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A classification result for helix surfaces with parallel mean curvature in product spaces

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Abstract. We determine all helix surfaces with parallel mean curvature vector field which are not minimal or pseudo-umbilical in spaces of type $M^n(c)\times\mathbb{R}$, where $M^n(c)$ is a simply connected n-dimensional manifold with constant sectional curvature c.

1. Preliminaries and the main result

Let us consider a space form $M^n(c)$, i.e., a simply connected n-dimensional manifold with constant sectional curvature c, the product manifold $\overline{M}=M^n(c)\times\mathbb{R}$, and an isometrically immersed surface Σ^2 in $\overline{M}=M^n(c)\times\mathbb{R}$.

The second fundamental form σ of the surface Σ^2 is defined by the equation of Gauss

$$\overline{\nabla}_X Y=\nabla_X Y+\sigma(X,Y),$$

for any tangent vector fields X and Y, where $\overline{\nabla}$ and ∇ are the Levi-Civita connections on $\overline{M}$ and Σ^2, respectively. Then the mean curvature vector field H of Σ^2 is given by $H=(1/2)\operatorname{trace}\sigma$. The shape operator A and the normal connection $\nabla^\perp$ are defined by the equation of Weingarten

$$\overline{\nabla}_X V=-A_V X+\nabla^\perp_X V,$$

for any tangent vector field X and any normal vector field V.

Definition 1.1. A surface Σ^2 is called a *helix surface* (or a *constant angle surface*) if the angle function $\theta\in[0,\pi)$ between its tangent spaces and the unit vector field ξ tangent to $\mathbb{R}$ is constant on Σ^2.

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A helix surface is characterized by the fact that the tangent part T of ξ has constant length.

Definition 1.2. If the mean curvature vector field H of a surface Σ^2 is parallel in the normal bundle, i.e., $\nabla^{\perp}H=0$, then Σ^2 is called a *pmc surface.*

In our paper, we consider non-minimal pmc helix surfaces in $M^n(c)\times\mathbb{R}$ and prove the following classification theorem.

Theorem 1.3. *Let Σ^2 be a non-minimal pmc helix surface with Gaussian curvature K and mean curvature vector field H in $M^n(c)\times\mathbb{R}$, with $c\neq 0$, and let T be the tangent part of the unit vector field ξ tangent to $\mathbb{R}$. Then one of the following holds*:

(1) *Σ^2 is a minimal surface in a non-minimal totally umbilical hypersurface of $M^n(c)$;*

(2) *Σ^2 is a surface with constant mean curvature in a 3-dimensional totally umbilical or totally geodesic submanifold of $M^n(c)$;*

(3) *Σ^2 is a vertical cylinder over a circle in $M^n(c)$ with curvature $\varkappa=2|H|$;*

(4) *$c<0$ and Σ^2 lies in $M^2(c)\times\mathbb{R}$. Also $4|H|^2+c|T|^2=0$, $K=c(1-|T|^2)<0$, and the Abresch–Rosenberg differential vanishes on Σ^2. If the surface is complete, then it is a cmc surface of type P^2_H in $M^2(c)\times\mathbb{R}$;*

(5) *$c>0$ and locally Σ^2 is the standard product $\gamma_1\times\gamma_2$, where $\gamma_1\colon I\subset\mathbb{R}\to M^4(c)\times\mathbb{R}$ is a helix in $M^4(c)\times\mathbb{R}$ with curvatures $\varkappa_1=\sqrt{c(1-|T|^2)}$ and $\varkappa_2=|T|\sqrt{c}$, and $\gamma_2\colon I\subset\mathbb{R}\to M^4(c)\subset M^4(c)\times\mathbb{R}$ is a circle in $M^4(c)$ with curvature $\varkappa=\sqrt{4|H|^2+c(1-|T|^2)}$. If the surface is complete, then the above decomposition holds globally.*

Next, we will briefly recall some notions and results that will be used in the proof of our theorem.

