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A modification of the Hodge star operator on manifolds with boundary

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Abstract. For smooth compact oriented Riemannian manifolds M of dimension $4k+2$, $k\geq 0$, with or without boundary, and a vector bundle F on M with an inner product and a flat connection, we construct a modification of the Hodge star operator on the middle-dimensional (parabolic) cohomology of M twisted by F. This operator induces a canonical complex structure on the middle-dimensional cohomology space that is compatible with the natural symplectic form given by integrating the wedge product. In particular, when $k=0$ we get a canonical almost complex structure on the non-singular part of the moduli space of flat connections on a Riemann surface, with monodromies along boundary components belonging to fixed conjugacy classes when the surface has boundary, that is compatible with the standard symplectic form on the moduli space.

1. Introduction

Let M be a smooth compact oriented Riemannian manifold of dimension n, with or without boundary. Let F be a smooth real vector bundle over M, of finite fiber dimension, equipped with a positive-definite inner product B and a flat connection. We denote by $H^*(M;F)$ the (de Rham) cohomology of M with coefficients in the local system given by F.

Let $*\colon H^*(M;F)\to H^*(M;F)$ be the Hodge star operator given by the orientation and the Riemannian metric on M (see Section 3).

For $n=2m$ the wedge product of forms and the inner product B define a bilinear form $\omega\colon H^m(M;F)\otimes H^m(M;F)\to\mathbb{R}$. If $n=4k+2$, the form ω is skew-symmetric.

If the boundary of M is empty, then the form ω is non-degenerate and gives a symplectic structure on the vector space $H^{2k+1}(M;F)$. It is well known that in this case the Hodge star operator $*$ gives a complex structure on $H^{2k+1}(M;F)$ compatible with the symplectic form ω.

In the general case, when M may have a non-empty boundary, we replace $H^*(M;F)$ by the *parabolic cohomology* $H^*_{\mathrm{par}}(M;F)$ of M with coefficients in the local system given by F (see Section 3). Thus $H^*_{\mathrm{par}}(M;F)$ is the kernel of the homomorphism of restriction to the boundary,

$$H^*_{\mathrm{par}}(M\,;F)=\mathrm{Ker}(r\colon\, H^*(M\,;F)\to H^*(\partial M\,;F)).$$

If $n{=}4k{+}2$, the restriction of the skew-symmetric form ω to the parabolic cohomology $H^{2k+1}_{\mathrm{par}}(M;F)$ is again non-degenerate and equips it with a structure of a symplectic vector space.

It is the aim of this note to show that, if the boundary of M is non-empty and $n{=}4k{+}2$, then there is a canonical modification of the Hodge star operator which gives an operator on parabolic cohomology, denoted here by J_{par},

$$J_{\mathrm{par}}\colon\, H^{2k+1}_{\mathrm{par}}(M\,;F)\longrightarrow H^{2k+1}_{\mathrm{par}}(M\,;F).$$

The operator J_{par} satisfies $J^2_{\mathrm{par}}{=}{-}\mathrm{Id}$ and gives a complex structure on the vector space $H^{2k+1}_{\mathrm{par}}(M;F)$ compatible with the symplectic form ω on it. When the boundary of M is empty then $H^*_{\mathrm{par}}(M;F){=}H^*(M;F)$ and J_{par} is equal to the ordinary Hodge star operator.

If $n{=}2$, i.e. if M is a compact oriented surface one can consider the moduli space $\mathscr{M}$ of flat connections on the trivial principal bundle $M{\times}\mathrm{G}$, G being a compact Lie group with a Lie algebra $\mathfrak{g}$. The flat connections have monodromies along boundary components restricted to fixed conjugacy classes in G. We choose a real-valued invariant positive-definite inner product on $\mathfrak{g}$. The moduli space $\mathscr{M}$ is a manifold with singularities. Away from the singular points, the tangent spaces to $\mathscr{M}$ can be identified with the parabolic cohomology $H^1_{\mathrm{par}}(M;\mathfrak{g}_\phi)$, where $\mathfrak{g}_\phi$ is the trivial vector bundle over M with fiber $\mathfrak{g}$ and connection ϕ. Let $\Sigma{\subset}\mathscr{M}$ denote the singular locus. The symplectic form ω is closed as a 2-form on $\mathscr{M}\backslash\Sigma$ and turns it into a symplectic manifold [3].

