Ark. Mat., **53** (2015), 177–202
DOI: 10.1007/s11512-014-0196-5

The Alexander polynomial as quantum invariant of links

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Abstract. In these notes we collect some results about finite-dimensional representations of $U_q(\mathfrak{gl}(1|1))$ and related invariants of framed tangles, which are well-known to experts but difficult to find in the literature. In particular, we give an explicit description of the ribbon structure on the category of finite-dimensional $U_q(\mathfrak{gl}(1|1))$-representations and we use it to construct the corresponding quantum invariant of framed tangles. We explain in detail why this invariant vanishes on closed links and how one can modify the construction to get a non-zero invariant of framed closed links. Finally we show how to obtain the Alexander polynomial by considering the vector representation of $U_q(\mathfrak{gl}(1|1))$.

1. Introduction

The Alexander polynomial is a classical invariant of links in $\mathbb{R}^3$, defined first in the 1920s by Alexander [1]. Constructed originally in combinatorial terms, it can be defined also in modern language using the homology of a cyclic covering of the link complement (see for example [16]).

The Alexander polynomial can also be defined using the Burau representation of the braid group (see for example [13, Chapter 3]). As is well-known to experts, this representation can be constructed using a solution of the Yang–Baxter equation, which comes from the action of the R-matrix of $U_q(\mathfrak{gl}(1|1))$ [14] (or alternatively of $U_q(\mathfrak{sl}_2)$ for a root of unity q; see [22] for the parallel between $\mathfrak{gl}(1|1)$ and $\mathfrak{sl}_2$).

In other words, the key point of the construction is the *braided* structure of the monoidal category of finite-dimensional representations of $U_q(\mathfrak{gl}(1|1))$, that is, there is an action of an R-matrix satisfying the braid relation. This can obviously be used to construct representations of the braid group. Considering tensor powers of the vector representation of $U_q(\mathfrak{gl}(1|1))$, one obtains in this way the Burau

This work has been supported by the Graduiertenkolleg 1150, funded by the Deutsche Forschungsgemeinschaft.

representation of the braid group. Given a representation of the braid group, one can extend it to an invariant of links considered as closures of braids by defining a Markov trace.

In these notes, we exploit this construction a bit further, proving that the category of finite-dimensional $U_q(\mathfrak{gl}(1|1))$-representations is not only braided, but actually *ribbon*. A ribbon category is exactly what one needs to use the Reshetikhin–Turaev construction [19] to get invariants of oriented framed tangles. The advantage of the ribbon structure is that one can consider arbitrary diagrams of links, and not just braid diagrams.

To construct a ribbon structure on the category of modules over some algebra, a possible strategy is to prove that the algebra is actually a ribbon Hopf algebra. Unfortunately, similarly to the case of a classical semisimple Lie algebra, the Hopf algebra $U_q(\mathfrak{gl}(1|1))$ is not ribbon. We hence consider another version of the quantum enveloping algebra, which we call $U_\hbar(\mathfrak{gl}(1|1))$, and which is a topological algebra over $\mathbb{C}[\![\hbar]\!]$. Roughly speaking, the relation between $U_q(\mathfrak{gl}(1|1))$ and $U_\hbar(\mathfrak{gl}(1|1))$ is given by setting $q{=}e^\hbar$. The price for working with power series pays off, since $U_\hbar(\mathfrak{gl}(1|1))$ is in fact a ribbon Hopf algebra. By a standard argument, we see that the R-matrix and the ribbon element of $U_\hbar(\mathfrak{gl}(1|1))$ act on finite-dimensional representations of $U_q(\mathfrak{gl}(1|1))$ and hence deduce the ribbon structure of this category.

Given an oriented framed tangle T and a labeling $\boldsymbol{\ell}$ of the strands of T by finite-dimensional irreducible $U_q(\mathfrak{gl}(1|1))$-representations, we then get an invariant $Q^{\boldsymbol{\ell}}(T)$, which is a certain $U_q(\mathfrak{gl}(1|1))$-equivariant map. In particular, restricting to oriented framed links (viewed as special cases of tangles), we obtain a $\mathbb{C}(q)$-valued invariant.

If we label all the strands by the vector representation of $U_q(\mathfrak{gl}(1|1))$, an easy calculation shows that the corresponding invariant of oriented framed tangles is actually independent of the framing and hence is an invariant of oriented tangles (as is well-known, the same happens for the ordinary $\mathfrak{sl}_n$-invariant).

Unfortunately, when considering invariants of closed links, there is a little problem we have to take care of. Namely, it follows from the fact that the category of finite-dimensional $U_q(\mathfrak{gl}(1|1))$-modules is not semisimple (this is true even in the non-quantized case and well-known, see for example [4] where the blocks of the category of finite-dimensional $\mathfrak{gl}(m|n)$-representations are studied in detail) that the invariant $Q^{\boldsymbol{\ell}}(L)$ is zero for all closed links L (see Proposition 4.4). The work-around to avoid this problem is to choose a strand of the link L, cut it and consider the invariant of the framed 1-tangle that is obtained in this way (Theorem 4.6). The resulting invariant will be an element of the endomorphism ring of an irreducible representation (the one that labels the strand being cut); since this ring can be naturally identified with $\mathbb{C}(q)$, the invariant that we obtain in this way is actu-

ally a rational function. The construction does not depend on the strand we cut, but rather on the representation labeling the strand. In particular for a constant labeling $\boldsymbol{\ell}$ of all the components of L we get a true invariant of framed links.

Applying this construction to the constant labeling by the vector representation, one obtains as before an invariant of links. In fact, it is easy to prove that this coincides with the Alexander polynomial (see Theorem 4.10).

The structure of these notes is the following. In Section 2 we define the quantum enveloping superalgebras U_q and $U_\hbar$ and explicitly describe the ribbon structure of the latter. In Section 3 we study in detail the category of finite-dimensional representations of U_q. In Section 4 we construct the invariants of framed tangles and of links and finally recover the Alexander polynomial as a consequence of the construction. In the appendix we collect two technical results about $U_\hbar$.

We want to stress that the content of this short note is well-known to experts. Relations between $U_q(\mathfrak{gl}(1|1))$, or more generally $U_q(\mathfrak{gl}(n|n))$, and the Alexander polynomial have been noticed, studied and generalized by lots of authors (see for example [7]–[11], [14] and [20]). In particular, almost everything we write here is a special case of what is analyzed in [22], where quantum $\mathfrak{gl}(1|1)$- and $\mathfrak{sl}_2$-invariants associated with arbitrary coloring of tangles are studied in detail. Our aim is to provide, from a purely representation theoretical point of view, a short but complete and self-contained explanation of how the Alexander polynomial arises as quantum invariant corresponding to the vector representation of $U_q(\mathfrak{gl}(1|1)$, including a full proof of the ribbon structure of $U_q(\mathfrak{gl}(1|1))$.

Acknowledgements. The present work is part of the author's Ph.D. thesis. The author would like to thank his advisor Catharina Stroppel for her help and support. The author would also like to thank the anonymous referee for many helpful comments.

2. The quantum enveloping superalgebras U_q and $U_\hbar$

We recall the definition of the quantum enveloping algebra $U_q{=}U_q(\mathfrak{gl}(1|1))$ and of its Hopf superalgebra structure. We then define the $\hbar$-version $U_\hbar{=}U_\hbar(\mathfrak{gl}(1|1))$ and prove that it is a ribbon Hopf superalgebra.

In the following, as usual, by a *super*object (for example vector space, algebra, Lie algebra, module) we will mean a $\mathbb{Z}/2\mathbb{Z}$-graded object. If X is such a superobject we will use the notation $|x|$ to indicate the degree of a homogeneous element $x{\in}X$. Elements of degree 0 are called *even*, while elements of degree 1 are called *odd*. We stress that whenever we write $|x|$ we will always be assuming x to be homogeneous.

Throughout the section we will use some standard facts about Hopf superalgebras. The analogous statements in the non-super setting can be found for example in [5], [12] or [18]. The proofs carry directly over to the super case.

