

# Equivariant Poincaré series of filtrations and topology

Antonio Campillo, Félix Delgado and Sabir M. Gusein-Zade

**Abstract.** Earlier, for an action of a finite group  $G$  on a germ of an analytic variety, an equivariant  $G$ -Poincaré series of a multi-index filtration in the ring of germs of functions on the variety was defined as an element of the Grothendieck ring of  $G$ -sets with an additional structure. We discuss to which extent the  $G$ -Poincaré series of a filtration defined by a set of curve or divisorial valuations on the ring of germs of analytic functions in two variables determines the (equivariant) topology of the curve or of the set of divisors.

## 1. Introduction

The Poincaré series of a multi-index filtration (say, on the ring of germs of functions on a variety) was defined in [5]. It was computed for filtrations on the ring  $\mathcal{O}_{\mathbb{C}^2,0}$  of germs of analytic functions in two variables corresponding to plane curve singularities with several branches [1] and for divisorial ones [7]. In [1] it was found that the Poincaré series of the filtration defined by a plane curve singularity  $(C,0) \subset (\mathbb{C}^2,0)$  coincides with the Alexander polynomial in several variables of the corresponding link  $C \cap S_\varepsilon^3 \subset S_\varepsilon^3$  ( $S_\varepsilon^3$  is the sphere of a small radius  $\varepsilon$  centered at the origin in  $\mathbb{C}^2$ ). Therefore it defines the (embedded) topology of the plane curve singularity [9]. Identifying all the variables in the Alexander polynomial one gets the monodromy zeta function of the singularity. In [3] it was shown that the Poincaré series of a divisorial filtration in the ring  $\mathcal{O}_{\mathbb{C}^2,0}$  of germs of functions in two variables also defines the topology of the corresponding set of divisors (more precisely the topology of its minimal resolution). The corresponding statement for the divisorial

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filtration defined by *all* components of the exceptional divisor of a resolution of a normal surface singularity was obtained in [6].

The intention to generalize connections between Poincaré series of filtrations and monodromy zeta functions to an equivariant context led to the desire to define equivariant analogues of Poincaré series and of monodromy zeta functions. In particular, in [4] there was defined an equivariant analogue of the Poincaré series of a multi-index filtration on the ring of functions on a complex-analytic space singularity with an action of a finite group  $G$ . This  $G$ -Poincaré series is an element of the Grothendieck ring  $K_0((G, r)\text{-sets})$  of  $G$ -sets with an additional structure. (For the trivial group  $G$  this ring coincides with the ring  $\mathbb{Z}[[t_1, \dots, t_r]]$  of power series in several variables.) It was computed for the filtrations defined by plane curve singularities and for divisorial filtrations in the plane (see Section 3 below). Here we discuss to which extent the  $G$ -Poincaré series of these filtrations determine the (equivariant) topology of the curve or of the set of divisors. We show that the  $G$ -Poincaré series of a collection of divisorial valuations determines the topology of the set of divisors. This is not, in general, the case for curve valuations. We describe some conditions on curves under which the corresponding statement holds. It remains unclear whether the (equivariant) topology of a collection of curves always determines the  $G$ -Poincaré series of the collection.

## 2. $G$ -equivariant resolutions

It is well known that two plane curve singularities are topologically equivalent if and only if they have combinatorially equivalent (embedded) resolutions: see e.g. [8]. The formula for the Poincaré series of a plane curve singularity in terms of a resolution from [1] implies, in particular, that the Poincaré series of a plane curve singularity is a topological invariant. The notion of topological equivalence of two sets of divisors in [3] was in fact formulated in terms of topologically equivalent resolutions.

*Remark 2.1.* A divisorial valuation can be defined by a (generic) pair of curvettes corresponding to the divisor (see below). In this way a set of divisors can be defined by a curve (the union of the corresponding pairs of curvettes) and two sets of divisors are topologically equivalent if and only if the corresponding curves are topologically equivalent.

Here we discuss the concept of topologically equivalent resolutions in an equivariant setting.

Let a finite group  $G$  act on  $(\mathbb{C}^2, 0)$  (by complex-analytic transformations). Without loss of generality one can assume that this action is faithful and is defined by a two-dimensional representation of the group  $G$  (i.e. that  $G$  acts on  $\mathbb{C}^2$  by linear transformations).

Let  $(C_i, 0) \subset (\mathbb{C}^2, 0)$ ,  $i=1, \dots, r$ , be (different) irreducible plane curve singularities, i.e. branches.

*Remark 2.2.* Here we do not assume, in general, that the curve  $\bigcup_{i=1}^r C_i$  is  $G$ -invariant (i.e. that the set  $\{C_i\}_{i=1}^r$  contains all  $G$ -shifts of its elements) or that all the branches  $C_i$  belong to different orbits of the  $G$ -action. Restrictions of this sort could be required for particular statements.

*Definition 2.3.* A  $G$ -equivariant resolution (or simply a  $G$ -resolution) of the set  $\{C_i\}_{i=1}^r$  (or of the curve  $\bigcup_{i=1}^r C_i$ ) is a proper complex-analytic map  $\pi: (\mathcal{X}, \mathcal{D}) \rightarrow (\mathbb{C}^2, 0)$  from a smooth surface  $\mathcal{X}$  with an action of the group  $G$  such that

- (1)  $\pi$  is an isomorphism outside of the origin in  $\mathbb{C}^2$ ;
- (2)  $\pi$  commutes with the  $G$ -actions on  $\mathcal{X}$  and on  $\mathbb{C}^2$ ;
- (3) the total transform  $\pi^{-1}(\bigcup_{i=1}^r C_i)$  of the curve  $\bigcup_{i=1}^r C_i$  is a normal crossing divisor on  $\mathcal{X}$ ;

(4) for each branch  $C_i$  its strict transform  $\tilde{C}_i$  is a germ of a smooth curve transversal to the exceptional divisor  $\mathcal{D} = \pi^{-1}(0)$  at a smooth point  $x$  of it and is invariant with respect to the isotropy subgroup  $G_x = \{g \in G: gx = x\}$  of the point  $x$ .

*Remark 2.4.* (a) The resolution  $\pi$  can be obtained by a sequence of blow-ups of points (preimages of the origin). The exceptional divisor  $\mathcal{D}$  is the union of its irreducible components  $E_\sigma$ ,  $\sigma \in \Gamma$ . The set  $\Gamma$  inherits the partial order defined by a representation of  $\pi$  as a sequence of blow-ups: a component  $E_{\sigma'}$  is greater than another component  $E_\sigma$ ,  $\sigma' > \sigma$ , if the exceptional divisor of any modification which contains  $E_{\sigma'}$  also contains  $E_\sigma$ . The condition (2) means that this sequence of blow-ups should be  $G$ -equivariant, i.e., if a point  $x$  is blown-up, the point  $gx$  should be blown-up for each  $g \in G$  as well. In particular, the set  $\Gamma$  of the components of the exceptional divisor is a  $G$ -set.