The expression of the curvature tensor $\overline{R}$ of $\overline{M}=M^n(c)\times\mathbb{R}$ can be deduced from

$$\langle\overline{R}(X,Y)Z,W\rangle=c(\langle d\pi Y,d\pi Z\rangle\langle d\pi X,d\pi W\rangle-\langle d\pi X,d\pi Z\rangle\langle d\pi Y,d\pi W\rangle),$$

where $\pi\colon\overline{M}=M^n(c)\times\mathbb{R}\to M^n(c)$ is the projection map (see [2]). We obtain

$$\overline{R}(X,Y)Z=c(\langle Y,Z\rangle X-\langle X,Z\rangle Y-\langle Y,\xi\rangle\langle Z,\xi\rangle X+\langle X,\xi\rangle\langle Z,\xi\rangle Y+\langle X,Z\rangle\langle Y,\xi\rangle\xi-\langle Y,Z\rangle\langle X,\xi\rangle\xi). \tag{1.1}$$

We will also need the Gauss equation of a surface Σ^2 in $\overline{M}=M^n(c)\times\mathbb{R}$,

$$\langle R(X,Y)Z,W\rangle = \langle \overline{R}(X,Y)Z,W\rangle+\langle\sigma(Y,Z),\sigma(X,W)\rangle - \langle\sigma(X,Z),\sigma(Y,W)\rangle, \tag{1.2}$$

and its Codazzi equation

$$(\overline{R}(X,Y)Z)^{\perp} = (\nabla^{\perp}_X\sigma)(Y,Z)-(\nabla^{\perp}_Y\sigma)(X,Z), \tag{1.3}$$

where X, Y, Z, and W are tangent vector fields and R is the curvature tensor corresponding to ∇.

Definition 1.4. A surface Σ^2 in $M^n(c)\times\mathbb{R}$ is called a *vertical cylinder* over γ if $\Sigma^2=\pi^{-1}(\gamma)$, where $\pi\colon M^n(c)\times\mathbb{R}\to M^n(c)$ is the projection map and $\gamma\colon I\subset\mathbb{R}\to M^n(c)$ is a curve in $M^n(c)$.

It is easy to see that vertical cylinders $\Sigma^2=\pi^{-1}(\gamma)$ are characterized by the fact that ξ is tangent to Σ^2.

Definition 1.5. Let $\gamma\colon I\subset\mathbb{R}\to\overline{M}^{n+1}$ be a curve parametrized by arc-length. Then γ is called a *Frenet curve of osculating order* r, $1\le r\le n+1$, if there exist r orthonormal vector fields $\{X_1=\gamma',...,X_r\}$ along γ such that

$$\overline{\nabla}_{X_1}X_1=\varkappa_1X_2,\quad \overline{\nabla}_{X_1}X_i=-\varkappa_{i-1}X_{i-1}+\varkappa_iX_{i+1},\quad ...,\quad \overline{\nabla}_{X_1}X_r=-\varkappa_{r-1}X_{r-1},$$

for all $i\in\{2,...,r-1\}$, where $\{\varkappa_1,\varkappa_2,...,\varkappa_{r-1}\}$ are positive functions on I called the *curvatures* of γ. A Frenet curve of osculating order r is called a *helix of order* r if $\varkappa_i$=constant>0 for $1\le i\le r-1$. A helix of order 2 is called a *circle*, and a helix of order 3 is simply called *helix*.

When the ambient space is $M^2(c)\times\mathbb{R}$, a pmc surface Σ^2 is a surface with constant mean curvature (a *cmc surface*). In order to study such surfaces, U. Abresch and H. Rosenberg introduced a holomorphic differential, now called the *Abresch–Rosenberg differential*, and determined all complete cmc surfaces in $M^2(c)\times\mathbb{R}$ on which it vanishes (see [1]). They proved that there are four classes of such surfaces, denoted by S^2_H, D^2_H, C^2_H, and P^2_H, all of them described in detail in [1]. This holomorphic differential is the $(2,0)$-part of a quadratic form Q defined on a cmc surface by

$$Q(X,Y)=2\langle\sigma(X,Y),H\rangle-c\langle X,\xi\rangle\langle Y,\xi\rangle.$$

Complete cmc helix surfaces in $M^2(c)\times\mathbb{R}$, and actually in all 3-dimensional homogeneous spaces, were determined in [5, Theorem 2.2], while [3, Theorem 1] shows that there are no non-minimal pmc helix surfaces with $0<|T|<1$ in $\mathbb{S}^3(1)\times\mathbb{R}$.