2.1. The Lie superalgebra $\mathfrak{gl}(1|1)$

Let $\mathbb{C}^{1|1}$ be the 2-dimensional complex vector space with basis $\{u_0, u_1\}$ viewed as a super vector space by setting $|u_0|=0$ and $|u_1|=1$. The space of linear endomorphisms of $\mathbb{C}^{1|1}$ inherits a $\mathbb{Z}/2\mathbb{Z}$-grading and turns into a Lie superalgebra $\mathfrak{gl}(1|1)$ with the supercommutator

$$[a,b] = ab - (-1)^{|a|\,|b|}ba. \tag{2.1}$$

As a Lie superalgebra, $\mathfrak{gl}(1|1)$ is four-dimensional and generated by the elements

$$h_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{2.2}$$

with $|h_1|=|h_2|=0$ and $|e|=|f|=1$, subject to the defining relations

$$\begin{aligned} &[h_1,e] = e, \quad [h_1,f] = -f, \quad [h_1,h_2] = 0, \quad [e,e] = 0, \\ &[h_2,e] = -e, \quad [h_2,f] = f, \quad [e,f] = h_1 + h_2, \quad [f,f] = 0. \end{aligned} \tag{2.3}$$

Let $\mathfrak{h} \subset \mathfrak{gl}(1|1)$ be the Cartan subalgebra consisting of all diagonal matrices. In $\mathfrak{h}^*$ let $\{e_1, e_2\}$ be the basis dual to $\{h_1, h_2\}$. On $\mathfrak{h}^*$ we define a non-degenerate symmetric bilinear form by setting on the basis

$$(\varepsilon_i, \varepsilon_j) = \begin{cases} 1, & \text{if } i=j=1, \\ -1, & \text{if } i=j=2, \\ 0, & \text{if } i \neq j. \end{cases} \tag{2.4}$$

The *roots* of $\mathfrak{gl}(1|1)$ are $\alpha = \varepsilon_1 - \varepsilon_2$ and $-\alpha$; we choose α to be the positive *simple root*. Denote by $\mathsf{P} = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2 \subset \mathfrak{h}^*$ the *weight lattice* and by $\mathsf{P}^* = \mathbb{Z}h_1 \oplus \mathbb{Z}h_2 \subset \mathfrak{h}$ its dual.

2.2. The quantum enveloping superalgebra

The *quantum enveloping superalgebra* $U_q = U_q(\mathfrak{gl}(1|1))$ is defined to be the unital superalgebra over $\mathbb{C}(q)$ with generators E, F and $\mathbf{q}^h$ ($h \in \mathsf{P}^*$) in degrees $|\mathbf{q}^h| = 0$ and $|E| = |F| = 1$ subject to the relations for $h, h' \in \mathsf{P}^*$,

$$\begin{gathered} \mathbf{q}^0 = 1, \qquad \mathbf{q}^h E = q^{\langle h,\alpha\rangle} E \mathbf{q}^h, \qquad EF + FE = \frac{K - K^{-1}}{q - q^{-1}}, \\ \mathbf{q}^h \mathbf{q}^{h'} = \mathbf{q}^{h+h'}, \quad \mathbf{q}^h F = q^{-\langle h,\alpha\rangle} F \mathbf{q}^h, \qquad E^2 = F^2 = 0, \end{gathered} \tag{2.5}$$

where $K = \mathbf{q}^{h_1 + h_2}$. The elements $\mathbf{q}^h$, which for the moment are formal symbols, can be interpreted in terms of exponentials in $U_\hbar(\mathfrak{gl}(1|1))$ (see below). Notice that all elements $\mathbf{q}^h$ for $h \in \mathsf{P}^*$ are linear combination of $\mathbf{q}^{h_1}$ and $\mathbf{q}^{h_2}$, so that U_q is finitely generated. Note also that K is a central element of U_q, very much in contrast to $U_q(\mathfrak{sl}_2)$.

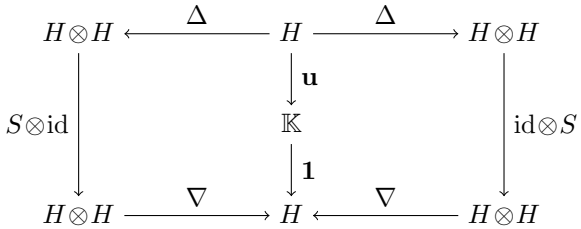
2.3. Hopf superalgebras

We recall that if A is a superalgebra then $A \otimes A$ can be given a superalgebra structure by declaring $(a \otimes b)(c \otimes d) = (-1)^{|b|\,|c|} ac \otimes bd$. If M and N are A-supermodules, then $M \otimes N$ becomes an $A \otimes A$-supermodule with action

$$(a \otimes b) \cdot (m \otimes n) = (-1)^{|b|\,|m|} am \otimes bn \quad \text{for } a, b \in A,\ m \in M \text{ and } n \in N.$$

A *superbialgebra* B over a field $\mathbb{K}$ is then a unital superalgebra, which is also a coalgebra, such that the counit $\mathbf{u} \colon B \to \mathbb{K}$ and the comultiplication $\Delta \colon B \to B \otimes B$ are homomorphisms of superalgebras (and are homogeneous of degree 0). A *Hopf superalgebra* H is a superbialgebra equipped with a $\mathbb{K}$-linear *antipode* $S \colon H \to H$ (homogeneous of degree 0) such that the usual diagram

(2.6)



commutes, where $\nabla \colon H \otimes H \to H$ and $\mathbf{1} \colon \mathbb{K} \to H$ are the multiplication and unit of the algebra structure.

If H is a Hopf superalgebra and M and N are (finite-dimensional) H-supermodules, then the comultiplication Δ makes it possible to give $M\otimes N$ an H-module structure by letting

$$x\cdot(m\otimes n)=\Delta(x)(m\otimes n)=\sum_{(x)}(-1)^{|x_{(2)}|\,|m|}x_{(1)}m\otimes x_{(2)}n \tag{2.7}$$

for $x\in H$ and $m\otimes n\in M\otimes N$, where we use Sweedler notation $\Delta(x)=\sum_{(x)}x_{(1)}\otimes x_{(2)}$. Notice in particular that signs appear. The antipode S, moreover, allows us to turn $M^*=\mathrm{Hom}_{\mathbb{K}}(M,\mathbb{K})$ into an H-module via

$$(x\varphi)(v)=(-1)^{|\varphi|\,|x|}\varphi(S(x)v) \tag{2.8}$$

for $x\in H$ and $\varphi\in M^*$. Again, notice that a sign appears. A good rule to keep in mind is that a sign appears whenever an odd element steps over some other odd element. A good reference for sign issues is [17, Chapter 3].

2.4. The Hopf superalgebra structure on U_q

Let us now go back to U_q. We define a *comultiplication* $\Delta\colon U_q\to U_q\otimes U_q$, a *counit* $\mathbf{u}\colon U_q\to\mathbb{C}(q)$ and an *antipode* $S\colon U_q\to U_q$ by setting on the generators

$$\begin{aligned}
&\Delta(E)=E\otimes K^{-1}+1\otimes E, && \Delta(F)=F\otimes 1+K\otimes F,\\
&S(E)=-EK, && S(F)=-K^{-1}F,\\
&\Delta(\mathbf{q}^h)=\mathbf{q}^h\otimes\mathbf{q}^h, && S(\mathbf{q}^h)=\mathbf{q}^{-h},\\
&\mathbf{u}(E)=\mathbf{u}(F)=0, && \mathbf{u}(\mathbf{q}^h)=1,
\end{aligned} \tag{2.9}$$

and extending Δ and $\mathbf{u}$ to algebra homomorphisms and S to an algebra anti-homomorphism. We have the following result.

Proposition 2.1. *The maps Δ, $\mathbf{u}$ and S turn U_q into a Hopf superalgebra.*

Proof. This is a straightforward calculation. □

Notice that from the centrality of K it follows that $S^2=\mathrm{id}$; this is a special property of U_q, which for instance does not hold in $U_q(\mathfrak{gl}(m|n))$ for general m and n (see [2] for a definition of the general linear quantum supergroup).

We define a *bar involution* on U_q by setting

$$\overline{E}=E,\quad \overline{F}=F,\quad \overline{\mathbf{q}^h}=\mathbf{q}^{-h}\quad\text{and}\quad \bar{q}=q^{-1}. \tag{2.10}$$

Note that $\bar{\Delta}=({}^-\otimes{}^-)\circ\Delta\circ{}^-$ defines another comultiplication on U_q, and by definition $\bar{\Delta}(\bar{x})=\overline{\Delta(x)}$ for all $x\in U_q$.

2.5. The Hopf superalgebra $U_\hbar$

Our goal is to construct a ribbon category of representations of U_q, so that we can define link invariants. The main ingredient is the R-matrix. Unfortunately, as usual, it is not possible to construct a universal R-matrix for U_q; instead, we need to consider the $\hbar$-version of the quantum enveloping superalgebra, which we will denote by $U_\hbar$, and which is a $\mathbb{C}[\![\hbar]\!]$-superalgebra completed with respect to the $\hbar$-adical topology. We will prove that $U_\hbar$ is a ribbon algebra. Then, using a standard argument of Tanisaki [21], we obtain a ribbon structure on the category of finite-dimensional U_q-representations. For details about topological $\mathbb{C}[\![\hbar]\!]$-algebras we refer to [12, Chapter XVI]. We will denote by the symbol $\widehat{\otimes}$ the *completed* tensor product of topological $\mathbb{C}[\![\hbar]\!]$-algebras.

We define $U_\hbar$ to be the unital $\mathbb{C}[\![\hbar]\!]$-algebra topologically generated by the elements E, F, H_1 and H_2 in degrees $|H_1|=|H_2|=0$ and $|E|=|F|=1$ subject to the relations

$$\begin{aligned} &H_1H_2 = H_2H_1, \\ &H_iE - EH_i = \langle H_i,\alpha\rangle E, \qquad H_iF - FH_i = -\langle H_i,\alpha\rangle F, \\ &EF + FE = \frac{e^{\hbar(H_1+H_2)} - e^{-\hbar(H_1+H_2)}}{e^\hbar - e^{-\hbar}}, \qquad E^2 = F^2 = 0. \end{aligned} \tag{2.11}$$

Note that although $e^\hbar - e^{-\hbar}$ is not invertible, it is the product of $\hbar$ with an invertible element of $\mathbb{C}[\![\hbar]\!]$, and hence the fourth relation makes sense.

