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On the order map for hypersurface coamoebas

Jens Forsgård and Petter Johansson

In memory of Mikael Passare, who continues to inspire.

Abstract. Given a hypersurface coamoeba of a Laurent polynomial f, it is an open problem to describe the structure of the set of connected components of its complement. In this paper we approach this problem by introducing the lopsided coamoeba. We show that the closed lopsided coamoeba comes naturally equipped with an order map, i.e. a map from the set of connected components of its complement to a translated lattice inside the zonotope of a Gale dual of the point configuration $\operatorname{supp}(f)$. Under a natural assumption, this map is a bijection. Finally we use this map to obtain new results concerning coamoebas of polynomials of small codimension.

1. Introduction

The amoeba $\mathcal{A}(f)$ of a Laurent polynomial

$$f(z)=\sum_{\alpha\in A} c_\alpha z^\alpha \in \mathbb{C}[z_1^{\pm1},\dots,z_n^{\pm1}] \tag{1}$$

is defined as the image of the zero locus $V(f)\subset(\mathbb{C}^*)^n$ under the componentwise logarithm mapping, i.e. $\mathcal{A}(f)=\operatorname{Log}(V(f))$, where $\operatorname{Log}\colon(\mathbb{C}^*)^n\to\mathbb{R}^n$ is given by $z\mapsto(\log|z_1|,\dots,\log|z_n|)$. An important step in the study of amoebas was taken in [3] with the introduction of the so-called *order map*. This is an injective map, here denoted by ord, from the set of connected components of the complement of the amoeba $\mathcal{A}(f)$, to the set of integer points in the Newton polytope $\Delta_f=\operatorname{Conv}(A)$. If E denotes a connected component of the amoeba complement $\mathcal{A}(f)^c$, then the jth component of $\operatorname{ord}(E)$ is given by the integral

$$\operatorname{ord}(E)_j=\frac{1}{(2\pi i)^n}\int_{\operatorname{Log}^{-1}(x)}\frac{z_j f_j'(z)}{f(z)}\frac{dz_1\dots dz_n}{z_1\dots z_n},\quad x\in E.$$

Evaluating $\operatorname{ord}(E)$ in the univariate case amounts to counting zeros of f by the argument principle, yielding an analogous interpretation of ord for multivariate polynomials. With this in mind, it is not hard to see that the vertex set $\operatorname{vert}(\Delta_f)$ is always contained in the image of ord, and furthermore it was shown in [17] that any subset of $\mathbb{Z}^n \cap \Delta_f$ that contains $\operatorname{vert}(\Delta_f)$ appears as the image of the order map for some polynomial with the given Newton polytope. Thus, even though the image of ord is non-trivial to determine, this map gives a good understanding of the structure of the set of connected components of the complement of the amoeba $\mathcal{A}(f)$. In particular, we have the sharp lower and upper bounds on the cardinality of this set given by $|\operatorname{vert}(\Delta_f)|$ and $|\mathbb{Z}^n \cap \Delta_f|$ respectively. See [9] and [15] for an overview of amoeba theory.

The coamoeba $\mathcal{A}'(f)$ of f is defined as the image of $V(f)$ under the componentwise argument mapping, i.e. $\mathcal{A}'(f) = \operatorname{Arg}(V(f))$, where $\operatorname{Arg}\colon (\mathbb{C}^*)^n \to \mathbf{T}^n$ is given by $\operatorname{Arg}(z) = (\arg(z_1), ..., \arg(z_n))$. It is sometimes useful to consider the multivalued Arg-mapping, which yields the coamoeba as a multiple periodic subset of $\mathbb{R}^n$. The starting point of this paper is the problem of describing the structure of the set of connected components of the complement of the closed coamoeba. The progress so far is restricted to the upper bound on the cardinality of this set given by the normalized volume $n!\operatorname{Vol}(\Delta_f)$, see [11]. However, there is no known analogy of the order map for amoebas. Our approach to this problem is to introduce the *lopsided coamoeba.* As the name is chosen to emphasize the analogy with amoebas, let us briefly recall the notion of *lopsided amoeba* as introduced in [16].

For a point $x \in \mathbb{R}^n$, consider the list of the moduli of the monomials of f at x,

$$f\{x\} = [e^{\log|c_{\alpha_1}| + \langle \alpha_1, x\rangle}, ..., e^{\log|c_{\alpha_N}| + \langle \alpha_N, x\rangle}],$$

where $N = |A|$. This list is said to be *lopsided* if one component is greater than the sum of the others. If $f\{x\}$ is lopsided, then $x \notin \mathcal{A}(f)$. The lopsided amoeba $\mathcal{LA}(f)$ is defined as the set of points $x \in \mathbb{R}^n$ such that $f\{x\}$ is *not* lopsided. There is an inclusion $\mathcal{A}(f) \subset \mathcal{LA}(f)$, and in particular each connected component of $\mathcal{LA}(f)^c$ is contained in a unique connected component of $\mathcal{A}(f)^c$. Let us consider the relation between the lopsided amoeba and the order map. If the list $f\{x\}$ is dominated in the sense of lopsidedness by the monomial with exponent α, then it follows by Rouché's theorem that $\operatorname{ord}(E) = \alpha$. Hence, while ord can map connected components of the complement of $\mathcal{A}(f)$ to elements in the set $(\mathbb{Z}^n \cap \Delta_f) \setminus A$, when restricted to the set of connected components of $\mathcal{LA}(f)^c$ it becomes an injective map into the point configuration A. In this sense, the structure of the set of connected components of the complement of the lopsided amoeba is better captured by A than by its Newton polytope Δ_f.

We always assume that a half-space $H\subset\mathbb{C}$ is open and contains the origin in its boundary, that is $H=H_\phi=\{z\in\mathbb{C}|\mathrm{Re}(e^{i\phi}z)>0\}$ for some $\phi\in\mathbb{R}$. For each point $\theta\in\mathbf{T}^n$, consider the list

$$f\langle\theta\rangle=[e^{i(\arg(c_{\alpha_1})+\langle\alpha_1,\theta\rangle)},\ldots,e^{i(\arg(c_{\alpha_N})+\langle\alpha_N,\theta\rangle)}],$$

which we by abuse of notation also view as a set $f\langle\theta\rangle\subset S^1\subset\mathbb{C}$. We say that the list $f\langle\theta\rangle$ is *lopsided* if there exist a half-space $H\subset\mathbb{C}$ such that, as a set, $f\langle\theta\rangle\subset\overline{H}$ but $f\langle\theta\rangle\not\subset\partial H$.

Definition 1.1. The lopsided coamoeba $\mathcal{LA}'(f)$ is the set of points $\theta\in\mathbf{T}^n$ such that $f\langle\theta\rangle$ is *not* lopsided.

When necessary we will consider $\mathcal{LA}'(f)$ as a subset of $\mathbb{R}^n$.

The main result of this paper is that we provide an order map for lopsided coamoebas. That is, we provide a map from the set of connected components of the complement of the closed lopsided coamoeba, to a translated lattice inside a certain zonotope, related to a *Gale dual* of A, see Theorem 4.3.

As noted above, the image of the order map of the (lopsided) amoeba depends in an intricate manner on the coefficients of the polynomial f. The order map, which we provide for the lopsided coamoeba will, under a natural assumption, be a bijection. That is, the dependency on the coefficients of f lies only in the translation of the lattice, and this dependency is explicitly given in Theorem 4.3. As a consequence, we are able to use this map to obtain new results concerning the geometry of coamoebas. In particular, we give an affirmative answer to a special case of a conjecture by Passare, see Corollary 5.3.

Let us give a brief outline of the paper. Section 2 contains fundamental results in coamoeba theory, most of which are previously known. In Section 3 we will turn to lopsided coamoebas, considering their fundamental properties and their relation to ordinary coamoebas. In Section 4 we provide the order map for the lopsided coamoeba. In the last section we consider coamoebas of polynomials of codimension one and two, using the results of the previous sections.

1.1. Notation

We will use $\mathrm{CC}(S)$ to denote the set of connected components of the complement of a set S, in its natural ambient space. That is, $\mathrm{CC}(\mathcal{A}(f))$ denotes the set of connected components of the complement of the amoeba, which always are subsets of $\mathbb{R}^n$, while $\mathrm{CC}(\mathcal{A}'(f))$ denotes the set of connected components of the complement of the coamoeba viewed on the real n-torus $\mathbf{T}^n$. The transpose of a matrix M is denoted by M^t. By g_M we denote the greatest common divisor of the maximal minors of M. We use e_j for the jth vector of the standard basis in any vector space, and $\langle \cdot, \cdot \rangle$ for the standard scalar product. I_m denotes the unit matrix of size $m \times m$. We use the convention that $X \subset Y$ includes the case $X = Y$.

