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On the weak-type $(1,1)$ of the uncentered Hardy–Littlewood maximal operator associated with certain measures on the plane

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Abstract. Suppose μ is a positive measure on $\mathbb{R}^2$ given by $\mu=\nu\times\lambda$, where ν and λ are Radon measures on $\mathcal{S}^1$ and $\mathbb{R}^+$, respectively, which do not vanish on any open interval. We prove that if for either ν or λ there exists a set of positive measure A in its domain for which the upper and lower s-densities, $0<s\leq 1$, are positive and finite for every $x\in A$ then the uncentered Hardy–Littlewood maximal operator M_μ is weak-type $(1,1)$ if and only if ν is doubling and λ is doubling away from the origin. This generalizes results of Vargas concerning rotation-invariant measures on $\mathbb{R}^n$ when $n=2$.

1. Introduction

Let μ be a positive Borel measure on $\mathbb{R}^n$, finite on compact sets and with $\mu(B)>0$ for all Euclidean balls B. We define the *uncentered Hardy–Littlewood maximal operator* associated with μ by

$$M_\mu f(x) = \sup_{x\in B} \frac{1}{\mu(B)} \int_B |f(y)|\,d\mu(y) \quad \text{for } f\in L^1_\mu(\mathbb{R}^n),$$

where the B are open balls.

The case when $\mu=\nu\times\lambda+h\delta_{(0,0)}$, where ν is Lebesgue measure on the unit sphere $\mathcal{S}^{n-1}$ and λ is a measure on $\mathbb{R}^+$ are the rotation-invariant measures on $\mathbb{R}^n$ which have been studied in several papers. In [3] it was shown that M_μ is weak-type $(1,1)$ for all μ when $n=1$, and that if $n=2$ and $d\mu=e^{-x^2-y^2}\,dx\,dy$ then M_μ is not weak-type $(1,1)$. In [4] the main result is that M_μ is weak-type $(1,1)$ if and only if λ is doubling away from the origin (see Section 3 for the definition) when $n\geq 2$. This includes the result concerning the Gaussian on $\mathbb{R}^2$ as a special case as it is easy to see that this measure is not doubling away from the origin.

In the present paper we will restrict our attention to product measures on $\mathbb{R}^2$ of the form $\mu=\nu\times\lambda$, where ν is a measure on the circle $\mathbb{S}^1$ and λ is a measure on $\mathbb{R}^+$. Technically we must extend μ to all of $\mathbb{R}^2$ by writing $\mu=\nu\times\lambda+h\delta_{(0,0)}$. However in what follows there is no loss of generality in assuming that $h=0$ i.e. that $\mu(\{(0,0)\})=0$, as the weak-type $(1,1)$ for $\nu\times\lambda$ will be preserved if $h\delta_{(0,0)}$ is added to μ, and the analysis of Section 5 takes place away from the origin. Our results will generalize those of [4] when $n=2$, by taking ν equal to Lebesgue measure on $\mathbb{S}^1$.

After some preliminaries we prove that M_μ is weak-type $(1,1)$, when ν is a doubling measure and λ is doubling away from the origin. We will next turn to establishing a partial converse of this statement.

The following are our main results. It is assumed that μ is a Borel measure which is positive on Euclidean balls and finite on compact sets.

Theorem 1.1. *Let $\mu=\nu\times\lambda$ be a measure for which ν is doubling and λ is doubling away from the origin. Then M_μ is weak-type $(1,1)$.*

We will use the upper and lower s-densities of Radon measures to provide a characterization of some measures $\mu=\nu\times\lambda$ where the converse to Theorem 1.1 holds. We recall the following definitions.

Definition 1.2. For a Radon measure σ defined on $\mathbb{R}^n$ we define the *lower and upper s-densities* respectively by

$$\theta_*^s(\sigma,x)=\liminf_{r\downarrow 0}\frac{\sigma(B(x,r))}{(2r)^s},$$

$$\theta^{*s}(\sigma,x)=\limsup_{r\downarrow 0}\frac{\sigma(B(x,r))}{(2r)^s},$$

where $B(x,r)$ is the closed ball centered at x with radius r and $s>0$.