Although the relation between U_q and $U_\hbar$ is technically not easy to formalize (see [5] for details), one should keep in mind the picture

$$\begin{aligned} q &\longleftrightarrow e^\hbar, \\ \mathbf{q}^{h_i} &\longleftrightarrow e^{\hbar H_i}. \end{aligned} \tag{2.12}$$

This also explains why we use the symbols $\mathbf{q}^h$ as generators for U_q. In the following, we set $q=e^\hbar$ as an element of $\mathbb{C}[\![\hbar]\!]$ and $K=e^{\hbar(H_1+H_2)}$ as an element of $U_\hbar$.

As before, we define a *comultiplication* $\Delta\colon U_\hbar \to U_\hbar \mathbin{\widehat{\otimes}} U_\hbar$, a *counit* $\mathbf{u}\colon U_\hbar \to \mathbb{C}[\![\hbar]\!]$ and an *antipode* $S\colon U_\hbar \to U_\hbar$ by setting for the generators

$$\begin{aligned} \Delta(E) &= E\otimes K^{-1} + 1\otimes E, \quad & \Delta(F) &= F\otimes 1 + K\otimes F, \\ S(E) &= -EK, & S(F) &= -K^{-1}F, \\ \Delta(H_i) &= H_i\otimes 1 + 1\otimes H_i, & S(H_i) &= -H_i, \\ \mathbf{u}(E) &= \mathbf{u}(F) = 0, & \mathbf{u}(H_i) &= 0, \end{aligned} \tag{2.13}$$

and extending Δ and $\mathbf{u}$ to algebra homomorphisms and S to an algebra anti-homomorphism. We have the following result.

Proposition 2.2. *The maps* Δ, $\mathbf{u}$ *and* S *turn* $U_\hbar$ *into a Hopf superalgebra.*

The proof requires precisely the same calculations as the proof of Proposition 2.1.

As for U_q, we define a *bar involution* on $U_\hbar$ by setting

$$\overline{E}=E, \quad \overline{F}=F, \quad \overline{H}_i=H_i \quad \text{and} \quad \bar{\hbar}=-\hbar. \tag{2.14}$$

As before, $\bar{\Delta}=(^-\otimes^-)\circ\Delta\circ^-$ defines another comultiplication on $U_\hbar$, and by definition $\bar{\Delta}(\bar{x})=\overline{\Delta(x)}$ for all $x\in U_\hbar$.

2.6. The braided structure

We are going to recall the *braided Hopf superalgebra structure* (cf. [18] and [23]) of $U_\hbar$. The main ingredient is the universal R-matrix, which has been explicitly computed by Khoroshkin and Tolstoy (cf. [15]). We adapt their definition to our notation.([1])

We define $R=\Theta\Upsilon\in U_\hbar\,\widehat{\otimes}\,U_\hbar$, where

$$\Upsilon=e^{\hbar(H_1\otimes H_1-H_2\otimes H_2)}, \tag{2.15}$$

$$\Theta=1+(q-q^{-1})F\otimes E. \tag{2.16}$$

Notice that the expression for Υ makes sense as an element of the completed tensor product $U_\hbar\,\widehat{\otimes}\,U_\hbar$. Recall that a vector w in some representation W of $U_\hbar$ is said to be a *weight vector* of *weight* μ if $H_iw=\langle H_i,\mu\rangle w$ for $i=1,2$. The element Υ is then characterized by the property that it acts on a weight vector $w_1\otimes w_2$ by $q^{(\mu_1,\mu_2)}=e^{\hbar(\mu_1,\mu_2)}$, if w_1 and w_2 have weights μ_1 and μ_2 respectively.

The element Θ is called the *quasi R-matrix*; it is easy to check that it satisfies

$$\Theta\overline{\Theta}=\overline{\Theta}\Theta=1\otimes 1. \tag{2.17}$$

It follows in particular that R is invertible with inverse $R^{-1}=\Upsilon^{-1}\Theta^{-1}=\Upsilon^{-1}\overline{\Theta}$.

Recall that a bialgebra B is called *quasi-cocommutative* [12, Definition VIII.2.1] if there exists an invertible element $R\in B\otimes B$ such that for all $x\in B$ we have $\Delta^{\mathrm{op}}(x)=R\Delta(x)R^{-1}$, where Δ^{op} is the opposite comultiplication $\Delta^{\mathrm{op}}=\sigma\circ\Delta$ with $\sigma(a\otimes b)=(-1)^{|a|\,|b|}(b\otimes a)$.

Lemma 2.3. *For all* $x\in U_\hbar$ *we have*

$$R\Delta(x)=\Delta^{\mathrm{op}}(x)R. \tag{2.18}$$

Hence the Hopf algebra $U_\hbar$ *is quasi-cocommutative.*

([1]) Our comultiplication is the opposite of [15], and hence we have to take the opposite R-matrix, cf. also [12, Chapter 8].

Proof. Using Lemma A.1 we compute

$$R\Delta(x) = \Theta\Upsilon\Delta(x) = \Theta\bar{\Delta}^{\mathrm{op}}(x)\Upsilon = \Delta^{\mathrm{op}}(x)\Theta\Upsilon = \Delta^{\mathrm{op}}(x)R. \quad \square$$

A quasi-cocommutative Hopf algebra is *braided* or *quasi-triangular* if the following quasi-triangularity identities hold:

$$(\Delta\otimes\mathrm{id})(R) = R_{13}R_{23} \quad \text{and} \quad (\mathrm{id}\otimes\Delta)(R) = R_{13}R_{12}. \tag{2.19}$$

In this case, the element R is a *universal R-matrix*.

Proposition 2.4. *The Hopf superalgebra $U_\hbar$ is braided.*

Proof. Since

$$(\Delta\otimes\mathrm{id})(\Upsilon) = e^{\hbar(H_1\otimes 1\otimes H_1 + 1\otimes H_1\otimes H_1 - H_2\otimes 1\otimes H_2 - 1\otimes H_2\otimes H_2)} = \Upsilon_{13}\Upsilon_{23}$$

we can compute using Lemma A.2,

$$(\Delta\otimes\mathrm{id})(R) = (\Delta\otimes\mathrm{id})(\Theta)(\Delta\otimes\mathrm{id})(\Upsilon) = \Theta_{13}\Upsilon_{13}\Theta_{23}\Upsilon_{13}^{-1}\Upsilon_{13}\Upsilon_{23} = R_{13}R_{23}.$$

Similarly we get $(\mathrm{id}\otimes\Delta)(R) = R_{13}R_{12}$. $\square$

As an easy consequence of the braided structure, the following Yang–Baxter equation holds (see [12, Theorem VIII.2.4] or [5, Proposition 4.2.7]):

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{2.20}$$

2.7. The ribbon structure

Write $R = \sum_r a_r \otimes b_r$ and define

$$u = \sum_r (-1)^{|a_r|\,|b_r|} S(b_r)a_r \in U_\hbar. \tag{2.21}$$

Then (cf. [5, Proposition 4.2.3]) u is invertible and

$$S^2(x) = uxu^{-1} \quad \text{for all } x \in U_\hbar. \tag{2.22}$$

In our case, in particular, since $S^2 = \mathrm{id}$, the element u is central. By an easy explicit computation, we have

$$u = (1 + (q - q^{-1})EKF)e^{\hbar(H_2^2 - H_1^2)} \tag{2.23}$$

and

$$S(u)=e^{\hbar(H_2^2-H_1^2)}(1-(q-q^{-1})FK^{-1}E). \tag{2.24}$$

We recall that a braided Hopf superalgebra A is *ribbon* (cf. [18, Chapter 4] or [5, Section 4.2.C]) if there is an even central element $v\in A$ such that

$$v^2=uS(u),\quad \mathbf{u}(v)=1,\quad S(v)=v\quad\text{and}\quad \Delta(v)=(R_{21}R_{12})^{-1}(v\otimes v). \tag{2.25}$$

In $U_\hbar$ let

$$v=K^{-1}u=uK^{-1}=(K^{-1}+(q-q^{-1})EF)e^{\hbar(H_2^2-H_1^2)}. \tag{2.26}$$

Then we have the following result.

Proposition 2.5. *With v as above, $U_\hbar$ is a ribbon Hopf superalgebra.*

Proof. Since both u and K^{-1} are central, so is v. Let us check that $S(u)=uK^{-2}$. Indeed we have

$$\begin{aligned} u&=(1+(q-q^{-1})EFK)e^{\hbar(H_2^2-H_1^2)}\\ &=e^{\hbar(H_2^2-H_1^2)}(1+(q-q^{-1})EFK)\\ &=e^{\hbar(H_2^2-H_1^2)}(1+(K-K^{-1})K-(q-q^{-1})FEK)\\ &=e^{\hbar(H_2^2-H_1^2)}(K^2-(q-q^{-1})FEK)\\ &=S(u)K^2. \end{aligned} \tag{2.27}$$

It then follows immediately that $v^2=u^2K^{-2}=uS(u)$ and that $S(v)=S(u)K=uK^{-1}=v$.

