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Riemann's zeta-function and the divisor problem. III

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Abstract. In two earlier papers with the same title, we studied connections between Voronoi's formula in the divisor problem and Atkinson's formula for the mean square of Riemann's zeta-function. Now we consider this correspondence in terms of segments of sums appearing in these formulae and show that a certain arithmetic conjecture concerning the divisor function implies best possible bounds for the classical error terms $\Delta(x)$ and $E(T)$.

1. Introduction

This paper is a continuation of [6] and [10], where we considered connections between the square $\left|\zeta\left(\frac{1}{2}+it\right)\right|^2$ of Riemann's zeta-function and the divisor function $d(n)$, the number of positive divisors of n. This analogy was pointed out by Atkinson [1] in his classical paper, the main result of which was a formula for the function

$$E(T)=\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^2 dt-\left(\log \frac{T}{2\pi}+2\gamma-1\right)T,$$

where γ is Euler's constant. Atkinson's formula is of the form

$$E(T)=\Sigma_1(T)+\Sigma_2(T)+O(\log^2 T), \tag{1.1}$$

where $\Sigma_1(T)$ and $\Sigma_2(T)$ are sums of length $\asymp T$ involving coefficients $d(n)$. The notation $A\asymp B$ for positive A and B means that $A\ll B\ll A$. As references to Atkinson's formula, in addition to his original paper, see [4], [3], and [11]. The significant one of the sums in (1.1) is

$$\Sigma_1(T)=\sqrt{2}\left(\frac{T}{2\pi}\right)^{1/4}\sum_{n\le N}(-1)^n d(n)n^{-3/4}e(T,n)\cos f(T,n), \tag{1.2}$$

where $N \asymp T$,

$$e(T,n) = 1 + O\left(\frac{n}{T}\right), \tag{1.3}$$

and

$$f(T,n) = -\frac{\pi}{4} + 2\sqrt{2\pi nT} + An^{3/2} + O(n^{5/2}T^{-3/2}) \tag{1.4}$$

with $A = A(T) = \frac{1}{6}\sqrt{2}\pi^{3/2}T^{-1/2}$. The explicit formulae for $e(T,n)$ and $f(T,n)$, which can be found in the above-mentioned references, will be irrelevant in the following. The second sum in (1.1) is

$$\Sigma_2(T) = -2 \sum_{n \le N'} \frac{d(n)}{\sqrt{n}\log\dfrac{T}{2\pi n}} \cos\left(T\log\frac{T}{2\pi n} - T + \frac{\pi}{4}\right), \tag{1.5}$$

where $N' = T/2\pi + N/2 - \sqrt{N^2/4 + NT/2\pi}$.

Turning to the divisor problem, recall an identity of the Voronoi type for the sum

$$D\left(x, \frac{h}{k}\right) = \sum_{n \le x} d(n) e\left(\frac{nh}{k}\right),$$

where $e(\alpha) = e^{2\pi i\alpha}$ as usual. For $x > 0$, $k \ge 1$, and $(h,k) = 1$ we have (see [9, Theorem 1.6])

$$D\left(x, \frac{h}{k}\right) = \frac{x}{k}(\log x + 2\gamma - 1 - 2\log k) + E\left(0, \frac{h}{k}\right) + \Delta\left(x, \frac{h}{k}\right) \tag{1.6}$$

with

(1.7)

$$\Delta\left(x, \frac{h}{k}\right) = -\sqrt{x}\sum_{n=1}^{\infty} \frac{d(n)}{\sqrt{n}}\left(e\left(-\frac{n\bar{h}}{k}\right)Y_1\left(\frac{4\pi\sqrt{nx}}{k}\right) + \frac{2}{\pi}e\left(\frac{n\bar{h}}{k}\right)K_1\left(\frac{4\pi\sqrt{nx}}{k}\right)\right),$$

where $h\bar{h} \equiv 1 \pmod k$, $E(s, h/k)$ is the Estermann zeta-function, and the convention of summation is that if x is an integer, then the term for $n = x$ is to be halved. Here Y_1 and K_1 stand for Bessel functions in the standard notation. As a reference for Bessel functions, see e.g. [12]. We are going to need only the cases $h/k = 1$ and $h/k = \frac{1}{2}$ of (1.6), and then $E(0,1) = E\left(0, \frac{1}{2}\right) = \frac{1}{4}$ (see [9, (1.1.10)]).

