Uniform exponential stability of the Ekman spiral

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Abstract. This paper studies stability of the Ekman boundary layer. We utilize a new approach, developed by the authors in a precedent paper, based on Fourier transformed finite vector Radon measures, which yields exponential stability of the Ekman spiral. By this method we can also derive very explicit bounds for solutions of the linearized and the nonlinear Ekman systems. For example, we can prove these bounds to be uniform with respect to the angular velocity of rotation, which has proved to be relevant for several aspects. Another advantage of this approach is that we obtain well-posedness in classes containing nondecaying vector fields such as almost periodic functions. These outcomes give respect to the nature of boundary layer problems and cannot be obtained by approaches in standard function spaces such as Lebesgue, Bessel-potential, Hölder or Besov spaces.

1. Introduction and main results

In this note we apply a new approach to rotating boundary layers developed in [12] in order to examine stability of the Ekman boundary layer problem

$$
(1) \quad
\begin{cases}
\partial_t v - \nu \Delta v + \omega \mathbf{e}_3 \times v + (v \cdot \nabla) v = -\nabla q & \text{in } (0,T) \times G, \\
\operatorname{div} v = 0 & \text{in } (0,T) \times G, \\
v = \mathbf{U}^E|_{\partial G} & \text{on } (0,T) \times \partial G, \\
v|_{t=0} = v_0 & \text{in } G.
\end{cases}
$$

Here $\mathbf{e}_3 = (0,0,1)^T$, $\nu > 0$ is the viscosity coefficient, and $\omega \in \mathbb{R}$ is the Coriolis parameter, which equals twice the angular velocity of rotation. For G we will consider

simultaneously the half-space $\mathbb{R}^3_+$ or a layer, i.e., we have $G=\mathbb{R}^2\times D$ with $D=(0,d)$ and either a finite fixed $d\in(0,\infty)$ or $d=\infty$. The vector field $\mathbf{U}^E$ is the so-called *Ekman spiral* (introduced by the geophysicist Ekman [8]) given as

$$(2) \qquad \mathbf{U}^E(x_3) = U_\infty \Big(1 - e^{-x_3/\delta}\cos\frac{x_3}{\delta}, e^{-x_3/\delta}\sin\frac{x_3}{\delta}, 0 \Big)^T, \quad x_3\geq 0.$$

System (1) is known to be a well-established model for the layer arising in a rotating system (e.g. the earth) between a straight geostrophic flow (e.g. wind) and the surface on which the no slip condition is imposed.

Observe that in the above model rotation about the x_3-axis is assumed, whereas U_∞ denotes the total velocity of the flow, blowing in direction of the x_1-axis. The parameter δ denotes the layer thickness given by $\delta=\sqrt{2\nu/|\omega|}$. By geostrophic approximation (see [23]) (1) is a reasonable model at least for the upper part of the northern hemisphere. The couple $(\mathbf{U}^E, p^E)$ with pressure

$$p^E(x_2) = -\omega U_\infty x_2$$

represents a stationary solution of system (1). Note that $\mathbf{U}^E(0)=0$, i.e. system (1) is subject to Dirichlet conditions at the lower boundary, and that $\mathbf{U}^E$ is oscillating and nondecaying in the tangential direction. We note that remarkable persistent stability of $\mathbf{U}^E$ is observed in geophysical literature.

As for the Ekman problem, the tangentially nondecaying and oscillating behavior is typical for geostrophic boundary layer problems. To give respect to this fact, it seems natural to consider this type of boundary layer problems in classes containing nondecaying functions. Hence, the frequently performed L^p approach for $1<p<\infty$ to the corresponding mathematical models fails in this situation. Giving account to this fact, in [12] an operator theory on spaces of Fourier transformed finite vector-valued Radon measures is developed. These spaces in particular include nondecaying such as almost periodic functions (see Remark 2.9).

A further advantage in dealing with Fourier transformed quantities lies in the fact that all performed calculations and estimations become rather explicit. As a consequence we can derive detailed information on how the solution depends on involved parameters such as time, viscosity, layer thickness, and angular velocity of rotation. In particular, we obtain that the corresponding bounds are uniform in the angular velocity of rotation. This turned out to be relevant for several reasons such as, for instance, the investigation of statistical properties of turbulence, cf. [22] and [25]. It also represents the basis for the examination of rapidly oscillating limits as $\omega\to\infty$, cf. [1]–[3], [5], [16], [18]–[20] and [26]. We refer to [12] for a more extensive motivation of the importance of the uniformity in ω and the functional setting chosen here in general.

Mathematically, an approach to stability in time is given in [6] and to asymptotic stability in [15]. These two papers consider the problem in the L^2 setting, which, of course, does not include nondecaying perturbations of $\mathbf{U}^E$. We refer to [4], [5], [14], [17] and [24] for more mathematical literature on the Ekman problem dealing also with vanishing Rossby and Ekman numbers and with stratification. For a spectral analysis of the linearized problem we refer to [13] and [21]. Local-in-time well-posedness in the homogeneous L^p-valued Besov space $\dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^2, L^p(\mathbb{R}_+))$ is obtained in [10]. By the fact that almost periodic functions are contained in $\dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^2)$, this represents the first result in a space including nondecaying functions. This local-in-time well-posedness is recovered in [12] in the Fourier transformed Radon measure setting, however, with much more explicit bounds on the solution. In particular, the derived quantities such as the existence interval of the solution as well as its bound are uniform in ω, which is not possible in $\dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^2, L^p(\mathbb{R}_+))$. The approach developed in [12] also gives access to instability of the Ekman problem for large Reynolds numbers as demonstrated in [9].

Next, we formulate our main results. For a rigorous definition of the appearing spaces we refer to Section 2. Let $M_0(\mathbb{R}^2, L^2(D)^3)$ denote the space of finite $L^2(D)^3$-valued Radon measures with no point mass at the origin. We consider its Fourier image, i.e.,

$$\mathrm{FM}_0(\mathbb{R}^2, L^2(D)^3) := \{\mathcal{F}\mu : \mu \in M_0(\mathbb{R}^2, L^2(D)^3)\}$$

and equip it with its canonical norm. It can be shown that $\mathrm{FM}_0(\mathbb{R}^2, L^2(D)^3) \subset \mathrm{BUC}(\mathbb{R}^2, L^2(D)^3)$ (see Lemma 2.8(iii)) and that this space includes almost periodic functions (see Remark 2.9). Let $\ell \geq 0$. Then $\mathrm{FM}_\ell(\mathbb{R}^2, L^2(D)^3) := \mathrm{FM}_0(\mathbb{R}^2, L^2(D)^3)$ for $\ell = 0$. For $\ell > 0$ and a sum-closed frequency set F_ℓ (see Definition 2.4) we also consider the space

$$\mathrm{FM}_\ell(\mathbb{R}^2, L^2(D)^3) := \{v \in \mathrm{FM}_0(\mathbb{R}^2, L^2(D)^3) : \mathrm{supp}\,\hat{v} \in F_\ell\}.$$

Note that the Helmholtz projection P is bounded on $\mathrm{FM}_\ell(\mathbb{R}^2, L^2(D)^3)$ for $\ell \geq 0$ (see Proposition 3.3(a)). Thus we may define its solenoidal part as

$$\mathrm{FM}_{\ell,\sigma}(\mathbb{R}^2, L^2(D)^3) := P(\mathrm{FM}_\ell(\mathbb{R}^2, L^2(D)^3)), \quad \ell \geq 0.$$

Next, we set $u_0 := v_0 - \mathbf{U}^E$, $u = v - \mathbf{U}^E$ and $p := q - p^E$. Then (v, q) solves (1) if and only if (u, p) solves the transformed system

$$(3) \quad \begin{cases} \partial_t u - \nu \Delta u + \omega \mathbf{e}_3 \times u + (\mathbf{U}^E \cdot \nabla)u + u^3 \partial_3 \mathbf{U}^E + (u \cdot \nabla)u = -\nabla p & \text{in } (0,T) \times G, \\ \mathrm{div}\,u = 0 & \text{in } (0,T) \times G, \\ u = 0 & \text{on } (0,T) \times \partial G, \\ u|_{t=0} = u_0 & \text{in } G. \end{cases}$$

The Stokes–Coriolis–Ekman operator A_{SCE} is defined as the full linear operator of the linearized Cauchy problem associated with (3) (see Section 3 for a rigorous definition, in particular Proposition 3.4). For A_{SCE} we prove the following theorem, which is our main result for the linearized Ekman problem.