The relations $\Delta(v)=(R_{21}R_{12})^{-1}(v\otimes v)$ and $\mathbf{u}(v)=1$ follow from analogous relations for u, which hold for every quasi-triangular Hopf superalgebra (see [18, Proposition 4.3]). □

3. Representations

We define a parity function $|\cdot|\colon \mathsf{P}\to\mathbb{Z}/2\mathbb{Z}$ on the weight lattice by setting $|\varepsilon_1|=0$ and $|\varepsilon_2|=1$ and extending additively. By a *representation* of U_q we mean from now on a finite-dimensional U_q-supermodule with a decomposition into weight spaces $M=\bigoplus_{\lambda\in\mathsf{P}}M_\lambda$ with integral weights $\lambda\in\mathsf{P}$, such that $\mathbf{q}^h$ acts as $q^{\langle h,\lambda\rangle}$ on M_λ. We suppose further that M is $\mathbb{Z}/2\mathbb{Z}$-graded, and that the grading is uniquely determined by the requirement that M_λ is in degree $|\lambda|$.

3.1. Irreducible representations

It is not difficult to find all simple representations: up to isomorphism they are indexed by their highest weight $\lambda\in\mathsf{P}$. If $\lambda\in\mathrm{Ann}(h_1+h_2)$, then the simple representation with highest weight λ is one-dimensional, generated by a vector v^λ in degree $|v^\lambda|=|\lambda|$ with

$$Ev^\lambda=0,\quad Fv^\lambda=0,\quad \mathbf{q}^h v^\lambda=q^{\langle h,\lambda\rangle}v^\lambda\quad\text{and}\quad Kv^\lambda=v^\lambda. \tag{3.1}$$

We will denote this representation by $\mathbb{C}(q)_\lambda$, to emphasize that it is just a copy of $\mathbb{C}(q)$ on which the action is twisted by the weight λ. In particular for $\lambda=0$ we have the trivial representation $\mathbb{C}(q)_0$, that we will simply denote by $\mathbb{C}(q)$ below.

If $\lambda\notin\mathrm{Ann}(h_1+h_2)$ then the simple representation $L(\lambda)$ with highest weight λ is two-dimensional; we denote by v_0^λ its highest weight vector. Let us also introduce the following notation that will be useful later:

$$q^\lambda=q^{\langle h_1+h_2,\lambda\rangle}\quad\text{and}\quad[\lambda]=[\langle h_1+h_2,\lambda\rangle], \tag{3.2}$$

where, as usual, $[k]$ is the quantum number defined by

$$[k]=\frac{q^k-q^{-k}}{q-q^{-1}}=q^{-k+1}+q^{-k+3}+\ldots+q^{k-3}+q^{k-1}. \tag{3.3}$$

Even though the second equality holds only for $k>0$, we define $[k]$ for all integers k using the first equality; in particular we have $[-k]=-[k]$.

Then $L(\lambda)=\mathbb{C}(q)\langle v_0^\lambda\rangle\oplus\mathbb{C}(q)\langle v_1^\lambda\rangle$ with $|v_0^\lambda|=|\lambda|$, $|v_1^\lambda|=|\lambda|+1$ and

$$\begin{aligned} &Ev_0^\lambda=0, && Fv_0^\lambda=[\lambda]v_1^\lambda, && \mathbf{q}^h v_0^\lambda=q^{\langle h,\lambda\rangle}v_0^\lambda, && Kv_0^\lambda=q^\lambda v_0^\lambda,\\ &Ev_1^\lambda=v_0^\lambda, && Fv_1^\lambda=0, && \mathbf{q}^h v_1^\lambda=q^{\langle h,\lambda-\alpha\rangle}v_1^\lambda, && Kv_1^\lambda=q^\lambda v_1^\lambda. \end{aligned} \tag{3.4}$$

Remark 3.1. As a remarkable property of U_q, we notice that since $E^2=F^2=0$ *all* simple U_q-modules (even the ones with non-integral weights) are finite-dimensional. In fact, formulas (3.4) define two-dimensional simple U_q-modules for all complex weights $\lambda\in\mathbb{C}\varepsilon_1\oplus\mathbb{C}\varepsilon_2$ such that $\langle h_1+h_2,\lambda\rangle\neq0$.

From now on, we set

$$\mathsf{P}'=\{\lambda\in\mathsf{P}\mid\lambda\notin\mathrm{Ann}(h_1+h_2)\} \tag{3.5}$$

and we will mostly consider two-dimensional simple representations $L(\lambda)$ for $\lambda\in\mathsf{P}'$. Also, $\mathsf{P}^\pm=\{\lambda\in\mathsf{P}\mid\langle h_1+h_2,\lambda\rangle\gtrless0\}$ will be the set of positive/negative weights and $\mathsf{P}'=\mathsf{P}^+\sqcup\mathsf{P}^-$.

Remark 3.2. Note that in analogy with the classical Lie algebra situation, we can set $\alpha^\vee = h_1 + h_2$. Then e, f and $\alpha^\vee$ generate the Lie superalgebra $\mathfrak{sl}(1|1)$ inside $\mathfrak{gl}(1|1)$. We work with $\mathfrak{gl}(1|1)$ and not with $\mathfrak{sl}(1|1)$ since the latter is not reductive, but nilpotent.

3.2. Decomposition of tensor products

The following lemma is the first step to decompose a tensor product of U_q-representations.

Lemma 3.3. *Let $\lambda, \mu \in \mathsf{P}'$, and suppose also $\lambda+\mu \in \mathsf{P}'$. Then*

$$L(\lambda) \otimes L(\mu) \cong L(\lambda+\mu) \oplus L(\lambda+\mu-\alpha). \tag{3.6}$$

Proof. Under our assumptions, the vectors

$$E(v_1^\lambda \otimes v_1^\mu) = v_0^\lambda \otimes q^{-\mu} v_1^\mu + (-1)^{|\lambda|+1} v_1^\lambda \otimes v_0^\mu, \tag{3.7}$$

$$F(v_0^\lambda \otimes v_0^\mu) = [\lambda] v_1^\lambda \otimes v_0^\mu + (-1)^{|\lambda|} q^\lambda v_0^\lambda \otimes [\mu] v_1^\mu \tag{3.8}$$

are linearly independent. One can easily verify that $v_1^\lambda \otimes v_1^\mu$ and $E(v_1^\lambda \otimes v_1^\mu)$ span a module isomorphic to $L(\lambda+\mu-\alpha)$, while $v_0^\lambda \otimes v_0^\mu$ and $F(v_0^\lambda \otimes v_0^\mu)$ span a module isomorphic to $L(\lambda+\mu)$. □

On the other hand, we have the following result.

Lemma 3.4. *Let $\lambda, \mu \in \mathsf{P}'$ and suppose that $\lambda+\mu \in \mathrm{Ann}(h_1+h_2)$. Then the representation $M = L(\lambda) \otimes L(\mu)$ is indecomposable and has a filtration*

$$0 = M_0 \subset M_1 \subset M_2 \subset M \tag{3.9}$$

with successive quotients

$$M_1 \cong \mathbb{C}(q)_\nu, \quad M_2/M_1 \cong \mathbb{C}(q)_{\nu-\alpha} \oplus \mathbb{C}(q)_{\nu+\alpha} \quad \text{and} \quad M/M_2 \cong \mathbb{C}(q)_\nu, \tag{3.10}$$

where $\nu = \lambda+\mu-\alpha$.

Moreover, $L(\lambda') \otimes L(\mu') \cong L(\lambda) \otimes L(\mu)$ for any $\lambda', \mu' \in \mathsf{P}'$ such that $\lambda'+\mu' = \lambda+\mu$.

Proof. Since $\lambda+\mu \in \mathrm{Ann}(h_1+h_2)$ we have $q^\lambda = q^{-\mu}$ and $[\lambda] = -[\mu]$. Using (3.7) and (3.8) we get that

$$F(v_0^\lambda \otimes v_0^\mu) = (-1)^{|\lambda|+1} [\lambda] E(v_1^\lambda \otimes v_1^\mu). \tag{3.11}$$

In particular, since $E^2=F^2=0$, the vector $F(v_0^\lambda\otimes v_0^\mu)$ generates a one-dimensional submodule $M_1\cong\mathbb{C}(q)_{\lambda+\mu-\alpha}$ of M. It then follows that the images of $v_0^\lambda\otimes v_0^\mu$ and $v_1^\lambda\otimes v_1^\mu$ in M/M_1 generate two one-dimensional submodules isomorphic to $\mathbb{C}(q)_{\lambda+\mu}$ and $\mathbb{C}(q)_{\lambda+\mu-2\alpha}$ respectively. Let therefore M_2 be the submodule of M generated by $v_0^\lambda\otimes v_0^\mu$ and $v_1^\lambda\otimes v_1^\mu$. Then M/M_2 is a one-dimensional representation isomorphic to $\mathbb{C}(q)_\nu$.