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2. The proof of Theorem 1.3

The map $p\in\Sigma^2\mapsto(A_H-\mu\,\mathrm{I})(p)$ is analytic, and, therefore, either Σ^2 is a pseudo-umbilical surface, i.e., $A_H=|H|^2\,\mathrm{I}$; or H is an umbilical direction on a closed set without interior points. We shall denote by W the set of points where H is not an umbilical direction, which, in the second case, is an open dense set in Σ^2.

If Σ^2 is a pmc surface in $M^n(c)\times\mathbb{R}$, with mean curvature vector field H, then either Σ^2 is pseudo-umbilical, i.e., H is an umbilical direction everywhere; or, at any point in W, there exists a local orthonormal frame field that diagonalizes A_U for any normal vector field U defined on W (see [2, Lemma 1]). If Σ^2 is a pseudo-umbilical pmc surface in $M^n(c)\times\mathbb{R}$, then it lies in $M^n(c)$, i.e., $|T|=0$ (see [2, Lemma 3]).

From [2, Remark 1], we also know that, since H is parallel, the immersion of Σ^2 in $M^n(c)\times\mathbb{R}$ is analytic, i.e., the functions of two variables that locally define the immersion are real-analytic. Therefore, it satisfies a principle of unique continuation and, as a consequence, it cannot vanish on an open connected subset of Σ^2 unless it vanishes identically.

Now, when $|T|=0$, our surface lies in $M^n(c)$ and we obtain the first two items of the theorem using [9, Theorem 4].

When $|T|=1$, the vector field ξ is tangent to the surface and this means that Σ^2 is a pmc vertical cylinder over a curve γ in $M^n(c)$. Since ξ is parallel in $M^n(c)\times\mathbb{R}$, it follows that $\sigma(\xi,\xi)=0$ and then we easily get that γ is a circle in $M^n(c)$ with curvature $\varkappa=2|H|$.

Henceforth, let us assume that $0<|T|<1$. It follows that H is not an umbilical direction on an open dense set W. We will work on this set and then extend our results throughout Σ^2 by continuity.

Consider a global orthonormal frame field $\{E_1=T/|T|,E_2\}$ on Σ^2, and let N be the normal part of ξ. Then, since Σ^2 is a helix surface, it follows that $\nabla_{E_1}E_1=\nabla_{E_1}E_2=0$ and, as $\bar\nabla_X\xi=0$ implies $\nabla_XT=A_NX$ and $\sigma(T,X)=-\nabla^\perp_XN$, that $A_NE_1=0$ (see [8, Proposition 2.1]). We also have

$$\begin{aligned}\langle\nabla_{E_2}E_2,E_1\rangle&=-\langle E_2,\nabla_{E_2}E_1\rangle=-\frac{1}{|T|}\langle E_2,A_NE_2\rangle=-\frac{1}{|T|}\langle\sigma(E_2,E_2),N\rangle\\&=-\frac{1}{|T|}\langle 2H-\sigma(E_1,E_1),N\rangle=-\frac{2\langle H,N\rangle}{|T|},\end{aligned}$$

which means that

$$\nabla_{E_2}E_2 = -\frac{2\langle H,N\rangle}{|T|}E_1 \quad \text{and} \quad \nabla_{E_2}E_1 = \frac{2\langle H,N\rangle}{|T|}E_2. \tag{2.1}$$