Theorem 1.1. *Let $\nu, U_\infty > 0$, $\delta = \sqrt{2\nu/|\omega|}$, $d > 0$, $\ell \geq 0$ and set $X_2 := L^2(0,d)^3$. Assume that the Reynolds number Re of the Ekman spiral satisfies*

$$(4) \qquad\qquad \mathrm{Re} := \frac{U_\infty \delta}{\nu} < \frac{1}{\sqrt{2}}.$$

Further we assume that $\ell = 0$ implies $d < \infty$, and that $\ell > 0$ implies $d = \infty$. (This means that either we consider a layer $(d < \infty)$ or the half-space $\mathbb{R}^3_+$ and Fourier transformed initial data supported in a sum-closed frequency set F_ℓ.) According to the two cases we set

$$\varepsilon_\ell := \begin{cases} \dfrac{\nu(1 - \sqrt{2}\,\mathrm{Re})}{d^2}, & \ell = 0, \\[2mm] \nu\big(1 - \sqrt{2}\,\mathrm{Re}\big)\ell^2, & \ell > 0. \end{cases}$$

Then the C_0-semigroup $(\exp(-tA_{\mathrm{SCE}}))_{t \geq 0}$ on $\mathrm{FM}_{\ell,\sigma}(\mathbb{R}^2, X_2)$ satisfies
   (i) $\|\exp(-tA_{\mathrm{SCE}})\|_{\mathscr{L}(\mathrm{FM}_{\ell,\sigma}(\mathbb{R}^2, X_2))} \leq e^{-2\varepsilon_\ell t}$, $t \geq 0$;
   (ii)

$$\|\nabla \exp(-(A_{\mathrm{SCE}} - \varepsilon_\ell)\cdot)u_0\|_{L^2(\mathbb{R}_+, \mathrm{FM}_\ell(\mathbb{R}^2, X_2))} \leq \frac{\|u_0\|_{\mathrm{FM}_\ell(\mathbb{R}^2, X_2)}}{\sqrt{\nu(1 - \sqrt{2}\,\mathrm{Re})}};$$

   (iii)

$$\|\nabla \exp(-(A_{\mathrm{SCE}} - \varepsilon_\ell)\cdot)\star f\|_{L^2(\mathbb{R}_+, \mathrm{FM}_\ell(\mathbb{R}^2, X_2))} \leq \frac{\|f\|_{L^1(\mathbb{R}_+, \mathrm{FM}_\ell(\mathbb{R}^2, X_2))}}{\sqrt{\nu(1 - \sqrt{2}\,\mathrm{Re})}};$$

for all $u_0 \in \mathrm{FM}_{\ell,\sigma}(\mathbb{R}^2, X_2)$ and $f \in L^1(\mathbb{R}_+, \mathrm{FM}_{\ell,\sigma}(\mathbb{R}^2, X_2))$, where

$$g \star f(t) := \int_0^t g(t-s)f(s)\,ds.$$

In particular, all estimates are uniform in $\omega \in \mathbb{R}$.

Remark 1.2. The physically most relevant case for the layer height is $d > \delta$.

Based on Theorem 1.1 and a fixed point argument, in Section 4 we derive the following main result for the full nonlinear Ekman problem (1).

Theorem 1.3. *Let the assumptions of Theorem* 1.1 *be satisfied, and let* $\mathbf{U}^E$ *be the Ekman spiral given in* (2). *Then for every* $v_0 \in \mathrm{FM}_{l,\sigma}(\mathbb{R}^2, X_2) + \mathbf{U}^E$ *such that*

$$\|v_0 - \mathbf{U}^E\|_{\mathrm{FM}_0(L^2)} < \begin{cases} \dfrac{\pi\nu(1 - \sqrt{2}\,\mathrm{Re})}{2^{1/4}3\sqrt{d}}, & \text{if } \ell = 0 \text{ and } d < \infty, \\[3mm] \dfrac{\pi\nu(1 - \sqrt{2}\,\mathrm{Re})\sqrt{\ell}}{2^{1/4}3}, & \text{if } \ell > 0 \text{ and } d = \infty, \end{cases}$$

there is a unique global (*mild*) *solution* v *of* (1) *satisfying*

$$\exp(2\varepsilon_\ell \cdot)(v - \mathbf{U}^E) \in \mathrm{BC}((0,\infty), \mathrm{FM}_{0,\sigma}(\mathbb{R}^2, X_2)),$$

$$\exp(\varepsilon_\ell \cdot)\nabla(v - \mathbf{U}^E) \in L^2((0,\infty), \mathrm{FM}_0(\mathbb{R}^2, X_2)).$$

In particular, the Ekman spiral is exponentially stable. More precisely, we have

$$\|v(t) - \mathbf{U}^E\|_{\mathrm{FM}_0(\mathbb{R}^2, X_2)} \le 2\exp(-2\varepsilon_\ell t)\|v_0 - \mathbf{U}^E\|_{\mathrm{FM}_0(\mathbb{R}^2, X_2)}, \quad t \ge 0.$$

In addition, all estimates above are uniform in $\omega \in \mathbb{R}$, *i.e., with respect to the angular velocity of rotation.*

Remark 1.4. By standard bootstrap arguments it can be proved that the solution v given by Theorem 1.3 enjoys higher regularity in $\mathrm{FM}_\ell(\mathbb{R}^2, X_2)$. By this fact we can recover the pressure via

$$\nabla q = (I - P)(\nu\Delta v - \omega\mathbf{e}_3 \times v - (v\cdot\nabla)v).$$

Then it can be shown that

$$(v, q) \in C^\infty((0,\infty) \times \mathbb{R}^2 \times (0, d)),$$

i.e., (v, q) is the unique classical solution of problem (1).

2. Vector-valued Radon measures

Here we introduce notation and recall basic ingredients on X-valued Radon measures. For basic theory we refer to [7]. The theory related to boundary layers is developed in [12]. Therein also the proofs of the results listed below can be found.

We use standard notation throughout this article. The symbols $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Z}$ denote the sets of reals, complex numbers and integers, respectively. We also write $\mathbb{N} = \{1, 2, 3, ...\}$ for the natural numbers and set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The symbols X, Y and Z usually denote Banach spaces, whereas $\mathscr{L}(X, Y)$ stands for the set of bounded linear operators from X to Y. If $X = Y$, we write $\mathscr{L}(X)$. Let $G \subset \mathbb{R}^n$

be a domain. As usual $L^2(G,X)$ and $H^k(G,X)=W^{k,2}(G,X)$ denote X-valued Lebesgue and Sobolev space respectively. The space $C_c^\infty(G,X)$ is the set of smooth and compactly supported functions. We will also write $C(G,X)$, $BC(G,X)$ and $\mathrm{BUC}(G,X)$ for the spaces of continuous, bounded and continuous, and bounded and uniformly continuous functions, respectively. The ball in $\mathbb{R}^n$ centered at x_0 with radius $R>0$ is denoted by $B(x_0,R)$.

The Fourier transformation on the space of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^n)$ in this note is defined as

$$\hat{u}(\xi)=\mathcal{F}u(\xi)=\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}e^{-i\xi x}u(x)\,dx, \quad u\in\mathcal{S}(\mathbb{R}^n).$$

As usual, its extension by duality to the space $\mathcal{S}'(\mathbb{R}^n,X):=\mathcal{L}(\mathcal{S}(\mathbb{R}^n),X)$ of tempered distributions is again denoted by $\mathcal{F}u$ or $\hat{u}$ for $u\in\mathcal{S}'(\mathbb{R}^n,X)$.

Next we recall some basic definitions related to X-*valued measures*, cf. [7].

Definition 2.1. Let X be a Banach space, Ω be a set, $\mathscr{A}$ be a σ-algebra over Ω, and $\mu\colon\mathscr{A}\to X$ be a set function.

(i) The function μ is a *vector-valued* (or X-*valued*) *measure* if it is σ-additive, that is, if it satisfies

$$\mu\left(\bigcup_{j=1}^\infty A_j\right)=\sum_{j=1}^\infty\mu(A_j)$$

for all pairwise disjoint sets $A_j\in\mathscr{A}$, $j=1,2,\dots$.

(ii) The variation of an X-valued measure μ is defined as

$$|\mu|(\mathcal{O}):=\sup\left\{\sum_{A\in\Pi(\mathcal{O})}\|\mu(A)\|_X:\Pi(\mathcal{O})\subset\mathscr{A}\text{ is a finite decomposition of }\mathcal{O}\right\}$$

for $\mathcal{O}\in\mathscr{A}$. (Note that $\Pi(\mathcal{O})$ is a *decomposition* of $\mathcal{O}\in\mathscr{A}$ if $A\cap B=\varnothing$ for all $A,B\in\Pi(\mathcal{O})$ with $A\neq B$, and $\bigcup_{A\in\Pi(\mathcal{O})}A=\mathcal{O}$.)

(iii) The quantity $|\mu|(\Omega)$ is the *total variation* of μ. If $|\mu|(\Omega)<\infty$, then μ is called *finite or of bounded variation*.