The last assertion follows easily since both $L(\lambda)\otimes L(\mu)$ and $L(\lambda')\otimes L(\mu')$ are isomorphic, as left U_q-modules, to U_q/I, where I is the left ideal generated by the elements $\mathbf{q}^h-q^{\langle h,\nu\rangle}$ for $h\in\mathsf{P}$. □

3.3. The dual of a representation

Let us now consider the dual $L(\lambda)^*$ of the representation $L(\lambda)$, and let $\{(v_0^\lambda)^*, (v_1^\lambda)^*\}$ be the basis dual to the standard basis $\{v_0^\lambda, v_1^\lambda\}$, with $|(v_0^\lambda)^*|=|v_0|=|\lambda|$ and $|(v_1^\lambda)^*|=|v_1|=|\lambda|+1$. By explicit computation, the action of U_q on $L(\lambda)^*$ is given by

$$\begin{aligned} &E(v_0^\lambda)^*=-(-1)^{|\lambda|}q^\lambda(v_1^\lambda)^*, && E(v_1^\lambda)^*=0,\\ &F(v_0^\lambda)^*=0, && F(v_1^\lambda)^*=(-1)^{|\lambda|}[\lambda]q^{-\lambda}(v_0^\lambda)^*,\\ &\mathbf{q}^h(v_0^\lambda)^*=q^{-\langle h,\lambda\rangle}(v_0^\lambda)^*, && \mathbf{q}^h(v_1^\lambda)^*=q^{-\langle h,\lambda-\alpha\rangle}(v_1^\lambda)^*. \end{aligned} \tag{3.12}$$

The assignment

$$\begin{aligned} L(\alpha-\lambda)&\longrightarrow L(\lambda)^*,\\ v_0^{\alpha-\lambda}&\longmapsto -(-1)^{|\lambda|}q^\lambda(v_1^\lambda)^*,\\ v_1^{\alpha-\lambda}&\longmapsto (v_0^\lambda)^*, \end{aligned} \tag{3.13}$$

defines a $\mathbb{Q}(q)$-linear map, which is in fact an isomorphism of U_q-modules

$$L(\lambda)^*\cong L(\alpha-\lambda). \tag{3.14}$$

Remark 3.5. Together with Lemma 3.4 it follows that $L(\lambda)\otimes L(\lambda)^*$ is an indecomposable representation. In the filtration (3.9), the submodule M_1 is the image of the coevaluation map $\mathbb{C}(q)\to L(\lambda)\otimes L(\lambda)^*$ while the submodule M_2 is the kernel of the evaluation map $L(\lambda)\otimes L(\lambda)^*\to\mathbb{C}(q)$, see (4.1) and (4.2) below.

Remark 3.6. At this point it is probably useful to recall that the natural isomorphism $V\cong V^{**}$ for a super vector space is given by $x\mapsto(\varphi\mapsto(-1)^{|x|\,|\varphi|}\varphi(x))$.

3.4. The vector representation

The vector representation of U_q is isomorphic to $L(\varepsilon_1)$ with its standard basis $\{v_0^{\varepsilon_1}, v_1^{\varepsilon_1}\}$, and grading given by $|v_0^{\varepsilon_1}|=0$ and $|v_1^{\varepsilon_1}|=1$, and the action of U_q is given by

$$\begin{aligned} &Ev_0^{\varepsilon_1}=0, \quad Fv_0^{\varepsilon_1}=v_1^{\varepsilon_1}, \quad \mathbf{q}^h v_0^{\varepsilon_1}=q^{\langle h,\varepsilon_1\rangle}v_0^{\varepsilon_1}, \quad Kv_0^{\varepsilon_1}=qv_0^{\varepsilon_1}, \\ &Ev_1^{\varepsilon_1}=v_0^{\varepsilon_1}, \quad Fv_1^{\varepsilon_1}=0, \quad \mathbf{q}^h v_1^{\varepsilon_1}=q^{\langle h,\varepsilon_2\rangle}v_1^{\varepsilon_1}, \quad Kv_1^{\varepsilon_1}=qv_1^{\varepsilon_1}. \end{aligned} \tag{3.15}$$

For $L(\varepsilon_1)^{\otimes n}$ we obtain directly from Lemma 3.3 the following decomposition.

Proposition 3.7. ([3, Theorem 6.4]) *The tensor powers of $L(\varepsilon_1)$ decompose as*

$$L(\varepsilon_1)^{\otimes m} \cong \bigoplus_{l=0}^{m-1} \binom{m-1}{l} L(m\varepsilon_1 - l\alpha). \tag{3.16}$$

Let us now consider mixed tensor products, involving also the dual $L(\varepsilon_1)^*$. By (3.14) we have that $L(\varepsilon_1)^*$ is isomorphic to $L(-\varepsilon_2)$. The following generalizes Proposition 3.7.

Theorem 3.8. *Suppose $m\neq n$. Then we have the decomposition*

$$L(\varepsilon_1)^{\otimes m}\otimes L(\varepsilon_1)^{*\otimes n} \cong \bigoplus_{l=0}^{m+n-1} \binom{m+n-1}{l} L(m\varepsilon_1 - n\varepsilon_2 - l\alpha). \tag{3.17}$$

On the other hand, we have

$$L(\varepsilon_1)^{\otimes n}\otimes L(\varepsilon_1)^{*\otimes n} \cong \bigoplus_{l=0}^{2n-2} \binom{2n-2}{l} L((n-l)\alpha+\varepsilon_2)\otimes L(-\varepsilon_2) \tag{3.18}$$

and each summand $L((n-l)\alpha+\varepsilon_2)\otimes L(-\varepsilon_2)$ is indecomposable but not irreducible.

Proof. The decomposition (3.17) follows from Lemma 3.3 by induction. To obtain (3.18) write

$$L(\varepsilon_1)^{\otimes n}\otimes L(\varepsilon_1)^{*\otimes n} = (L(\varepsilon_1)^{\otimes n}\otimes L(\varepsilon_1)^{*\otimes n-1})\otimes L(\varepsilon_1)^*$$

and use (3.17) together with Lemma 3.4. □

In particular, notice that $L(\varepsilon_1)^{\otimes m}\otimes L(\varepsilon_1)^{*\otimes n}$ is semisimple as long as $m\neq n$.

4. Invariants of links

In this section we define the ribbon structure on the category of representations of U_q and derive the corresponding invariants of oriented framed tangles and links.

Recall that if W is an n-dimensional complex super vector space the *evaluation maps* are defined by

$$\begin{aligned} &\mathrm{ev}_W : W^*\otimes W \longrightarrow \mathbb{C}(q), \quad \widehat{\mathrm{ev}}_W : W\otimes W^* \longrightarrow \mathbb{C}(q), \\ &\varphi\otimes w \longmapsto \varphi(w), \qquad\qquad w\otimes\varphi \longmapsto (-1)^{|\varphi|\,|w|}\varphi(w), \end{aligned} \tag{4.1}$$

and the *coevaluation maps* are defined by

$$\begin{aligned} &\mathrm{coev}_W : \mathbb{C}(q) \longrightarrow W\otimes W^*, \qquad \widehat{\mathrm{coev}}_W : \mathbb{C}(q) \longrightarrow W^*\otimes W, \\ &1 \longmapsto \sum_{i=1}^{n} w_i\otimes w_i^*, \qquad\qquad 1 \longmapsto \sum_{i=1}^{n} (-1)^{|w_i|} w_i^*\otimes w_i, \end{aligned} \tag{4.2}$$

where $\{w_i\}_{i=1}^n$ is a basis of W and $\{w_i^*\}_{i=1}^n$ is the corresponding dual basis of W^*. Note that if $\sigma_{V,W}$ denotes the map

$$\begin{aligned} &\sigma_{V,W} : V\otimes W \longrightarrow W\otimes V, \\ &v\otimes w \longmapsto (-1)^{|v|\,|w|} w\otimes v, \end{aligned} \tag{4.3}$$

then $\widehat{\mathrm{ev}}_W = \mathrm{ev}_W\circ\sigma_{W^*,W}$ and $\widehat{\mathrm{coev}}_W = \sigma_{W,W^*}\circ\mathrm{coev}_W$.

4.1. Ribbon structure on U_q-representations

Following the arguments of Tanisaki [21] (see also [5, Section 10.1.D]), we can construct a ribbon structure on the category of U_q-representations using the ribbon superalgebra structure on $U_\hbar$. We now indicate the main steps of those arguments.

The key observation is that, although Υ does not make sense as an element of $U_q\otimes U_q$, it acts on every tensor product $V\otimes W$ of two finite-dimensional U_q-modules. In other words, there is a well-defined operator $\Upsilon_{V,W}\in\mathrm{End}_{\mathbb{C}(q)}(V\otimes W)$ determined by setting $\Upsilon_{V,W}(v_\lambda\otimes w_\mu) = q^{(\lambda,\mu)}(v_\lambda\otimes w_\mu)$ if v_λ and w_μ have weights λ and μ respectively. Note however that $\Upsilon_{V,W}$ is not U_q-equivariant, since Υ satisfies $\Upsilon\Delta(x) = \bar{\Delta}^{\mathrm{op}}(x)\Upsilon$ (see Lemma A.1).