The formula

$$\begin{aligned} D\left(x, \tfrac{1}{2}\right) &= \sum_{n \le x} (-1)^n d(n) \\ &= \tfrac{1}{2} x(\log x + 2\gamma - 1 - 2\log 2) + \tfrac{1}{4} \\ &\quad - \sqrt{x} \sum_{n=1}^{\infty} (-1)^n \frac{d(n)}{\sqrt{n}} \left(Y_1\left(2\pi\sqrt{nx}\right) + \frac{2}{\pi} K_1\left(2\pi\sqrt{nx}\right)\right) \end{aligned}$$

is interesting in view of Atkinson's formula. As an application, we have

$$D^*(x) = \tfrac{1}{2} D\left(4x, \tfrac{1}{2}\right) = x(\log x + 2\gamma - 1) + \tfrac{1}{8} + \Delta^*(x) \tag{1.8}$$

with

$$\Delta^*(x) = -\sqrt{x} \sum_{n=1}^{\infty} (-1)^n \frac{d(n)}{\sqrt{n}} \left(Y_1\left(4\pi\sqrt{nx}\right) + \frac{2}{\pi} K_1\left(4\pi\sqrt{nx}\right)\right). \tag{1.9}$$

This is an analogue of Voronoi's formula (1.6) for the ordinary divisor sum $D(x) = D(x, 1)$, up to the constant term and the signs $(-1)^n$ in (1.9), which are missing in the formula (1.7) for $\Delta(x) = \Delta(x, 1)$. In addition to this analogue, there is the following concrete connection between $D^*(x)$ and $D(x)$:

$$D^*(x) = -D(x) + 2D(2x) - \tfrac{1}{2} D(4x); \tag{1.10}$$

we gave an approximate version of this relation in [6]. Namely, the leading terms on both sides coincide, and the same holds even for the error terms:

$$\Delta^*(x) = -\Delta(x) + 2\Delta(2x) - \tfrac{1}{2}\Delta(4x), \tag{1.11}$$

as can be verified by the argument following (15.69) in [4].

In applications, approximate formulae for $\Delta(x)$, $\Delta^*(x)$, and $\Delta(x, h/k)$ are often more convenient than the precise ones. In particular, we have

$$\Delta^*(x) = \frac{x^{1/4}}{\pi\sqrt{2}} \sum_{n \le N} (-1)^n d(n) n^{-3/4} \cos\left(4\pi\sqrt{nx} - \frac{\pi}{4}\right) + O(x^{1/2+\varepsilon} N^{-1/2}) \tag{1.12}$$

with $1 \le N \ll x$. We let ε generally stand for a small positive constant, not necessarily the same at each occurrence. Constants implied by notation like $O(\ldots)$ and $\ll$ will depend on ε whenever ε is involved in the estimations, so actually we should write $O_\varepsilon(\ldots)$ and $\ll_\varepsilon$, but we omit ε here for simplicity. The formula (1.12) can be verified either as a corollary of the similar well-known result for $\Delta(x)$ with the coefficients $(-1)^n$ removed, or directly as an application of an approximate formula for $\Delta(x, h/k)$ (see [9, Theorem 1.1]).

Comparison of (1.12) with the sum $\Sigma_1(T)$ in Atkinson's formula reveals an analogue

$$E(T) \approx 2\pi\Delta^*\left(\frac{T}{2\pi}\right).$$

Thus the function

$$E^*(t) = E(t) - 2\pi\Delta^*\left(\frac{t}{2\pi}\right)$$

is expected to be"small" in some sense or another. In fact, this is true at least in a mean-value sense since

$$\int_2^T E^*(t)^2\,dt \ll T^{4/3}\log^3 T, \tag{1.13}$$

whereas

$$\int_2^T E(t)^2\,dt \sim cT^{3/2} \tag{1.14}$$

for a constant c. We proved (1.13) in [7], and (1.14) is due to Heath Brown [2]. Proofs of (1.13) and (1.14) can be found in [4], Sections 15.4 and 15.5. The estimate (1.13) has been improved by Ivić [5] to an asymptotic formula with an error term, and similar refinements of (1.14) have been considered by several authors. Moreover, estimates for higher moments of $E^*(t)$ have been obtained by Ivić.