Given a Riemannian metric on M, the modified Hodge star operator J_{par} on $H^1_{\mathrm{par}}(M;\mathfrak{g}_\phi)$ constructed in Section 4 gives a canonical almost complex structure on the non-singular part of the moduli space $\mathscr{M}\backslash\Sigma$ compatible with the symplectic form ω. This applies both when the boundary of M is empty and when it is non-empty.

2. A linear problem

Let V be a finite-dimensional vector space over the field of complex numbers $\mathbb{C}$, equipped with a real-valued positive-definite inner product $(\,\cdot\,,\cdot\,)$ such that the operator of multiplication by the complex number $i=\sqrt{-1}$ is an isometry. We denote this operator by J. (In other words, $(\cdot,\cdot)$ is the real part of a hermitian inner product on V.)

Let U be a real subspace of V satisfying

$$J(U)\cap U^{\perp}=\{0\}. \tag{1}$$

Here $U^{\perp}$ denotes the orthogonal complement of U in V with respect to the inner product $(\cdot,\cdot)$. The condition (1) is equivalent to the requirement that the alternating 2-form $\omega(u,v)=(Ju,v)$ is non-degenerate on U and, hence, equips U with a structure of a symplectic space.

The aim of this section is to show that the complex structure of V induces a specific complex structure on every real subspace U satisfying (1). This complex structure will be compatible with the symplectic 2-form $\omega(u,v)=(Ju,v)$ on U.

Let U be a real subspace of V. We denote by $p_U\colon V\to U$ the orthogonal projection of V onto U and define $G\colon U\to U$ by $G(u)=p_U(J(u))$ for $u\in U$.

Lemma 2.1. (i) *For every real subspace U of V, the real linear operator $G\colon U\to U$ is skew-symmetric with respect to the inner product $(\cdot,\cdot)$.*

(ii) *If U satisfies the condition* (1) *then G is invertible and the symmetric operator $G^2=G\circ G\colon U\to U$ is negative definite.*

Proof. (i) Let $u,v\in U$. Since p_U is symmetric, while J is skew-symmetric with respect to $(\cdot,\cdot)$ on V, it follows that

$$\begin{aligned}(G(u),v)&=(p_UJ(u),v)=(J(u),p_U(v))=(J(u),v)\\&=-(u,J(v))=-(p_U(u),J(v))=-(u,p_UJ(v))=-(u,G(v)).\end{aligned}$$

Thus $G\colon U\to U$ is skew-symmetric.

(ii) If U satisfies the condition (1) then $\mathrm{Ker}(p_U)$ intersects the image of $J|_U$ trivially and G is injective and hence invertible. For $u\in U$, $u\neq 0$, we have

$$(G^2(u),u)=-(G(u),G(u))<0$$

and thus G^2 is negative definite. □

Let U satisfy the condition (1) and let $R\colon U\to U$ be the positive square root of the positive-definite symmetric operator $-G^2\colon U\to U$, $R=(-G^2)^{1/2}$. The operator G commutes with $-G^2$ and maps its eigenspaces to themselves. It follows that G commutes with R. We define the operator $J_U\colon U\to U$ by $J_U=R^{-1}G$.

Let $\omega(u,v)=(Ju,v)$ for $u,v\in U$.

Proposition 2.2. *If U is a real subspace of V satisfying the condition (1) then the operator $J_U : U \to U$ satisfies*

(i) $J_U^2 = -\mathrm{Id}$;

(ii) $(J_U(u), J_U(v)) = (u, v)$ *for* $u, v \in U$;

(iii) $\omega(J_U(u), J_U(v)) = \omega(u, v)$ *for* $u, v \in U$;

(iv) $\omega(u, J_U(u)) > 0$ *for all* $u \in U, u \neq 0$;

that is, J_U is a complex structure and an isometry on U, and it is compatible with the symplectic form ω.