Next we define X-valued Radon measures. For this purpose let $\Omega\subset\mathbb{R}^n$ be open, $\mathscr{A}$ be a σ-algebra over Ω, and denote by $\mathscr{B}(\Omega)$ the Borel σ-algebra over Ω. Recall that $\eta\colon\mathscr{A}\to[0,\infty)$ is a *Radon measure* if it is Borel regular, that is, if $\mathscr{B}(\Omega)\subseteq\mathscr{A}$

and if for each $A\subseteq\Omega$ there exists a $B\in\mathscr{B}(\Omega)$ such that $A\subseteq B$ and $\eta^*(A)=\eta^*(B)$, where η^* denotes the outer measure associated with η given by

$$(5) \qquad \eta^*(A):=\inf\left\{\sum_{j=1}^{\infty}\eta(E_j):\{E_j\}_{j\in\mathbb{N}}\subseteq\mathscr{A} \text{ and } A\subseteq\bigcup_{j=1}^{\infty}E_j\right\}$$

Also observe that in the sequel we identify a measure η by its outer measure, so that η is complete in the sense that all subsets B of a set $A\in\mathscr{A}$ satisfying $\eta(A)=0$ belong to $\mathscr{A}$.

Definition 2.2. Let $\Omega\subset\mathbb{R}^n$ be open, X be a Banach space, and $\mathscr{A}$ be a σ-algebra over Ω. The set function $\mu\colon\mathscr{A}\to X$ is a *finite X-valued Radon measure* if μ is an X-valued measure and if the variation $|\mu|$ is a finite Radon measure. The set of all finite X-valued Radon measures is denoted by $M(\Omega,X)$.

From now on assume X to have the Radon–Nikodým property and $\Omega\subset\mathbb{R}^n$ to be open. By $\rho_\mu\in L^1(\Omega,X,|\mu|)$ we denote the Radon–Nikodým derivative of a measure $\mu\in M(\Omega,X)$ with respect to $(\Omega,\mathscr{A},|\mu|)$. Then we have the representation

$$\mu(\mathcal{O})=\int_{\mathcal{O}}\rho_\mu\,d|\mu|,\quad \mathcal{O}\in\mathscr{A}.$$

Note that by definition each vector Radon measure is well-defined on $\mathscr{B}(\Omega)$. By this fact, for every $\psi\in\mathrm{BC}(\Omega,\mathscr{L}(X,Y))$, where Y is another Banach space, its multiplication by an arbitrary $\mu\in M(\Omega,X)$ can be defined as

$$(6) \qquad \mu\lfloor\psi(\mathcal{O}):=\int_{\mathcal{O}}\psi\rho_\mu\,d|\mu|,\quad \mathcal{O}\in\mathscr{A}.$$

The properties of this quantity are summarized in the following lemma.

Lemma 2.3. ([12, Lemma 2.6]) *Let $\Omega\subset\mathbb{R}^n$ be open and let X, Y and Z be Banach spaces having the Radon–Nikodým property. Furthermore, let $\mu\in M(\Omega,X)$ and the functions $\psi,\phi\in\mathrm{BC}(\Omega,\mathscr{L}(Y,Z))$ be given. Then*
(i) $|\mu\lfloor\psi|=|\mu|\lfloor\|\psi\rho_\mu\|_Y\leq|\mu|\lfloor\|\psi\|_{\mathscr{L}(X,Y)}$,
(ii) $\mu\lfloor\psi\in M(\Omega,Y)$,
(iii) $\left(\mu\lfloor\psi\right)\lfloor\phi=\mu\lfloor(\phi\psi)$.

By the intention to introduce the Fourier transform of vector Radon measures, from now on we assume $\Omega=\mathbb{R}^n$. Note that the fact that the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing functions with its canonical topology is continuously and densely embedded in $C_\infty(\mathbb{R}^n):=\{v\in C(\mathbb{R}^n):\lim_{R\to0}\sup_{\mathbb{R}^n\setminus B(0,R)}|v(x)|=0\}$ gives us

$$\mathscr{L}(C_\infty(\mathbb{R}^n),X)\hookrightarrow\mathscr{L}(\mathcal{S}(\mathbb{R}^n),X)=\mathcal{S}'(\mathbb{R}^n,X).$$

Thus, in the sense of the identification

$$\mu \longmapsto T_\mu, \quad T_\mu f := \mu \lfloor f(\mathbb{R}^n),$$

we have the embedding

$$M(\mathbb{R}^n, X) \hookrightarrow \mathcal{S}'(\mathbb{R}^n, X).$$

This allows for the definition of the space of Fourier transformed Radon measures

$$\mathrm{FM}(\mathbb{R}^n, X) := \{\hat{\mu} : \mu \in M(\mathbb{R}^n, X)\},$$

which we equip with the canonical norm

$$\|u\|_{\mathrm{FM}} := \|\mathcal{F}^{-1} u\|_M.$$

Observe that replacing the Fourier transform by its inverse in the definition does not change the value of the norm, i.e., we have $\| \cdot \|_{\mathrm{FM}} = \|\mathcal{F} \cdot \|_M = \|\mathcal{F}^{-1} \cdot \|_M$. In order to define multipliers with symbols not necessarily continuous at the origin, we also introduce the spaces

$$M_0(\mathbb{R}^n, X) := \{\mu \in M(\mathbb{R}^n, X) : \mu(\{0\}) = 0\},$$

that is, the subspace of Radon measures with no point mass at the origin and

$$\mathrm{FM}_0(\mathbb{R}^n, X) := \{\hat{\mu} : \mu \in M_0(\mathbb{R}^n, X)\}.$$

Related to exponential stability we introduce further subspaces. These rely on sum-closed frequency sets, which are defined as follows.

Definition 2.4. We say that $F \subseteq \mathbb{R}^n$ is a *sum-closed frequency set* in $\mathbb{R}^n$, if
(i) F is closed;
(ii) $0 \notin F$;
(iii) $F + F := \{x + y : x, y \in F\} \subseteq F \cup \{0\}$.
For a sum-closed frequency set with distance $\ell\sqrt{2} > 0$ from zero in the sequel we write F_ℓ.[1] For consistency we also set $F_\ell = \mathbb{R}^n \setminus \{0\}$ if $\ell = 0$. The class of all sum-closed frequency sets in $\mathbb{R}^n$ is denoted by $\mathscr{F}^n$.

Typical examples of sum-closed frequency sets are (see also [11]):
(i) Countable sum-closed frequency sets in $\mathbb{R}^n$ for which pairwise distances between frequency vectors are uniformly bounded away from zero. This case corresponds to almost periodic initial data.

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[1] Observe that this differs from the definition of F_ℓ in [11] by the factor $\sqrt{2}$.

(ii) The set $\mathbb{Z}^n \setminus \{0\}$ or more general

$$F := \left\{ \sum_{j=1}^n m_j a_j : m = (m_1, ..., m_n) \in \mathbb{Z}^n \right\} \setminus \{0\},$$

where $a = \{a_1, ..., a_n\}$ represents a basis of $\mathbb{R}^n$. This case corresponds to periodic initial data. Indeed, supp $\hat{u}_0$ is contained in the above F for some a if and only if u_0 is periodic. Clearly, this is a special case of (i).

(iii) The set $\{x \in \mathbb{R}^n : x_j \geq \varepsilon\}$ for $j \in \{1, ..., n\}$ and $\varepsilon > 0$. Note that this example provides non-real-valued initial data only.

We also rigorously clarify what we mean by the support of a vector measure.

Definition 2.5. Let X and Y be Banach spaces having the Radon–Nikodým property. For $\mu \in M_0(\mathbb{R}^n, X)$ we set

$$N_\mu := \bigcup_{\substack{\mathcal{O} \subset \mathbb{R}^n \text{ open} \\ \mu = 0 \text{ on } \mathcal{O}}} \mathcal{O},$$

where we recall that vanishing of a vector-valued measure is canonically defined as

$$\mu = 0 \text{ on } \mathcal{O} \quad :\Longleftrightarrow \quad \mu(E) = 0 \text{ for } E \subset \mathcal{O}.$$

The *support* of μ is defined as

$$\text{supp}\, \mu := \overline{\mathbb{R}^n \setminus N_\mu}.$$

Remark 2.6. Note that the support of μ defined above coincides with the support of μ regarded as a tempered distribution.

In the sequel we will frequently make use of the following observation, mostly without any further notice.