As to concrete connections between the divisor function and the zeta-function, we proved in [10] that if $\Delta(x) \ll x^\alpha$, then

$$E(T) \ll T^{(1+2\alpha)/5}(\log T)^{12/5}. \tag{1.15}$$

In Section 4 we are going to reprove this by a somewhat modified argument. In particular, for the best possible value $\alpha = \frac{1}{4} + \varepsilon$ we have

$$E(T) \ll T^{3/10+\varepsilon} \quad \text{and} \quad \zeta\left(\tfrac{1}{2}+it\right) \ll t^{3/20+\varepsilon}.$$

As far as applications to the zeta-function are concerned, the *behavior* of Δ is more significant than its *order*, and therefore we introduce the function

$$\Delta(x,y) = \sup_{|\xi|\le y} |\Delta(x+\xi) - \Delta(x)|.$$

The function $\Delta^*(x,y)$ is defined analogously in terms of $\Delta^*(x)$. Note that $\Delta(x,y)$ and $\Delta^*(x,y)$ are closely related by (1.11). We showed in [10] that if

$$\Delta(x,y) \ll \sqrt{y}\,x^\varepsilon \tag{1.16}$$

for $y \ge 1$, then $\zeta\left(\frac{1}{2}+it\right) \ll t^{1/8+\varepsilon}$.

Actually, the following weaker conjecture suffices in place of (1.16) since the range $x^{1/4} \ll y \ll \sqrt{x}$ will be relevant for our purposes, as will be seen in the context of Theorem 4 and Corollary 5.

Conjecture. *For* $1 \le y \ll \sqrt{x}$, *we have*

$$\Delta(x,y) \ll \max\bigl(x^{1/8}, \sqrt{y}\bigr)x^{\varepsilon}. \tag{1.17}$$

This conjecture can be motivated by mean-value considerations. Namely, we have (see [8], Corollary to Theorem 1)

$$\int_X^{2X} (\Delta(x+\xi)-\Delta(x))^2\,dx \asymp X\xi\log^3\frac{\sqrt{X}}{\xi}$$

for $X^{\varepsilon} \ll \xi \le \frac{1}{2}\sqrt{X}$, so typically $\Delta(x+\xi)-\Delta(x) \ll \sqrt{\xi}x^{\varepsilon}$.

The conjecture (1.17) would imply best possible estimates for $\Delta(x)$ and $E(T)$.

Theorem 1. *On the assumption of the Conjecture, we have*

$$\Delta(x) \ll x^{1/4+\varepsilon} \quad \textit{and} \quad E(T) \ll T^{1/4+\varepsilon}. \tag{1.18}$$

For a proof, we analyze segments of the approximate Voronoi formula and the sum $\Sigma_1(T)$ in Atkinson's formula.

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2. Voronoi sums

Let $V^*(N_1,N_2;x)$ be a weighted subsum of the approximate Voronoi formula (1.12) for $\Delta^*(x)$, that is the sum of the form

$$V^*(N_1,N_2;x) = \frac{x^{1/4}}{\pi\sqrt{2}} \sum_{N_1 \le n \le N_2} (-1)^n d(n)w(n)n^{-3/4}\cos\Bigl(4\pi\sqrt{nx}-\frac{\pi}{4}\Bigr),$$

and let $V(N_1,N_2;x)$ be an analogous sum related to $\Delta(x)$, thus without the coefficients $(-1)^n$. The weight function $w(x)$, supported in the interval $[N_1,N_2]$ with $N_1 < N_2 \ll N_1$, is assumed to be smooth in the sense that $w^{(j)}(x) \ll_j N_1^{-j}$ for $j=0,1,\ldots$. These assumptions will be adopted in the sequel, also for the weighted Atkinson sums to be considered in the next section.