Proof. (i) $J_U^2 = R^{-1}GR^{-1}G = R^{-2}G^2 = (-G^2)^{-1}G^2 = -\mathrm{Id}$.

(ii) Since R is symmetric, G is skew-symmetric and R and G commute, we have for $u, v \in U$,

$$(J_U(u), J_U(v)) = (R^{-1}G(u), R^{-1}G(v)) = (G(u), R^{-2}G(v))$$
$$= (u, -GR^{-2}G(v)) = (u, R^{-2}(-G^2)(v)) = (u, v).$$

(iii) Furthermore, we have $GJ_U = GR^{-1}G = J_U G$ and $J_U(v) = p_U J_U(v)$ since $J_U(v) \in U$. Therefore

$$\omega(J_U(u), J_U(v)) = (JJ_U(u), J_U(v)) = (JJ_U(u), p_U J_U(v))$$
$$= (p_U JJ_U(u), J_U(v)) = (GJ_U(u), J_U(v)) = (J_U G(u), J_U(v))$$
$$= (G(u), v) = (p_U J(u), v) = (J(u), v) = \omega(u, v).$$

(iv) Finally, if $u \in U$, $u \neq 0$, then

$$\omega(u, J_U(u)) = (J(u), J_U(u)) = (p_U J(u), J_U(u)) = (G(u), J_U(u))$$
$$= (u, -GJ_U(u)) = (u, -GR^{-1}G(u))$$
$$= (u, R^{-1}(-G^2)(u)) = (u, R^{-1}R^2(u)) = (u, R(u)) > 0$$

since R is a positive-definite symmetric operator on U. □

Example 2.3. Let $V = \mathbb{C}^2$ equipped with the standard inner product on $\mathbb{C}^2$ identified with $\mathbb{R}^4$. Choose a real number $r \in \mathbb{R}$. Let $u_1 = (1, 0)$, $u_2(r) = (i, r) \in V$ and $U_r = \mathrm{span}_{\mathbb{R}}\{u_1, u_2(r)\}$. Thus $n = \dim_{\mathbb{R}} U_r = 2$. Identifying $\mathbb{C}^2$ with $\mathbb{R}^4$ via $\mathbb{C}^2 \ni (z_1, z_2) \leftrightarrow (\mathrm{Re}(z_1), \mathrm{Im}(z_1), \mathrm{Re}(z_2), \mathrm{Im}(z_2)) \in \mathbb{R}^4$ we obtain $U_r = \{(a, b, br, 0) \mid a, b \in \mathbb{R}\}$, $J(U_r) = \{(-b, a, 0, br) \mid a, b \in \mathbb{R}\}$ and $U_r^{\perp} = \{(0, -cr, c, d) \mid c, d \in \mathbb{R}\}$. It follows that for every $r \in \mathbb{R}$, the real subspace U_r satisfies the condition (1): $J(U_r) \cap U_r^{\perp} = \{0\}$. If $r \neq 0$, then U_r satisfies the additional property

$$J(U_r) \cap U_r = \{0\}, \tag{2}$$

that is, U_r is a *totally real subspace* of V. Taking direct sums of pairs (V, U_r) one gets examples of subspaces U satisfying the condition (1) in every even dimension n. The skew-symmetric operator $G\colon U_r \to U_r$ is given by $G(u_1) = (1/(1+r^2))u_2(r)$ and $G(u_2(r)) = -u_1$. Hence, $G^2 = -(1/(1+r^2))\,\mathrm{Id}_{U_r}$, $R = (1/\sqrt{1+r^2})\,\mathrm{Id}_{U_r}$, and the complex structure $J_{U_r}\colon U_r \to U_r$ is given by $J_{U_r}(u_1) = (1/\sqrt{1+r^2})u_2(r)$ and $J_{U_r}(u_2(r)) = -\sqrt{1+r^2}u_1$.