(b) The condition (4) is equivalent to saying that  $\pi$  is a resolution of the curve  $C = \bigcup_{i,g} gC_i$  where  $g$  runs through all the elements of  $G$  and  $i=1, \dots, r$ . In particular,  $\pi^{-1}(C)$  is a normal crossing divisor on  $\mathcal{X}$ . A smooth irreducible curve  $(C_1, 0) \subset (\mathbb{C}^2, 0)$  has a trivial  $G$ -resolution  $(\mathbb{C}^2, 0) \dashrightarrow (\mathbb{C}^2, 0)$  if and only if it is  $G$ -invariant.

The group  $G$  acts both on the space  $\mathcal{X}$  of the resolution and on the exceptional divisor  $\mathcal{D}$ . For  $\sigma \in \Gamma$ , let  $G_\sigma$  be the isotropy subgroup of the component  $E_\sigma$ ,

i.e.  $\{g \in G : gE_\sigma = E_\sigma\}$ . Note that the isotropy subgroups  $G_\sigma$  are Abelian for all  $\sigma$  except possibly the first (minimal) one. (In the latter case  $G_\sigma$  coincides with the group  $G$  itself.) The group  $G_\sigma$  acts on the component  $E_\sigma$ . Let  $G_\sigma^* \subset G_\sigma$  be the isotropy subgroup  $G_x = \{g \in G : gx = x\}$  of a generic point  $x \in E_\sigma$ . (The group  $G_\sigma^*$  is always Abelian.) A point  $x \in E_\sigma$  will be called *special* if its isotropy subgroup  $G_x$  is different from  $G_\sigma^*$ . (In this case  $G_x \supset G_\sigma^*$  and  $G_x \neq G_\sigma^*$ .) One has a one-dimensional representation  $\beta_{x\sigma}$  of the isotropy subgroup  $G_x$  of a point  $x \in E_\sigma$  in the normal space to  $E_\sigma$  in  $\mathcal{X}$  at the point  $x$ .

Let  $E_\sigma$  be a component of the exceptional divisor  $\mathcal{D}$ , let  $x \in E_\sigma$  be a smooth point of  $\mathcal{D}$ , i.e. a point which is not a point of intersection with other components of  $\mathcal{D}$ , and let  $\tilde{L}$  be a germ of a smooth curve on  $\mathcal{X}$  at the point  $x$  invariant with respect to the isotropy group  $G_x$  of the point  $x$ . The curve  $L = \pi(\tilde{L})$  is called a *curvette* corresponding to the component  $E_\sigma$  (and/or to the point  $x$ ).

A  $G$ -resolution of a curve  $(\bigcup_{i=1}^r C_i, 0) \subset (\mathbb{C}^2, 0)$  can be described by its dual resolution graph in the following way: The vertices of this graph correspond to the components  $E_\sigma$  of the exceptional divisor  $\mathcal{D}$  (i.e. to the elements of the partially ordered  $G$ -set  $\Gamma$ ) and to the strict transforms of the curves  $C_i$  and of their  $G$ -shifts  $gC_i$ ,  $g \in G$ . These vertices will be depicted by dots and arrows respectively. There is a natural  $G$ -action on the set of vertices of the graph (preserving dots and arrows). Two vertices are connected by an edge if and only if the corresponding components of the total transform  $\pi^{-1}(C)$  of the curve  $C = \bigcup_{i,g} gC_i$  intersect.

*Remark 2.5.* Observe that the described information (namely the  $G$ -action on the set of vertices of the graph) determines the isotropy subgroups  $G_\sigma$  of the components and the isotropy subgroups of all the intersection points of the components of the total transform  $\pi^{-1}(C)$  of the curve  $C$  (all the latter subgroups are Abelian).

The same definition and description apply to a set of divisorial valuations with the only difference that in this case the corresponding divisors should be indicated and there are no strict transforms of branches. Also Remark 2.4(a) is valid.

In what follows we shall use the following description of the behavior of representations under blow-ups. Assume that one has the complex plane  $\mathbb{C}^2$  with a representation of a finite Abelian group  $H$ . This representation is the sum of two irreducible ones, say  $\gamma_1$  and  $\gamma_2$ . Let  $p: (Y, E) \rightarrow (\mathbb{C}^2, 0)$  be the blow-up of the origin ( $E \simeq \mathbb{C}\mathbb{P}^1$ ). The group  $H$  acts on  $Y$  as well.

If the representations  $\gamma_1$  and  $\gamma_2$  are different then the  $H$ -action on  $E$  has two special points invariant with respect to  $H$ . At one of them the representation of  $H$  in the normal space to  $E$  is  $\gamma_1$  and in the tangent space to  $E$  is  $\gamma_2\gamma_1^{-1}$ . At the other one they are  $\gamma_2$  and  $\gamma_1\gamma_2^{-1}$  respectively. The isotropy subgroup  $H_x$  of a non-special

point  $x$  of  $E$  is  $H_x = \{h \in H : \gamma_1(h) = \gamma_2(h)\}$ . The representation of  $H_x$  in the normal space to  $E$  is  $\gamma_1|_{H_x} = \gamma_2|_{H_x}$ . (The representation of  $H_x$  in the tangent space to  $E$  is trivial.)

If the representations  $\gamma_1$  and  $\gamma_2$  coincide, there are no special points on  $E$ , the action of  $H$  on  $E$  is trivial and the representation of  $H_x = H$  in the normal space to  $E$  is  $\gamma_1 = \gamma_2$ .

Thus the representation of  $H$  on  $\mathbb{C}^2$  determines the representations in the tangent and in the normal spaces to  $E$  at all points.

Let  $\{C_i\}_{i=1}^r$  and  $\{C'_i\}_{i=1}^r$  be two collections of branches in the complex plane  $(\mathbb{C}^2, 0)$  with an action of a finite group  $G$ . We say that these collections are *G-topologically equivalent* if there exists a  $G$ -invariant germ of a homeomorphism  $\psi : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$  such that  $\psi(C_i) = C'_i$  for  $i=1, \dots, r$ . A version of this definition can be applied to collections of divisorial valuations as well. A divisorial valuation  $v$  on  $\mathcal{O}_{\mathbb{C}^2, 0}$  can be described by a generic pair of curvettes corresponding to the divisor. (Genericity means that the strict transforms of the curvettes intersect the divisor at different points.) Two collections of divisorial valuations  $\{v_i\}_{i=1}^r$  and  $\{v'_i\}_{i=1}^r$  are said to be *G-topologically equivalent* if the corresponding collections of curvettes  $\{L_{ij}\}_{i=1, j=1, 2}^r$  and  $\{L'_{ij}\}_{i=1, j=1, 2}^r$  are *G-topologically equivalent*.