On the other hand, notice that the definition (2.16) of Θ makes sense also in U_q, and (A.1) holds in U_q. Moreover, one has the following counterpart of equations (A.3) and (A.4):

$$(\Delta\otimes\mathrm{id})(\Theta) = \Theta_{13}(\Upsilon_{V,Z})_{13}\Theta_{23}(\Upsilon_{V,Z}^{-1})_{13}, \tag{4.4}$$

$$(\mathrm{id}\otimes\Delta)(\Theta) = \Theta_{13}(\Upsilon_{V,Z})_{13}\Theta_{12}(\Upsilon_{V,Z}^{-1})_{13}. \tag{4.5}$$

This is now an equality of linear endomorphisms of $V\otimes W\otimes Z$ for all finite-dimensional U_q-representations V, W and Z. Setting

$$R_{V,W}=\Theta\Upsilon_{V,W}\in\mathrm{End}_{\mathbb{C}(q)}(V\otimes W) \tag{4.6}$$

one gets an operator that satisfies the Yang–Baxter equation. Note that $R_{V,W}$ is invertible, since Θ and $\Upsilon_{V,W}$ both are. Because of (2.18), if we define $\check{R}_{V,W}=\sigma\circ R_{V,W}$, where $\sigma\colon V\otimes W\to W\otimes V$ is defined by $\sigma(v\otimes w)=(-1)^{|v|\,|w|}w\otimes v$, then we get an U_q-equivariant isomorphism $\check{R}_{V,W}\in\mathrm{Hom}_{U_q}(V\otimes W,W\otimes V)$.

Analogously, although the elements u and v do not make sense in U_q, they act on each finite-dimensional U_q-representation V as operators $u_V,v_V\in\mathrm{End}_{U_q}(V)$ (they are U_q-equivariant because u and v are central in $U_\hbar$). Below, we will forget the subscripts of the operators $\check{R}$, u and v.

For convenience, we give explicit formulas for the (inverse of the) operator $\check{R}_{L(\lambda),L(\mu)}$ for $\lambda,\mu\in\mathsf{P}'$,

$$\begin{aligned}
\check{R}^{-1}(v_1^\lambda\otimes v_1^\mu)&=(-1)^{(|\lambda|+1)(|\mu|+1)}q^{-(\mu-\alpha,\lambda-\alpha)}v_1^\mu\otimes v_1^\lambda,\\
\check{R}^{-1}(v_1^\lambda\otimes v_0^\mu)&=(-1)^{(|\lambda|+1)|\mu|}\\
&\qquad\times(q^{-(\mu,\lambda-\alpha)}v_0^\mu\otimes v_1^\lambda+(-1)^{|\mu|}q^{-(\mu-\alpha,\lambda)}(q^{-1}-q)[\mu]v_1^\mu\otimes v_0^\lambda),\\
\check{R}^{-1}(v_0^\lambda\otimes v_1^\mu)&=(-1)^{|\lambda|(|\mu|+1)}q^{-(\mu-\alpha,\lambda)}v_1^\mu\otimes v_0^\lambda,\\
\check{R}^{-1}(v_0^\lambda\otimes v_0^\mu)&=(-1)^{|\lambda|\,|\mu|}q^{-(\mu,\lambda)}v_0^\mu\otimes v_0^\lambda.
\end{aligned} \tag{4.7}$$

4.2. Invariants of tangles

Let D be an oriented framed tangle diagram. We will not draw the framing because we will always suppose that it is the *blackboard framing*. (Recall that a framing is a trivialization of the normal bundle: since the tangle is oriented, such a trivialization is uniquely determined by a section of the normal bundle; the blackboard framing is the trivialization determined by the unit vector orthogonal to the plane—or to the blackboard—pointing outwards.)

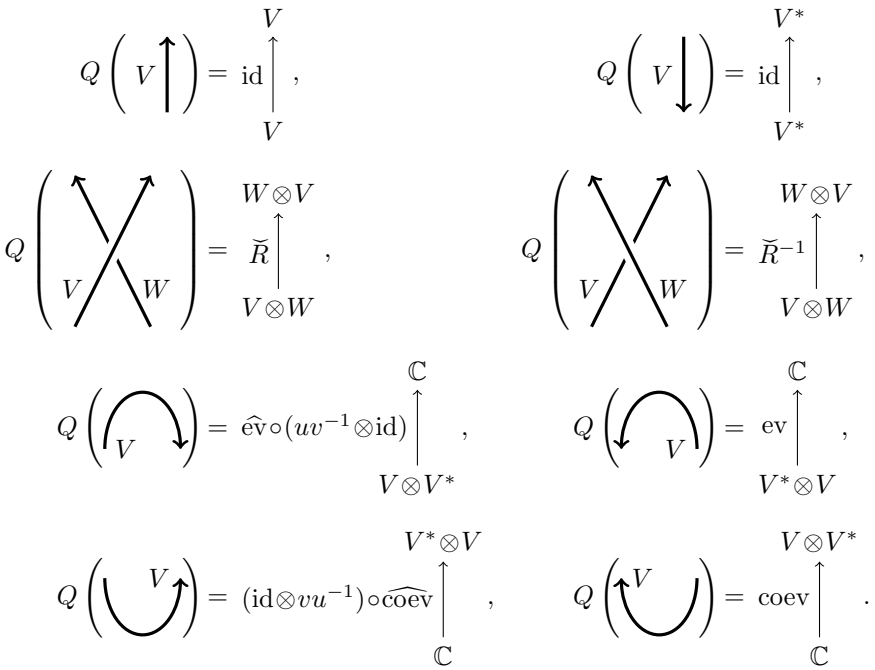
We assume that $D\subset\mathbb{R}\times[0,1]$ and we let $\boldsymbol{s}(D)=D\cap(\mathbb{R}\times\{0\})=\{s_1^D,\dots,s_a^D\}$ with $s_1^D<\dots<s_a^D$ be the source points of D and $\boldsymbol{t}(D)=D\cap(\mathbb{R}\times\{1\})=\{t_1^D,\dots,t_b^D\}$ with $t_1^D<\dots<t_b^D$ be the target points of D. Let also $\boldsymbol{\ell}$ be a labeling of the strands of D by simple two-dimensional representations of U_q (that is, a map from the set of strands of D to P'). We indicate by $\ell_1^s,\dots,\ell_a^s$ the labeling of the strands at the source points of D and by $\ell_1^t,\dots,\ell_b^t$ the labeling at the target points. Moreover, we let $\gamma_1^s,\dots,\gamma_a^s$ and $\gamma_1^t,\dots,\gamma_b^t$ be the signs corresponding to the orientations of the

strands at the source and target points (where $+1$ corresponds to a strand oriented upwards and -1 to a strand oriented downwards).

Given these data, one can define a U_q-equivariant map

$$Q^{\boldsymbol{\ell}}(D)\colon L(\ell_1^s)^{\gamma_1^s}\otimes\ldots\otimes L(\ell_a^s)^{\gamma_a^s}\longrightarrow L(\ell_1^t)^{\gamma_1^t}\otimes\ldots\otimes L(\ell_b^t)^{\gamma_b^t}, \tag{4.8}$$

where $L(\lambda)^{-1}=L(\lambda)^*$, by decomposing D into elementary pieces as shown below and assigning the corresponding morphisms as follows:



As already mentioned, although U_q itself is not a ribbon superalgebra, its representation category is a ribbon category. As in [18, Theorem 4.7] one can prove the following result.

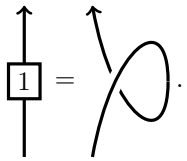
Theorem 4.1. *The map $Q^{\boldsymbol{\ell}}(D)$ just defined is an isotopy invariant of oriented framed tangles.*

The proof, for which we refer to [18, Theorem 4.7], is a direct check of the Reidemeister moves (or, more precisely, of the analogues of the Reidemeister moves for framed tangles). In fact, the axioms of a ribbon category are equivalent to the validity of these moves.

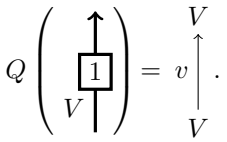
If all strands are labeled by the same simple representation $L(\lambda)$ (i.e. $\boldsymbol{\ell}$ is the constant map with value λ), then we write $Q^{\lambda}(D)$ instead of $Q^{\boldsymbol{\ell}}(D)$.

Let us indicate a full $+1$ twist by the symbol

(4.9)



Then we have (cf. [18, Section 4.2])



Lemma 4.2. *The element v acts by the identity on the vector representation $L(\varepsilon_1)$ and on its dual $L(\varepsilon_1)^*$.*

Proof. Recall that we denote by $\{v_0^{\varepsilon_1}, v_1^{\varepsilon_1}\}$ the standard basis of $L(\varepsilon_1)$. We have

$$\begin{aligned} vv_0^{\varepsilon_1} &= (K^{-1}+(q-q^{-1})EF)\mathbf{q}^{-(h_1+h_2)(h_1-h_2)}v_0^{\varepsilon_1} \\ &= (K^{-1}+(q-q^{-1})EF)q^{-\langle h_1+h_2,\varepsilon_1\rangle\langle h_1-h_2,\varepsilon_1\rangle}v_0^{\varepsilon_1} \\ &= (q^{-1}+q-q^{-1})q^{-1}v_0^{\varepsilon_1} \\ &= v_0^{\varepsilon_1}. \end{aligned} \tag{4.10}$$

As $L(\varepsilon_1)$ is irreducible and v acts in an U_q-equivariant way, it follows that v acts by the identity on $L(\varepsilon_1)$. Since $S(v)=v$, the element v acts by the identity also on $L(\varepsilon_1)^*$. □

As a consequence, if we label all strands of our tangles by the vector representation then we need not worry about the framing any more.