Real subspaces U satisfying both properties (1) and (2) are typical of the geometric context in which the observations of the present section will be applied.

3. Hodge theory on manifolds with boundary

This section is devoted to a recollection of background material on Hodge theory on manifolds with boundary that will be used in the following sections.

Let M be a smooth compact oriented Riemannian manifold of dimension n, with or without boundary. Let F be a smooth real vector bundle over M, of finite fiber dimension, equipped with a positive-definite inner product $B(\,\cdot\,,\cdot\,)$ and a flat connection A. Let $d_A\colon \Omega^0(F) \to \Omega^1(F)$ be the covariant derivative operator corresponding to A. Here we use $\Omega^p(F)$ to denote smooth sections of $\Lambda^p T^*M \otimes F$, the p-forms with values in F. We also write $d_A\colon \Omega^p(F) \to \Omega^{p+1}(F)$ for the unique extension of the covariant derivative that satisfies the Leibniz rule. Since A is a flat connection, we have $d_A d_A = 0$ and get a cochain complex

$$(3) \qquad 0 \longrightarrow \Omega^0(F) \xrightarrow{d_A} \Omega^1(F) \xrightarrow{d_A} \dots \xrightarrow{d_A} \Omega^p(F) \xrightarrow{d_A} \Omega^{p+1}(F) \longrightarrow \dots .$$

The Riemannian metric, the orientation on M and the inner product B on F give rise to the L^2 inner product $(\,\cdot\,,\cdot\,)$ on $\Omega^*(F)$,

$$(\alpha, \beta) = \int_M B(\alpha \wedge *\beta),$$

where $*$ denotes the Hodge star operator. (The Hodge star operator $*$ on $\Lambda^* T^*M \otimes F$ is defined as the tensor product of the usual Hodge star operator on $\Lambda^* T^*M$ with the identity on F.) We have also the codifferential

$$\delta_A = (-1)^{n(p+1)+1} * d_A * \colon \Omega^p(F) \longrightarrow \Omega^{p-1}(F),$$

which on closed manifolds is the L^2-adjoint of the operator d_A.

From now on the operators d_A and δ_A will be denoted by d and δ respectively.

For the Hodge decomposition theorem on manifolds with boundary we refer to [4] and [1].

A form $\omega\in\Omega^p(F)$ is called *closed* if it satisfies $d\omega=0$ and *coclosed* if it satisfies $\delta\omega=0$. We denote by C^p and cC^p the spaces of closed respectively coclosed p-forms. We define $E^p=d(\Omega^{p-1}(F))$ and $cE^p=\delta(\Omega^{p+1}(F))$.

Along the boundary ∂M every p-form $\omega\in\Omega^p(F)$ can be decomposed into tangential and normal components (depending on the Riemannian metric on M). For $x\in\partial M$, one has

$$\omega(x)=\omega_{\tan}(x)+\omega_{\rm norm}(x), \tag{4}$$

where $\omega_{\rm norm}(x)$ belongs to the kernel of the restriction homomorphism

$$r^*\colon \Lambda^*T_x^*M\otimes F_x\longrightarrow \Lambda^*T_x^*(\partial M)\otimes F_x,$$

while $\omega_{\tan}(x)$ belongs to the orthogonal complement of that kernel,

$$\omega_{\tan}(x)\in \operatorname{Ker}(r^*)^{\perp}\subset \Lambda^*T_x^*M\otimes F_x.$$

Note that r^* maps the orthogonal complement $\operatorname{Ker}(r^*)^{\perp}$ of the kernel isomorphically onto $\Lambda^*T_x^*(\partial M)\otimes F_x$.