It is clear that the  $G$ -resolution graph of a collection of curve or divisorial valuations does not determine the  $G$ -equivariant topology of the collection. Moreover the  $G$ -resolution graphs of collections can be the same for different actions of the group  $G$  on  $\mathbb{C}^2$  (e.g. if  $G$  is Abelian and all blow-ups are performed at points with the isotropy subgroups  $G_x = G$ ). Even if the representation of  $G$  is fixed, the  $G$ -resolution graph of a collection of curve or divisorial valuations does not determine the  $G$ -topology of the collection.

*Example 2.6.* Let  $\mathbb{C}^2$  be the complex plane with the action of the cyclic group  $G = \mathbb{Z}_{15}$  defined by  $\sigma^*(x, y) = (\sigma^3 x, \sigma^5 y)$ , where  $\sigma = \exp(2\pi i/15)$  is the generator of the group  $\mathbb{Z}_{15}$  (and  $i$  here, but not elsewhere, denotes the imaginary unit). Let  $C_i$ ,  $i=1, 2, 3$ , be the curves (given by their parameterizations)  $(t, 0)$ ,  $(0, t)$  and  $(t, t^2)$ , respectively and let  $C'_i$ ,  $i=1, 2, 3$ , be the curves  $(0, t)$ ,  $(t, 0)$  and  $(t^2, t)$ . An important property of these curves is that no element of  $G$  different from 1 sends the curve  $C_3$  (or the curve  $C'_3$ ) to itself and that, in the minimal  $G$ -resolution, the strict transform of the curve  $C_3$  (or the curve  $C'_3$ ) intersects a component  $E_\sigma$  of the exceptional divisor with  $G_\sigma = G$  and  $G_\sigma^* = (e)$ . (Thus the strict transforms of all the  $G$ -shifts of the curve intersect one and the same component of the exceptional divisor at different points.)

The minimal  $G$ -resolution graph is shown on Figure 1 (it is one and the same for both cases). The partial order on  $\Gamma$  is defined by the numbering of the elements

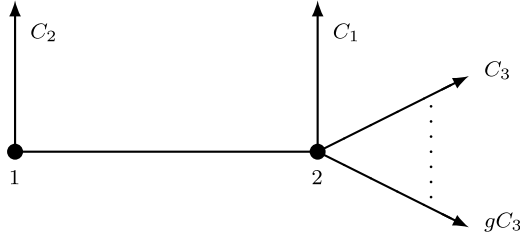


Figure 1. Example 2.6.

of  $\Gamma$  (vertices). The action of  $G$  on the set  $E_\sigma$  of components of the exceptional divisor is trivial; the curves with the numbers 1 and 2 are  $G$ -invariant and all the  $G$ -shifts of the third curve are different.

A local  $G$ -equivariant homeomorphism  $(\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$  which sends the curve  $C_i$  to the curve  $C'_i$  does not exist since the isotropy groups of the curves  $C_1 \setminus \{0\}$  and  $C'_1 \setminus \{0\} = C_2 \setminus \{0\}$  are different. Moreover there is no homeomorphism  $(\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$  which sends  $C_3$  to  $C'_3$ ,  $C_1$  to  $C'_2$  and  $C_2$  to  $C'_1$ , because a homeomorphism has to preserve the intersection multiplicities of branches.

*Example 2.7.* The fact that the action of the group  $G = \mathbb{Z}_{15}$  on  $\mathbb{C}^2 \setminus \{0\}$  has different isotropy subgroups of different points is not really essential. Let  $G = \mathbb{Z}_7$  with the action  $\sigma * (x, y) = (\sigma x, \sigma^3 y)$ , where  $\sigma = \exp(2\pi i/7)$  (and  $i$  here, but not elsewhere, denotes the imaginary unit). Let  $C_i$  and  $C'_i$ ,  $i=1, 2, 3$ , be defined as in Example 2.6. The same arguments (based on the intersection multiplicities) imply that a local homeomorphism  $(\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$  which sends  $C_3$  to  $C'_3$  should send  $C_1$  to  $C'_1 = C_2$ . However there is no  $G$ -equivariant local homeomorphism from  $(\mathbb{C}, 0)$  with the  $G$ -action  $\sigma * z = \sigma z$  to  $(\mathbb{C}, 0)$  with the  $G$ -action  $\sigma * z = \sigma^3 z$ .

The argument in Example 2.6 can be easily adapted to the case of divisorial valuations.

*Example 2.8.* Let  $G$  be the group  $\mathbb{Z}_{15}$  with the same action on  $(\mathbb{C}^2, 0)$  as in Example 2.6. The divisorial valuation  $v$  (resp.  $v'$ ) is defined by the following two curvettes:  $C_1 := \{(t, t^2)\}$  and  $C_2 := \{(t, -t^2)\}$  (resp.  $C'_1 := \{(t^2, t)\}$  and  $C'_2 := \{(-t^2, t)\}$ ). The minimal resolution graph for both cases is shown in Figure 2. The component of the exceptional divisor corresponding to the valuation is distinguished (marked by the circle). The  $G$ -action on it is trivial.

A local  $G$ -equivariant homeomorphism  $(\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$  should preserve the  $x$ - and  $y$ -axes (because of the isotropy subgroups). However such a homeomorphism

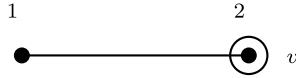


Figure 2. Example 2.8.

cannot send a curvette corresponding to  $v$  to a curvette corresponding to  $v'$  since it should preserve the intersection multiplicities of branches (in other words, since  $v(x) \neq v'(x)$ ).

For an Abelian group  $G$ , special points on the first component  $E_1$  of the exceptional divisor (obtained by blowing up the origin in  $\mathbb{C}^2$ ) exist if and only if the representation of  $G$  is the sum of two different one-dimensional representations of  $G$  and they correspond to these representations. If the group  $G$  is not Abelian, special points on  $E_1$  correspond to Abelian subgroups of  $G$  and their one-dimensional representations.