The next lemma gives transformation formulae for the Voronoi sums defined above.

Lemma 2. *Let* $x^\varepsilon \ll N_1 \ll x$, $n_0 = [4x]$, *and*

$$\nu_0 = N_1^{-1/2} x^{1/2+\varepsilon}. \tag{2.1}$$

Then

$$V^*(N_1,N_2;x) = -\frac{(-1)^{n_0}}{8\pi} \sum_{|\nu|\le\nu_0} (-1)^\nu (d(n_0+\nu) - d(n_0-\nu))$$
$$\times \int_{N_1}^{N_2} \sin\left(\frac{\pi\nu\sqrt{y}}{2\sqrt{x}}\right) \frac{w(y)}{y}\, dy + O(x^\varepsilon). \tag{2.2}$$

Analogously, with $n_1 = [x]$, *we have*

$$V(N_1,N_2;x) = -\frac{1}{4\pi} \sum_{|\nu|\le\nu_0} (d(n_1+\nu) - d(n_1-\nu))$$
$$\times \int_{N_1}^{N_2} \sin\left(\frac{2\pi\nu\sqrt{y}}{\sqrt{x}}\right) \frac{w(y)}{y}\, dy + O(x^\varepsilon). \tag{2.3}$$

Proof. Consider the sum $V^*(N_1,N_2;x)$; the sum $V(N_1,N_2;x)$ can be treated analogously. We apply the Voronoi sum formula with oscillating signs (see [9], Theorem 1.7): for $0<a<b$ and $f \in C^1[a,b]$, we have

$$\sum_{a\le n\le b} (-1)^n d(n) f(n) = \frac{1}{2} \int_a^b (\log y + 2\gamma - 2\log 2) f(y)\, dy + \frac{1}{2} \sum_{n=1}^\infty (-1)^n d(n)$$
$$\times \int_a^b \left(-2\pi Y_0\left(2\pi\sqrt{ny}\right) + 4K_0\left(2\pi\sqrt{ny}\right)\right) f(y)\, dy \tag{2.4}$$

with a similar convention of summation as above if a or b is an integer. For the sum $V^*(N_1,N_2;x)$, the function f is

$$f(y) = w(y) y^{-3/4} \cos\left(4\pi\sqrt{xy} - \frac{\pi}{4}\right).$$

By repeated integration by parts, we see that the leading term in (2.4) is negligible. The same holds for the terms involving the K_0-Bessel function by the asymptotic formula $K_0(x) \sim \sqrt{\pi/2x}\, e^{-x}$ for $x \ge 1$. For the Y_0-Bessel function, we have an asymptotic expansion starting with the first approximation

$$Y_0(x) = \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi}{4}\right) + O(x^{-3/2}) \quad \text{for } x \ge 1, \tag{2.5}$$

where the error term can be made more precise by subsequent explicit terms. Then, integrating again by parts, we see that those terms on the right of (2.4) with $n=$

$n_0+\nu$ and $|\nu|>\nu_0$ are negligible if ν_0 is as in (2.1). Further, the contribution of the error term in (2.5) to $V^*(N_1,N_2;x)$ is $\ll x^{\varepsilon}$. Hence

$$V^*(N_1,N_2;x)=-\frac{(-1)^{n_0}x^{1/4}}{\sqrt{2}\pi}\sum_{|\nu|\le\nu_0}(-1)^{\nu}d(n_0+\nu)(n_0+\nu)^{-1/4}$$
$$\times\int_{N_1}^{N_2}\sin\Big(2\pi\sqrt{(n_0+\nu)y}-\frac{\pi}{4}\Big)\cos\Big(4\pi\sqrt{xy}-\frac{\pi}{4}\Big)\frac{w(y)}{y}\,dy+O(x^{\varepsilon}).$$