Remark 2.7. Let X and Y be Banach spaces having the Radon–Nikodým property. For $\sigma \in \mathrm{BC}(\mathbb{R}^n \setminus \{0\}, \mathscr{L}(X, Y))$ and $\mu \in M_0(\mathbb{R}^n)$ we have

$$\text{supp}\, \mu \lfloor \sigma \subset \text{supp}\, \mu.$$

Proof. Let $\mathcal{O} \subset \mathbb{R}^n$ be open and such that $\mu = 0$ on $\mathcal{O}$. Since $M_0(\mathbb{R}^n, X)$ is a Banach space (see Lemma 2.8(i)), the latter is equivalent to $|\mu|(\mathcal{O}) = 0$. As $\mu \lfloor \sigma$ by Lemma 2.3(i) is obviously continuous with respect to $|\mu|$, this yields $\mu \lfloor \sigma = 0$ on $\mathcal{O}$. Consequently, we have $N_\mu \subset N_{\mu \lfloor \sigma}$, which implies that

$$\text{supp}\, \mu \lfloor \sigma = \overline{\mathbb{R}^n \setminus N_{\mu \lfloor \sigma}} \subset \overline{\mathbb{R}^n \setminus N_\mu} = \text{supp}\, \mu. \qquad \square$$

Now let $\ell \geq 0$. For $\ell = 0$ we set $\mathrm{FM}_\ell(\mathbb{R}^n, X) := \mathrm{FM}_0(\mathbb{R}^n, X)$. For $\ell > 0$ and a sum-closed frequency set $F_\ell \in \mathscr{F}^n$ we also define the space

$$\mathrm{FM}_\ell(\mathbb{R}^n, X) := \{u \in \mathrm{FM}_0(\mathbb{R}^n, X) : \operatorname{supp} \hat{u} \subset F_\ell\}.$$

Next, we list some useful properties of the spaces just introduced.

Lemma 2.8. ([12, Lemma 2.12]) *Suppose X, X_1 and X_2 are Banach spaces having the Radon–Nikodým property and that $X_1 \cdot X_2 \hookrightarrow X$ with an embedding constant less or equal to one. Then the following assertions hold:*

(i) *The spaces $M(\mathbb{R}^n, X)$ and $M_0(\mathbb{R}^n, X)$, and thus also the spaces $\mathrm{FM}(\mathbb{R}^n, X)$ and $\mathrm{FM}_\ell(\mathbb{R}^n, X)$ for $\ell \geq 0$, are Banach spaces.*

(ii) *For all $u \in \mathrm{FM}(\mathbb{R}^n, X_2)$ and $v \in \mathrm{FM}(\mathbb{R}^n, X_1)$ we have that*

$$\|u \cdot v\|_{\mathrm{FM}(\mathbb{R}^n, X)} \leq (2\pi)^{-n/2} \|u\|_{\mathrm{FM}(\mathbb{R}^n, X_2)} \|v\|_{\mathrm{FM}(\mathbb{R}^n, X_1)},$$

i.e., $\mathrm{FM}(\mathbb{R}^n, X_2) \cdot \mathrm{FM}(\mathbb{R}^n, X_1) \hookrightarrow \mathrm{FM}(\mathbb{R}^n, X)$. In particular, $(\mathrm{FM}(\mathbb{R}^n, X), \cdot)$ is an (abelian) algebra (with unit), if $(X, \cdot)$ is an (abelian) algebra (with unit).

(iii) *We have*

$$(7) \qquad \mathcal{F}L^1(\mathbb{R}^n, X) \hookrightarrow \mathrm{FM}_0(\mathbb{R}^n, X) \hookrightarrow \dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^n, X) \hookrightarrow \mathrm{BUC}(\mathbb{R}^n, X),$$

where $\dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^n, X)$ denotes the homogeneous Besov space.

Observe that the space $\mathrm{FM}_\ell(\mathbb{R}^n, X)$, $\ell \geq 0$, is not an algebra, but at least we obtain by Lemma 2.8(ii) that

$$\mathrm{FM}_\ell(\mathbb{R}^n, X_2) \cdot \mathrm{FM}_\ell(\mathbb{R}^n, X_1) \hookrightarrow \mathrm{FM}(\mathbb{R}^n, X),$$

which is crucial to handle nonlinear terms.

Remark 2.9. The fact that the product $\delta_{t_0} a \in M_0(\mathbb{R}^n, X)$ for Dirac measures δ_{t_0}, $t_0 \in \mathbb{R}^n \setminus \{0\}$, and $a \in X$, gives rise to another interesting class of functions contained in the space $\mathrm{FM}_0(\mathbb{R}^n, X)$. In fact, every sequence $\{a_j\}_{j \in \mathbb{N}} \subseteq X$ satisfying $\sum_{j=1}^\infty \|a_j\|_X < \infty$ defines for each sequence of frequencies $\{\lambda_j\}_{j \in \mathbb{N}} \subseteq F_\ell$ an element

$$\left(x \mapsto \sum_{j=1}^\infty a_j e^{-i\lambda_j x}\right) \in \mathrm{FM}_0(\mathbb{R}^n, X),$$

by the fact that $\sum_{j=1}^\infty \delta_{\lambda_j} a_j \in M_0(\mathbb{R}^n, X)$. This class of almost periodic functions is significant for applications to rotating boundary layers as explained in the introduction.

For $\sigma \in \mathrm{BC}(\mathbb{R}^n \setminus \{0\}, \mathscr{L}(X, Y))$ we define

$$(8) \qquad \mathrm{op}(\sigma) f := \mathcal{F}^{-1} \hat{f} \lfloor \sigma, \quad f \in \mathrm{FM}_0(\mathbb{R}^n, X).$$

We recall three results from [12], which allow for a transfer of L^2-boundedness to the FM-setting. First, as a consequence of the theory for vector measures developed above we obtain the following multiplier result.

Proposition 2.10. ([12, Proposition 2.13]) *Let $\ell \geq 0$. Let X and Y be Banach spaces having the Radon–Nikodým property and suppose that*

$$\sigma \in \mathrm{BC}(\mathbb{R}^n \setminus \{0\}, \mathscr{L}(X, Y)).$$

Then $\mathrm{op}(\sigma)$ as defined in (8) is bounded from $\mathrm{FM}_0(\mathbb{R}^n, X)$ to $\mathrm{FM}_0(\mathbb{R}^n, Y)$ and we have

$$\|\mathrm{op}(\sigma)\|_{\mathscr{L}(\mathrm{FM}_\ell(\mathbb{R}^n, X), \mathrm{FM}_\ell(\mathbb{R}^n, Y))} = \|\sigma\|_{L^\infty(\mathbb{R}^n, \mathscr{L}(X, Y))}.$$

Remark 2.11. If H_1 and H_2 are Hilbert spaces, Plancherel's theorem implies that the right-hand side of the equality in Proposition 2.10 equals the operator norm of $\mathrm{op}(\sigma)$ in $\mathscr{L}(L^2(\mathbb{R}^n, H_1), L^2(\mathbb{R}^n, H_2))$. Hence in this case we have

$$\|\mathrm{op}(\sigma)\|_{\mathscr{L}(\mathrm{FM}_0(\mathbb{R}^n, H_1), \mathrm{FM}_0(\mathbb{R}^n, H_2))} = \|\sigma\|_\infty = \|\mathrm{op}(\sigma)\|_{\mathscr{L}(L^2(\mathbb{R}^n, H_1), L^2(\mathbb{R}^n, H_2))}.$$

In the last part of this section the domain $\mathbb{R}^n$ is essentially fixed. So we occasionally suppress $\mathbb{R}^n$ and simply write $\mathrm{FM}(X)$ instead of $\mathrm{FM}(\mathbb{R}^n, X)$ and so on. In applications we will often use the fact noted in the above remark for the case that H_1 and H_2 are certain L^2-spaces. In the same spirit the following lemma will turn out to be helpful.

Lemma 2.12. ([12, Lemma 2.15]) *Let $\ell \geq 0$, $J \subset \mathbb{R}$ be an interval, and let H_1 and H_2 be Hilbert spaces. Assume that*

$$L \in \mathscr{L}(L^2(\mathbb{R}^n, H_1), L^2(J, L^2(\mathbb{R}^n, H_2))) \quad \text{with } \|L\|_{\mathscr{L}(L^2(H_1), L^2(J, L^2(H_2)))} \leq M$$

is an operator with a symbol σ_L satisfying

$$\sigma_L \in C(\mathbb{R}^n \setminus \{0\}, \mathscr{L}(H_1, L^2(J, H_2))).$$

Then

$$L \in \mathscr{L}(\mathrm{FM}_\ell(\mathbb{R}^n, H_1), L^2(J, \mathrm{FM}_\ell(\mathbb{R}^n, H_2))), \quad \|L\|_{\mathscr{L}(\mathrm{FM}_\ell(H_1), L^2(J, \mathrm{FM}_\ell(H_2)))} \leq M.$$

Remark 2.13. In [12] Proposition 2.10 and Lemma 2.12 are established for the case $\ell = 0$. However, in view of Remark 2.7, under the action of operators having a suitable symbol representation, $FM_\ell(\mathbb{R}^n, X)$ is a closed invariant subspace of $FM_0(\mathbb{R}^n, X)$. Therefore these two results readily generalize to the case $\ell \geq 0$.