**Theorem 2.9.** *Assume that the initial action (representation) of the group  $G$  on  $\mathbb{C}^2$  is fixed. Then a  $G$ -resolution graph of a collection of curve or divisorial valuations with the correspondence between the “tails” of the graph (i.e. the connected components of the graph without the first vertex 1) and the special points on  $E_1$  determines the  $G$ -topology of the collection of the curves or of the divisorial valuations.*

*Proof.* The case of divisorial valuations is formulated in terms of curves (via pairs of corresponding curvettes) and thus follows from the curve case. We shall show that if  $G$ -resolutions of the collections of curves  $\{C_i\}_{i=1}^r$  and  $\{C'_i\}_{i=1}^r$  are described by the same data (i.e. they lie in the same  $\mathbb{C}^2$  with a  $G$ -representation, have the same  $G$ -resolution graphs and the same correspondences between the tails of the graphs and the special points on  $E_1$ ), then there exists a  $G$ -equivariant homeomorphism (in fact a  $C^\infty$ -diffeomorphism)  $\psi: (\mathcal{X}, \mathcal{D}) \rightarrow (\mathcal{X}', \mathcal{D}')$  of a neighborhood of the exceptional divisor  $\mathcal{D}$  in  $\mathcal{X}$  to a neighborhood of  $\mathcal{D}'$  in  $\mathcal{X}'$  sending the strict transforms  $\tilde{C}_i$  of the curves  $C_i$  to the strict transforms  $\tilde{C}'_i$  of the curves  $C'_i$ . Blowing down this diffeomorphism one obtains the required homeomorphism  $(\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ .

Such a diffeomorphism can be constructed inductively following the processes of the  $G$ -resolutions. We shall show that after each blow-up (or rather after a set of blow-ups at the points from one  $G$ -orbit) one can construct a  $G$ -equivariant diffeomorphism of a neighborhood of the new exceptional divisor(s) in the first resolution to a neighborhood of the one(s) in the second resolution which sends intersection points of the strict transforms of the branches  $C_i$  with the exceptional divisor to the corresponding points for the branches  $C'_i$ . Moreover we shall show that

this can be done in such a way that the diffeomorphism remains complex-analytic in neighborhoods of all these intersection points.

After the first blow-up one has an identification (an analytic one) of the first exceptional divisors and of their neighborhoods in both resolutions. This identification preserves special points. There may exist intersection points of the strict transforms of the branches  $C_i$  with the exceptional divisor which are not special ones and the corresponding points for the branches  $C'_i$ . The diffeomorphism has to send the former points to the latter ones. This can be done by a (smooth) isotopy of the initial diffeomorphism (the identification in this case), say, with the help of a  $G$ -invariant vector field which brings the images of the points to the required ones. Moreover this can be done so that the diffeomorphism remains complex-analytic in neighborhoods of these points.

The description of the behavior of a representation under the blow-up shows that, for each point  $x$  in the exceptional divisor of the first resolution, its isotropy subgroup  $G_x$  and its representations in the tangent and in the normal spaces to the exceptional divisor coincide with those for the corresponding image  $x'=\psi(x)$ .

The same construction takes place at each step of the resolution process. Assume that a point  $x$  of the exceptional divisor of the first resolution has to be blown-up (and thus its image  $x'$  under the constructed diffeomorphism as well).

One has two somewhat different situations. The point  $x$  (and thus the point  $x'$ ) is either the intersection point of two components of the exceptional divisor or a point of only one component. In the first case one has the representations of the group  $G_x=G_{x'}$  in the tangent spaces to the components. In the second case one has representations of this group in the tangent space to the component and in the normal one. If one fixes local coordinates at the point  $x$  in which the representation of  $G_x$  is diagonal, then the diffeomorphism (being complex-analytic by the induction hypothesis) defines local coordinates at the point  $x'=\psi(x)$  identifying (complex-analytically) neighborhoods of  $x$  and of  $x'$ . This gives complex-analytic isomorphisms of the new born components and of their tubular neighborhoods. As above, this isomorphism, in general, does not send intersection points of the new born components of the first resolution with the strict transforms of the curves  $C_i$  to the corresponding points on the component in the second resolution. However this can be corrected by a smooth isotopy which remains complex-analytic in neighborhoods of the points under consideration.

In this way one gets a diffeomorphism of neighborhoods of the exceptional divisors  $\mathcal{D}$  and  $\mathcal{D}'$  of the resolutions  $\pi$  and  $\pi'$  which sends the intersection points of the strict transforms of the curves  $C_i$  with  $\mathcal{D}$  (and thus of their shifts  $gC_i$ ) to the corresponding points for the curves  $C'_i$ . Generally speaking the strict transforms of the curves  $C_i$  do not go to the strict transforms of the curves  $C'_i$ . (In fact this may



happen only if the isotropy group of the corresponding point on  $\mathcal{D}$  is trivial.) This can be corrected by a local diffeomorphism in a neighborhood of the intersection point which should be duplicated at all the points of the  $G$ -orbit.  $\square$

### 3. $G$ -equivariant Poincaré series

In [4] the  $G$ -equivariant Poincaré series of a multi-index filtration defined by a set of valuations or order functions was defined as an element of the Grothendieck ring of  $G$ -sets with an additional structure.

Let  $(V, 0)$  be a germ of a complex-analytic variety with an action of a finite group  $G$ . The group  $G$  acts on the ring  $\mathcal{O}_{V,0}$  of germs of functions on  $(V, 0)$ , i.e.  $g^*f(x) = f(g^{-1}x)$  for  $f \in \mathcal{O}_{V,0}$ ,  $g \in G$  and  $x \in V$ . A function  $v: \mathcal{O}_{V,0} \rightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$  is called an *order function* if  $v(\lambda f) = v(f)$  for a non-zero  $\lambda \in \mathbb{C}$  and  $v(f_1 + f_2) \geq \min\{v(f_1), v(f_2)\}$ . (If moreover  $v(f_1 f_2) = v(f_1) + v(f_2)$ , the function  $v$  is a *valuation*.) A multi-index filtration of the ring  $\mathcal{O}_{V,0}$  is defined by a collection  $v_1, \dots, v_r$  of order functions:

$$(1) \quad J(\underline{v}) = \{f \in \mathcal{O}_{V,0} : \underline{v}(f) \geq \underline{v}\},$$

where  $\underline{v} = (v_1, \dots, v_r) \in \mathbb{Z}_{\geq 0}^r$ ,  $\underline{v}(f) = (v_1(f), \dots, v_r(f))$ , and  $(v'_1, \dots, v'_r) \geq (v''_1, \dots, v''_r)$  if and only if  $v'_i \geq v''_i$  for all  $i = 1, \dots, r$ . We assume that the filtration  $J(\underline{v})$  is *finitely determined*, i.e., for any  $\underline{v} \in \mathbb{Z}_{\geq 0}^r$ , there exists an integer  $k$  such that  $\mathfrak{m}^k \subset J(\underline{v})$ , where  $\mathfrak{m}$  is the maximal ideal in  $\mathcal{O}_{V,0}$ .