When the trigonometric product here is written in terms of the exponential function, those functions involving the exponent $\pm 2\pi\sqrt{y}\big(\sqrt{n_0+\nu}-\sqrt{4x}\big)i$ will be relevant and the others will give a small contribution, again by integration by parts. In this way, the preceding formula becomes

$$V^*(N_1,N_2;x)=-\frac{(-1)^{n_0}x^{1/4}}{2\sqrt{2}\pi}\sum_{|\nu|\le\nu_0}(-1)^{\nu}d(n_0+\nu)(n_0+\nu)^{-1/4}$$
$$\times\int_{N_1}^{N_2}\sin\big(2\pi\sqrt{y}\big(\sqrt{n_0+\nu}-\sqrt{4x}\big)\big)\frac{w(y)}{y}\,dy+O(x^{\varepsilon}).$$

We make here the simplifications

$$(n_0+\nu)^{-1/4}=(4x)^{-1/4}+O(N_1^{-1/2}x^{-3/4+\varepsilon}),$$
$$\sqrt{y}\big(\sqrt{n_0+\nu}-\sqrt{4x}\big)=\frac{\nu\sqrt{y}}{4\sqrt{x}}+O(\sqrt{N_1}x^{-1/2+\varepsilon})$$

and omit the error terms with an admissible error. Now replacing ν by $-\nu$, then adding this new sum to the original one, and finally dividing the result by 2, we end up with the desired formula. $\square$

To deal with the ν-sums in Lemma 2, we need the following simple lemma.

Lemma 3. *For* $1\le\xi\le\frac{1}{2}n$, *we have*

$$\sum_{1\le\nu\le\xi}(d(n+\nu)-d(n-\nu))\ll\Delta(n,\xi)+\frac{\xi^2}{n}+n^{\varepsilon}, \tag{2.6}$$

$$\sum_{1\le\nu\le\xi}(-1)^{\nu}(d(n+\nu)-d(n-\nu))\ll\Delta^*\Big(\frac{n}{4},\frac{\xi}{4}\Big)+\frac{\xi^2}{n}+n^{\varepsilon}. \tag{2.7}$$

Proof. The estimate (2.6) follows from the calculations

$$
\begin{aligned}
\sum_{1\le\nu\le\xi}(d(n+\nu)-d(n-\nu)) &= D(n+\xi)+D(n-\xi)-2D(n)+O(n^{\varepsilon})\\
&= (\Delta(n+\xi)-\Delta(n))+(\Delta(n-\xi)-\Delta(n))+O\left(\frac{\xi^2}{n}\right)+O(n^{\varepsilon})\\
&\ll \Delta(n,\xi)+\frac{\xi^2}{n}+n^{\varepsilon},
\end{aligned}
$$

and the proof of (2.7) is analogous, based on the formula (1.8). □

Estimating the ν-sums in Lemma 2 by summation by parts and applying the above lemma, we get immediately the following bounds for the Voronoi sums.

Theorem 4. *For $x^{\varepsilon}\ll N_1\ll x$, we have*

$$
\begin{aligned}
V^*(N_1,N_2;x) &\ll (\Delta^*(x,N_1^{-1/2}x^{1/2+\varepsilon})+1)x^{\varepsilon},\\
V(N_1,N_2,x) &\ll (\Delta(x,N_1^{-1/2}x^{1/2+\varepsilon})+1)x^{\varepsilon}.
\end{aligned}
$$

We note two applications of this theorem.

Corollary 5. *On the assumption of the Conjecture (1.17), we have*

$$V(N_1,N_2;x),V^*(N_1,N_2;x)\ll\left(\frac{x}{N_1}\right)^{1/4+\varepsilon}$$

for $x^{\varepsilon}\ll N_1\ll\sqrt{x}$.

Corollary 6. *If $\Delta(x)\ll x^{\alpha}$ with $\frac{1}{4}<\alpha<\frac{1}{3}$ for all $x\ge1$, then*

$$V(N_1,N_2;x),V^*(N_1,N_2;x)\ll X^{\alpha+\varepsilon}$$

for $x\asymp X$ and $X^{\varepsilon}\ll N_1\ll X$.