Since $A_N E_1 = 0$, we also get that $A_N E_2 = 2\langle H,N\rangle E_2$. From the Ricci equation

$$\langle R^\perp(X,Y)U,V\rangle = \langle [A_U,A_V]X,Y\rangle + \langle \bar{R}(X,Y)U,V\rangle,$$

where X and Y are tangent vector fields and U and V are normal vector fields, we obtain $[A_H,A_N]=0$, which means that $\langle H,N\rangle\langle A_H E_1,E_2\rangle = 0$. As we also have

$$E_2(\langle H,N\rangle) = \langle H,\nabla^\perp_{E_2}N\rangle = -|T|\langle H,\sigma(E_1,E_2)\rangle = -|T|\langle A_H E_1,E_2\rangle,$$

one sees that $\langle A_H E_1,E_2\rangle = 0$. Moreover, again using the Ricci equation, we have $[A_H,A_U]=0$ for any normal vector field U and then, since H is not umbilical, $\sigma(E_1,E_2)=0$ and $\nabla^\perp_{E_2}N=0$.

Next, let us denote by λ_i the eigenfunctions $\langle A_H E_i,E_i\rangle$ of A_H and in the following we will compute $E_j(\lambda_i)$, $i,j\in\{1,2\}$.

Using (1.1), (2.1), and $\sigma(E_1,E_2)=0$, we get

$$\nabla^\perp_{E_2}\sigma(E_1,E_1) = (\bar{R}(E_2,E_1)E_1)^\perp + 2\sigma(E_1,\nabla_{E_2}E_1) = 0$$

and then, from the Codazzi equation (1.3),

$$\begin{aligned}\nabla^\perp_{E_1}\sigma(E_2,E_2) &= (\bar{R}(E_1,E_2)E_2)^\perp + 2\sigma(E_2,\nabla_{E_1}E_2) + \nabla^\perp_{E_2}\sigma(E_1,E_2) \\ &\quad -\sigma(\nabla_{E_2}E_1,E_2) - \sigma(E_1,\nabla_{E_2}E_2) \\ &= -c|T|N + \frac{2\langle H,N\rangle}{|T|}(\sigma(E_1,E_1)-\sigma(E_2,E_2)).\end{aligned} \tag{2.2}$$

Therefore, as H is parallel, we have

$$E_1(\lambda_1) = -E_1(\lambda_2) = -\langle \nabla^\perp_{E_1}\sigma(E_2,E_2),H\rangle = \frac{\langle H,N\rangle}{|T|}(4|H|^2 + c|T|^2 - 4\lambda_1) \tag{2.3}$$

and

$$E_2(\lambda_1) = -E_2(\lambda_2) = \langle \nabla^\perp_{E_2}\sigma(E_1,E_1),H\rangle = 0. \tag{2.4}$$

From [6, Proposition 1.4], we know that

$$\tfrac{1}{2}\Delta|T|^2 = |A_N|^2 + K|T|^2 - 2\langle A_H T,T\rangle$$

and then, since $|T|=\text{constant}\neq 0$, the Gaussian curvature K of Σ^2 is given by

$$K = 2\lambda_1 - \frac{4\langle H,N\rangle^2}{|T|^2}. \tag{2.5}$$

Since $\nabla^{\perp}_X N = -\sigma(X,T)$ implies that

$$E_1(\langle H,N\rangle) = -|T|\lambda_1 \quad \text{and} \quad E_2(\langle H,N\rangle) = 0, \tag{2.6}$$

from (2.3) and (2.4), we obtain

$$E_1(K) = \frac{2\langle H,N\rangle}{|T|}(4|H|^2 + c|T|^2) \quad \text{and} \quad E_2(K) = 0. \tag{2.7}$$

The fact that Σ^2 is not pseudo-umbilical implies that it lies in $M^4(c)\times\mathbb{R}$ (see [2, Theorem 1]).

In order to describe our surface by taking advantage of the above formulas, we will first consider the case when $H\|N$ on an open connected subset W_0 of Σ^2. This means that $\langle H,N\rangle = \pm|H|\,|N| = \text{constant}$ and, from (2.6), one obtains that $\lambda_1 = 0$. Then, from (2.3) and (2.5), we get $4|H|^2 + c|T|^2 = 0$ and $K = c(1-|T|^2)$ on W_0.