Let  $\mathbb{P}\mathcal{O}_{V,0}$  be the projectivization of the ring  $\mathcal{O}_{V,0}$ . In [2] there were defined the notions of cylindric subsets of  $\mathbb{P}\mathcal{O}_{V,0}$ , their Euler characteristics and the integral with respect to the Euler characteristic over  $\mathbb{P}\mathcal{O}_{V,0}$ . In the same way these notions can be defined for the quotient  $\mathbb{P}\mathcal{O}_{V,0}/G$  of  $\mathbb{P}\mathcal{O}_{V,0}$  by the action of  $G$ . The (usual) Poincaré series of the multi-index filtration can be defined as

$$P_{\{v_i\}}(t_1, \dots, t_r) = \int_{\mathbb{P}\mathcal{O}_{V,0}} \underline{t}^{\underline{v}(f)} d\chi,$$

where  $\underline{t} = (t_1, \dots, t_r)$  and  $\underline{t}^{\underline{v}} = t_1^{v_1} \dots t_r^{v_r}$ ;  $\underline{t}^{\underline{v}(f)}$  is considered as a function on  $\mathbb{P}\mathcal{O}_{V,0}$  with values in the ring (Abelian group)  $\mathbb{Z}[[t_1, \dots, t_r]]$ , see [2].

*Definition 3.1.* A (“locally finite”)  $(G, r)$ -set  $A$  is a triple  $(X, \underline{v}, \alpha)$  where

- $X$  is a  $G$ -set, i.e. a set with a  $G$ -action;
  - $\underline{v}$  is a function on  $X$  with values in  $\mathbb{Z}_{\geq 0}^r$ ;
  - $\alpha$  associates to each point  $x \in X$  a one-dimensional representation  $\alpha_x$  of the isotropy group  $G_x = \{a \in G : ax = x\}$  of the point  $x$ ;
- satisfying the following conditions:

- (1)  $\alpha_{ax} = a\alpha_x a^{-1}$  for  $x \in X$  and  $a \in G$ ;
- (2) for any  $\underline{w} \in \mathbb{Z}_{\geq 0}^r$  the set  $\{x \in X : \underline{w}(x) \leq \underline{w}\}$  is finite.

All (locally finite)  $(G, r)$ -sets form an Abelian semigroup in which the sum is defined as the disjoint union. The Cartesian product, appropriately defined (see [4]), makes the semigroup of  $(G, r)$ -sets a semiring. Let  $K_0((G, r)\text{-sets})$  be the corresponding Grothendieck ring—the Grothendieck ring of (locally finite)  $(G, r)$ -sets.

Let the filtration  $\{J(\underline{v})\}$  be defined by the order functions  $v_1, \dots, v_r$  by (1). For an element  $f \in \mathbb{P}\mathcal{O}_{V,0}$ , let  $G_f$  be the isotropy group of the corresponding point of  $\mathbb{P}\mathcal{O}_{V,0}$ , i.e.  $G_f = \{a \in G : a^*f = \lambda_f(a)f\}$ . The map  $a \mapsto \lambda_f(a)$  defines a one-dimensional representation  $\lambda_f$  of the group  $G_f$ . For an element  $f \in \mathbb{P}\mathcal{O}_{V,0}$ , let  $T_f$  be the element of the Grothendieck ring  $K_0((G, r)\text{-sets})$  represented by the orbit  $Gf$  of  $f$  (as a  $G$ -set) with  $\underline{w}^{T_f}(a^*f) = \underline{v}(a^*f)$  and  $\alpha_{a^*f}^{T_f} = \lambda_{a^*f}$ ,  $a \in G$ .

Let us consider  $T: f \mapsto T_f$  as a function on  $\mathbb{P}\mathcal{O}_{V,0}/G$  with values in the Grothendieck ring  $K_0((G, r)\text{-sets})$ . This function is cylindric and integrable (with respect to the Euler characteristic).

*Definition 3.2.* ([4]) The equivariant Poincaré series  $P_{\{v_i\}}^G$  of the filtration  $\{J(\underline{v})\}$  is defined by

$$P_{\{v_i\}}^G = \int_{\mathbb{P}\mathcal{O}_{V,0}/G} T_f d\chi \in K_0((G, r)\text{-sets}).$$

Let all the order functions  $v_i$  defining the filtration  $\{J(\underline{v})\}$ ,  $i=1, 2, \dots, r$ , be either curve or divisorial valuations. (In general it is possible that some of them are curve valuations and some of them are divisorial ones, but we shall not consider this case here.) Let  $\pi: (\mathcal{X}, \mathcal{D}) \rightarrow (\mathbb{C}^2, 0)$  be a  $G$ -resolution of this set of valuations, see above.

For a collection of divisorial valuations  $\{v_i\}_{i=1}^r$ , let  $\dot{E}_\sigma$  be the “smooth part” of the component  $E_\sigma$  in  $\mathcal{D}$ , i.e.  $E_\sigma$  itself minus intersection points with all other components of the exceptional divisor  $\mathcal{D}$ . For a collection of curve valuations  $\{v_i\}_{i=1}^r$  corresponding to branches  $C_i$ , let  $\dot{E}_\sigma$  be the “smooth part” of the component  $E_\sigma$  in the total transform  $\pi^{-1}(C)$ , i.e.  $E_\sigma$  itself minus intersection points of all the components of  $\pi^{-1}(C)$ ,  $C = \bigcup_{i,g} gC_i$ . Let  $\dot{\mathcal{D}} = \bigcup_\sigma \dot{E}_\sigma$  and let  $\widehat{\mathcal{D}} = \dot{\mathcal{D}}/G$  be the corresponding quotient space, i.e. the space of orbits of the action of the group  $G$  on  $\dot{\mathcal{D}}$ . Let  $p: \dot{\mathcal{D}} \rightarrow \widehat{\mathcal{D}}$  be the quotient map.