Following [1], we define Ω_N^p to be the space of smooth p-forms from $\Omega^p(F)$ satisfying *Neumann boundary conditions* at every point of ∂M,

$$\Omega_N^p=\{\omega\in\Omega^p(F)\mid \omega_{\rm norm}=0\},$$

and Ω_D^p to be the space of smooth p-forms from $\Omega^p(F)$ satisfying *Dirichlet boundary conditions* at every point of ∂M,

$$\Omega_D^p=\{\omega\in\Omega^p(F)\mid \omega_{\tan}=0\}.$$

Furthermore, we define $cE_N^p=\delta(\Omega_N^{p+1})$ and $E_D^p=d(\Omega_D^{p-1})$ and let

$$CcC^p=C^p\cap cC^p=\{\omega\in\Omega^p(F)\mid d\omega=0 \text{ and } \delta\omega=0\},$$

$$CcC_N^p=\{\omega\in\Omega^p(F)\mid d\omega=0,\ \delta\omega=0 \text{ and } \omega_{\rm norm}=0\},$$

$$CcC_D^p=\{\omega\in\Omega^p(F)\mid d\omega=0,\ \delta\omega=0 \text{ and } \omega_{\tan}=0\}.$$

If the boundary is empty, $\partial M=\varnothing$, then, trivially, every form ω satisfies $\omega_{\rm norm}=\omega_{\tan}=0$, the space $CcC^p=CcC_N^p=CcC_D^p$ consists of all forms which are both closed and coclosed, and this space is equal to the space of harmonic p-forms, that is, to the kernel of the Laplacian $\Delta=\delta d+d\delta$ acting on $\Omega^p(F)$.

If, on the other hand, the boundary is non-empty, $\partial M\neq\varnothing$ and M is connected then the intersection $CcC_N^p\cap CcC_D^p=\{0\}$ ([1], Lemma 2) and the kernel of the Laplacian Δ contains all forms which are both closed and coclosed but can be strictly larger than the space of such forms ([1], Example).

In the following the symbol $\oplus$ will denote an orthogonal direct sum.

Theorem 3.1. (Hodge decomposition theorem) *Let M be a compact connected oriented smooth Riemannian n-manifold, with or without boundary and let F be a smooth real vector bundle over M, of finite fiber dimension, equipped with an inner product and a flat connection A. Then the space $\Omega^p(F)$ of F-valued smooth p-forms decomposes into the orthogonal direct sum*

$$\Omega^p(F) = cE^p_N \oplus CcC^p \oplus E^p_D. \tag{5}$$

Furthermore, we have the orthogonal direct sum decompositions

$$CcC^p = CcC^p_N \oplus (E^p \cap cC^p) = (C^p \cap cE^p) \oplus CcC^p_D. \tag{6}$$

For the proof of Theorem 3.1 see [4].

We denote by $H^*(M;F)$ the cohomology of the complex (3) and define $H^*(\partial M;F|_{\partial M})$ and $H^*(M,\partial M;F)$ accordingly.

It follows from (5) that the space C^p of closed p-forms decomposes as $C^p = CcC^p \oplus E^p_D$. Hence, from (6), we get $C^p = CcC^p \oplus E^p_D = CcC^p_N \oplus (E^p \cap cC^p) \oplus E^p_D$. Using (6) once again we see that $(E^p \cap cC^p) \oplus E^p_D = E^p$. Therefore,

$$C^p = CcC^p \oplus E^p_D = CcC^p_N \oplus (E^p \cap cC^p) \oplus E^p_D = CcC^p_N \oplus E^p. \tag{7}$$

Thus, CcC^p_N is the orthogonal complement of the exact p-forms within the closed ones, so $CcC^p_N \cong H^p(M;F)$. In a similar way, the space cC^p of coclosed p-forms decomposes as

$$cC^p = cE^p_N \oplus CcC^p = cE^p_N \oplus (C^p \cap cE^p) \oplus CcC^p_D = cE^p \oplus CcC^p_D. \tag{8}$$

It follows again from (5) and (6) that $CcC^p_D \cong H^p(M,\partial M;F)$.

4. A modified Hodge star operator on parabolic cohomology

The main aim of this section is to define a modified Hodge star operator on the parabolic cohomology (the definition of parabolic cohomology is recalled below).