We will also make use of the following general result on operator-valued convolution.

Lemma 2.14. ([12, Lemma 2.16]) *Let X and Y be Banach spaces, $1 \leq p \leq \infty$, $T \in (0, \infty]$, and set $J = (0, T)$. For $g \in \mathscr{L}(X, L^p(J, Y))$ and $f \in L^1(J, X)$ we have*

$$\left(t \mapsto g \star f(t) := \int_0^t g(t-s)f(s)\, ds \right) \in L^p(J, Y)$$

and

(9) $$\|g \star f\|_{L^p(J,Y)} \leq \|g\|_{\mathscr{L}(X, L^p(J,Y))} \|f\|_{L^1(J,X)}.$$

Remark 2.15. Observe that (9) is sharper than the usual Young inequality by the fact that $\| \cdot \|_{\mathscr{L}(X, L^p(J,Y))} \leq \| \cdot \|_{L^p(J, \mathscr{L}(X,Y))}$, but the converse in general is not true. Indeed, (9) provides an estimate for singular integral operators, which is not possible with the standard Young's inequality.

3. Proof of Theorem 1.1

The major advantage of working with $d < \infty$ or $\ell > 0$ lies in the fact that exponential decay of the Stokes–Coriolis–Ekman semigroup can be provided. This follows as a consequence of Poincaré's inequality, which is valid in these situations. Let $\ell \geq 0$, $d \in (0, \infty]$ and as before we set $G = \mathbb{R}^2 \times (0, d)$ and $X_2 = L^2(0, d)^3$. Similarly to the FM-setting we put

$$L^2_\ell(G)^3 = L^2_\ell(\mathbb{R}^2, X_2) := \{v \in L^2(\mathbb{R}^2, X_2) : \operatorname{supp} \hat{v} \in F_\ell\}.$$

We emphasize that here v is regarded as a function $v \colon \mathbb{R}^2 \to X_2$ with $\operatorname{supp} \hat{v} \subset \mathbb{R}^2$. Also note that $L^2_\ell(G) = L^2(G)$ if $\ell = 0$, see Definition 2.4.

Lemma 3.1. *Let either $\ell = 0$ and $0 < d < \infty$, or $\ell > 0$ and $d = \infty$. Then*

$$\|u\|_{L^2_\ell(G)^3} \leq \begin{cases} \dfrac{d}{\sqrt{2}} \|\nabla u\|_{L^2_\ell(G)^3}, & \ell = 0, \\[2mm] \dfrac{1}{\ell \sqrt{2}} \|\nabla u\|_{L^2_\ell(G)^3}, & \ell > 0, \end{cases}$$

for all $u \in L^2_\ell(G)^3$. The assertion remains true, if we replace $L^2_\ell(G)^3$ by the space $FM_\ell(\mathbb{R}^2, X_2)$.

Proof. The inequalities in case when $\ell=0$ (i.e. $d<\infty$) are standard, since we can apply the one-dimensional Poincaré inequality in the x_3-direction. For $\ell>0$ we give a proof in the FM-setting. By Plancherel's theorem the L^2-case follows similarly.

We observe that by definition supp $\hat{u}\subset\mathbb{R}^2\setminus B\big(0,\ell\sqrt{2}\big)$. Since $1\leq|\xi|/\ell\sqrt{2}$ on $\mathbb{R}^2\setminus B\big(0,\ell\sqrt{2}\big)$ this gives us in view of Lemma 2.3(i),

$$
\begin{aligned}
|\hat{u}|(B(0,R)) &= \big(|\hat{u}|\lfloor 1\big)\big(\{\ell\sqrt{2}\leq|\xi|\leq R\}\big)\\
&\leq \left(|\hat{u}|\lfloor\frac{|\xi|}{\ell\sqrt{2}}\right)\big(\{\ell\sqrt{2}\leq|\xi|\leq R\}\big) = \frac{1}{\ell\sqrt{2}}|\hat{u}\lfloor i\xi|(B(0,R))
\end{aligned}
$$

for $R>\ell\sqrt{2}$ and $u\in\mathrm{FM}_\ell(\mathbb{R}^2,X_2)$. Thus we have, for $u\in\mathrm{FM}_\ell(\mathbb{R}^2,X_2)$,

$$
\begin{aligned}
\|u\|_{\mathrm{FM}(X_2)} &= \lim_{R\to\infty}|\hat{u}|(B(0,R))\\
&\leq \frac{1}{\ell\sqrt{2}}\lim_{R\to\infty}\big|\hat{u}\lfloor i\xi\big|(B(0,R)) = \frac{1}{\ell\sqrt{2}}\|\nabla u\|_{\mathrm{FM}(X_2)}. \quad \square
\end{aligned}
$$

In order to estimate the crucial perturbation arising from the Ekman spiral, also the following Poincaré type inequality will be used, cf. [12, Lemma 3.7]. Its proof is a simple consequence of the fundamental theorem of calculus.

Lemma 3.2. *We have, for $\alpha, d>0$ and $v\in C_c^\infty([0,d))$,*

$$
\|e^{-(\cdot)/\alpha}v\|_{L^2(0,d)}\leq\alpha\left(\int_0^{d/\alpha}e^{-2x}x\,dx\right)^{1/2}\|v'\|_{L^2(0,d)}\leq\frac{\alpha}{2}\|v'\|_{L^2(0,d)}.
$$

Next, we recall existence of the Helmholtz decomposition and the Stokes–Coriolis–Ekman semigroup in the FM-setting. We define solenoidal fields as

$$
\mathrm{FM}_{\ell,\sigma}(\mathbb{R}^{n-1},X_p)
$$

$$
(10) \qquad := \overline{\left\{u\in\mathrm{FM}_0^\infty\left(\mathbb{R}^{n-1},\bigcap_{k=0}^\infty W^{k,p}(D)\right):\mathrm{div}\,u=0 \text{ and } u^3|_{\partial(\mathbb{R}^{n-1}\times D)}=0\right\}}^{\|\cdot\|_{\mathrm{FM}}}
$$

and gradient fields as

$$
(11) \qquad G_{\mathrm{FM}} = \{\nabla p: p\in L^1_{\mathrm{loc}}(\mathbb{R}^{n-1}\times D) \text{ and } \nabla p\in\mathrm{FM}_\ell(\mathbb{R}^{n-1},X_p)\}.
$$

In [12, Lemma 3.4] the following was established.

Proposition 3.3. *Let* $d \in (0, \infty]$ *and* $\ell \geq 0$. *We have the Helmholtz decomposition*

$$\mathrm{FM}_\ell(\mathbb{R}^2, X_2) = \mathrm{FM}_{\ell,\sigma}(\mathbb{R}^2, X_2) \oplus G_{\mathrm{FM}}.$$

The associated Helmholtz projector $P \colon \mathrm{FM}_\ell(\mathbb{R}^2, X_2) \to \mathrm{FM}_{\ell,\sigma}(\mathbb{R}^2, X_2)$ *admits a symbol representation* $\sigma_P = \mathcal{F} P \mathcal{F}^{-1} \in \mathrm{BC}(\mathbb{R}^2 \setminus \{0\}, \mathscr{L}(X_2))$ *such that*

$$\|P\|_{\mathscr{L}(\mathrm{FM}(X_2))} = \|\sigma\|_{L^\infty(\mathbb{R}^2, \mathscr{L}(X_2))} = 1.$$

Applying P to the first line of (3), the resulting linear operator is given as $A_{\mathrm{SCE}} = A_\nu + B_\omega + B_E$ with

$$\begin{aligned}
A_\nu u &= -P\Delta u, & &\text{Stokes operator,} \\
B_\omega u &= \omega P e_3 \times u, & &\text{Coriolis part,} \\
B_E u &= P(\mathbf{U}^E \cdot \nabla) u + P u^3 \partial_3 \mathbf{U}^E, & &\text{Ekman part.}
\end{aligned}$$

We call A_{SCE} with the domain

$$\mathscr{D}(A_{\mathrm{SCE}}) = \{ u \in \mathrm{FM}_{\ell,\sigma}(\mathbb{R}^{n-1}, X_2) : \partial^\alpha u \in \mathrm{FM}_\ell(\mathbb{R}^{n-1}, X_2),$$
$$\alpha \in \mathbb{N}_0^n, \ |\alpha| \leq 2 \text{ and } u|_{\partial(\mathbb{R}^{n-1} \times D)} = 0 \}$$

the *Stokes–Coriolis–Ekman operator*. In [12, Theorem 3.6] the following proposition was proved.