For  $x \in \dot{\mathcal{D}}$ , let the corresponding curvette  $L_x$  be given by an equation  $h'_x = 0$ ,  $h'_x \in \mathcal{O}_{\mathbb{C}^2,0}$ . Let  $h_x = \sum_{g \in G_x} (h'_x/g^*h'_x)(0)g^*h'_x$ . The germ  $h_x$  is  $G_x$ -equivariant and

$\{z:h_x(z)=0\}=L_x$ . Moreover, in what follows we assume that the germ  $h_x$  is fixed in this way for one point  $x$  of each  $G$ -orbit and is defined by  $h_{gx}=gh_xg^{-1}$  for other points of the orbit. (This defines  $h_{gx}$  modulo a constant factor.)

Let  $\{\Xi\}$  be a stratification of the space (in fact of a smooth curve)  $\widehat{\mathcal{D}}$  ( $\widehat{\mathcal{D}}=\coprod\Xi$ ) such that

- (1) each stratum  $\Xi$  is connected;
- (2) for each point  $\hat{x}\in\Xi$  and for each point  $x$  from its preimage  $p^{-1}(\hat{x})$ , the conjugacy class of the isotropy subgroup  $G_x$  of the point  $x$  is the same, i.e. depends only on the stratum  $\Xi$ .

The last condition is equivalent to saying that, over each stratum  $\Xi$ , the map  $p: \dot{\mathcal{D}}\rightarrow\widehat{\mathcal{D}}$  is a covering.

For a component  $E_\sigma$  of the exceptional divisor  $\mathcal{D}$ , let  $v_\sigma$  be the corresponding divisorial valuation on the ring  $\mathcal{O}_{\mathbb{C}^2,0}$ : for  $f\in\mathcal{O}_{\mathbb{C}^2,0}$ ,  $v_\sigma(f)$  is the order of zero of the lifting  $f\circ\pi$  of the function  $f$  along the component  $E_\sigma$ . Let  $\{\sigma_1,\dots,\sigma_r\}$  be a subset of  $\Gamma$ , and let  $v_1,\dots,v_r$  be the corresponding divisorial valuations. They define the multi-index filtration (1).

For a point  $x\in\dot{\mathcal{D}}$ , let  $T_x$  be the element of the Grothendieck ring  $K_0((G,r)\text{-sets})$  defined by  $T_x=T_{h_x}$ , where  $h_x$  is a  $G_x$ -equivariant function defining a curvette at the point  $x$ . The element  $T_x$  is well-defined, i.e. does not depend on the choice of the function  $h_x$ . One can see that the element  $T_x$  is one and the same for all points from the preimage of a stratum  $\Xi$  and therefore it will be denoted by  $T_\Xi$ .

**Theorem 3.3.** ([4])

$$(2) \quad P_{\{v_i\}}^G = \prod_{\{\Xi\}} (1-T_\Xi)^{-\chi(\Xi)}.$$

**Proposition 3.4.** *The initial action (representation) of the group  $G$  on  $\mathbb{C}^2$ , the  $G$ -resolution graph of a collection of curve or divisorial valuations plus the correspondence between the tails of the graph and the special points on  $E_1$  determine the  $G$ -Poincaré series of the set of valuations.*

*Proof.* In order to compute the  $G$ -Poincaré series  $P_{\{v_i\}}^G$  using (2) one has to describe the stratification  $\{\Xi\}$  and to determine, for each stratum  $\Xi$ , its Euler characteristic, the corresponding isotropy subgroup  $G_\Xi$  (the  $G$ -set representing  $T_\Xi$  is just  $G/G_\Xi$ ), the function  $\underline{w}_\Xi$  and the one-dimensional representation  $\alpha_x$  of the isotropy subgroup  $G_x$ ,  $x\in p^{-1}(\Xi)$ . Theorem 2.9 implies that the stratification and the corresponding isotropy subgroups are determined by the described data. The function  $\underline{w}_\Xi$ , i.e., the values for the corresponding curvettes, is computed from

the resolution graph in the standard way (it depends only on the corresponding component of the exceptional divisor). Thus the only remaining problem is to determine the representation  $\alpha_x$  for  $x \in \Xi$ . Let  $u$  and  $v$  be local  $G_x$ -equivariant coordinates in a neighborhood of the point  $x$  such that the corresponding component of the exceptional divisor is given by the equation  $u=0$ . By Theorem 2.9, the action of  $G_x$  on the function  $u$  (or rather the dual action in the normal space) is determined by the described data. Let  $\omega=dx \wedge dy$  be a  $G$ -equivariant 2-form on  $\mathbb{C}^2$ . The representation of  $G$  on  $\mathbb{C}^2$  determines the action of  $G$  on (the one-dimensional space generated by)  $\omega$ . One has  $(\pi^*\omega)_x = \varphi(u, v)u^\nu du \wedge dv$ , where  $\varphi(0, 0) \neq 0$  and the multiplicity  $\nu$  is determined by the resolution graph (see, e.g., [8, Section 8.3]). From the action of  $G_x$  on  $\omega$  and on  $u$  one gets the action on  $v$ . If  $h_x=0$  is a  $G$ -equivariant equation of the curvette  $\pi(\{z:v(z)=0\})$  at the point  $x$  one has  $\pi^*h_x = \psi(u, v)u^m v$ , where  $\psi(0, 0) \neq 0$  and  $m$  can be computed from the resolution graph. Thus the action  $\alpha_x$  of  $G_x$  on  $h_x$  is also determined.  $\square$

In what follows we shall use the following property of the Grothendieck ring  $K_0((G, r)$ -sets). The ring  $K_0((G, r)$ -sets) has the maximal ideal  $\mathfrak{M}$  which is generated by all irreducible  $(G, r)$ -sets different from 1. Let an element  $P$  belong to  $1 + \mathfrak{M}$ . Then it has a unique representation in the ‘‘A’Campo type form’’:  $P = \prod (1 - T)^{s_T}$ , where  $T$  runs over all irreducible elements of  $K_0((G, r)$ -sets) and the  $s_T$  are integers. In general this product contains infinitely many factors. Theorem 3.3 implies that if  $P$  is the  $G$ -Poincaré series of a set of curve or divisorial valuations on  $\mathcal{O}_{\mathbb{C}^2, 0}$ , then this product is finite.

#### 4. $G$ -Poincaré series and $G$ -topology of sets of divisors

Let  $(\mathbb{C}^2, 0)$  be endowed by a  $G$ -action and let  $\{v_i\}_{i=1}^r$  be a set of divisorial valuations on  $\mathcal{O}_{\mathbb{C}^2, 0}$ .