As in Section 3, M is a smooth compact oriented Riemannian manifold of dimension n, with or without boundary and F is a smooth real vector bundle over M with a positive-definite inner product $B(\cdot,\cdot)$ and a flat connection A.

Let now $r^*\colon H^*(M;F) \to H^*(\partial M;F|_{\partial M})$ be the homomorphism of the restriction to the boundary.

We define *the parabolic cohomology $H^*_{\mathrm{par}}(M;F)$ of the manifold M with coefficients in the bundle F with the flat connection A* to be the kernel of the restriction homomorphism r^*,

$$H^*_{\mathrm{par}}(M;F) := \mathrm{Ker}\,(r^*\colon H^*(M;F) \to H^*(\partial M;F|_{\partial M}))$$

(cf. [5] and [3], Section 3).

Of course, the parabolic cohomology $H^*_{\mathrm{par}}(M;F)$ is equal to the image of $j^*\colon H^*(M,\partial M;F)\to H^*(M;F)$.

We assume now that the manifold M has dimension $n=4k+2$. When $p=2k+1$, the Hodge star operator $*$ maps $\Omega^p(F)$ onto itself, $*\colon \Omega^p(F)\to\Omega^p(F)$, and satisfies $**=-\mathrm{Id}$. Moreover, it maps CcC^p onto itself, mapping CcC^p_N onto CcC^p_D and vice versa. Thus $*$ gives a complex structure on $\Omega^p(F)$ and on CcC^p. For the rest of this section we shall denote the Hodge star operator $*$ on $\Omega^p(F)$ by J. We have

$$J(CcC^p)=CcC^p,\quad J(CcC^p_N)=CcC^p_D\quad\text{and}\quad J(CcC^p_D)=CcC^p_N. \tag{9}$$

Since M is compact, the cohomology groups $H^p(M;F)$ and $H^p(M,\partial M;F)$ and, hence, CcC^p_N and CcC^p_D are finite-dimensional vector spaces. Denote by $P_N\colon CcC^p\to CcC^p_N$ and $P_D\colon CcC^p\to CcC^p_D$ the orthogonal projections of CcC^p onto CcC^p_N and CcC^p_D respectively. By (6) the kernel $\mathrm{Ker}(P_N)$ is equal to $E^p\cap cC^p$, while the kernel $\mathrm{Ker}(P_D)$ is equal to $C^p\cap cE^p$. Since J is an isometry of CcC^p, it follows from (9) that $P_N\circ J=J\circ P_D$. Let $\mathscr{P}_N\colon CcC^p_D\to CcC^p_N$ be the restriction of P_N to CcC^p_D and let $\mathscr{P}_D\colon CcC^p_N\to CcC^p_D$ be the restriction of P_D to CcC^p_N. We have

$$\mathscr{P}_N\circ J=J\circ\mathscr{P}_D. \tag{10}$$

When $H^p(M,\partial M;F)$ is identified with CcC^p_D and $H^p(M;F)$ with CcC^p_N, the homomorphism $j^*\colon H^p(M,\partial M;F)\to H^p(M;F)$ is identified with $\mathscr{P}_N\colon CcC^p_D\to CcC^p_N$. The parabolic cohomology group $H^p_{\mathrm{par}}(M;F)$ is thus identified with the image of $\mathscr{P}_N\colon CcC^p_D\to CcC^p_N$ which we denote by U, $U=\mathrm{Im}(\mathscr{P}_N)\subset CcC^p_N$.

It follows then from (10) that $J(U)$ is equal to the image of $\mathscr{P}_D\colon CcC^p_N\to CcC^p_D$. We denote this image by T, $T=\mathrm{Im}(\mathscr{P}_D)=J(U)\subset CcC^p_D$.

Let $T^\perp$ be the orthogonal complement of T in CcC^p_D.