Proposition 3.4. *Let* $d \in (0, \infty]$ *and* $\ell \geq 0$. *Then the Stokes–Coriolis–Ekman operator* A_{SCE} *is the generator of the holomorphic* C_0-*semigroup* $(\exp(-tA_{\mathrm{SCE}}))_{t \geq 0}$ *on* $\mathrm{FM}_{\ell,\sigma}(\mathbb{R}^2, X_2)$ *having the symbol representation*

$$\sigma_{\exp(-t(A_{\mathrm{SCE}} + \lambda_0))} = \mathcal{F} \exp(-t(A_{\mathrm{SCE}} + \lambda_0)) \mathcal{F}^{-1} \in \mathrm{BC}(\mathbb{R}^2 \setminus \{0\}, \mathscr{L}(X_2)), \quad t \geq 0,$$

for $\lambda_0 > 0$ *sufficiently large.*

Remark 3.5. Again we note that, concerning the statements in Propositions 3.3 and 3.4, in [12] only the case $\ell = 0$ is treated. However, since we have suitable symbol representations, also here the argument performed in Remark 2.13 applies to the general case.

The space $L^2_{\ell,\sigma}(G)$ and the Stokes–Coriolis–Ekman operator A_{SCE} in $L^2_{\ell,\sigma}(G)$ are defined accordingly. Now we prove Theorem 1.1 in the L^2-setting.

Proposition 3.6. *Let ν, U_∞, d, ℓ, δ, Re and ε_ℓ satisfy the relations given in Theorem 1.1 and set $G=\mathbb{R}^2\times(0,d)$. Then the C_0-semigroup $(\exp(-tA_{\mathrm{SCE}}))_{t\geq0}$ on $L^2_{l,\sigma}(G)$ satisfies*

(i) $\|\exp(-tA_{\mathrm{SCE}})\|_{\mathscr{L}(L^2_{\ell,\sigma}(G))}\leq e^{-2\varepsilon_\ell t}$,

(ii) $\|\nabla\exp(-(A_{\mathrm{SCE}}-\varepsilon_\ell)\cdot)u_0\|_{L^2(\mathbb{R}_+,L^2_\ell(G))}\leq\|u_0\|_{L^2_\ell(G)}/\sqrt{\nu(1-\sqrt{2}\,\mathrm{Re})}$,

for all $u_0\in L^2_{\ell,\sigma}(G)$ and $t\geq0$. In particular, all estimates are uniform in $\omega\in\mathbb{R}$.

Proof. For $u_0\in L^2_{\ell,\sigma}(G)$, according to Proposition 3.3(b) we may set $u(t):=e^{-tA_{\mathrm{SCE}}}u_0$. Then u solves

$$\begin{cases} u'+A_{\mathrm{SCE}}u=0 & \text{in } (0,\infty), \\ u(0)=u_0. \end{cases}$$

Multiplying the above equation with u, integrating with respect to x, and taking into account the skew-symmetry of B_ω and B^1_E we obtain

(12) $$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_2+\nu\|\nabla u(t)\|^2_2+(u^3(t)\partial_3U^E,u(t))=0, \quad t>0.$$

Note that by (2), for the derivative of the Ekman spiral we obtain

$$\partial_3\mathbf{U}^E(x_3)=\frac{U_\infty}{\delta}e^{-x_3/\delta}\begin{pmatrix} \cos\dfrac{x_3}{\delta}+\sin\dfrac{x_3}{\delta} \\ \cos\dfrac{x_3}{\delta}-\sin\dfrac{x_3}{\delta} \\ 0 \end{pmatrix}.$$

The third term in (12) we estimate as

$$|(u^3(t)\partial_3U^E,u(t))| \leq \sum_{j=1}^{2}\|u^3(t)e^{(\cdot)/2\delta}(\partial_3U^E)^j\|_2\|e^{-(\cdot)/2\delta}u^j(t)\|_2$$

$$\leq \frac{\sqrt{2}U_\infty}{\delta}\sum_{j=1}^{2}\|e^{-(\cdot)/2\delta}u^3(t)\|_2\|e^{-(\cdot)/2\delta}u^j(t)\|_2$$

$$\leq \sqrt{2}U_\infty\delta\|\nabla u(t)\|^2_2,$$

where we twice applied Lemma 3.2 with $\alpha=2\delta$. Inserting this into (12) we deduce

(13) $$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_2+\nu\big(1-\sqrt{2}\,\mathrm{Re}\big)\|\nabla u(t)\|^2_2\leq0, \quad t>0.$$

Combining estimate (13) with Poincaré's inequality from Lemma 3.1 and keeping in mind that $\nu-\sqrt{2}U_\infty\delta>0$ due to assumption (4), we deduce

$$(14) \qquad \frac{d}{dt}\|u(t)\|_2^2 \le -2\nu\big(1-\sqrt{2}\,\mathrm{Re}\big)\|\nabla u(t)\|_2^2 \le -4\varepsilon_\ell\|u(t)\|_2^2, \quad t>0.$$

By virtue of Grönwall's lemma we therefore obtain

$$(15) \qquad \|u(t)\|_2 \le e^{-2\varepsilon_\ell t}\|u_0\|_2, \quad t\ge 0.$$

Thus (i) is proved. Multiplying (13) with $e^{2\varepsilon_\ell t}$ and integrating over $t\in\mathbb{R}_+$ yields due to (15) that for $t\ge 0$,

$$
\int_0^\infty \|e^{\varepsilon_\ell t}\nabla u(s)\|_2^2\,ds \le -\frac{1}{2\nu\big(1-\sqrt{2}\,\mathrm{Re}\big)}\int_0^\infty e^{2\varepsilon_\ell t}\frac{d}{dt}\|u(t)\|_2^2\,dt
$$
$$
\le \frac{1}{2\nu\big(1-\sqrt{2}\,\mathrm{Re}\big)}\left(\|u_0\|_2^2+2\varepsilon_\ell\int_0^\infty e^{2\varepsilon_\ell t}\|u(t)\|_2^2\,dt\right)
$$
$$
\le \frac{1}{\nu\big(1-\sqrt{2}\,\mathrm{Re}\big)}\|u_0\|_2^2.
$$

Thus the proposition is proved. $\square$

We turn to the proof of Theorem 1.1.

Proof of Theorem 1.1. The proof is completely analogous to [12, Theorem 7.10]. For the reader's convenience we repeat the steps. First recall that in [12, Lemma 3.9] it is proved that
(1) $(\xi'\mapsto\sigma_{T_{\mathrm{SCE}}}(t,\xi'))\in C(\mathbb{R}^2\setminus\{0\},\mathscr{L}(X_2))$, $t\ge 0$, and
(2) $(\xi'\mapsto\sigma_{T_{\mathrm{SCE}}}(\cdot,\xi'))\in C(\mathbb{R}^2\setminus\{0\},\mathscr{L}(X_2,L^2((0,T),X_2)))$
for every $T\in(0,\infty)$, where $\sigma_{T_{\mathrm{SCE}}}=\mathcal{F}\exp(-tA_{\mathrm{SCE}})\mathcal{F}^{-1}$ denotes the symbol of the Stokes–Coriolis–Ekman semigroup. Note that by Plancherel's theorem we have

$$\|\sigma\|_{L^\infty(\mathbb{R}^n,\mathscr{L}(H_1,H_2))} = \|\mathrm{op}(\sigma)\|_{\mathscr{L}(L^2(\mathbb{R}^n,H_1),L^2(\mathbb{R}^n,H_1))}$$

for Hilbert spaces H_1 and H_2. Using this fact, Proposition 3.6(i), Lemma 3.3, and (1) we observe that

$$\sigma_{T_{\mathrm{SCE}}}(t)\sigma_P \in \mathrm{BC}(\mathbb{R}^2\setminus\{0\},\mathscr{L}(X_2))$$

and

$$\|\sigma_{T_{\mathrm{SCE}}}(t)\sigma_P\|_{L^\infty(\mathbb{R}^2,\mathscr{L}(X_2))} \le e^{-2\varepsilon_\ell t}, \quad t\ge 0.$$

Proposition 2.10 then yields (i).

Next, from Proposition 3.6(ii) we infer

$$\|\nabla \exp(-(A_{\mathrm{SCE}}-\varepsilon_\ell)\cdot)P\|_{\mathscr{L}(L^2(\mathbb{R}^2,X_2),L^2((0,T),L^2(\mathbb{R}^2,X_2)))} \leq \frac{1}{\sqrt{\nu(1-\sqrt{2}\,\mathrm{Re})}}$$

for all $T>0$. Setting $H_1=H_2=X_2$, $J=(0,T)$, $L=\exp(-(A_{\mathrm{SCE}}-\varepsilon_\ell)\cdot)P$ and $M=1/\sqrt{\nu(1-\sqrt{2}\,\mathrm{Re})}$ we therefore see that relation (ii) is obtained as a consequence of Lemma 2.12 and (2).