Theorem 1.3. *Let $\mu=\nu\times\lambda$ be a positive measure on $\mathbb{R}^2$ with $\mu(B)>0$ for all Euclidean balls B and $\mu(K)<\infty$ for all compact sets K. Suppose either of the following holds:*

(1) ν is a Radon measure for which there exists a set $A\subset\mathbb{S}^1$, $\nu(A)>0$, with $0<\theta_*^s(x,\nu)\le\theta^{*s}(x,\nu)<\infty$ for all $x\in A$; or

(2) λ is a Radon measure for which there exists a set $A\subset\mathbb{R}^+$, $\lambda(A)>0$, with $0<\theta_*^s(x,\lambda)\le\theta^{*s}(x,\lambda)<\infty$ for all $x\in A$;

then M_μ is weak-type $(1,1)$ if and only if ν is doubling and λ is doubling away from the origin.

When $s=1$ we have the following slightly stronger result.

Corollary 1.4. *If the Lebesgue decomposition of either ν or λ has a non-zero absolutely continuous part, then M_μ is weak-type $(1,1)$ if and only if ν is doubling and λ is doubling away from the origin.*

We also obtain the following under the assumption that one of the factors ν and λ satisfies the somewhere doubling property. See Section 3 for the definition.

Theorem 1.5. *If either ν or λ is somewhere doubling, then M_μ is weak-type $(1,1)$ if and only if ν is doubling and λ is doubling away from the origin.*

The converse of Theorem 1.1 does not hold in full generality. In Example 5.9 we give an example of a measure $\mu=\nu\times\lambda$ where M_μ is weak-type $(1,1)$ and ν is not doubling and λ is not doubling away from the origin.

2. Some basic inequalities

We begin by defining a few fundamental objects, and then state some simple geometric propositions. The proofs may be established using basic trigonometry and will therefore either be sketched or left to the reader. The notation, however, will be employed throughout the paper.

The following object appeared in [4].

Definition 2.1. Given a ball $B=B(x_0,R)$ we define its *associated sector* S_B as

$$S_B=\left\{x\in\mathbb{R}^n : |x_0|-R<|x|<|x_0|+R \text{ and } \operatorname{ang}(x,x_0)<\arcsin\left(\frac{R}{|x_0|}\right)\right\}$$

if $|x_0|\geq R$, where $\operatorname{ang}(x,x_0)$ is the angle between the rays emanating from 0 and ending at x and x_0.

When $|x_0|<R$ we have

$$S_B=\{x\in\mathbb{R}^n : |x|<|x_0|+R\}.$$

Definition 2.2. We define the *axis* of the ball B to be the ray emanating from the origin which passes through the center of B.

Notation 2.3. (1) By F_θ we will mean a ray emanating from the origin making an angle θ with some other specified ray.

For (2)–(4) let B be a ball of radius $R<|x_0|$.

(2) By $r_1(\theta)$ and $r_2(\theta)$ we will denote the lengths along F_θ, if any, where F_θ intersects the boundary of B. We will set $r(\theta)=r_2(\theta)-r_1(\theta)$. Thus $r(\theta)$ is the length of the segment of F_θ which is inside B.

(3) By A_B we will denote the annulus

$$\{x\colon |x_0|-R<|x|<|x_0|+R\}.$$

(4) By $g_1(\theta)$ and $g_2(\theta)$ we denote the lengths of the two line segments in F_θ contained in $A_B\setminus B$ when such segments exist for θ. They are thus the lengths of the gaps between the boundary of the annulus and the boundary of B.

(5) We will let $g(\theta)=|x_0|+R-r(\theta)$, when $R\geq|x_0|$.

In the following propositions θ is an angle made with respect to the axis of the ball.

Proposition 2.4. *Let B be a ball with center x_0 and radius R, where $|x_0|/4\leq R<|x_0|$. Then for small enough θ, independent of x_0 and R, we have*

$$\frac{2g_1(\theta)}{|x_0|-R}\leq 10, \tag{1}$$

$$\frac{2g_2(\theta)}{(|x_0|+R)-4g_2(\theta)}\leq 10, \tag{2}$$

$$\frac{r(\theta)}{g_1(\theta)}, \frac{r(\theta)}{g_2(\theta)}\geq 100. \tag{3}$$

Proof. We prove (1), leaving (2) and (3) to the reader.