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2. Preliminaries

As implicitly stated in the introduction, the coamoeba of a hypersurface is in general not closed. Let Γ be a (not necessarily proper) subface of Δ_f. The truncated polynomial with respect to Γ is defined as

$$f_\Gamma(z) = \sum_{\alpha \in A \cap \Gamma} c_\alpha z^\alpha.$$

It was shown in [7] and [12] that the closure of a coamoeba is the union of all the coamoebas of its truncated polynomials, that is

$$\overline{\mathcal{A}'}(f) = \bigcup_{\Gamma \subset \Delta_f} \mathcal{A}'(f_\Gamma). \tag{2}$$

We will refer to $\mathcal{A}'(f_\Gamma)$ as the coamoeba of the face Γ. If the above union is taken only over the proper subfaces Γ of Δ_f one obtains the *phase limit set* $\mathcal{P}^\infty(f)$ (see [12]), and similarly if the union is taken only over the edges of Δ_f one obtains the *shell* $\mathcal{H}(f)$ of $\mathcal{A}'(f)$ (see [6] and [11]). For the latter we note that the coamoeba of an edge $\Gamma \subset \Delta_f$ consists of a family of parallel hyperplanes, whose normal is in turn parallel to Γ. It is natural to focus on $\overline{\mathcal{A}'}(f)$ rather than $\mathcal{A}'(f)$, the main reason being that the components of the complement of $\overline{\mathcal{A}'}(f)$, when viewed in $\mathbb{R}^n$, are convex. To see this, we give the following argument due to Passare. If $\Theta \subset \mathbb{R}^n$ is a connected component of the complement of $\overline{\mathcal{A}'}(f)$, then the function $g(w) = 1/f(e^{iw})$ is holomorphic on the tubular domain $\Theta + i\mathbb{R}^n$. As it cannot be extended to a holomorphic function on any larger tubular domain, the convexity follows from Bochner's tube theorem [2].

By abuse of notation one identifies the index set A with the matrix

$$A = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \alpha_1 & \alpha_2 & \dots & \alpha_N \end{pmatrix}. \tag{3}$$

We will restrict the term *integer affine transformation* of A to refer to a matrix $T \in \mathrm{GL}_n(\mathbb{Q})$ such that

$$\begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix} A \in \mathbb{Z}^{(n+1)\times N}.$$

The transformation T induces a function $\mathbb{C}^A \to \mathbb{C}^{TA}$ by the monomial change of variables

$$z_j \longmapsto z^{T_j},$$

where T_j denotes the jth row of T. With the notation $e^{x+i\theta} = (e^{x_1+i\theta_1}, \dots, e^{x_n+i\theta_n})$ we find that

$$T(f_j)(e^{(x+i\theta)T^{-1}}) = \langle c_j, e^{(x+i\theta)T^{-1}TA_j} \rangle = \langle c_j, e^{(x+i\theta)A_j} \rangle = f_j(e^{x+i\theta}).$$

Thus, a point $\theta \in \mathbb{R}^n$ belongs to $\mathcal{A}'(f_j)$ if and only if $(T^{-1})^t\theta$ belongs to $\mathcal{A}'(T(f_j))$. We conclude the following relation previously described in [13].

Proposition 2.1. *As subsets of $\mathbb{R}^n$, we have that $\mathcal{A}'(T(f))$ is the image of $\mathcal{A}'(f)$ under the linear transformation $(T^{-1})^t$.*

Corollary 2.2. *As subsets of $\mathbf{T}^n$, the coamoeba $\mathcal{A}'(T(f))$ consists of $|\det(T)|$ linearly transformed copies of $\mathcal{A}'(f)$.*

Proof. The transformation $(T^{-1})^t$ acts with a scaling factor $1/|\det(T)|$ on $\mathbb{R}^n$. Now consider a fundamental domain. $\square$

Any point configuration A can be shrunk, by means of an integer affine transformation, to a point configuration whose maximal minors are relatively prime [5].

The polynomial f, and the point configuration A, is called *maximally sparse* if $A = \mathrm{vert}(\Delta_f)$. If in addition Δ_f is a simplex, then $V(f)$ is known as a *simple hypersurface*, and we will say that f is a *simple polynomial.* Let us describe the coamoeba of a simple hypersurface. Consider first when Δ_f is the standard 2-simplex. After a dilation of the variables, which corresponds to a translation of the coamoeba, we can assume that $f(z_1, z_2) = 1 + z_1 + z_2$. If the coamoebas of the

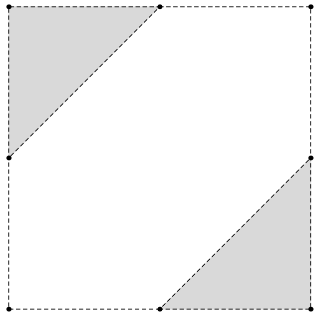

Figure 1. The coamoeba of $f(z_1, z_2)=1+z_1+z_2$ in the domain $[-\pi, \pi]^2$.

truncated polynomials of the edges of Δ_f are drawn, with orientations given by the outward normal vectors of Δ_f, then $\mathcal{A}'(f)$ consists of the interiors of the oriented regions, together with all intersection points. An arbitrary simple trinomial differs from the standard 2-simplex only by an integer affine transformation. Hence the coamoeba of any simple trinomial consists of a certain number of copies of $\mathcal{A}'(f)$, and is given by the same recipe as for the standard 2-simplex.

Consider now when Δ_f is the standard n-simplex, that is $f(z)=1+z_1+...+z_n$. Let $\mathrm{Tri}(f)$ denote the set of all trinomials one can construct from the set of monomials of f, which we still consider as polynomials in the n variables $z_1, ..., z_n$. It was shown in [6] that we have the identity

$$\overline{\mathcal{A}'}(f) = \bigcup_{g\in\mathrm{Tri}(f)} \overline{\mathcal{A}'}(g), \tag{4}$$

which also holds without taking closures if $n\neq 3$. Again, an arbitrary simple polynomial is only an integer affine transformation away, and hence the identity (4) holds for all simple hypersurfaces.

The complement of the closed coamoeba of $f(z)=1+z_1+...+z_n$, in the fundamental domain $[-\pi, \pi)^n$ in $\mathbb{R}^n$, consists of the convex hull of the open cubes $(0, \pi)^n$ and $(\pi, 0)^n$. In particular $\overline{\mathcal{A}'}(f)^c$ has exactly one connected component in $\mathbf{T}^n$. Thus, the number of connected components of $\overline{\mathcal{A}'}(f)^c$ equals the normalized volume $n!\,\mathrm{Vol}(\Delta_f)=1$ in this case. For each integer affine transformation T we have that $\mathrm{Vol}(\Delta_{T(f)})=|\det(T)|\,\mathrm{Vol}(\Delta_f)$. It follows that for any simple hypersurface, the number of connected components of the complement of its coamoeba will be equal to the normalized volume of its Newton polytope.

Let us end this section with a fundamental property of the shell $\mathcal{H}(f)$, which we have not seen a proof of elsewhere.

Lemma 2.3. *Let $l \subset \mathbb{R}^n$ be a line segment with endpoints in $\overline{\mathcal{A}'}(f)^c$ that intersect $\overline{\mathcal{A}'}(f)$. Then l intersect $\mathcal{A}'(f_\Gamma)$ for some edge $\Gamma \subset \Delta_f$. In particular, each cell of the hyperplane arrangement $\mathcal{H}(f)$ contains at most one connected component of $\overline{\mathcal{A}'}(f)^c$.*

Proof. We have divided this rather technical proof into three parts.

Part 1. Let us first present a slight modification of an argument given in [6, Lemma 2.10], when proving the inclusion $\overline{\mathcal{A}'}(f) \subset \bigcup_{\Gamma \subset \Delta_f} \mathcal{A}'(f_\Gamma)$. Assume that Δ_f has full dimension and that the sequence $\{z(j)\}_{j=1}^\infty \subset V(f)$ is such that

$$\lim_{j\to\infty} z(j) \notin (\mathbb{C}^*)^n \quad \text{and} \quad \lim_{j\to\infty} \operatorname{Arg}(z(j)) = \theta \in \mathbf{T}^n.$$

We claim that $\theta \in \mathcal{A}'(f_\Gamma)$ for some strict subface $\Gamma \subset \Delta_f$. As $V(f)$ is invariant under multiplication of f with a Laurent monomial, we can assume that the constant 1 is a monomial of f. We can also choose a subsequence of $\{z(j)\}_{j=1}^\infty$ such that, after possibly reordering A,

$$|z(j)^{\alpha_1}| \geq \ldots \geq |z(j)^{\alpha_N}|, \quad j = 1, 2, \ldots,$$

and in addition

$$\lim_{j\to\infty} \frac{|z(j)^{\alpha_k}|}{|z(j)^{\alpha_1}|} = d_k$$

for some $d_k \in [0, 1]$. It is shown in the proof of [6, Lemma 2.10] that $\Gamma = \{\alpha_k \,|\, d_k > 0\}$ is a face of Δ_f, and furthermore that $\theta \in \mathcal{A}'(f_\Gamma)$. With the above ordering of A, assume that the constant 1 is the pth monomial. We need to show that Γ is a strict subface of Δ_f. Assuming the contrary, we find that $d_k > 0$ for each k, and hence

$$\lim_{j\to\infty} |z(j)^{\alpha_k}| = \lim_{j\to\infty} \frac{|z(j)^{\alpha_k}|}{|z(j)^{\alpha_1}|} |z(j)^{\alpha_1}| = \frac{d_k}{d_p},$$

which in particular is finite and non-zero. As Δ_f has full dimension, this implies that $\lim_{j\to\infty} |z(j)_m|$ is finite and non-zero for each $m = 1, ..., n$. As $\operatorname{Arg}(z(j)) \to \theta$ when $j \to \infty$, we find that $\lim_{j\to\infty} z(j) \in (\mathbb{C}^*)^n$, which contradicts our initial assumptions. Hence, $d_N = 0$, and Γ is a strict subface of Δ_f.

Part 2. We now claim that if $n \geq 2$, then the set

$$P = \{z \in V(f) \,|\, \operatorname{Arg}(z) \in N(l) \cap \mathcal{A}'(f)\},$$

where $N(l)$ is an arbitrarily small neighborhood of l in $\mathbb{R}^n$, is such that $\operatorname{Log}(P)$ is unbounded. To see this, consider the function $g(w) = f(e^w)$, where $w_k = x_k + i\theta_k$.