Let $\{E_3 = H/|H|, E_4, E_5\}$ be a global orthonormal frame field in the normal bundle. Then, as $\sigma(E_1,E_2) = 0$, on W_0 we have

$$A_3 = \begin{pmatrix} 0 & 0 \\ 0 & 2|H| \end{pmatrix}, \quad A_4 = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \quad \text{and} \quad A_5 = \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix},$$

where $A_\alpha = A_{E_\alpha}$; and, from the Gauss equation (1.2), since $\langle \bar{R}(E_1,E_2)E_2,E_1\rangle = c(1-|T|^2)$, we have

$$K = c(1-|T|^2) - a^2 - b^2.$$

But, as we have seen, $K = c(1-|T|^2)$ on W_0, which implies that $a=b=0$ on W_0. Consider the subbundle $L = \operatorname{span}\{\operatorname{Im}\sigma\} = \operatorname{span}\{H\}$ in the normal bundle. It follows that $\xi \in T\Sigma^2 \oplus L$ and $T\Sigma^2\oplus L$ is parallel with respect to $\bar{\nabla}$ and invariant by $\bar{R}$. Therefore, we use [4, Theorem 2] to show that W_0 lies in $M^2(c)\times\mathbb{R}$. From the analyticity of the immersion of Σ^2 in $M^4(c)\times\mathbb{R}$, it follows that Σ^2 lies in $M^2(c)\times\mathbb{R}$ (see [2, Remark 1] for more details). As a direct consequence, we have $H\|N$ on Σ^2 and then $\lambda_1 = 0$, $4|H|^2 + c|T|^2 = 0$, and $K = c(1-|T|^2) < 0$ on Σ^2. Now, it is easy to verify that the Abresch–Rosenberg differential vanishes on the surface. Moreover, if Σ^2 is complete, all these properties of Σ^2 lead to the conclusion that our surface is a cmc surface of type P^2_H.

Next, we will consider the remaining case, when $H\|N$ only at isolated points. We can then define a local orthonormal frame field

$$\left\{E_3 = \frac{1}{|H|\sin\beta}H - \frac{\cot\beta}{|N|}N,\ E_4 = \frac{N}{|N|},\ E_5\right\}$$

in the normal bundle, where $\beta \in (0,\pi)$ is the angle between H and N. Therefore, since

$$A_H = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & 2|H|^2 - \lambda_1 \end{pmatrix},$$

$$A_N = \begin{pmatrix} 0 & 0 \\ 0 & 2\langle H, N\rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2|H|\,|N|\cos\beta \end{pmatrix},$$

with respect to $\{E_1, E_2\}$, we can write

$$A_3 = \begin{pmatrix} \dfrac{\lambda_1}{|H|\sin\beta} & 0 \\ 0 & 2|H|\sin\beta - \dfrac{\lambda_1}{|H|\sin\beta} \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{2\langle H, N\rangle}{|N|} \end{pmatrix},$$

$$A_5 = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix},$$

and then, from the Gauss equation (1.2), one obtains

$$K = c(1-|T|^2) + 2\lambda_1 - \frac{\lambda_1^2}{|H|^2\sin^2\beta} - \lambda^2. \tag{2.8}$$

Using equation (2.5), we have

$$\lambda^2 = c(1-|T|^2) - \frac{\lambda_1^2}{|H|^2\sin^2\beta} + \frac{4\langle H, N\rangle^2}{|T|^2} \tag{2.9}$$

and then

$$2\lambda E_1(\lambda) = E_1\left(-\frac{\lambda_1^2}{|H|^2\sin^2\beta} + \frac{4\langle H, N\rangle^2}{|T|^2}\right). \tag{2.10}$$

