Proof. It follows from (2.7) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\|_B \le \frac{1}{4}\varphi(x,x,0) \tag{2.18}$$

for all $x \in A$. So

$$\left\|\frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x)\right\|_B \le \sum_{j=l}^{m-1}\left\|\frac{1}{4^j}f(2^j x) - \frac{1}{4^{j+1}}f(2^{j+1}x)\right\|_B \le \sum_{j=l}^{m-1}\frac{1}{4^{j+1}}\varphi(2^j x, 2^j x, 0) \tag{2.19}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (2.19) that the sequence $\{(1/4^n)f(2^n x)\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\{(1/4^n)f(2^n x)\}$ converges. So one can define the mapping $Q : A \to B$ by

$$Q(x) := \lim_{n\to\infty}\frac{1}{4^n}f(2^n x) \tag{2.20}$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.19), we get (2.17).

It follows from (2.4) and the continuity of the ternary product that

$$\begin{aligned}
&\|Q([x,y,z]) - [Q(x),Q(y),Q(z)]\|_B \\
&\quad = \lim_{n\to\infty}\frac{1}{4^{3n}}\|f(2^{3n}[x,y,z]) - [f(2^n x), f(2^n y), f(2^n z)]\|_B \\
&\quad \le \lim_{n\to\infty}\frac{1}{4^{3n}}\varphi(2^n x, 2^n y, 2^n z) \\
&\quad \le \lim_{n\to\infty}\frac{1}{4^{n}}\varphi(2^n x, 2^n y, 2^n z) = 0
\end{aligned} \tag{2.21}$$

for all $x, y, z \in A$. So

$$Q([x,y,z]) = [Q(x),Q(y),Q(z)] \tag{2.22}$$

for all $x, y, z \in A$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Remark 2.3. For a Pexiderized quadratic functional equation

$$f(x+y) + g(x-y) = 2h(x) + 2k(y), \tag{2.23}$$

one can obtain similar results to Theorems 2.1 and 2.2.

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