In order to see assertion (iii) observe that (ii) implies that for $T>0$,

$$\|\nabla \exp(-(A_{\mathrm{SCE}}-\varepsilon_\ell)\cdot)\|_{\mathscr{L}(\mathrm{FM}_{\ell,\sigma}(\mathbb{R}^2,X_2),L^2((0,T),\mathrm{FM}_\ell(\mathbb{R}^2,X_2)))} \leq \frac{1}{\sqrt{\nu(1-\sqrt{2}\,\mathrm{Re})}}.$$

Thus, by setting $p=2$, $g=\nabla \exp(-(A_{\mathrm{SCE}}-\varepsilon_\ell)\cdot)$ as well as $X=\mathrm{FM}_{\ell,\sigma}(\mathbb{R}^2,X_2)$, and $Y=\mathrm{FM}_\ell(\mathbb{R}^2,X_2)$, Lemma 2.14 yields

$$\|\nabla \exp(-(A_{\mathrm{SCE}}-\varepsilon_\ell)\cdot)\star f\|_{L^2((0,T),\mathrm{FM}(X_2))} \leq \frac{1}{\sqrt{\nu(1-\sqrt{2}\,\mathrm{Re})}}\|f\|_{L^1((0,T),\mathrm{FM}(X_2))}$$

for $T>0$ and all $f\in L^1(\mathbb{R}_+,\mathrm{FM}_{\ell,\sigma}(\mathbb{R}^2,X_2))$. Hence Theorem 1.1 is proved. $\square$

4. Proof of Theorem 1.3

Again we assume that either $\ell=0$ and $0<d<\infty$, or $\ell>0$ and $d=\infty$. We define the space

$$\mathbb{E}:=\{v\in\mathrm{BC}(\mathbb{R}_+,\mathrm{FM}_{\ell,\sigma}(\mathbb{R}^2,X_2)): \nabla v\in L^2(\mathbb{R}_+,\mathrm{FM}_{\ell,\sigma}(\mathbb{R}^2,X_2))\}$$

equipped with the norm

$$\|v\|_{\mathbb{E}}:=\frac{\sup_{t>0}\|e^{2\varepsilon_\ell t}u(t)\|_{\mathrm{FM}(X_2)}}{\sqrt{\nu(1-\sqrt{2}\,\mathrm{Re})}}+\|e^{\varepsilon_\ell\cdot}\nabla u(t)\|_{L^2(\mathbb{R}_+,\mathrm{FM}(X_2))}.$$

Observe that for $u\in\mathbb{E}$ and $T\in(0,\infty)$ we have $u\in L^2((0,T),\mathrm{FM}_\ell(\mathbb{R}^2,H^1(0,d)^3))$, and hence $u(t,x',\cdot)|_{\partial(0,d)}$ and therefore also $u|_{\partial G}$ is well-defined. For fixed initial value $u_0\in\mathrm{FM}_{\ell,\sigma}(\mathbb{R}^2,X_2)$ we further set

$$B_{u_0}:=\{v\in\mathbb{E}: v|_{\partial G}=0 \text{ and } v|_{t=0}=u_0 \text{ and } \|v\|_{\mathbb{E}}\leq M\|u_0\|_{\mathrm{FM}(X_2)}\}.$$

On B_{u_0} we consider the map

$$Hu(t) := \exp(-tA_{\mathrm{SCE}})u_0 - \int_0^t \exp(-(t-s)A_{\mathrm{SCE}})P(u(s)\cdot\nabla)u(s)\,ds, \quad t > 0.$$

Observe that we have to show that the application of P and of the semigroup to the nonlinear term is well-defined in $\mathrm{FM}_{\ell,\sigma}(\mathbb{R}^2, X_2)$. This will be confirmed below after (17).

We will show that H is contractive on B_{u_0} for suitable $M > 0$. To this end, we estimate

$$\|e^{2\varepsilon_\ell t}Hu(t)\|_{\mathrm{FM}(X_2)}$$

$$\leq \|\exp(-t(A_{\mathrm{SCE}} - 2\varepsilon_\ell))u_0\|_{\mathrm{FM}(X_2)}$$

$$+ \int_0^\infty \|\exp(-(t-s)(A_{\mathrm{SCE}} - 2\varepsilon_\ell))Pe^{2\varepsilon_\ell s}(u(s)\cdot\nabla)u(s)\|_{\mathrm{FM}(X_2)}\,ds$$

$$(16) \qquad \leq \|u_0\|_{\mathrm{FM}(X_2)} + \int_0^\infty \|e^{2\varepsilon_\ell s}(u(s)\cdot\nabla)u(s)\|_{\mathrm{FM}(X_2)}\,ds,$$

where we applied twice Theorem 1.1(i) and then Proposition 3.3. Due to the relation $u(s, x', \cdot)|_{\partial(0,d)} = 0$ we obtain, for $\lambda > 0$,

$$\|u(s, x', \cdot)\|_{L^\infty(0,d)^3} \leq \sqrt{2}\|u(s, x', \cdot)\|_2^{1/2}\|\partial_3 u(s, x', \cdot)\|_{L^2(0,d)^3}^{1/2}$$

$$\leq \frac{1}{\sqrt{2}}\left(\lambda\|u(s, x', \cdot)\|_2 + \frac{1}{\lambda}\|\partial_3 u(s, x', \cdot)\|_{L^2(0,d)^3}\right)$$

$$=: \|u(s, x', \cdot)\|_\lambda.$$

By the fact that $H_0^1(0,d)^3$, equipped with $\|\cdot\|_\lambda$, and $L^2(0,d)^3$ enjoy the Radon–Nikodým property, we can estimate the nonlinear term due to Lemma 2.8(ii) as

$$\|e^{2\varepsilon_\ell s}(u(s)\cdot\nabla)u(s)\|_{\mathrm{FM}(X_2)}$$

$$\leq e^{2\varepsilon_\ell s}\sum_{j=1}^3 \|u(s)^j\partial_j u(s)\|_{\mathrm{FM}(X_2)}$$

$$\leq e^{2\varepsilon_\ell s}\frac{3}{2\sqrt{2}\pi}\left(\lambda\|u(s)\|_{\mathrm{FM}(L^2)} + \frac{1}{\lambda}\|\nabla u(s)\|_{\mathrm{FM}(L^2)}\right)\|\nabla u(s)\|_{\mathrm{FM}(X_2)}$$

$$\leq e^{2\varepsilon_\ell s}\frac{3}{2\sqrt{2}\pi}\|u(s)\|_{\mathrm{FM}(X_2)}^{1/2}\|\nabla u(s)\|_{\mathrm{FM}(X_2)}^{3/2},$$

where we have set $\lambda = \|\nabla u(s)\|_{\mathrm{FM}(X_2)}^{1/2}/\|u(s)\|_{\mathrm{FM}(X_2)}^{1/2}$.

In order to be able to further proceed in a unified way for the two different cases $\ell = 0$ and $\ell > 0$ we let

$$r := \begin{cases} \sqrt{d}, & \ell = 0, \\ \dfrac{1}{\sqrt{\ell}}, & \ell > 0. \end{cases}$$

By Lemma 3.1 we can then continue the above calculation to the result

$$(17) \qquad \|e^{2\varepsilon_\ell s}(u(s)\cdot\nabla)u(s)\|_{\mathrm{FM}(X_2)} \le \frac{3r}{2^{7/4}\pi}\|e^{\varepsilon_\ell s}\nabla u(s)\|_{\mathrm{FM}(X_2)}^2.$$

Now we confirm that the nonlinear term even belongs to $\mathrm{FM}_\ell(\mathbb{R}^2, X_2)$ so that application of P and afterwards of the Stokes–Coriolis–Ekman semigroup on the space $\mathrm{FM}_{\ell,\sigma}(\mathbb{R}^2, X_2)$ is well-defined. For, first we write

$$(u\cdot\nabla)u = \operatorname{div} uu^T.$$

Since $\operatorname{supp}\hat{u}\subset F_\ell$ and F_ℓ is a sum-closed frequency set, we have $\operatorname{supp}\mathcal{F}uu^T\subset F_\ell\cup\{0\}$, and hence

$$\operatorname{supp}\mathcal{F}\operatorname{div} uu^T \subset F_\ell.$$

(Observe that $\mathcal{F}uu^T$ is well-defined as a convolution, cf. [12, Lemma 2.11]; see also [11].) In other words, we have indeed $(u\cdot\nabla)u\in\mathrm{FM}_\ell(\mathbb{R}^2, X_2)$.