Notice that the w-space $\mathbb{C}^n$ is identified with the image of the z-space $(\mathbb{C}^*)^n$ under the multivalued *complex* logarithm. That is, the coamoeba $\mathcal{A}'(f)$ and the line l are considered as subsets of $\mathbb{R}^n$, which is the image of the w-space $\mathbb{C}^n$ under taking coordinatewise imaginary parts.

We can assume that l is parallel to the θ_1-axis and, by a translation of the coamoeba, that there are $\rho_1, ..., \rho_n > 0$ such that the set

$$S = [-\rho_1, \rho_1] \times ... \times [-\rho_n, \rho_n]$$

fulfills $l \subset S \subset N(l)$. Furthermore we can choose $0 < r < \rho_1$ such that, with

$$\widetilde{S} = [-r, r] \times [-\rho_2, \rho_2] \times ... \times [-\rho_n, \rho_n],$$

the set $S \setminus \widetilde{S}$ consists of two n-cells that are neighborhoods of the endpoints of l. Hence, we can assume that $S \setminus \widetilde{S} \subset \overline{\mathcal{A}'}(f)^c$. If we assume that $\operatorname{Log}(P)$ is bounded, then there exists a sufficiently large $R \in \mathbb{R}$ such that if

$$D = \{x \in \mathbb{R}^n \mid |x| > R\},$$

then $g(w)$ has no zeros in $D + iS \subset \mathbb{C}^n$. Let w' denote the vector $(w_2, ..., w_n)$, and let $(D + iS)'$ be the projection of $D + iS$ onto the last $n-1$ components. Then in particular, $g(w)$ has no zeros when $w' \in (D + iS)'$ and w_1 lies in the domain given by $\{w_1 | r < |\mathrm{Im}(w_1)| < \rho_1\} \cup (\{w_1 \mid |\mathrm{Re}(w_1)| > R\} \cap \{w_1 \mid |\mathrm{Im}(w_1)| < \rho_1\})$, see Figure 2. Consider a curve γ as in Figure 2, and the integral

$$k(w') = \frac{1}{2\pi i} \int_\gamma \frac{g_1'(w_1, w')}{g(w_1, w')} \, dw_1, \quad w' \in (D + iS)'.$$

By the argument principle, for a fixed w' this counts the number of roots of $g(w)$ inside the box in Figure 2. As it depends continuously on w' in the domain $(D + iS)'$ it is constant, and by considering w' with $|x'| > R$ (here it is essential that $n \geq 2$) we conclude that it is zero. However, this contradicts the assumption that l intersects $\mathcal{A}'(f)$. Hence, $\operatorname{Log}(P)$ is unbounded.

Part 3. We will now prove the lemma using induction on the dimension d of Δ_f. If $d = 1$, then there is nothing to prove. Consider the case of a fixed $d > 1$, assuming that the statement is proven for each smaller dimension. Notice that f has $n - d$ homogeneities, and hence it is essentially a polynomial in d variables. Dehomogenizing f corresponds to projecting $\mathbf{T}^n$ onto $\mathbf{T}^d$ such that the coamoeba $\mathcal{A}'(f) \subset \mathbf{T}^n$ consists precisely of the fibers over the coamoeba of the dehomogenized polynomial. The image of l under this projection will intersects the coamoeba of an edge of Δ_f in $\mathbf{T}^d$ if and only if l intersects the coamoeba of an edge of Δ_f in $\mathbf{T}^n$.

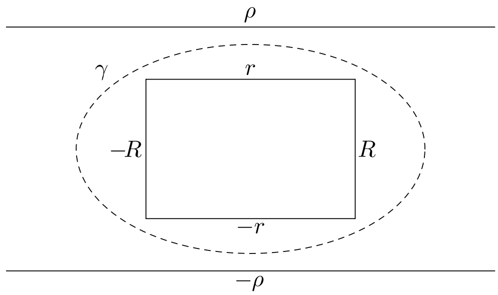

Figure 2. The curve $\gamma\subset\mathbb{C}$.

Hence, it is enough to prove the statement under the assumption that $d=n$. In particular, $n\geq 2$.

Choose a decreasing sequence $\{\varepsilon(k)\}_{k=1}^{\infty}$ of positive real numbers such that $\lim_{k\to\infty}\varepsilon(k)=0$, and consider the family of neighborhoods of l given by

$$N(l,k)=\left\{\theta\in\mathbb{R}^n \,\middle|\, \inf_{x\in l}|\theta-x|<\varepsilon(k)\right\},$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$. Define

$$P(k)=\{z\in V(f)\,|\,\operatorname{Arg}(z)\in N(l,k)\cap\mathcal{A}'(f)\}.$$

Since $n\geq 2$, part 2 shows that for each k, the set $\operatorname{Log}(P(k))$ is unbounded. That is, for each k, we can find a sequence $\{z(k,m)\}_{m=1}^{\infty}$ such that $z(k,m)\in V(f)$, with

$$\operatorname{Arg}(z(k,m))\in N(l,k)\cap\mathcal{A}'(f)\subset\overline{N(l,k)\cap\mathcal{A}'(f)},$$

but $\lim_{m\to\infty} z(k,m)\notin(\mathbb{C}^*)^n$. As $\overline{N(l,k)\cap\mathcal{A}'(f)}$ is compact, we can choose a subsequence such that $\operatorname{Arg}(z(k,m))$ converges to some $\theta(k)\in\overline{N(l,k)\cap\mathcal{A}'(f)}$ when $m\to\infty$. Then, part 1 gives a strict subface $\Gamma(k)$ of Δ_f such that $\theta(k)\in\mathcal{A}'(f_{\Gamma(k)})$. Since Δ_f has only finitely many strict subfaces, we can choose a subsequence of $\{\theta(k)\}_{k=1}^{\infty}$ such that $\Gamma=\Gamma(k)$ does not depend on k. As $\{\theta(k)\}_{k=1}^{\infty}\subset\overline{N(l,1)}$, which is compact, we can also choose this subsequence such that $\theta(k)$ converges to some $\theta\in\overline{N(l,1)}$ when $k\to\infty$. On the one hand, we have that $\theta\in l$ by construction of the sets $N(l,k)$. On the other hand, $\theta(k)\in\mathcal{A}'(f_\Gamma)$ implies that $\theta\in\overline{\mathcal{A}'}(f_\Gamma)$. In particular, l and $\overline{\mathcal{A}'}(f_\Gamma)$ intersect at θ.

The identity (2) shows that the endpoints of l is contained in the complement of $\overline{\mathcal{A}'}(f_\Gamma)$. As the dimension of Γ is strictly less than the dimension of Δ_f, the induction hypothesis shows that l intersects the coamoeba of an edge of Γ. As each edge of Γ is an edge of Δ_f, the lemma is proven. $\square$

3. Lopsided coamoebas

In this section we will investigate the basic properties of (closed) lopsided coamoebas. The formulation of Definition 1.1 was partly chosen to stress the analogy with the lopsided amoeba. A more natural description is perhaps the following; denote the components of $f\langle\theta\rangle$ by $t_1, ..., t_N$, and consider the convex cone

$$\mathbb{R}_+ f\langle\theta\rangle = \{r_1 t_1 + ... + r_N t_N \mid r_1, ..., r_N \in \mathbb{R}_+\}.$$

Lemma 3.1. *We have that $\theta \in \mathcal{LA}'(f)$ if and only if $0 \in \mathbb{R}_+ f\langle\theta\rangle$.*

Proof. If $\theta \in \mathcal{LA}'(f)^c$, then $\mathbb{R}_+ f\langle\theta\rangle \subset \mathrm{int}(H)$, where $H \subset \mathbb{C}$ is a half-space such that $f\langle\theta\rangle \subset H$ but $f\langle\theta\rangle \not\subset \partial H$. Conversely, if $\mathbb{R}_+ f\langle\theta\rangle$ does not contain the origin, then it follows from the convexity of $\mathbb{R}_+ f\langle\theta\rangle$ that there exist a half-space H such that $\mathbb{R}_+ f\langle\theta\rangle \subset \mathrm{int}(H)$. $\square$

Corollary 3.2. *We have the inclusion $\mathcal{A}'(f) \subset \mathcal{LA}'(f)$.*

Proof. If $f(re^{i\theta}) = 0$ then $0 \in \mathbb{R}_+ f\langle\theta\rangle$. $\square$

Corollary 3.3. *If A is simple, then $\mathcal{A}'(f) = \mathcal{LA}'(f)$.*

Proof. By considering integer affine transformations, we see that it is enough to prove this for the standard n-simplex $f(z) = 1 + z_1 + ... + z_n$. We have that $0 \in \mathbb{R}_+ f\langle\theta\rangle$ if and only if we can find $r_0, ..., r_n \in \mathbb{R}_+$ such that $r_0 + r_1 e^{i\theta_1} + ... + r_n e^{i\theta_n} = 0$, and this is equivalent to $\theta \in \mathcal{A}'(f)$. $\square$

Simple hypersurfaces are not the only ones for which the identity $\mathcal{A}'(f) = \mathcal{LA}'(f)$ holds. It will be the case as soon as $\mathcal{A}'(f) = \mathbf{T}^n$, and such examples are easy to construct by considering products of polynomials. An example of a non-simple polynomial f such that $\mathcal{A}'(f) = \mathcal{LA}'(f) \subsetneq \mathbf{T}^n$ is given by $f(z_1, z_2) = 1 + z_1 + z_2 - r z_1 z_2$ for any $r \in \mathbb{R}_+$.