**Theorem 4.1.** *The  $G$ -equivariant Poincaré series  $P_{\{v_i\}}^G$  of the set of divisorial valuations  $\{v_i\}_{i=1}^r$  determines the  $G$ -equivariant topology of this set.*

*Proof.* Let  $v_{ig}(\varphi) := v_i((g^{-1})^*\varphi)$  be the valuation defined by the shift  $gE_i$  of the component  $E_i$ . Let  $\{v_{ig}\}_{i=1, g \in G}^r$  be the corresponding set of valuations on the ring  $\mathcal{O}_{\mathbb{C}^2, 0}$  numbered by the set  $\{1, \dots, r\} \times G$ . One can easily see that the  $G$ -Poincaré series  $P_{\{v_i\}}^G$  of the set of valuations  $\{v_i\}_{i=1}^r$  determines the  $G$ -Poincaré series  $P_{\{v_{ig}\}}^G$  of the set  $\{v_{ig}\}_{i=1, g \in G}^r$ . Indeed,  $P_{\{v_{ig}\}}^G$  is represented by the same  $G$ -set  $X$  as  $P_{\{v_i\}}^G$  with the same function  $\alpha$  and with  $\underline{w}: X \rightarrow \mathbb{Z}_{\geq 0}^{|G|}$  defined by  $w_{ig}(x) = w_i((g^{-1})^*x)$ ,  $x \in X$ . As was explained in [4, Statement 2], the  $G$ -Poincaré

series  $P_{\{v_{ig}\}}^G$  determines the usual (non-equivariant) Poincaré series  $P_{\{v_{ig}\}}(\underline{t})$ . The (usual) Poincaré series  $P_{\{v_{ig}\}}(\underline{t})$  determines the minimal resolution graph of the set of valuations  $\{v_{ig}\}_{i=1,g \in G}^r$  [3]. (Formally speaking, in [3] it was assumed that the divisorial valuations in the set are different. However one can easily see that there is no difference if one permits repeated valuations.) Moreover the action of the group  $G$  on the set  $\{v_{ig}\}_{i=1,g \in G}^r$  of valuations induces a  $G$ -action on the minimal resolution graph. By Theorem 2.9 one has to show that the  $G$ -Poincaré series  $P_{\{v_{ig}\}}^G$  determines the representation of  $G$  on  $\mathbb{C}^2$  and the correspondence between “tails” of the resolution graph emerging from the special points on the first component of the exceptional divisor and these points. (If there are no special points on the first component of the exceptional divisor (this can happen only if  $G$  is cyclic), only the representation of  $G$  on  $\mathbb{C}^2$  has to be determined.)

Let us consider the case of an Abelian group  $G$  first. If there are no special points on the first component  $E_1$  of the exceptional divisor, all points of  $E_1$  are fixed with respect to the group  $G$ , the group  $G$  is cyclic and the representation is a scalar one. This (one-dimensional) representation is dual to the representation of the group  $G$  on the one-dimensional space generated by any linear function. The case when there are no more components in  $\mathcal{D}$ , i.e. if the resolution is achieved by the first blow-up, is trivial. Otherwise let us consider a maximal component  $E_\sigma$  among those components  $E_\tau$  of the exceptional divisor for which  $G_\tau = G$  and the corresponding curvette is smooth. (The last condition can be easily detected from the resolution graph.) The smooth part  $\dot{E}_\sigma$  of this component contains a special point with  $G_x = G$  (or all the points of  $\dot{E}_\sigma$  are such that  $G_x = G$ ). The point(s) from  $\dot{E}_\sigma$  with  $G_x = G$  bring(s) into equation (2) a factor of the form  $(1 - T_\underline{x})^{-1}$ , where  $T_\underline{x}$  is represented by the  $G$ -set consisting of one point, which cannot be eliminated by other factors. The ( $G$ -equivariant) curvette  $L$  at the described special point of the divisor is smooth. Therefore the representation of  $G$  on the one-dimensional space generated by a  $G$ -equivariant equation of  $L$  coincides with the representation on the space generated by a linear function.

Let us take all factors in the representation of the Poincaré series  $P_{\{v_i\}}^G$  of the form  $\prod_T (1 - T)^{s_T}$ , where  $T$  is represented by the  $G$ -set consisting of one point and with  $s_T = -1$ . For each of them, the corresponding  $\underline{w}$  of  $T$  determines the corresponding component of the exceptional divisor and therefore the topological type of the corresponding curvettes. Therefore one can choose a factor which corresponds to a component with a smooth curvette and the representation  $\alpha^T$  gives us the representation on the space generated by a linear function.

Let the first component  $E_1$  contain two special points. Without loss of generality we can assume that they correspond to the coordinate axes  $\{(x, y) : x = 0\}$  and

$\{(x, y): y=0\}$ . The representation of the group  $G$  on  $\mathbb{C}^2$  is defined by its action on the linear functions  $x$  and  $y$ . For each of them this action can be recovered from a factor of the form described above just in the same way. Moreover, a factor, which determines the action of the group  $G$  on the function  $x$ , corresponds to a component of the exceptional divisor from the tail emerging from the point  $\{(x, y): x=0\}$ .

Now let  $G$  be an arbitrary (not necessarily Abelian) group. For an element  $g \in G$  consider the action of the cyclic group  $\langle g \rangle$  generated by  $g$  on  $\mathbb{C}^2$ . One can see that the  $G$ -equivariant Poincaré series  $P_{\{v_i\}}^G$  determines the  $\langle g \rangle$ -Poincaré series  $P_{\{v_i\}}^{\langle g \rangle}$  just like in [4, Statement 2]. This implies that the  $G$ -equivariant Poincaré series determines the representation of the subgroup  $\langle g \rangle$ . (Another way is to repeat the arguments above adjusting them to the subgroup  $\langle g \rangle$ .) Therefore the  $G$ -Poincaré series  $P_{\{v_i\}}^G$  determines the value of the character of the  $G$ -representation on  $\mathbb{C}^2$  for each element  $g \in G$  and thus the representation itself.

Special points of the  $G$ -action on the first component  $E_1$  of the exceptional divisor correspond to some Abelian subgroups  $H$  of  $G$ . For each such subgroup  $H$  there are two special points corresponding to different one-dimensional representations of  $H$ . Again the construction above for an Abelian group permits us to identify tails of the dual resolution graph with these two points. This finishes the proof.  $\square$

## 5. $G$ -Poincaré series and $G$ -topology of curves

A connection between the equivariant Poincaré series of a set of curve valuations and the equivariant topology of the corresponding curve singularity is more involved than for divisorial valuations.