Lemma 4.1. *The kernel of $\mathscr{P}_N\colon CcC^p_D\to CcC^p_N$ is equal to $T^\perp$.*

Proof. Let $w\in T^\perp\subset CcC^p_D$. Let $x\in CcC^p_N$. As P_D is a symmetric mapping and since $\mathscr{P}_D(x)\in T$, we get that $(w,x)=(P_D(w),x)=(w,P_D(x))=(w,\mathscr{P}_D(x))=0$. Hence w is orthogonal to CcC^p_N and therefore $\mathscr{P}_N(w)=0$. Thus $T^\perp\subset\mathrm{Ker}(\mathscr{P}_N)$. On the other hand

$$\begin{aligned}\dim T^\perp&=\dim CcC^p_D-\dim T=\dim CcC^p_D-\dim U\\&=\dim CcC^p_D-\dim\mathrm{Im}(\mathscr{P}_N)=\dim\mathrm{Ker}(\mathscr{P}_N).\end{aligned}$$

Thus $T^\perp=\mathrm{Ker}(\mathscr{P}_N)$. □

Lemma 4.2. *Let $v \in T = J(U)$. If v is orthogonal to U then $v = 0$.*

Proof. Assume that $v \in T = J(U)$ is orthogonal to U. Since $v \in CcC_D^p$, we have $\mathscr{P}_N(v) \in U = \mathrm{Im}(\mathscr{P}_N)$. On the other hand, as $\mathscr{P}_N$ is a projection along a space orthogonal to CcC_N^p and, hence, orthogonal to U, we get that $\mathscr{P}_N(v)$ is also orthogonal to U. Since $\mathscr{P}_N(v)$ both belongs to U and is orthogonal to U, we must have $\mathscr{P}_N(v) = 0$. Thus v belongs to $\mathrm{Ker}(\mathscr{P}_N)$ which, by Lemma 4.1, is equal to $T^\perp$. Belonging to T and $T^\perp$ at the same time, v must be 0. □

Let V be the subspace of CcC^p spanned by CcC_D^p and CcC_N^p. Since both these spaces are finite-dimensional, so is V. Moreover, (9) implies that V is a complex subspace of CcC^p with respect to the complex structure J given by the Hodge star operator. V inherits the real inner product $(\cdot, \cdot)$ from CcC^p and J acts as an isometry. Finally, $U \subset V$ and, according to Lemma 4.2,

$$J(U) \cap U^\perp = 0, \tag{11}$$

where this time $U^\perp$ denotes the orthogonal complement of U in V.

The alternating 2-form $\omega(u, v) = (J(u), v)$ is a symplectic (non-degenerate) form on V. The property (11) implies that the restriction of ω to U is a symplectic (non-degenerate) form on U.

Since (11) is satisfied, we can now apply the construction of Section 2 to V, U and J and obtain a linear operator

$$J_U : U \longrightarrow U,$$

which equips the space U with a complex structure. When U is identified with the parabolic cohomology $H^p_{\mathrm{par}}(M; F)$ we denote the operator corresponding to J_U by J_{par},

$$J_{\mathrm{par}} : H^p_{\mathrm{par}}(M; F) \longrightarrow H^p_{\mathrm{par}}(M; F) \tag{12}$$

and call it *the modified Hodge star operator on the parabolic cohomology*. We have the real inner product $(\cdot, \cdot)$ and the symplectic form ω on $H^p_{\mathrm{par}}(M; F) = U$. Proposition 2.2 now gives the following result.

Theorem 4.3. *Let M be a smooth compact oriented Riemannian manifold of dimension $n = 4k+2$, with or without boundary, and F be a real finite-dimensional vector bundle over M equipped with an inner product and a flat connection. Let $p = 2k+1$. Then the modified Hodge star operator $J_{\mathrm{par}} : H^p_{\mathrm{par}}(M; F) \to H^p_{\mathrm{par}}(M; F)$ satisfies*

(i) $J^2_{\mathrm{par}} = -\mathrm{Id}$;

(ii) $\omega(J_{\text{par}}(u), J_{\text{par}}(v)) = \omega(u,v)$ *for* $u, v \in H^p_{\text{par}}(M;F)$;

(iii) $\omega(u, J_{\text{par}}(u)) > 0$ *for all* $u \in H^p_{\text{par}}(M;F)$, $u \neq 0$;

that is, J_{par} *is a complex structure on the parabolic cohomology* $H^p_{\text{par}}(M;F)$ *compatible with the symplectic form* ω.