Consider the polynomial

$$F(c, z) = \sum_{\alpha \in A} c_\alpha z^\alpha,$$

obtained by viewing the coefficients c as variables. This polynomial has a coamoeba $\mathcal{A}'(F) \subset \mathbf{T}^{N+n}$ which, as F is simple, coincides with its lopsided coamoeba $\mathcal{LA}'(F)$. As the convex cone $\mathbb{R}_+ f\langle\theta\rangle$ coincides with the cone $\mathbb{R}_+ F\langle \arg(c), \theta\rangle$, we see that $\mathcal{LA}'(f)$ is nothing but the intersection of $\mathcal{A}'(F)$ with the sub n-torus of $\mathbf{T}^{N+n}$ given by fixing $\mathrm{Arg}(c)$. In this manner, the lopsided coamoeba inherits some properties of simple coamoebas.

Proposition 3.4. *Let* $\mathrm{Tri}(f)$ *denote the set of all trinomials* g *one can construct from the set of monomials of* f*. Then*

$$\overline{\mathcal{LA}'}(f) = \bigcup_{g \in \mathrm{Tri}(f)} \overline{\mathcal{A}'}(g).$$

Proof. By the previous discussion we can view $\mathcal{LA}'(f)$ as the intersection of $\mathcal{A}'_F$ with the sub n-torus of $\mathbf{T}^{N+n}$ given by fixing $\mathrm{Arg}(c)$. This is of course also the case for each trinomial $g \in \mathrm{Tri}(f)$, and hence the identity follows from (4). □

As was the case in (4), this identity holds also without taking closures if $N \neq 4$. Lopsided coamoebas first appeared under this disguise in [6]. This proposition gives a naive algorithm for determining lopsided coamoebas, by determining the coamoebas of each trinomial in $\mathrm{Tri}(f)$.

Definition 3.5. Let $\mathrm{Bin}(f)$ denote the set of all binomials that can be obtained by removing all but two monomials of f. The *shell* $\mathcal{LH}(f)$ of the lopsided coamoeba $\mathcal{LA}'(f)$ is defined as the union

$$\mathcal{LH}(f) = \bigcup_{g \in \mathrm{Bin}(f)} \mathcal{A}'(g).$$

In the case $n \geq 2$, Proposition 3.4 states that $\overline{\mathcal{LA}'}(f)$ is the closure of the coamoeba of the polynomial $\prod_{g \in \mathrm{Tri}(f)} g(z)$. Recall that the ordinary shell of a coamoeba is defined as the union of all coamoebas of the edges of its Newton polytope. As the Newton polytope of each binomial in $\mathrm{Bin}(f)$ is an edge of the Newton polytope of some trinomial in $\mathrm{Tri}(f)$, we find that $\mathcal{LH}(f)$ is a subset of the ordinary shell of this product, which motivates the choice of name.

Proposition 3.6. *The boundary of* $\overline{\mathcal{LA}'}(f)$ *is contained in* $\mathcal{LH}(f)$*.*

Proof. The boundary of $\overline{\mathcal{LA}'}(f)$ consists of points θ for which $f\langle\theta\rangle$ contains (at least) two antipodal points, which implies that θ belongs to the coamoeba of the corresponding binomial. □

The focus on $\overline{\mathcal{A}'}(f)$ rather than $\mathcal{A}'(f)$ leads us naturally to consider $\overline{\mathcal{LA}'}(f)$ in more detail. Its complement has the following characterization.

Proposition 3.7. *We have that* $\theta \in \overline{\mathcal{LA}'}(f)^c$ *if and only if there is an open half-space* $H \subset \mathbb{C}$ *with* $f\langle\theta\rangle \subset H$*.*

Proof. The "if" part is clear. To show "only if", note that if $\theta \in \mathcal{LA}'(f)^c$, then there is an open half-space H with $f\langle\theta\rangle \subset \overline{H}$. If there is no open half-space H with $f\langle\theta\rangle \subset H$, then $f\langle\theta\rangle$ contains two antipodal points. Then we can find a simple trinomial $g \in \mathrm{Tri}(f)$ such that $\theta \in \overline{\mathcal{A}'}(g)$, and by the description of simple trinomials in the previous section there is a sequence $\{\theta_k\}_{k=1}^{\infty} \subset \mathrm{int}(\mathcal{A}'(g))$ such that $\lim_{k\to\infty} \theta_k = \theta$. As g is simple we have that $\mathcal{A}'(g) = \mathcal{LA}'(g)$, and hence for each θ_k the list $g\langle\theta_k\rangle$ is not lopsided. Then neither is $f\langle\theta_k\rangle$, showing that $\{\theta_k\}_{k=1}^{\infty} \subset \mathcal{LA}'(f)$, and as a consequence that $\theta \in \overline{\mathcal{LA}'}(f)$. □

Let us end this section by describing the relation between the sets $\mathrm{CC}(\overline{\mathcal{A}'}(f))$ and $\mathrm{CC}(\overline{\mathcal{LA}'}(f))$, beginning with yet another characterization of $\mathcal{LA}'(f)$.

Lemma 3.8. *Let $f_r(z)$ denote the polynomial $\sum_{\alpha\in A} r_\alpha c_\alpha z^\alpha$, where we have varied the moduli of the coefficients of f by $r = \{r_\alpha\}_{\alpha\in A} \in \mathbb{R}_+^N$. Then*

$$\mathcal{LA}'(f) = \bigcup_{r\in\mathbb{R}_+^N} \mathcal{A}'(f_r).$$

Proof. The statement follows from Lemma 3.1. If $\theta \in \mathcal{A}'(f_r)$, then $0 \in \mathbb{R}_+ f_r\langle\theta\rangle$. Conversely, if $0 \in \mathbb{R}_+ f\langle\theta\rangle$, then there exists an $r \in \mathbb{R}_+^N$ such that $f_r(e^{i\theta}) = 0$. □

Proposition 3.9. *Each connected component of $\overline{\mathcal{A}'}(f)^c$ contains at most one connected component of $\overline{\mathcal{LA}'}(f)^c$.*

Proof. It is clear that each connected component of $\overline{\mathcal{LA}'}(f)^c$ is included in some connected component of $\overline{\mathcal{A}'}(f)^c$, we only have to show that no two connected components of $\overline{\mathcal{LA}'}(f)^c$ are contained in the same connected component of $\overline{\mathcal{A}'}(f)^c$. We will show this by proving that any line segment l with endpoints in $\overline{\mathcal{LA}'}(f)^c$ that intersect $\mathcal{LA}'(f)$, also intersect $\overline{\mathcal{A}'}(f)$.

Consider first the case when $f(z)$ is a univariate polynomial. Let $\theta_1, \theta_2 \in \overline{\mathcal{LA}'}(f)^c$ be the endpoints of a line segment l, i.e. $l = [\theta_1, \theta_2]$, and assume that there exist a $\theta \in (\theta_1, \theta_2)$ with $\theta \in \mathcal{LA}'(f)$. Then Lemma 3.8 gives an $r \in \mathbb{R}_+^N$ such that $\theta \in \mathcal{A}'(f_r)$. Let γ be the path from c to rc in the coefficient space $(\mathbb{C}^*)^A$ given by

$$\gamma(t)_\alpha = r_\alpha^{1-t} c_\alpha, \quad t \in [0,1],$$

and let f_t denote the polynomial with coefficients $\gamma(t)$. Applying Lemma 3.8 once more, we find that for each $t \in [0,1]$ it holds that $\mathcal{A}'(f_t) \subset \mathcal{LA}'(f)$. In particular, for each t, we have that $\theta_1, \theta_2 \notin \mathcal{A}'(f_t)$. Let $z \in \mathbb{C}^*$ denote a root of $f_0(z) = f_r(z)$ such that $\arg(z) = \theta$. It is well known that the roots of f_t in $\mathbb{C}^*$ vary continuously with r. That is, we can find a continuous path $t\mapsto z(t)$ in $\mathbb{C}^*$ such that $z(0)=z$, and furthermore for each $t\in[0,1]$ we have that $z(t)$ is a root of the polynomial $f_t(z)$. Notice that if $f_t(z)$ has a root of higher multiplicity at $z(t)$, then the path $t\mapsto z(t)$ is neither smooth nor unique, however we need only that it is continuous. Indeed, the continuity of the path $t\mapsto z(t)$ in $\mathbb{C}^*$ implies continuity of the path $t\mapsto \arg(z(t))$. Finally, the continuity of the path $t\mapsto\arg(z(t))$, together with the facts that $\theta_1,\theta_2\notin\mathcal{A}'(f_t)$ for each $t\in[0,1]$ and that $\arg(z(0))=\theta\in(\theta_1,\theta_2)$, implies that $\arg(z(t))\in(\theta_1,\theta_2)$ for each t. In particular, $\arg(z(1))\in(\theta_1,\theta_2)$, which proves the proposition in this case.

Consider now the case when Δ_f is one-dimensional. Then $f(z)$ has $n-1$ quasi-homogeneities, and the coamoeba $\mathcal{A}'(f)$ consists of a family of parallel hyperplanes, each orthogonal to Δ_f. Dehomogenizing $f(z)$ corresponds to a projection $\mathbb{R}^n\to\mathbb{R}$ such that the hyperplanes in $\mathcal{A}'(f)$ are precisely the fibers over the points in the coamoeba of the dehomogenization $\tilde{f}$ of $f(z)$. This projection will map a line segment in $\mathbb{R}^n$ with endpoints in $\overline{\mathcal{LA}'}(f)^c$ that intersect $\mathcal{LA}'(f)$ to a line segment in $\mathbb{R}$ with endpoints in $\overline{\mathcal{LA}'}(\tilde{f})^c$ that intersect $\mathcal{LA}'(\tilde{f})$. Hence, this case follows from the univariate case.