Next, we shall compute $E_1(\lambda)$. Using (2.2), the fact that H is parallel and $\nabla^\perp_X N = -\sigma(X, T)$, we obtain

$$\begin{aligned} E_1(\lambda) &= -E_1(\langle \sigma(E_2, E_2), E_5\rangle) \\ &= -\langle \nabla^\perp_{E_1}\sigma(E_2, E_2), E_5\rangle - \langle \sigma(E_2, E_2), \nabla^\perp_{E_1} E_5\rangle \\ &= -\frac{4\langle H, N\rangle}{|T|}\lambda - \langle \sigma(E_2, E_2), E_3\rangle\langle \nabla^\perp_{E_1} E_5, E_3\rangle - \langle \sigma(E_2, E_2), E_4\rangle\langle \nabla^\perp_{E_1} E_5, E_4\rangle \end{aligned}$$

$$= -\frac{4\langle H,N\rangle}{|T|}\lambda + \left(2|H|\sin\beta - \frac{\lambda_1}{|H|\sin\beta}\right)\langle E_5, \nabla^{\perp}_{E_1}E_3\rangle + \frac{2\langle H,N\rangle}{|N|}\langle E_5, \nabla^{\perp}_{E_1}E_4\rangle$$
$$= -\frac{4\langle H,N\rangle}{|T|}\lambda - \frac{|T|}{|H||N|}\frac{\cos\beta}{\sin^2\beta}\lambda\lambda_1,$$

since

$$\langle E_5, \nabla^{\perp}_{E_1}E_3\rangle = \frac{|T|}{|N|}\lambda\cot\beta \quad \text{and} \quad \langle E_5, \nabla^{\perp}_{E_1}E_4\rangle = -\frac{|T|}{|N|}\lambda.$$

Replacing in (2.10) and using (2.3) and (2.6), we get, after a straightforward computation

$$\langle H,N\rangle\left(2\lambda_1 - \frac{\lambda_1^2}{|H|^2\sin^2\beta} - \lambda^2\right) = 0, \tag{2.11}$$

which, together with (2.8), leads to $\langle H,N\rangle(K-c(1-|T|^2))=0$, which, taking (2.7) into account, can be written as $E_1((K-c(1-|T|^2))^2)=0$. Again from (2.7), we have $E_2((K-c(1-|T|^2))^2)=0$. It follows that $K-c(1-|T|^2)=\text{constant}$ and then, using (2.5), one obtains

$$E_1\left(2\lambda_1 - \frac{4\langle H,N\rangle^2}{|T|^2}\right) = 0.$$

Now, from equations (2.3) and (2.6), we have $(4|H|^2+c|T|^2)\langle H,N\rangle=0$. We will consider two cases as $4|H|^2+c|T|^2=0$ or $4|H|^2+c|T|^2\neq 0$.

Case 1. $4|H|^2+c|T|^2=0$. Let us assume that $\langle H,N\rangle=0$ on an open connected set W_0. From (2.6) it follows that $\lambda_1=0$ and then, from (2.5), that $K=0$ on W_0. But, from (2.8), we have $K=c(1-|T|^2)-\lambda^2$, which means that $\lambda^2=c(1-|T|^2)$. This implies $c>0$ which is a contradiction, since $4|H|^2+c|T|^2=0$. Therefore $\langle H,N\rangle=0$ on a closed set without interior points and then, from (2.11), we have

$$2\lambda_1 - \frac{\lambda_1^2}{|H|^2\sin^2\beta} - \lambda^2 = 0$$

and $K=c(1-|T|^2)$ on an open dense set. Since $4|H|^2+c|T|^2=0$, from (2.5), one obtains $\sin^2\beta=2\lambda_1/c|N|^2$ and then $\lambda^2=2\lambda_1/|T|^2$, which means that $\lambda_1=0$, $\lambda=0$ and $\sin^2\beta=0$ on an open dense set. The last identity shows that $H\parallel N$ on an open dense set, which is a contradiction.