Remark 7. Corollary 5 is essentially best possible since

$$\int_X^{2X}V(N_1,N_2;x)^2\,dx\asymp X^{3/2}N_1^{-1/2}\log^3 N_1$$

if $N_2-N_1\asymp N_1$ and the weight function $w(x)$ is of a natural shape. This is seen by the usual “square and integrate” argument; the diagonal terms then give the dominating contribution.

3. Atkinson sums

The Atkinson sum

$$A(N_1,N_2;T)=\sqrt{2}\left(\frac{T}{2\pi}\right)^{1/4}\sum_{n=N_1}^{N_2}(-1)^n d(n)w(n)n^{-3/4}e(T,n)\cos f(T,n)$$

is a weighted segment of the sum $\Sigma_1(T)$. It is analogous to $2\pi V^*(N_1,N_2;T/2\pi)$, which was analyzed in the preceding section. We are actually going to represent the Atkinson sum in terms of Voronoi sums with $T+\tau$ in place of T, where τ runs over a certain interval.

Lemma 8. *We have*

$$A(N_1,N_2;T)\ll(1+N_1^{3/4}T^{-1/4})T^{\varepsilon}\max_{\tau}\left|V^*\left(N_1,N_2;\frac{T+\tau}{2\pi}\right)\right|+O(T^{1/8+\varepsilon}), \tag{3.1}$$

where $T^{\varepsilon}\ll N_1\ll\sqrt{T}$ and τ runs over the interval $-T^{1/3+\varepsilon}\le\tau\le(T^{1/3}+N_1)T^{\varepsilon}$.

Proof. To begin with, we simplify $e(T,n)$ and $f(T,n)$ omitting the error term in (1.3) and (1.4). Then

$$\begin{aligned}A(N_1,N_2;T)=\sqrt{2}\left(\frac{T}{2\pi}\right)^{1/4}&\sum_{n=N_1}^{N_2}(-1)^n d(n)w(n)n^{-3/4}\\ &\times\cos\left(2\sqrt{2\pi nT}+An^{3/2}-\pi/4\right)\\ &+O(N_1^{5/4}T^{-3/4+\varepsilon})+O(N_1^{11/4}T^{-5/4+\varepsilon}),\end{aligned} \tag{3.2}$$

where A is as in (1.4). The error terms here are $\ll T^{1/8+\varepsilon}$ for $N_1\ll\sqrt{T}$.

Next, as in [10], we use the formula

$$e^{iAy^3}=\int_{-\infty}^{\infty}\beta(u)e^{iuy}\,du, \tag{3.3}$$

where $\beta(u)$ is an Airy function. It can expressed by Bessel functions as (see [10])

$$\beta(u)=\frac{\sqrt{|u|}}{3\pi\sqrt{A}}K_{1/3}\left(\frac{2|u|^{3/2}}{3\sqrt{3A}}\right)\qquad\text{for } u<0,$$

$$\beta(u)=\frac{\sqrt{u}}{3\sqrt{3A}}\left(J_{1/3}\left(\frac{2u^{3/2}}{3\sqrt{3A}}\right)+J_{-1/3}\left(\frac{2u^{3/2}}{3\sqrt{3A}}\right)\right)\qquad\text{for } u>0.$$

We apply (3.3) for $y=\sqrt{n}$ with $N_1\le n\le N_2$. To separate a dominating part of the integral in (3.3), let

$$u_0=T^{-1/6+\varepsilon}\quad\text{and}\quad u_1=T^{-1/2+\varepsilon}(T^{1/3}+N_1), \tag{3.4}$$

and introduce a smooth weight function $v(u)$ supported in $[u_1/2,\infty)$ such that $v(u)=1$ for $u\ge u_1$ and $v^{(j)}(u)\ll_j u_1^{-j}$ for $j=0,1,2,\dots$. Then

$$\begin{aligned}e^{iAn^{3/2}}=&\int_{-\infty}^{-u_0}\beta(u)e^{iu\sqrt{n}}\,du+\int_{-u_0}^{u_1}(1-v(u))\beta(u)e^{iu\sqrt{n}}\,du\\&+\int_{u_1/2}^{\infty}v(u)\beta(u)e^{iu\sqrt{n}}\,du.\end{aligned}$$

We see, by familiar properties of Bessel functions, that the first and third integrals are negligibly small; in the first integral, the integrand itself is small since the K-Bessel function is exponentially small, and in the third integral the integrand is oscillating by the asymptotic expansion of the J-Bessel functions, so repeated integration by parts works for this part.