Now consider an arbitrary multivariate polynomial $f(z)$, and let l be a line segment in $\mathbb{R}^n$ with endpoints in $\overline{\mathcal{LA}'}(f)^c$ that intersect $\mathcal{LA}'(f)$. By Lemma 3.8 there exists an $r\in\mathbb{R}_+^N$ such that l intersect $\mathcal{A}'(f_r)$. Referring to Lemma 3.8 again, we find that $\overline{\mathcal{A}'}(f_r)\subset\overline{\mathcal{LA}'}(f)$, and hence the endpoints of l are contained in $\overline{\mathcal{A}'}(f_r)^c$. Applying Lemma 2.3 to the polynomial f_r, we find an edge $\Gamma\subset\Delta_{f_r}=\Delta_f$ such that l intersect $\mathcal{A}'((f_r)_\Gamma)$. This implies, by Lemma 3.8, that l intersect $\mathcal{LA}'((f_r)_\Gamma)=\mathcal{LA}'(f_\Gamma)$. As the identity (2) implies that $\overline{\mathcal{LA}'}(f_\Gamma)\subset\overline{\mathcal{LA}'}(f)$, we find that the endpoints of l are contained in $\overline{\mathcal{LA}'}(f_\Gamma)^c$. Since Γ is one-dimensional, we can conclude by the previous case that l intersects $\mathcal{A}'(f_\Gamma)$. The identity (2) yields that l intersects $\overline{\mathcal{A}'}(f)$. □

4. The order map for the lopsided coamoeba

The aim of this section is to provide an order map for the lopsided coamoeba. The role played by the point configuration $A\subset\mathbb{Z}^n$ for the order map of the lopsided amoeba, is here given to a so-called *dual matrix* B. Recall that $|A|=N$, that we are under the assumption that Δ_f is of full dimension, and that the integer $m=N-n-1$ is the codimension of A. A dual matrix of A is by definition an integer $N\times m$-matrix of full rank such that $AB=0$. If in addition the columns of B span the $\mathbb{Z}$-kernel of A, then B is known as a *Gale dual* of A. We denote by $\mathbb{Z}[B]\subset\mathbb{Z}^m$ the lattice generated by the *rows* of B, and note that B is a Gale dual of A if and only if $\mathbb{Z}[B]=\mathbb{Z}^m$. In this manner, assuming that B is a Gale dual will make our statements more streamlined, however it is not a necessary assumption in order to develop the theory. We will label the rows of B as $b_0, ..., b_{n+m}$. The zonotope $\mathcal{Z}_B$ is defined as the set

$$\mathcal{Z}_B = \left\{ \sum_{j=0}^{m+n} \frac{\pi}{2} \mu_j b_j \,\middle|\, |\mu_j| \leq 1,\ j=0, ..., m+n \right\}, \tag{5}$$

see also [1] and [10].

Fix a polynomial $f \in (\mathbb{C}^*)^A$, i.e. with notation as in (1) and (3) we fix a set of coefficients $c_{\alpha_1}, ..., c_{\alpha_N}$. Let us denote by $\arg_\pi \colon \mathbb{C}^* \to (-\pi, \pi]$ the principal branch of the arg-mapping, while Arg_π denotes the map acting on vectors componentwise by $\arg_\pi$.

Lemma 4.1. *For a fixed polynomial f, and a fixed point $\alpha \in A$ (that is, with the above notation $\alpha = \alpha_j$ for some j), consider the function $p_\alpha^k(\theta)$, with domain $\mathbb{R}^n$, given by*

$$p_\alpha^k(\theta) = \arg_\pi\left(\frac{c_{\alpha_k} e^{i\langle \alpha_k, \theta\rangle}}{c_\alpha e^{i\langle \alpha, \theta\rangle}}\right) - \arg_\pi(c_{\alpha_k}) + \arg_\pi(c_\alpha) - \langle \alpha_k - \alpha, \theta\rangle.$$

Then p_α^k maps $\mathbb{R}^n$ into $2\pi\mathbb{Z}$, and furthermore it is locally constant off the coamoeba of the binomial $c_\alpha z^\alpha + c_{\alpha_k} z^{\alpha_k}$, as viewed in $\mathbb{R}^n$.

Proof. For each θ we have that

$$\arg_\pi\left(\frac{c_{\alpha_k} e^{i\langle \alpha_k, \theta\rangle}}{c_\alpha e^{i\langle \alpha, \theta\rangle}}\right) = \arg_\pi(c_{\alpha_k}) - \arg_\pi(c_\alpha) + \langle \alpha_k - \alpha, \theta\rangle + 2\pi j(\theta),$$

where $j(\theta) \in \mathbb{Z}$. We see that $p_\alpha^k(\theta) = 2\pi j(\theta)$, and therefore p_α^k maps $\mathbb{R}^n$ into $2\pi\mathbb{Z}$. It is clear that $j(\theta)$ is locally constant, as a function of θ, off the set, where $\arg_\pi(c_{\alpha_k} e^{i\langle \alpha_k, \theta\rangle} / c_\alpha e^{i\langle \alpha, \theta\rangle}) = \pi$. This set is precisely the coamoeba of the binomial $c_\alpha z^\alpha + c_{\alpha_k} z^{\alpha_k}$, as viewed in $\mathbb{R}^n$, which proves the second statement. $\square$

In particular the vector-valued function

$$p_\alpha(\theta) = (p_\alpha^1(\theta), ..., p_\alpha^N(\theta))$$

is constant on each cell of the hyperplane arrangement $\mathcal{LH}(f)$, considered as subsets of $\mathbb{R}^n$.

Lemma 4.2. *With notation as in the previous lemma, define $v_\alpha \colon \mathbb{R}^n \to \mathbb{R}^m$ by*

$$v_\alpha(\theta) = (\mathrm{Arg}_\pi(c) + p_\alpha(\theta))B, \tag{6}$$

where the multiplication with B is usual matrix multiplication. Then v_α is well-defined on $\mathbf{T}^n$ (i.e. it is periodic in each θ_j with period 2π), it is invariant under *multiplication of f by a Laurent monomial, and furthermore if $\theta\in\overline{\mathcal{LA}'}(f)^c$, then $v_\alpha(\theta)\in\operatorname{int}(\mathcal{Z}_B)$.*

Proof. For any $\theta\in\mathbb{R}^n$ we have that $\operatorname{Arg}_\pi(c)+p_\alpha(\theta)$ equals the vector

$$\left(\arg_\pi\left(\frac{c_{\alpha_1}e^{i\langle\alpha_1,\theta\rangle}}{c_\alpha e^{i\langle\alpha,\theta\rangle}}\right),...,\arg_\pi\left(\frac{c_{\alpha_N}e^{i\langle\alpha_N,\theta\rangle}}{c_\alpha e^{i\langle\alpha,\theta\rangle}}\right)\right)+(\arg_\pi(c_\alpha)\langle\alpha,\theta\rangle,\theta_1,...,\theta_n)A,$$

where A denotes the matrix (3). It follows that

$$(\operatorname{Arg}_\pi(c)+p_\alpha(\theta))B=\left(\arg_\pi\left(\frac{c_{\alpha_1}e^{i\langle\alpha_1,\theta\rangle}}{c_\alpha e^{i\langle\alpha,\theta\rangle}}\right),...,\arg_\pi\left(\frac{c_{\alpha_N}e^{i\langle\alpha_N,\theta\rangle}}{c_\alpha e^{i\langle\alpha,\theta\rangle}}\right)\right)B. \tag{7}$$

We conclude that v_α is well-defined on $\mathbf{T}^n$, and that it is invariant under multiplication of f by a Laurent monomial.

Let us now turn to the last claim. Given a $\theta\in\overline{\mathcal{LA}'}(f)^c$, the components of $f\langle\theta\rangle$ are contained in one half-space $H\subset\mathbb{C}$. As v_α is invariant under multiplication of f with a Laurent monomial, we can assume that $\alpha=0$ and that $H=H_0$ is the right half space. That is

$$\arg_\pi(c_{\alpha_k}e^{i\langle\alpha_k,\theta\rangle})=\frac{\pi}{2}\mu_k$$

for some $\mu_k\in(-1,1)$. Since $\arg_\pi(x_1x_2)=\arg_\pi(x_1)+\arg_\pi(x_2)$ for any two elements $x_1,x_2\in H_0$, we find that

$$p_0^k(\theta)=\arg_\pi(c_{\alpha_k}e^{i\langle\alpha_k,\theta\rangle})-\arg_\pi(c_{\alpha_k})-\langle\alpha_k,\theta\rangle.$$

Thus, the following identities hold

$$\begin{cases}\arg_\pi(c_{\alpha_1})+\langle\alpha_1,\theta\rangle+p_0^1(\theta)=\frac{\pi}{2}\mu_1,\\ \vdots\\ \arg_\pi(c_{\alpha_N})+\langle\alpha_N,\theta\rangle+p_0^N(\theta)=\frac{\pi}{2}\mu_N.\end{cases} \tag{8}$$

Hence,

$$(\operatorname{Arg}_\pi(c)+p_\alpha(\theta))B=\left(\frac{\pi}{2}\mu-(0,\theta_1,...,\theta_n)A\right)B=\frac{\pi}{2}\mu B\in\operatorname{int}(\mathcal{Z}_B).\qquad\square$$

Theorem 4.3. *There is a well-defined map*

$$\text{co-ord}\colon\ \operatorname{CC}(\overline{\mathcal{LA}'}(f))\longrightarrow\operatorname{int}(\mathcal{Z}_B)\cap(\operatorname{Arg}_\pi(c)B+2\pi\mathbb{Z}[B]),$$

which for $\Theta\in\operatorname{CC}(\overline{\mathcal{LA}'}(f))$ is given by

$$\text{co-ord}(\Theta)=v_\alpha(\theta),\quad \theta\in\Theta,\ \alpha\in A. \tag{9}$$

Proof. Note first that by Lemma 4.1 we have that $v_\alpha(\theta)\in\mathrm{Arg}_\pi(c)B+2\pi\mathbb{Z}[B]$, and by Lemma 4.2 we have that $\theta\in\overline{\mathcal{LA}'}(f)^c$ implies that $v_\alpha(\theta)\in\mathrm{int}(\mathcal{Z}_B)$. Hence we only need to show that the right-hand side of (9) is independent of $\theta\in\Theta$ and $\alpha\in A$, so that the given map is well-defined.