Corollary 4.3. *The assignment $D\mapsto Q^{\varepsilon_1}(D)$ is an invariant of oriented tangles.*

4.3. Invariants of links

Since links are in particular tangles, we obtain from $Q^{\boldsymbol{\ell}}$ an invariant of oriented framed links; unfortunately, this invariant is always zero.

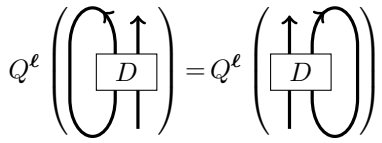
Proposition 4.4. *Let L be a closed link diagram and $\boldsymbol{\ell}$ be a labeling of its strands. Then $Q^{\boldsymbol{\ell}}(L)=0$.*

Proof. The invariant associated with L is some endomorphism φ of the trivial representation $\mathbb{C}(q)$. Up to isotopy, we can assume that there is some level at which the link diagram L has only two strands, one oriented upwards and the other one downwards, labeled by the same weight λ. Without loss of generality suppose that the leftmost is oriented upwards. Slice the diagram at this level, so that we can write φ as the composition $\varphi_2\circ\varphi_1$ of two U_q-equivariant maps $\varphi_1\colon \mathbb{C}(q)\to L(\lambda)\otimes L(\lambda)^*$ and $\varphi_2\colon L(\lambda)\otimes L(\lambda)^*\to\mathbb{C}(q)$. If $\varphi=\varphi_2\circ\varphi_1$ is not zero, then we have an inclusion φ_1 of $\mathbb{C}(q)$ inside $L(\lambda)\otimes L(\lambda)^*$ and a projection φ_2 of the latter onto $\mathbb{C}(q)$, so that $\mathbb{C}(q)$ would be a direct summand of $L(\lambda)\otimes L(\lambda)^*$. But this is not possible, since $L(\lambda)\otimes L(\lambda)^*$ is indecomposable (by Lemma 3.4). Hence $\varphi=0$. □

To get nontrivial invariants of closed links we need to cut the links, as we are going to explain now. First, we need the following result.

Proposition 4.5. *Let D be an oriented tangle diagram with two source points and two target points. Let $\boldsymbol{\ell}$ be a labeling of the strands of D such that $\ell_1^s=\ell_2^s=\ell_1^t=\ell_2^t$. Then*

(4.11)

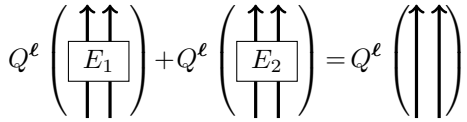


Proof. Let $\ell_1^s=\lambda$. Then $Q^{\boldsymbol{\ell}}(D)=\varphi$, where $\varphi\colon L(\lambda)\otimes L(\lambda)\to L(\lambda)\otimes L(\lambda)$. By Lemma 3.3 the representation $L(\lambda)\otimes L(\lambda)$ is isomorphic to the direct sum $L(2\lambda)\oplus L(2\lambda-\alpha)$. Let e_1 and e_2 be the two orthogonal idempotents corresponding to this decomposition.

We consider formal $\mathbb{C}(q)$-linear combinations of tangle diagrams, and we extend $Q^{\boldsymbol{\ell}}$ to them. Since $\mathrm{End}_{U_q}(L(\lambda)\otimes L(\lambda))$ is a two-dimensional $\mathbb{C}(q)$-vector space and $\widetilde{R}_{\lambda,\lambda}$ is not a multiple of the identity by (4.7), there are some $\mathbb{C}(q)$-linear combinations of tangle diagrams E_1 and E_2 such that $Q^{\boldsymbol{\ell}}(E_1)=e_1$ and $Q^{\boldsymbol{\ell}}(E_2)=e_2$. Hence

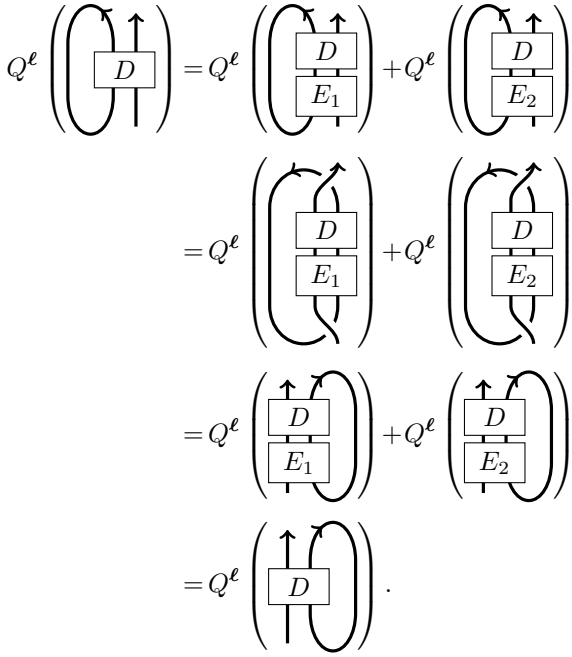
we can write

(4.12)



Now we have

(4.13)



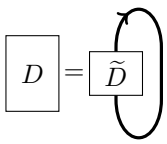
Here the second equality follows because we must have

$$\breve{R}e_1 = e_1\breve{R} = a_1e_1 \quad \text{and} \quad \breve{R}e_2 = e_2\breve{R} = a_2e_2 \tag{4.14}$$

for some $a_1, a_2 \in \mathbb{C}(q)$, since e_1 and e_2 project onto one-dimensional subspaces of $\mathrm{End}_{U_q}(L(\lambda)\otimes L(\lambda))$. The penultimate equality follows by isotopy invariance. □

Let now D be an oriented framed link diagram, $\boldsymbol{\ell}$ be a labeling of its strands and $\lambda \in \mathsf{P}'$ be some weight, which labels some strand of D. By cutting one of the strands labeled by λ, we can suppose that D is the closure of a tangle $\widetilde{D}$ with one source and one target point, as in the picture

(4.15)



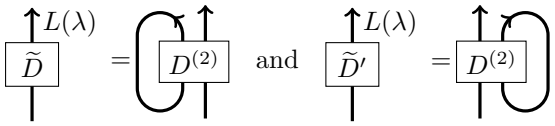
Then we define $\widehat{Q}^{\boldsymbol{\ell},\lambda}(D)=c\in\mathbb{C}(q)$, where

$$Q^{\boldsymbol{\ell}}\left(\begin{array}{c}\uparrow L(\lambda)\\ \boxed{\widetilde{D}}\\ |\end{array}\right)=c\,\mathrm{id}_{L(\lambda)}\,. \tag{4.16}$$

We have the following result.

Theorem 4.6. *The assignment $D\mapsto\widehat{Q}^{\boldsymbol{\ell},\lambda}(D)\in\mathbb{C}(q)$ is an invariant of oriented framed links.*

Proof. Since $Q^{\boldsymbol{\ell}}(\widetilde{D})$ is an invariant of oriented framed tangles, we need only show that $\widehat{Q}^{\boldsymbol{\ell},\lambda}$ is independent of how we cut D to get $\widetilde{D}$. If $\widetilde{D}'$ is obtained by some different cutting, but always along some strand labeled by λ, then after some isotopy we must have

(4.17) 

for some tangle $D^{(2)}$. By Proposition 4.5 we then have $Q^{\boldsymbol{\ell}}(\widetilde{D})=Q^{\boldsymbol{\ell}}(\widetilde{D}')$. □

If $\boldsymbol{\ell}$ is the constant labeling by the weight λ, we write $\widehat{Q}^{\lambda}$ instead of $\widehat{Q}^{\boldsymbol{\ell},\lambda}$. For $\lambda=\varepsilon_1$ we simply write $\widehat{Q}$. As a consequence of Corollary 4.3 and Theorem 4.6 we obtain the following result.