It remains to deal with the second integral. We have

$$\begin{aligned}\beta(u)&\ll T^{1/6}&&\text{for all } u\ne 0,\\ \beta(u)&\ll T^{1/8}u^{-1/4}&&\text{for } u\ge T^{-1/6},\end{aligned}$$

so

$$\int_{-u_0}^{u_1}|\beta(u)|\,du\ll(1+N_1^{3/4}T^{-1/4})T^{\varepsilon}. \tag{3.5}$$

We apply now, in (3.2), the truncated version of (3.3) with $y=n^{1/2}$ getting

$$\begin{aligned}A(N_1,N_2;T)=&\sqrt{2}\left(\frac{T}{2\pi}\right)^{1/4}\sum_{n=N_1}^{N_2}(-1)^n d(n)w(n)n^{-3/4}\\&\times\int_{-u_0}^{u_1}(1-v(u))\beta(u)\cos\left(4\pi\sqrt{n}\sqrt{\frac{T+\tau(u)}{2\pi}}-\frac{\pi}{4}\right)du+O(T^{1/8+\varepsilon}),\end{aligned}$$

where $\tau(u)=u\sqrt{T/2\pi}+u^2/8\pi$. This expression can be rewritten as

$$2\pi\int_{-u_0}^{u_1}(1-v(u))\beta(u)V^*\left(N_1,N_2;\frac{T+\tau(u)}{2\pi}\right)du+O(T^{1/8+\varepsilon}) \tag{3.6}$$

if $T^{1/4}$ is replaced by $(T+\tau(u))^{1/4}$, which can be done by an admissible error. Then the assertion of the lemma follows if the integral over u is estimated by (3.5) and $\tau(u)$ is estimated by (3.4). $\square$

Combining now (3.1) with Corollaries 5 and 6, we get another two corollaries.

Corollary 9. *On the Conjecture* (1.17), *we have*

$$A(N_1,N_2;T) \ll \left(\sqrt{N_1}+\left(\frac{T}{N_1}\right)^{1/4}\right)T^{\varepsilon}$$

for $T^{\varepsilon} \ll N_1 \ll \sqrt{T}$.

Corollary 10. *If* $\Delta(x) \ll x^{\alpha}$ *with* $\frac{1}{4}<\alpha<\frac{1}{3}$ *for all* $x \geq 1$ *and* $T^{\varepsilon} \ll N_1 \ll \sqrt{T}$, *then*

$$A(N_1,N_2;T) \ll (N_1^{3/4}T^{\alpha-1/4}+T^{\alpha})T^{\varepsilon}.$$

Remark 11. The range of the integral (3.6) can be specified more precisely if N_1 exceeds $T^{1/3}$. Then this integral is a sum of oscillatory integrals over $n \in [N_1,N_2]$, and each of these has a saddle point of size $\asymp N_1T^{-1/2}$. Therefore the range $u \asymp N_1T^{-1/2}$ gives the dominating contribution, and then $\tau(u) \asymp N_1$. In view of Lemma 2, this means that $A(N_1,N_2;T)$ depends on values of $d(n)$ lying at a distance about N_1 to the *right* of $2T/\pi$, whereas $V^*(N_1,N_2;T/2\pi)$ depends on values of $d(n)$ lying *symmetrically* near $2T/\pi$ at a distance at most $\ll N_1^{-1/2}T^{1/2+\varepsilon}$. Thus $A(N_1,N_2;T)$ and $V^*(N_1,N_2;T/2\pi)$ depend on values of $d(n)$ lying in *different* ranges if N_1 exceeds $T^{1/3}$ and this explains why the Voronoi–Atkinson analogy becomes weaker in this case.