The first claim of Lemma 4.2 says that v_α is well-defined on $\mathbf{T}^n$. As the function p_α is constant on the cells of the hyperplane arrangement $\mathcal{H}_f$, Proposition 3.6 tells us that v_α is constant on the connected components of the complement of the lopsided coamoeba of f. That is, $v_\alpha(\theta)$ is independent of the choice of $\theta\in\Theta$.

Finally, to see that $v_\alpha(\theta)$ is independent of the choice of α, we note again that $v_\alpha(\theta)$ is invariant under multiplication of f with a Laurent monomial. Hence we can assume that f contains the monomial $\alpha=0$, and that $H=H_0$. Then

$$p_0^k(\theta)-p_\alpha^k(\theta)=\arg_\pi(c_\alpha e^{i\langle\alpha,\theta\rangle})-\arg_\pi(c_\alpha)-\langle\alpha,\theta\rangle$$

is independent of k, and hence $(p_0(\theta)-p_\alpha(\theta))B=0$, which shows that $v_0(\theta)=v_\alpha(\theta)$ for each α. $\square$

Definition 4.4. The map co-ord from Theorem 4.3 is called the *order map* of the lopsided coamoeba $\mathcal{LA}'(f)$.

In order to show the statements on surjectivity and injectivity of co-ord, we have to use a more detailed notation. After multiplication with a Laurent monomial, which neither affects the map co-ord nor the lopsided coamoeba $\mathcal{LA}'(f)$, we can assume that A is of the form

$$A=\begin{pmatrix}1 & 1 & 1\\ 0 & A_{\mathbf{1}} & A_{\mathbf{2}}\end{pmatrix},$$

where $A_{\mathbf{1}}$ is a non-singular $n\times n$ matrix. We can also assume that $c_0=1$, i.e. that the constant 1 is a monomial of f.

The columns of any Gale dual of A is a basis for its $\mathbb{Z}$-kernel. Hence, if we fix a Gale dual $\widetilde{B}$, then any dual matrix can be presented in the form $B=\widetilde{B}T$, for some $T\in\mathrm{GL}_m(\mathbb{Q})$. This implies that any dual matrix B of A can be presented in the form

$$B=\begin{pmatrix}a_0\\ -A_{\mathbf{1}}^{-1}A_{\mathbf{2}}\\ I_m\end{pmatrix}T, \tag{10}$$

where $a_0\in\mathbb{Q}^m$ is defined by the property that each column of B should sum to zero, and $T\in\mathrm{GL}_m(\mathbb{Q})$.

Lemma 4.5. *Let A be under the assumptions imposed above. Let $c_{\mathbf{1}}$ and $c_{\mathbf{2}}$ denote the vectors $(c_1, ..., c_n)$ and $(c_{n+1}, ..., c_{n+m})$ respectively, and similarly for $l \in \mathbb{Z}^N$ and $\mu \in \mathbb{R}^N$. Consider the system*

$$
\left\{
\begin{aligned}
&\mathrm{Arg}_\pi(c_{\mathbf{1}}) + \theta A_{\mathbf{1}} + 2\pi l_{\mathbf{1}} = \frac{\pi}{2}\mu_{\mathbf{1}}, \\
&\mathrm{Arg}_\pi(c_{\mathbf{2}}) + \theta A_{\mathbf{2}} + 2\pi l_{\mathbf{2}} = \frac{\pi}{2}\mu_{\mathbf{2}}.
\end{aligned}
\right. \tag{11}
$$

Then $\theta \in \overline{\mathcal{LA}'}(f)^c$ if and only if θ solves (11) for some integers l and some numbers $\mu_0, ..., \mu_{n+m}$ such that $\mu_0, \mu_1 + \mu_0, ..., \mu_{n+m} + \mu_0 \in (-1, 1)$.

Proof. If $\theta \in \overline{\mathcal{LA}'}(f)^c$, then there is a half plane H_ϕ such that $f\langle\theta\rangle \subset H_\phi$. As the constant 1 is a term of f, we can choose $\phi \in (-\pi/2, \pi/2)$. Considering the polynomial $e^{-i\phi} f(z)$, we find that this is lopsided at θ for H_0. Thus, there are numbers $\lambda_1, ..., \lambda_{n+m} \in (-1, 1)$ and integers $l_1, ..., l_{n+m}$ such that

$$
\arg_\pi(c_k) + \langle\theta, \alpha_k\rangle + 2\pi l_k = \frac{\pi}{2}\lambda_k + \phi, \quad k = 1, ..., n+m.
$$

This shows that θ fulfills (11) with l as above, $\mu_0 = -2\phi/\pi$ and $\mu_k = \lambda_k + 2\phi/\pi$ for $k = 1, ..., n+m$. Conversely, if θ fulfills (11) for such l and μ, then $f\langle\theta\rangle \subset H_\phi$, where $\phi = -\pi\mu_0/2$. $\square$

Proposition 4.6. *The order map* co-ord *is a surjection.*

Proof. Let A be under the assumptions imposed above. Formally solving the first equation of (11) for θ by multiplication with $A_{\mathbf{1}}^{-1}$ and eliminating θ in the second equation, also applying the transformation T, one arrives at the equivalent system

$$
\left\{
\begin{aligned}
&\theta = \frac{\pi}{2}\mu_{\mathbf{1}} A_{\mathbf{1}}^{-1} - \mathrm{Arg}(c_{\mathbf{1}}) A_{\mathbf{1}}^{-1} - 2\pi l_{\mathbf{1}} A_{\mathbf{1}}^{-1}, \\
&\mathrm{Arg}_\pi(c) B + 2\pi(0, l_{\mathbf{1}}, l_{\mathbf{2}}) B = \frac{\pi}{2}(0, \mu_{\mathbf{1}}, \mu_{\mathbf{2}}) B.
\end{aligned}
\right. \tag{12}
$$

To see that co-ord is surjective, consider a point $\mathrm{Arg}_\pi(c)B + 2\pi l B = \pi\lambda B/2 \in \mathrm{int}(\mathcal{Z}_B)$, and note that we can assume that $l_0 = 0$. Define μ by $\mu_k = \lambda_k - \lambda_0$ for $k = 0, \ldots, n+m$. It follows that the pair (l, μ) fulfills the second equation of (12). Let $\theta \in \mathbb{R}^n$ be defined by the first equation of (12), it then follows that the triple (θ, l, μ) fulfills (11), and thus by Lemma 4.5 we have that $\theta \in \overline{\mathcal{LA}'}(f)^c$. By tracing backwards we find that the order of the component of $\overline{\mathcal{LA}'}(f)^c$ containing θ is $\mathrm{Arg}_\pi(c)B + 2\pi l B$, and hence the map co-ord is surjective. $\square$

Proposition 4.7. *If* $g_A=1$, *i.e. if the maximal minors of* A *are relatively prime, then* co-ord *is an injection.*

Proof. For any point $p\in\mathrm{int}(\mathcal{Z}_B)$, the set of all $\mu\in\mathbb{R}^N$ such that $2\pi\mu B=p$, is an affine space, and hence convex. It follows that the set of all $\mu\in(-1,1)^N$ such that $2\pi\mu B=p$, being the intersection of two convex sets, is also convex. This implies that for fixed integers l, the set of $\theta\in\mathbb{R}^n$ such that (11) is fulfilled with $\mu_0,\mu_1-\mu_0,...,\mu_N-\mu_0\in(-1,1)$ is in turn also convex, as it is the image of a convex set under an affine transformation. Since the right-hand side of (6) is constant on each cell of $\mathcal{LH}(f)$, this set is exactly one connected component of $\overline{\mathcal{LA}'}(f)^c$ in $\mathbb{R}^n$. Thus, if we consider two points θ and $\tilde{\theta}$ in $\mathbb{R}^n$, which both maps to $\mathrm{Arg}(c)B+2\pi lB$, then we can assume that θ and $\tilde{\theta}$ fulfills (11) for the same numbers μ, however possibly for different integers l. Under this assumption there are integers $s_1,...,s_N$ such that

$$\langle\alpha_k,\theta\rangle=\langle\alpha_k,\tilde{\theta}\rangle+2\pi s_k,\quad k=1,...,N.$$

The sublattice of $\mathbb{Z}^{n+1}$ generated by the columns of A has $n+1$ generators, and its index is given by the absolute value of their determinant. As the determinant is multilinear, this is a linear combination of the determinants of the maximal minors of A. It follows that the assumption that $g_A=1$ is equivalent to the fact that the columns of A span $\mathbb{Z}^{n+1}$ over $\mathbb{Z}$. Thus, for each vector e_j there are integers $r_j=(r_{j1},...,r_{jN})$ such that $e_j=\sum_{k=1}^N r_{jk}\alpha_k$. Hence,

$$\theta_j=\langle e_j,\theta\rangle=\sum_{k=1}^N r_{jk}\langle\alpha_k,\theta\rangle=\sum_{k=1}^N r_{jk}\langle\alpha_k,\tilde{\theta}\rangle+2\pi r_{jk}s_k=\tilde{\theta}_j+2\pi\langle r_j,s\rangle,$$

which shows that θ and $\tilde{\theta}$ correspond to the same point in $\mathbf{T}^n$. $\square$

Remark 4.8. In general, the map co-ord will be g_A-to-one. Thus, if one considers co-ord as a map from $\mathrm{CC}(\overline{\mathcal{LA}'}(f))$ into the full translated lattice $\mathrm{int}(\mathcal{Z}_B)\cap(\mathrm{Arg}_\pi(c)B+2\pi\mathbb{Z}^m)$, then injectivity is measured in terms of g_A, while surjectivity is measured in terms of g_B. In view of Corollary 2.2, if one is interested in the structure of the set of connected components of the complement of the closed coamoeba, it is natural to assume that co-ord is a bijection.