Plugging (17) into (16) gives us

$$\sup_{t>0}\|e^{2\varepsilon_\ell t}Hu(t)\|_{\mathrm{FM}(X_2)} \le \|u_0\|_{\mathrm{FM}(X_2)} + \frac{3r}{2^{7/4}\pi}\|e^{\varepsilon_\ell \cdot}\nabla u\|_{L^2(\mathbb{R}_+,\mathrm{FM}(X_2))}^2$$

$$(18) \qquad \le \|u_0\|_{\mathrm{FM}(X_2)}\left(1 + M^2\frac{3r}{2^{7/4}\pi}\|u_0\|_{\mathrm{FM}(X_2)}\right).$$

Next, we derive by utilizing Theorem 1.1(ii) and (iii) that

$$\|e^{\varepsilon_\ell \cdot}\nabla Hu\|_{L^2(\mathbb{R}_+,\mathrm{FM}(X_2))}$$

$$\le \|\nabla\exp(\cdot(A_{\mathrm{SCE}}-\varepsilon_\ell))u_0\|_{L^2(\mathbb{R}_+,\mathrm{FM}(X_2))}$$

$$+ \left(\int_0^\infty\left\|\int_0^t\exp(-(t-s)(A_{\mathrm{SCE}}-\varepsilon_\ell))Pe^{\varepsilon_\ell s}(u(s)\cdot\nabla)u(s)\,ds\right\|_{\mathrm{FM}(X_2)}^2 dt\right)^{1/2}$$

$$\le \frac{1}{\sqrt{\nu(1-\sqrt{2}\,\mathrm{Re})}}\left(\|u_0\|_{\mathrm{FM}(X_2)} + \int_0^\infty\|e^{2\varepsilon_\ell s}(u(s)\cdot\nabla)u(s)\|_{\mathrm{FM}(X_2)}\,ds\right)$$

$$\leq \frac{1}{\sqrt{\nu\left(1-\sqrt{2}\operatorname{Re}\right)}}\left(\|u_0\|_{\mathrm{FM}(X_2)} + \frac{3r}{2^{7/4}\pi}\|e^{\varepsilon\ell s}\nabla u\|^2_{L^2(\mathbb{R}_+,\mathrm{FM}(X_2))}\right)$$

$$(19) \qquad \leq \frac{1}{\sqrt{\nu\left(1-\sqrt{2}\operatorname{Re}\right)}}\|u_0\|_{\mathrm{FM}(X_2)}\left(1 + M^2\frac{3r}{2^{7/4}\pi}\|u_0\|_{\mathrm{FM}(X_2)}\right),$$

where we estimated the nonlinear term again using (17). Collecting (18) and (19) we arrive at

$$\|Hu\|_{\mathbb{E}} \leq \frac{1}{\sqrt{\nu\left(1-\sqrt{2}\operatorname{Re}\right)}}\|u_0\|_{\mathrm{FM}(X_2)}\left(1 + M^2\frac{3r}{2^{7/4}\pi}\|u_0\|_{\mathrm{FM}(X_2)}\right).$$

Thus, choosing $M = 2/\sqrt{\nu\left(1-\sqrt{2}\operatorname{Re}\right)}$ and $\|u_0\|_{\mathrm{FM}(X_2)} \leq \pi\nu\left(1-\sqrt{2}\operatorname{Re}\right)/2^{1/4}3r$ we see that $H(B_{u_0}) \subset B_{u_0}$. Utilizing the expansion

$$(u\cdot\nabla)u - (v\cdot\nabla)v = ((u-v)\cdot\nabla)u + (v\cdot\nabla)(u-v),$$

in a very similar way it can be shown that H is contractive, if the condition on u_0 is strict. The contraction mapping principle then proves Theorem 1.3.

References

1. BABIN, A., MAHALOV, A. and NICOLAENKO, B., Regularity and integrability of 3D Euler and Navier–Stokes equations for rotating fluids, *Asymptot. Anal.* **15** (1997), 103–150.
2. BABIN, A., MAHALOV, A. and NICOLAENKO, B., Global regularity of 3D rotating Navier–Stokes equations for resonant domains, *Indiana Univ. Math. J.* **48** (1999), 1133–1176.
3. BABIN, A., MAHALOV, A. and NICOLAENKO, B., 3D Navier–Stokes and Euler equations with initial data characterized by uniformly large vorticity, *Indiana Univ. Math. J.* **50** (2001), 1–35.
4. CHEMIN, J.-Y., DESJARDIN, B., GALLAGHER, I. and GRENIER, E., Ekman boundary layers in rotating fluids, *ESAIM Control Optim. Calc. Var.* **8** (2002), 441–466.
5. CHEMIN, J.-Y., DESJARDIN, B., GALLAGHER, I. and GRENIER, E., *Mathematical Geophysics*, Oxford Univ. Press, London, 2006.
6. DESJARDINS, B., DORMY, E. and GRENIER, E., Stability of mixed Ekman–Hartmann boundary layers, *Nonlinearity* **12** (1999), 181–199.
7. DIESTEL, J. and UHL, J. J., *Vector Measures*, *Mathematical Surveys* **15**, Amer. Math. Soc., Providence, RI, 1977.
8. EKMAN, V. W., On the influence of the earth's rotation on ocean-currents, *Ark. Mat. Astr. Fys.* **2**:11 (1905), 1–52.

9. Fischer, A. and Saal, J., On instability of the Ekman spiral, *Discrete Contin. Dyn. Syst. Ser. S* **6** (2013), 1225–1236.

10. Giga, Y., Inui, K., Mahalov, A., Matsui, S. and Saal, J., Rotating Navier–Stokes equations in a half-space with initial data nondecreasing at infinity: the Ekman boundary layer problem, *Arch. Ration. Mech. Anal.* **186** (2007), 177–224.

11. Giga, Y., Inui, K., Mahalov, A. and Saal, J., Global solvability of the Navier–Stokes equations in spaces based on sum-closed frequency sets, *Adv. Difference Equ.* **12** (2007), 721–736.

12. Giga, Y. and Saal, J., An approach to rotating boundary layers based on vector Radon measures, *J. Math. Fluid Mech.* **15** (2013), 89–127.

13. Greenberg, L. and Marletta, M., The Ekman flow and related problems: spectral theory and numerical analysis, *Math. Proc. Cambridge Philos. Soc.* **136** (2004), 719–764.

14. Grenier, E. and Masmoudi, N., Ekman layers of rotating fluids, the case of well prepared initial data, *Comm. Partial Differential Equations* **22** (1997), 953–975.

15. Hess, M., Hieber, M., Mahalov, A. and Saal, J., Nonlinear stability of Ekman layers, *Bull. Lond. Math. Soc.* **42** (2010), 691–706.

16. Iwabuchi, T. and Takada, R., Time periodic solutions to the Navier–Stokes equations in the rotational framework, *J. Evol. Equ.* **12** (2012), 985–1000.

17. Koba, H., Nonlinear stability of Ekman boundary layers in rotating stratified fluids, *Mem. Amer. Math. Soc.* **228** (2014).

18. Koba, H., Mahalov, A. and Yoneda, T., Global well-posedness for the rotating Navier–Stokes–Boussinesq equations with stratification effects, *Adv. Math. Sci. Appl.* **22** (2012), 61–90.

19. Lee, S., Koh, Y. and Takada, R., Dispersive estimates for Navier–Stokes equations in the rotational framework, *Preprint*.

20. Mahalov, A. and Nicolaenko, B., Global solubility of the three-dimensional Navier–Stokes equations with uniformly large initial vorticity, *Uspekhi Mat. Nauk.* **58**:2 (2003), 79–110 (Russian). English transl.: *Russian Math. Surveys* **58** (2003), 287–318.

21. Marletta, M. and Tretter, C., Essential spectra of coupled systems of differential equations and applications in hydrodynamics, *J. Differential Equations* **243** (2007), 36–69.

22. Monin, A. S. and Yaglom, A. M., *Statistical Fluid Mechanics*: *Mechanics of Turbulence*, Vols. **1** and **2**, MIT Press, Cambridge, MA, 1971.

23. Pedlosky, J., *Geophysical Fluid Dynamics*, 2nd ed., Springer, Berlin, 1987.

24. Rousset, F., Stability of large Ekman boundary layers in rotating fluids, *Arch. Ration. Mech. Anal.* **172** (2004), 213–245.

25. Vishik, M. J. and Fursikov, A. V., *Mathematical Problems of Statistical Hydromechanics*, Kluwer, Dordrecht, 1988.

26. Yoneda, T., Long-time solvability of the Navier–Stokes equations in a rotating frame with spatially almost periodic large data, *Arch. Ration. Mech. Anal.* **200** (2011), 225–237.

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