Case 2. $4|H|^2+c|T|^2\neq 0$. In this case, $\langle H,N\rangle=0$ on an open dense set and, from (2.6), it follows that $\lambda_1=0$ and then, from (2.5), we have $K=0$ and, from (2.9), $\lambda^2=c(1-|T|^2)$, which implies $c>0$. The shape operator is given by

$$A_3 = \begin{pmatrix} 0 & 0 \\ 0 & 2|H| \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad A_5 = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}.$$

From the equations (2.1), we see that $\nabla E_1 = \nabla E_2 = 0$ and we then can apply the de Rham decomposition theorem ([7]) to show that locally Σ^2 is the standard product $\gamma_1 \times \gamma_2$ of two Frenet curves $\gamma_1 \colon I \subset \mathbb{R} \to M^4(c) \times \mathbb{R}$ and $\gamma_2 \colon I \subset \mathbb{R} \to M^4(c) \times \mathbb{R}$ such that $\gamma_1' = E_1$ and $\gamma_2' = E_2$. We note that γ_2 actually lies in $M^4(c)$. If the surface is complete, then this decomposition holds globally.

Next, we will characterize γ_1 and γ_2 by using their Frenet equations. Let $\{X_1^1 = E_1, X_2^1, ..., X_r^1\}$ be the Frenet frame field of γ_1. We have that

$$\bar{\nabla}_{E_1} E_1 = \sigma(E_1, E_1) = \lambda E_5,$$

and then, from the first Frenet equation, it follows that $\varkappa_1 = |\lambda| = \sqrt{c(1-|T|^2)}$ and $X_2^1 = \pm E_5$.

Next, we have $\langle \nabla^\perp_{E_1} E_5, E_3 \rangle = 0$, since $E_3 = H/|H|$ is parallel, and

$$\langle \nabla^\perp_{E_1} E_5, E_4 \rangle = -\langle E_5, \nabla^\perp_{E_1} E_4 \rangle = \frac{|T|}{|N|} \langle E_5, \sigma(E_1, E_1) \rangle = \frac{|T|}{|N|} \lambda.$$

Therefore, one obtains

$$\bar{\nabla}_{E_1} X_2^1 = \pm \bar{\nabla}_{E_1} E_5 = \mp A_5 E_1 \pm \nabla^\perp_{E_1} E_5 = \mp \lambda E_1 \pm \frac{|T|}{|N|} \lambda E_4.$$

From the second Frenet equation, we get

$$\varkappa_2 = \frac{|T|}{|N|} |\lambda| = |T| \sqrt{c} \quad \text{and} \quad X_3^1 = \pm E_4.$$

It follows that

$$\bar{\nabla}_{E_1} X_3^1 = \pm \bar{\nabla}_{E_1} E_4 = \pm \nabla^\perp_{E_1} E_4 = \mp \frac{|T|}{|N|} \sigma(E_1, E_1) = \mp \frac{|T|}{|N|} \lambda E_5 = -\varkappa_2 X_2^1$$

and we have just proved that γ_1 is a helix.

Now, let $\{X_1^2 = E_2, X_2^2, ..., X_r^2\}$ be the Frenet frame field of γ_2. Then, from

$$\bar{\nabla}_{E_2} E_2 = \sigma(E_2, E_2) = 2|H| E_3 - \lambda E_5,$$

and the first Frenet equation of γ_2, one obtains

$$\varkappa = \sqrt{4|H|^2 + \lambda^2} = \sqrt{4|H|^2 + c(1-|T|^2)} \quad \text{and} \quad X_2^2 = \frac{1}{\varkappa}(2|H| E_3 - \lambda E_5).$$

It is easy to verify, using $\bar{\nabla}\xi = 0$, that $\nabla^\perp_{E_2} E_5 = 0$. Then, since E_3 is parallel, we have

$$\bar{\nabla}_{E_2} X_2^2 = -\frac{1}{\varkappa}(2|H| A_3 E_2 + \lambda A_3 E_2) = -\varkappa E_2$$

and we conclude.

Remark 2.1. From the proof of Theorem 1.3 it is easy to see that the only non-minimal pmc helix surfaces in $M^3(c)\times\mathbb{R}$ are those given by the first four cases of our theorem.

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