Remark 4.9. The order of a component Θ of the complement of $\overline{\mathcal{LA}'}(f)$ is most easily determined using the right-hand side of (7). In particular, if the constant 1 is a monomial of f, then

$$\text{co-ord}(\Theta)=v_0(\theta)=(\arg_\pi(c_{\alpha_1}e^{i\langle\alpha_1,\theta\rangle}),...,\arg_\pi(c_{\alpha_N}e^{i\langle\alpha_N,\theta\rangle}))B,\quad\theta\in\Theta.$$

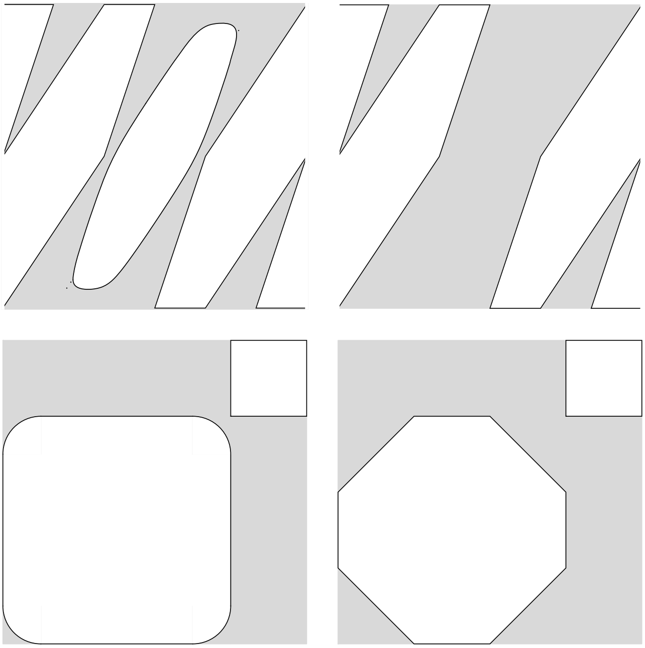

Figure 3. *Above*: the coamoeba and lopsided coamoeba of $f(z_1, z_2) = z_1^3 + z_2 + z_2^2 - z_1 z_2$.
Below: the coamoeba and lopsided coamoeba of $f(z_1, z_2) = 1 + z_1 + z_2 + i z_1 z_2$.

Example 4.10. Let us determine the map co-ord explicitly in the first example shown in Figure 3, that is we consider the polynomial $f(z_1, z_2) = z_1^3 + z_2 + z_2^2 - z_1 z_2$. The point configuration is

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 \end{pmatrix},$$

and a Gale dual of A is given by

$$B = (-1, -1, -1, 3)^t.$$

The corresponding zonotope is the interval $\mathcal{Z}_B = [-3\pi, 3\pi]$. As the translation $\operatorname{Arg}_\pi(c)B = 3 \arg_\pi(-1) = 3\pi$, the image of the map co-ord will be the doubleton $\{-\pi, \pi\}$. To determine co-ord, it is enough to evaluate v_α for some α and one point in each of the two connected components of $\overline{\mathcal{LA}'}(f)^c$, and we see from the picture in Figure 3 that a natural choice of points is $\theta_1 = (-2\pi/3, 0)$ and $\theta_2 = (2\pi/3, 0)$. We

Figure 4. $\mathcal{LA}'(f)$ in the fundamental domain $[-\pi, \pi]$.

find that

$$v_{\alpha_1}(\theta_1) = (0, -2\pi, -2\pi, -\pi)B = \pi,$$
$$v_{\alpha_1}(\theta_2) = (0, 2\pi, 2\pi, \pi)B = -\pi.$$

Example 4.11. Let us also consider a univariate case of codimension 1, namely

$$f(z) = 1 + z^3 + iz^5.$$

A Gale dual of A is given by $B=(2, -5, 3)^t$, and hence the zonotope is the interval $\mathcal{Z}_B = [-5\pi, 5\pi]$. As the translation term is $(0, 0, \pi/2)B = 3\pi/2$, the image of co-ord is $\{-9\pi/2, -5\pi/2, -\pi/2, 3\pi/2, 7\pi/2\}$. The lopsided coamoeba $\mathcal{LA}'(f)$ can be seen in Figure 4. We choose one point from each connected component, namely

$$\theta_1 = -\frac{7\pi}{8}, \quad \theta_2 = -\frac{\pi}{2}, \quad \theta_3 = 0, \quad \theta_4 = \frac{5\pi}{16} \quad \text{and} \quad \theta_5 = \frac{3\pi}{4},$$

and find that

$$v_0(\theta_1) = \left(0, -\frac{5\pi}{8}, \frac{\pi}{8}\right)B = \frac{7\pi}{2},$$
$$v_0(\theta_2) = \left(0, \frac{\pi}{2}, 0\right)B = -\frac{5\pi}{2},$$
$$v_0(\theta_3) = \left(0, 0, \frac{\pi}{2}\right)B = \frac{3\pi}{2},$$
$$v_0(\theta_4) = \left(0, \frac{15\pi}{16}, \frac{\pi}{16}\right)B = -\frac{9\pi}{2},$$
$$v_0(\theta_5) = \left(0, \frac{\pi}{4}, \frac{\pi}{4}\right)B = -\frac{\pi}{2}.$$

It is notable that the orders do not reflect the positions of the connected components of the complement on $\mathbf{T}$.

Let us make a short sidestep and consider the non-closed lopsided coamoeba $\mathcal{LA}'(f)$. The map co-ord extends to a map on $\mathrm{CC}(\mathcal{LA}'(f))$ if one allows the image to contain points on the boundary of $\mathcal{Z}_B$. However, the vertices of $\mathcal{Z}_B$ will not lie in the image of this map.

Theorem 4.12. *Let f be a Laurent polynomial, and let B be a dual matrix of A. Then the map* co-ord *can be extended to a surjective map*

$$\text{co-ord}\colon \operatorname{CC}(\mathcal{LA}'(f)) \longrightarrow (\mathcal{Z}_B \setminus \operatorname{vert}(\mathcal{Z}_B)) \cap (\operatorname{Arg}(c)B + 2\pi\mathbb{Z}[B]),$$

where $\operatorname{vert}(\mathcal{Z}_B)$ *denotes the set of vertices of* $\mathcal{Z}_B$. *If* $g_A=1$ *then this map is an injection.*

Proof. The proof is by following the same steps as in the proofs of Theorem 4.3, and Propositions 4.6 and 4.7, with the only difference that we allow for $|\mu_j|\le 1$. We only note that p is a vertex of $\mathcal{Z}_B$ if and only if any $\mu\in[-1,1]^N$ such that $p=\pi\mu B/2$ has $|\mu_k|=1$ for each k. This implies that $f\langle\theta\rangle$ is contained in one line (but not in an open half-space), and hence that $\theta\in\mathcal{LA}'(f)$. $\square$

Hence we also have a description of the set $\operatorname{CC}(\mathcal{LA}'(f))$, where we note especially that the bound $n!\operatorname{Vol}(\Delta_f)$ does not hold for $|\operatorname{CC}(\mathcal{LA}'(f))|$, as shown in the following example.

Example 4.13. Considering the point configuration A, with Gale dual B, given by

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 2 \\ -2 & -3 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is straightforward to check that the coefficients $c=(1,1,1,1,-1)$ yield that the set $(\mathcal{Z}_B \setminus \operatorname{vert}(\mathcal{Z}_B)) \cap (\operatorname{Arg}(c)B + 2\pi\mathbb{Z}^2)$ contains 6 elements, while $2!\operatorname{Vol}(\Delta_f)=5$.

However, we should remark that the result corresponding to Proposition 3.9 also fails, leaving the question of whether the normalized volume of the Newton polytope is the correct bound also for $|\operatorname{CC}(\mathcal{A}'(f))|$ as an open problem.

5. Coamoebas of polynomials of small codimension

When A is simple the coamoeba $\mathcal{A}'(f)$ is well known, and as noted earlier $\mathcal{A}'(f)=\mathcal{LA}'(f)$. Let us now consider coamoebas of polynomials of codimension one and two.

5.1. Circuits

Consider the case of codimension one, imposing also the assumption that all maximal minors of A are non-vanishing. In particular A is a *circuit*, an important special case treated exhaustively in [5, Chapter 7.1.B]. As before, we can write A in the form

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & A_{\mathbf{1}} & \alpha_{n+1} \end{pmatrix},$$

where $\det(A_{\mathbf{1}}) \neq 0$.

Lemma 5.1. *If A is a circuit, then a dual matrix of A is given by the column vector*

$$B = (\det(A_{\hat{0}}), -\det(A_{\hat{1}}), \dots, (-1)^n \det(A_{\widehat{n}}), (-1)^{n+1} \det(A_{\widehat{n+1}}))^t,$$

where $A_{\hat{\jmath}}$ denotes the $(n+1)\times(n+1)$-matrix obtained by removing the j-th column from A.