Corollary 4.7. *The assignment $D\mapsto\widehat{Q}(D)\in\mathbb{C}(q)$ is an invariant of oriented links.*

4.4. Recovering the Alexander polynomial

If we compute the action of the R-matrix on $L(\varepsilon_1)\otimes L(\varepsilon_1)$, we get by (4.7), setting $v_1=v_1^{\varepsilon_1}$ and $v_0=v_0^{\varepsilon_1}$,

$$\begin{aligned}&\breve{R}^{-1}(v_1\otimes v_1)=-qv_1\otimes v_1,\quad &&\breve{R}^{-1}(v_1\otimes v_0)=v_0\otimes v_1+(q^{-1}-q)v_1\otimes v_0,\\ &\breve{R}^{-1}(v_0\otimes v_1)=v_1\otimes v_0,\quad &&\breve{R}^{-1}(v_0\otimes v_0)=q^{-1}v_0\otimes v_0.\end{aligned} \tag{4.18}$$

On can easily check that

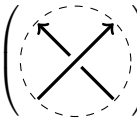
$$(\breve{R}^{-1})^2=(q^{-1}-q)\breve{R}^{-1}+\mathrm{id} \tag{4.19}$$

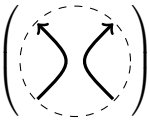
and hence

$$\check{R} = \check{R}^{-1} + (q - q^{-1})\mathrm{id}. \tag{4.20}$$

Proposition 4.8. *The invariant of links $\widehat{Q}$ satisfies the following skein relation*

$$\widehat{Q}\left(\right) - \widehat{Q}\left(\right) = (q - q^{-1})\widehat{Q}\left(\right), \tag{4.21}$$





where the pictures represent three links that differ only inside a small neighborhood of a crossing.

We recall one of the equivalent definitions of the Alexander–Conway polynomial [1], [6].

Definition 4.9. The *Alexander–Conway polynomial* is the value of the assignment

$$\Delta\colon \mathrm{Links} \longrightarrow \mathbb{Z}[t^{1/2}, t^{-1/2}] \tag{4.22}$$

defined by the skein relations

$$\Delta\left(\bigcirc\right) = 1, \tag{4.23}$$

$$\Delta\left(\right) - \Delta\left(\right) = (t^{1/2} - t^{-1/2})\Delta\left(\right). \tag{4.24}$$


Notice that obviously $\widehat{Q}(\bigcirc) = 1$, since $Q^{\varepsilon_1}(\uparrow) = \mathrm{id}$. As a consequence, we have that Q is essentially the Alexander–Conway polynomial.

Theorem 4.10. *For all oriented links L in $\mathbb{R}^3$ we have*

$$\Delta(L) = \widehat{Q}(L)|_{q=t^{1/2}}. \tag{4.25}$$

In particular, $\widehat{Q}(L) \in \mathbb{Z}[q, q^{-1}]$ is a Laurent polynomial in q.

Appendix A

We collect here two technical lemmas, which were used in Section 2 to construct the ribbon structure on $U_\hbar$. Both results follow from the explicit construction of the R-matrix (cf. [15]); we include the two proofs for completeness, although they are easy calculations.

Lemma A.1. *The following properties hold for all* $x \in U_\hbar$,

$$\Theta \bar{\Delta}^{\mathrm{op}}(x) = \Delta^{\mathrm{op}}(x)\Theta, \tag{A.1}$$

$$\Upsilon \Delta(x) = \bar{\Delta}^{\mathrm{op}}(x)\Upsilon. \tag{A.2}$$

Proof. It is enough to check (A.1) and (A.2) on the generators. We have

$$\begin{aligned}
\Theta\bar{\Delta}^{\mathrm{op}}(E) &= \Theta(K\otimes E + E\otimes 1)\\
&= K\otimes E + E\otimes 1 + (q-q^{-1})FK\otimes E^2 - (q-q^{-1})FE\otimes E\\
&= K\otimes E + E\otimes 1 + (q-q^{-1})EF\otimes E - (K-K^{-1})\otimes E\\
&= K^{-1}\otimes E + E\otimes 1 + (q-q^{-1})EF\otimes E\\
&= (K^{-1}\otimes E + E\otimes 1)\Theta\\
&= \Delta^{\mathrm{op}}(E)\Theta
\end{aligned}$$

and

$$\begin{aligned}
\Theta\bar{\Delta}^{\mathrm{op}}(F) &= \Theta(1\otimes F + F\otimes K^{-1})\\
&= 1\otimes F + F\otimes K^{-1} + (q-q^{-1})F\otimes EF - (q-q^{-1})F^2\otimes EK^{-1}\\
&= 1\otimes F + F\otimes K^{-1} - (q-q^{-1})F\otimes FE + F\otimes (K-K^{-1})\\
&= 1\otimes F + F\otimes K - (q-q^{-1})F\otimes FE\\
&= (1\otimes F + F\otimes K)\Theta\\
&= \Delta^{\mathrm{op}}(F)\Theta
\end{aligned}$$

and for $i=1,2$,

$$\begin{aligned}
\Theta\bar{\Delta}^{\mathrm{op}}(H_i) &= \Theta(1\otimes H_i + H_i\otimes 1)\\
&= 1\otimes H_i + H_i\otimes 1 + (q-q^{-1})F\otimes EH_i + (q-q^{-1})FH_i\otimes E\\
&= 1\otimes H_i + H_i\otimes 1 - (q-q^{-1})\langle H_i,\alpha\rangle F\otimes E + (q-q^{-1})F\otimes H_iE\\
&\quad + (q-q^{-1})\langle H_i,\alpha\rangle F\otimes E + (q-q^{-1})H_iF\otimes E
\end{aligned}$$

$$
\begin{aligned}
&= 1\otimes H_i + H_i\otimes 1 + (q-q^{-1})F\otimes H_iE + (q-q^{-1})H_iF\otimes E\\
&= (1\otimes H_i + H_i\otimes 1)\Theta\\
&= \Delta^{\mathrm{op}}(H_i)\Theta.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\Upsilon\Delta(E) &= e^{\hbar(H_1\otimes H_1 - H_2\otimes H_2)}(E\otimes K^{-1} + 1\otimes E)\\
&= (E\otimes K^{-1})e^{\hbar((H_1+1)\otimes H_1 - (H_2-1)\otimes H_2)} + (1\otimes E)e^{\hbar(H_1\otimes(H_1+1) - H_2\otimes(H_2-1))}\\
&= (E\otimes 1 + K\otimes E)e^{\hbar(H_1\otimes H_1 - H_2\otimes H_2)}\\
&= \bar{\Delta}^{\mathrm{op}}(E)\Upsilon
\end{aligned}
$$

and

$$
\begin{aligned}
\Upsilon\Delta(F) &= e^{\hbar(H_1\otimes H_1 - H_2\otimes H_2)}(F\otimes 1 + K\otimes F)\\
&= (F\otimes 1)e^{\hbar((H_1-1)\otimes H_1 - (H_2+1)\otimes H_2)} + (K\otimes F)e^{\hbar(H_1\otimes(H_1-1) - H_2\otimes(H_2+1))}\\
&= (F\otimes K^{-1} + 1\otimes F)e^{\hbar(H_1\otimes H_1 - H_2\otimes H_2)}\\
&= \bar{\Delta}^{\mathrm{op}}(F)\Upsilon.
\end{aligned}
$$

Finally, for $i=1,2$ we have $\Upsilon\Delta(H_i)=\Delta(H_i)\Upsilon$ since the elements H_1 and H_2 commute with each other. As $\bar{\Delta}^{\mathrm{op}}(H_i)=\Delta(H_i)$ we get $\Upsilon\Delta(H_i)=\bar{\Delta}^{\mathrm{op}}(H_i)\Upsilon$ and we are done. □

Lemma A.2. *In $U_\hbar$ the following identities hold*:

$$(\Delta\otimes\mathrm{id})(\Theta) = \Theta_{13}\Upsilon_{13}\Theta_{23}\Upsilon_{13}^{-1}, \tag{A.3}$$

$$(\mathrm{id}\otimes\Delta)(\Theta) = \Theta_{13}\Upsilon_{13}\Theta_{12}\Upsilon_{13}^{-1}. \tag{A.4}$$

Proof. The two computations are similar, so let us check (A.3) and leave (A.4) to the reader. The left-hand side is simply

$$(\Delta\otimes\mathrm{id})(\Theta) = 1 + (q-q)^{-1}F\otimes 1\otimes E + (q-q^{-1})K\otimes F\otimes E. \tag{A.5}$$

We will now compute the right-hand side. First we have

$$
\begin{aligned}
\Upsilon_{13}(1\otimes F\otimes E)\Upsilon_{13}^{-1} &= \Upsilon_{13}(1\otimes F\otimes E)e^{-\hbar(H_1\otimes 1\otimes H_1)}e^{\hbar(H_2\otimes 1\otimes H_2)}\\
&= \Upsilon_{13}e^{-\hbar(H_1\otimes 1\otimes(H_1-1))}(1\otimes F\otimes E)e^{\hbar(H_2\otimes 1\otimes H_2)}\\
&= \Upsilon_{13}e^{-\hbar(H_1\otimes 1\otimes(H_1-1))}e^{\hbar(H_2\otimes 1\otimes(H_2+1))}(1\otimes F\otimes E)\\
&= K\otimes F\otimes E.
\end{aligned} \tag{A.6}
$$

Therefore

$$\Theta_{13}\Upsilon_{13}\Theta_{23}\Upsilon_{13}^{-1} = (1+(q-q)^{-1}F\otimes 1\otimes E)(1+(q-q)^{-1}K\otimes F\otimes E) \tag{A.7}$$

coincides with (A.5) since $E^2=0$. □

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Received August 9, 2013
in revised form December 20, 2013
published online March 20, 2014