*Example 5.1.* One can see that the collections  $\{C_i\}_{i=1}^3$  and  $\{C'_i\}_{i=1}^3$  from Examples 2.6 and 2.7 in Section 1 have the same  $G$ -Poincaré series. Namely  $P_{\{C_i\}}^G = P_{\{C'_i\}}^G = (1-T)$ , where  $T$  is the  $(G, 3)$ -set defined by  $X = G/(e)$  ( $G = \mathbb{Z}_{15}$  or  $G = \mathbb{Z}_7$  in Examples 2.6 and 2.7 respectively),  $w$  is the constant function on  $X$  with  $w(x) = (2, 1, 2)$ . (The representation  $\alpha(x)$  is trivial being a representation of the trivial group  $(e)$ .) Thus the  $G$ -Poincaré series does not, in general, determine the topology of the set of curves.

*Remark 5.2.* One can see that the  $G$ -Poincaré series of the set of divisorial valuations from Example 2.8 are different since the factors corresponding to the special points on the second divisors are different. These factors do not appear in the  $G$ -Poincaré series of curves in Example 2.6 since the strict transforms of branches pass through these points.

We shall show that the effect like the one in Example 5.1 occurs only if, among the branches of the curve, there are smooth branches invariant with respect to a non-trivial element  $g$  of the group  $G$  whose action on  $\mathbb{C}^2$  is not a scalar one (i.e. the representation of the cyclic subgroup  $\langle g \rangle$  generated by  $g$  is the sum of two different one-dimensional representations).

**Theorem 5.3.** *Let  $\mathbb{C}^2$  be equipped with a faithful action of a finite group  $G$  and let  $\{C_i\}_{i=1}^r$  be a collection of irreducible curve singularities in  $(\mathbb{C}^2, 0)$  such that it does not contain curves from the same  $G$ -orbit and it does not contain a smooth curve invariant with respect to a non-trivial element of  $G$  whose action on  $\mathbb{C}^2$  is not a scalar one. Let  $\{v_i\}_{i=1}^r$  be the corresponding collection of valuations. Then the  $G$ -equivariant Poincaré series  $P_{\{v_i\}}^G$  of the collection  $\{v_i\}_{i=1}^r$  determines the minimal  $G$ -resolution graph of the curve  $\bigcup_{i=1}^r C_i$  and the  $G$ -equivariant topology of the pair  $(\mathbb{C}^2, \bigcup_{i=1}^r C_i)$ .*

*Proof.* Let  $\{v_{ig}\}_{i=1, g \in G}^r$ , where  $v_{ig}(\varphi) = v_i((g^{-1})^*\varphi)$  is the set of “ $G$ -shifts” of the valuations  $v_i$  (it is possible that  $v_{i_1 g_1} = v_{i_2 g_2}$  and even that  $v_{i g_1} = v_{i g_2}$ ). The  $G$ -Poincaré series  $P_{\{v_i\}}^G$  determines the  $G$ -Poincaré series  $P_{\{v_{ig}\}}^G$  just in the same way as in the divisorial case in Theorem 4.1.

Since, with every valuation from the collection  $\{v_{ig}\}_{i=1, g \in G}^r$  this collection contains also its  $G$ -shifts, the remark after Statement 2 from [4] implies that the usual (non-equivariant) Poincaré series  $P_{\{v_{ig}\}}(\{t_{ig}\}_{i=1, g \in G}^r)$  is determined by the  $G$ -Poincaré series  $P_{\{v_i\}}^G$ .

The collection  $\{v_{ig}\}_{i=1, g \in G}^r$  contains repeated (curve) valuations. Therefore we need a (somewhat more precise) version of Theorem 2 from [4] for collections of curve valuations with (possibly) repeated ones. Assume that  $\{C_i\}_{i=1}^r$  and  $\{C'_i\}_{i=1}^r$  are collections of branches in  $(\mathbb{C}^2, 0)$  possibly with  $C_{i_1} = C_{i_2}$  for  $i_1 \neq i_2$  (and/or  $C'_{j_1} = C'_{j_2}$  for  $j_1 \neq j_2$ ) and let  $\{v_i\}_{i=1}^r$  and  $\{v'_i\}_{i=1}^r$  be the collections of the corresponding valuations. Essentially the same proof as in [4, Theorem 2] gives the following proposition.

**Proposition 5.4.** *If  $P_{\{v_i\}}(t_1, \dots, t_r) = P_{\{v'_i\}}(t_1, \dots, t_r)$  then there exists a combinatorial equivalence of the minimal resolution graphs of the reductions of the curves  $\bigcup_{i=1}^r C_i$  and  $\bigcup_{i=1}^r C'_i$ , i.e. components of the exceptional divisors corresponding to equivalent vertices intersect the same numbers of the strict transforms of different curves from the collections  $\{C_i\}_{i=1}^r$  and  $\{C'_i\}_{i=1}^r$ . (The distribution of these curves into groups of equal ones can be different.)*

The knowledge of the usual Poincaré series of the collection  $\{v_{ig}\}_{i=1, g \in G}^r$  gives the minimal resolution graph (the usual, not the  $G$ -equivariant one) of the collection.

Moreover the action of the group  $G$  on the collection  $\{v_{ig}\}_{i=1, g \in G}^r$  of valuations determines the  $G$ -action on the resolution graph.

As in Theorem 3.3 one has to show that the  $G$ -Poincaré series  $P_{\{v_i\}}^G$  determines the representation of  $G$  on  $\mathbb{C}^2$ , and the correspondence between the “tails” of the resolution graph. This is done just as in Theorem 3.3 for divisorial valuations since there are no strict transforms of the branches  $gC_i$  at points of the exceptional divisor with  $G_x = G$  (and thus the corresponding curve being smooth) and therefore the corresponding factor  $(1 - T_{\Xi})^{-1}$  (which permits one to restore the action of  $G$  on the corresponding linear function, viz. a coordinate) is contained in the A’Campo type decomposition of the Poincaré series.  $\square$

*Remark 5.5.* One can see that the  $G$ -Poincaré series of a collection of curve valuations determines whether or not the collection of curves satisfies the conditions of Theorem 5.3.

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Antonio Campillo  
Instituto de Investigación en Matemáticas  
Universidad de Valladolid  
Paseo de Belén, 7  
ES-47011 Valladolid  
Spain  
[campillo@agt.uva.es](mailto:campillo@agt.uva.es)

Sabir M. Gusein-Zade  
Faculty of Mathematics and Mechanics  
Moscow State University  
Moscow 119991  
Russia  
[sabir@mccme.ru](mailto:sabir@mccme.ru)

Félix Delgado  
Instituto de Investigación en Matemáticas  
Universidad de Valladolid  
Paseo de Belén, 7  
ES-47011 Valladolid  
Spain  
[fdelgado@agt.uva.es](mailto:fdelgado@agt.uva.es)

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