4. Conditional estimates for $\Delta(x)$ and $E(T)$

We now prove Theorem 1 using Corollaries 5 and 9. As to $\Delta(x)$, we use the approximate Voronoi formula (1.12) (the version for $\Delta(x)$) choosing $N=\sqrt{x}$. We may equip the sum in that formula with a smooth weight, and the weighted sum can be split up into a sum of Voronoi sums of type $V(N_1,N_2;x)$, up to the part over $n \ll x^{\varepsilon}$, which can be estimated trivially. The conditional estimate $\Delta(x) \ll x^{1/4+\varepsilon}$ now follows from Corollary 5.

Turning to the estimates of $E(T)$, we use the inequality (10.1) in [6]:

$$E(t_1)+O((T-t_1)\log T) \leq E(T) \leq E(t_2)+O((t_2-T)\log T), \tag{4.1}$$

valid for $1 \le t_1 \le T \le t_2 \le 2T$. Let U be a parameter to be chosen suitably, and let

$$t_1 = T - U - u \text{ and } t_2 = T + U + u, \quad 0 \le u \le U,$$

where u is a variable. Averaging over u means smoothing of $E(t_1)$ and $E(t_2)$. The averages of the $\Sigma_2(t_j)$ are negligibly small by (1.5), and the sums $\Sigma_1(t_j)$ can be truncated to a length $T^{1+\varepsilon}U^{-2}$ owing to the smoothing. When we choose $U = T^{1/4+\varepsilon}$, the error terms in (4.1) are $\ll T^{1/4+\varepsilon}$, and the same holds for the relevant Atkinson sums by Corollary 9. Thus we get the asserted conditional estimate for $E(T)$.

Finally we note that the present argument gives again the bound

$$E(T) \ll T^{(1+2\alpha)/5+\varepsilon},$$

that is essentially the estimate (1.15), under the assumption $\Delta(x) \ll x^{\alpha}$. Namely, by Corollary 10 and the preceding discussion, the optimal value of U is determined by the condition

$$T^{\alpha-1/4}(TU^{-2})^{3/4} = U$$

giving $U = T^{(1+2\alpha)/5}$, and an estimate essentially like this then follows for $E(T)$.

References

1. Atkinson, F. V., The mean-value of the Riemann zeta function, *Acta Math.* **81** (1949), 353–376.
2. Heath-Brown, D. R., The mean value theorem for the Riemann zeta-function, *Mathematika* **25** (1978), 177–184.
3. Ivić, A., *Lectures on Mean Values of the Riemann Zeta Function*, Tata Institute of Fundamental Research Lectures on Math. Phys. **82**, Springer, Berlin, 1991.
4. Ivić, A., *The Riemann zeta-function*, 2nd ed., Dover, Mineola, NY, 2003.
5. Ivić, A., On the mean square of the zeta-function and the divisor problem, *Ann. Acad. Sci. Fenn. Math.* **32** (2007), 269–277.
6. Jutila, M., Riemann's zeta-function and the divisor problem, *Ark. Mat.* **21** (1983), 75–96.
7. Jutila, M., On a formula of Atkinson, in *Topics in Classical Number Theory*, Coll. Math. Soc. János Bolyai **34**, pp. 807–823, North-Holland, Amsterdam, 1984.
8. Jutila, M., On the divisor problem for short intervals, *Ann. Univ. Turku. Ser. A I* **186** (1984), 23–30.
9. Jutila, M., *Lectures on a Method in the Theory of Exponential Sums*, Tata Institute of Fundamental Research Lectures on Math. Phys. **80**, Springer, Berlin, 1987.
10. Jutila, M., Riemann's zeta-function and the divisor problem. II, *Ark. Mat.* **31** (1993), 61–70.
11. Jutila, M., Atkinson's formula revisited, in *Voronoi's Impact on Modern Science, Book* **1**, pp. 137–154, National Academy of Sciences, Kyiv, 1998.
12. Lebedev, N. N., *Special Functions and Their Applications*, Dover, New York, 1972.

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