Remark 4.4. (i) The symplectic form ω on $H^p_{\text{par}}(M;F) = U$ is the restriction of the form ω on $H^p(M;F) = CcC^p_N$ which in turn is given by

$$\omega(u,v) = (Ju, v) = (*u, v) = (v, *u) = \int_M B(v \wedge **u) = \int_M B(u \wedge v) = \\ = ([u] \cup [v])[M; \partial M],$$

where $[u]$ and $[v]$ denote the cohomology classes of the closed forms u and v. Thus the symplectic form ω is given by the cup (wedge) product composed with B.

(ii) When M is without boundary, $\partial M = \varnothing$, then $CcC^p_N = CcC^p_D = U = J(U)$ above and $J_{\text{par}} = J = *$. Thus, in that case, the parabolic cohomology $H^p_{\text{par}}(M;F)$ is equal to the ordinary cohomology $H^p(M;F)$ and the modified Hodge star operator is equal to the ordinary Hodge star operator.

(iii) If M is not connected then it is obvious from the construction above that the parabolic cohomology $H^p_{\text{par}}(M;F)$ and the modified Hodge star operator J_{par} are direct sums of their counter-parts on the components.

(iv) The modified Hodge star operator J_{par} is canonically determined by the choice of the Riemannian metric and the orientation on M, and the choice of the inner product and the flat connection on F.

5. The moduli space of flat connections on a Riemann surface with boundary

Let G be a compact Lie group with a Lie algebra $\mathfrak{g}$ equipped with a real-valued positive-definite invariant inner product. Let S be a smooth compact oriented surface, with or without boundary. We consider the moduli space $\mathscr{M} = \mathscr{M}(S; G, C_1, ..., C_k)$ of gauge equivalence classes of flat connections in the trivial principal G-bundle over S with monodromies along boundary components belonging to some fixed conjugacy classes $C_1, ..., C_k$ in G, k being the number of boundary components of S (see [3]).

The space $\mathscr{M}$ is a finite-dimensional manifold with singularities. We denote by $\Sigma \subset \mathscr{M}$ the singular locus. Every point of $\mathscr{M}$ can be represented by a group homomorphism $\phi\colon \pi_1(S) \to G$ such that ϕ maps elements of $\pi_1(S)$ given by the boundary components into the corresponding conjugacy classes C_j. Let G act on $\mathfrak{g}$ through the adjoint representation. To every such group homomorphism ϕ we can associate a bundle over S with fiber $\mathfrak{g}$ equipped with a flat connection and an $\mathbb{R}$-valued positive-definite inner product in the fibers. We denote that flat vector bundle by $\mathfrak{g}_\phi$. The tangent space to $\mathscr{M}$ at a non-singular point $[\phi]\in\mathscr{M}$ is naturally identified with the parabolic cohomology group $H^1_{\mathrm{par}}(S;\mathfrak{g}_\phi)$ (see [3], Section 3, Propositions 4.4 and 4.5 and pp. 409–410).

In [3] the manifold $\mathscr{M}\setminus\Sigma$ is equipped with a symplectic structure given by -1 times the wedge product of forms and the inner product on the bundle $\mathfrak{g}_\phi$ ([3], Section 3, pp. 386–387, and Theorem 10.5). Hence, this symplectic structure is the negative of the one given by the form ω in our paper.

It follows now from Theorem 4.3 that a choice of a Riemannian metric on the surface S gives, via the modified Hodge star operator J_{par}, a canonical almost complex structure on the moduli space $\mathscr{M}\setminus\Sigma$ compatible with the symplectic form ω. To get an almost complex structure on $\mathscr{M}\setminus\Sigma$ compatible with the symplectic form of [3] one has to take the operator $-J_{\mathrm{par}}$.

Note added in proof. The property (11) has also been proven in [2].

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