Proof. Let us use the notation

$$\det(\hat{A}) = (\det(A_{\hat{0}}), -\det(A_{\hat{1}}), \dots, (-1)^n \det(A_{\widehat{n}}), (-1)^{n+1} \det(A_{\widehat{n+1}}))^t.$$

We can write $A = TM$, where

$$T = \begin{pmatrix} 1 & 0 \\ 0 & A_{\mathbf{1}} \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} 1 & 1 & 1 \\ 0 & I_n & \beta \end{pmatrix},$$

with $\beta = A_{\mathbf{1}}^{-1}\alpha_{n+1} \in \mathbb{Q}^n$. It is straightforward to check that $M \det(\widehat{M}) = 0$, which implies that

$$A \det(\hat{A}) = TM \det(\widehat{M}) \det(T) = 0.$$

As $\det(\hat{A})$ is an integer vector, it follows that it is a dual matrix of A. $\square$

Theorem 5.2. *Let A be a circuit. Then $\overline{\mathcal{L}\mathcal{A}'}(f)$, and hence also $\overline{\mathcal{A}'}(f)$, has $n!\,\mathrm{Vol}(\Delta_f)$ many complement components for generic coefficients.*

Proof. Let $\mathrm{Vol}(A_{\hat{\jmath}})$ denote the normalized volume of the simplex $A_{\hat{\jmath}}$, that is $n!$ times its Euclidean volume. Then $|\det(A_{\hat{\jmath}})| = \mathrm{Vol}(A_{\hat{\jmath}})$. Using the dual matrix given in Lemma 5.1, we find that the zonotope $\mathcal{Z}_B$ is an interval of length

$$\pi(\mathrm{Vol}(A_{\hat{0}}) + \dots + \mathrm{Vol}(A_{\widehat{n+1}})),$$

and it follows from [5, Chapter 7, Proposition 1.2, p. 217] that

$$\pi(\mathrm{Vol}(A_{\hat{0}}) + \dots + \mathrm{Vol}(A_{\widehat{n+1}})) = 2\pi n!\,\mathrm{Vol}(\Delta_f).$$

The components of B are the maximal minors of A, and hence $g_A = g_B$, both which we can assume equals 1. We see that for generic coefficients

$$|\mathrm{int}(\mathcal{Z}_B)\cap(\mathrm{Arg}(c)B+2\pi\mathbb{Z})| = n!\,\mathrm{Vol}(\Delta),$$

and conclude the theorem from Propositions 4.6 and 4.7. □

It was conjectured by Passare [8, Conjecture 8.1] that if A is maximally sparse, then the maximal number of connected components of the complement of the closed coamoeba is obtained for generic coefficients. In general this conjecture is false, with counterexamples given already in the text [8]. However, we can conclude that the conjecture is true in the following special case.

Corollary 5.3. *If the Newton polytope Δ_f has $n+2$ vertices, then the upper bound $n!\,\mathrm{Vol}(\Delta_f)$ on the number of connected components of the complement of the coamoeba $\overline{\mathcal{A}'}(f)$ is obtained for maximally sparse polynomials with generic coefficients.*

Proof. Using the previous theorem, it is enough to show that if f is maximally sparse, then A is a circuit. Indeed, as all points in A are vertices of Δ_f, we find that any choice of $n+1$ points will span a simplex of full dimension, whence the corresponding determinant is non-vanishing. □

When $n \geq 2$, and for generic coefficients, the topological equivalence between $\overline{\mathcal{A}'}(f)$ and $\overline{\mathcal{LA}'}(f)$ implied by Theorem 5.2 also yields a method to construct a set of *base points* for the set of connected components of the complement of the coamoeba, by which we mean a set with exactly one element in each such component. Given a polynomial

$$f(z) = c_0 + c_1 z^{\alpha_1} + \ldots + c_n z^{\alpha_n} + c_{n+1} z^{\alpha_{n+1}},$$

under the above assumptions, consider the n polynomials given by

$$f_j(z) = f(z) - n c_j z^{\alpha_j} - 2c_{n+1} z^{\alpha_{n+1}}, \quad j = 1, \ldots, n,$$

and the system

$$f_1(z) = \ldots = f_n(z) = 0.$$

Note that since $n \geq 2$ we have that $\Delta_{f_j} = \Delta_f$ for each j. Avoiding the discriminant locus of this system, the Bernstein–Kushnirenko theorem [5, Chapter 6, Theorem 2.2, p. 201] tells us that such a system has exactly $n!\,\mathrm{Vol}(\Delta_f)$ distinct solutions in $(\mathbb{C}^*)^n$. Let S be the set of arguments of these solutions. The above system is equivalent to

$$\begin{cases} c_1 z^{\alpha_1} - c_j z^{\alpha_j} = 0, & j = 2, \ldots, n, \\ c_0 - c_{n+1} z^{\alpha_{n+1}} = 0, \end{cases} \tag{13}$$

which shows that for each $\theta \in S$ the set $f\langle\theta\rangle$ contains at most two points. Thus, under the genericity assumption $f\langle\theta\rangle$ is lopsided for each $\theta \in S$. It also follows that $|S| = n!\,\mathrm{Vol}(\Delta_f)$, and that the numbers

$$\phi_\theta = \arg_\pi\Big(\frac{c_1 e^{i\langle\alpha_1,\theta\rangle}}{c_0}\Big) = \ldots = \arg_\pi\Big(\frac{c_n e^{i\langle\alpha_n,\theta\rangle}}{c_0}\Big), \quad \theta \in S,$$

are distinct. Hence, the orders

$$\text{co-ord}(\Theta) = \phi_\theta(0, 1, \ldots, 1, 0)B$$

are also distinct. We conclude that S has exactly one element in each connected component of $\overline{\mathcal{A}'}(f)^c$.

5.2. The case $m=2$ and a relation to discriminants

Let us move up one step in the complexity chain and consider the case when $m=2$. We will assume that $g_A=1$. Related to the point configuration A is the so-called A-*discriminant* $D_A(c)$, which is a polynomial in the coefficients c vanishing if and only if the hypersurface $V(f) \subset (\mathbb{C}^*)^n$ is singular, see [5]. The polynomial $D_A(c)$ enjoys one homogeneity relation for each row of the matrix A, and choosing a Gale dual of A yields a dehomogenization of $D_A(c)$ in the following manner; introducing the variables

$$x_j = c_1^{b_{1j}} \ldots c_N^{b_{Nj}}, \quad j = 1, \ldots, m,$$

there is a Laurent monomial $g(c)$ such that $g(c)D_A(c) = D_B(x)$. This relation is described in more detail in [10], where it was first shown that the zonotope $\mathcal{Z}_B$ together with the coamoeba $\mathcal{A}'(D_B)$ of the dehomogenized discriminant generically covers $\mathbf{T}^2$ precisely $n!\,\mathrm{Vol}(\Delta_f)$ many times. Hence, if $\overline{\mathcal{A}'}(D_B) \neq \mathbf{T}^2$, then there is a choice of coefficients c such that the set $(\mathrm{Arg}(c) + 2\pi\mathbb{Z}^2) \cap \mathrm{int}(\mathcal{Z})$ has $n!\,\mathrm{Vol}(\Delta_f)$ many elements. If so, then we can find a coamoeba whose complement has the maximal number of connected components. As the next example shows this is not always the case.

Example 5.4. Consider the point configuration

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 1 & 2 \\ 0 & 0 & 3 & 3 & 2 \end{pmatrix},$$

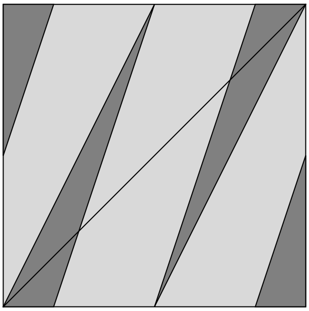

Figure 5. The coamoeba of $D_B(x)$ drawn with multiplicity, *darker areas* are covered twice.

where we note that $2!\,\mathrm{Vol}(\Delta_f)=11$. The dehomogenized discriminant related to the Gale dual

$$B=\begin{pmatrix} 1 & 2 \\ -1 & -3 \\ -2 & -2 \\ 2 & 0 \\ 0 & 3 \end{pmatrix}$$

is

$$\begin{aligned} D_B(x) = {} & 729x_1^2+2187x_1^3+2187x_1^4+729x_1^5+1728x_2+4752x_1x_2+5400x_1^2x_2 \\ & -1404x_1^3x_2-864x_1^4x_2+3456x_2^2-5616x_1x_2^2+576x_1^2x_2^2+256x_1^3x_2^2+1728x_2^3. \end{aligned}$$

Its coamoeba covers the torus $\mathbf{T}^2$ completely, and hence the complement of the closed lopsided coamoeba cannot have more than 10 connected components.

The connection between the zonotope $\mathcal{Z}_B$ and the dehomogenized discriminant $D_B(x)$ is believed to be true also in higher codimensions, however this is still an open problem. For the latest development, we refer the reader to [14].

The fact that we cannot always construct a coamoeba whose complement has $n!\,\mathrm{Vol}(\Delta_f)$ many connected components is of course a source of just criticism. However, let us note that it has not been proved that this upper bound is sharp. To the contrary, recent examples suggest that this is not the case [4].

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Jens Forsgård
Department of Mathematics
Stockholm University
SE-106 91 Stockholm
Sweden
jensf@math.su.se

Petter Johansson
Department of Mathematics
Stockholm University
SE-106 91 Stockholm
Sweden
petterj@math.su.se

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