(1) Assume without loss of generality that the axis of B is the positive y-axis. Thus we may write $x_0=(0,y_0)$. It follows that

$$r_1(\theta)=y_0\cos(\theta)-\sqrt{R^2-y_0^2\sin^2\theta} \quad\text{and}\quad r_2(\theta)=y_0\cos(\theta)+\sqrt{R^2-y_0^2\sin^2\theta}.$$

As $g_1(\theta)=r_1(\theta)-(y_0-R)$, we have

$$5(y_0-R)-g_1(\theta)=6(y_0-R)+\sqrt{R^2-y_0^2\sin^2\theta}-y_0\cos\theta.$$

Set $R=\sigma y_0$ for $\frac{1}{4}\leq\sigma<1$ to obtain

$$5(y_0-R)-g_1(\theta)=y_0\big(6(1-\sigma)+\sqrt{\sigma^2-\sin^2\theta}-\cos\theta\big).$$

As a function of σ, using a derivative argument, the right-hand side is seen to be greater than or equal to 0 over the interval $\left[\frac{1}{4},1\right]$ when θ is small enough. $\square$

The following two propositions follow similarly and the proofs are omitted.

Proposition 2.5. *Let B be a ball with center x_0 and radius R with $R<|x_0|/4$. Then there exists an N independent of x_0 and R such that when $\theta \le \arcsin(R/|x_0|)/N$ we have*

$$R \le r(\theta).$$

Proposition 2.6. *Let B be a ball with center x_0 and radius R with $R \ge |x_0|$. Then for θ small enough, independent of x_0 and R, we have*

$$\frac{r(\theta)}{g(\theta)} \ge 100. \tag{4}$$

3. Doubling measures

Throughout the remainder of the paper μ will be a measure on $\mathbb{R}^2$ which is the product of a positive measure ν on $\mathbb{S}^1$, with $\nu(\mathbb{S}^1)<\infty$ and $\nu(I)>0$ for every interval in $\mathbb{S}^1$, and a positive measure λ on $\mathbb{R}^+$ with $\lambda(J)>0$ for all intervals J in $\mathbb{R}^+$ and $\lambda(K)<\infty$ for all compact sets K in $\mathbb{R}^+$. We will let $|I|$ denote the Lebesgue measure of an interval I, $\bar{I}$ the closure of the interval, and I° the interior of the interval. As usual $A \sim B$ implies there exist constants $C_1>0$ and $C_2>0$ such that $C_1 \le A/B \le C_2$. By kI or kB for some interval I or ball B we mean the ball or interval concentric with the original one and having radius k-times that of the original radius. We are identifying the circle with $[0,2\pi)$ where arithmetic is mod 2π. Geometrically θ increases as we move counterclockwise in the plane.

Definition 3.1. For a measure ν as above on $\mathbb{S}^1$ we say ν is *doubling* if $\nu(2I) \le C\nu(I)$ for some $C>0$ and every interval $I \subseteq \mathbb{S}^1$. We say that ν is *doubling somewhere* or *somewhere doubling* if there exists an interval $I \subseteq \mathbb{S}^1$ such that $\nu(2J) \le C\nu(J)$ for some $C>0$ whenever J and $2J$ are contained in I. We say that ν is *nowhere doubling* if it is not somewhere doubling.

Definition 3.2. For a measure λ as above on $\mathbb{R}^+$ we say that λ is *doubling away from the origin* if there exists a constant $C>0$ such that

$$\lambda([a, a+2r]) \le C\lambda\big(\big[a+\tfrac{1}{2}r, a+\tfrac{3}{2}r\big]\big)$$

whenever $r \le 10a$.

We define λ to be somewhere doubling analogously to the definition given for ν.

Remark 3.3. Verifying the doubling conditions on closed intervals implies them for all intervals. This is because $C\nu(I^\circ)>\nu(I)$ and $C\lambda(I^\circ)>\lambda(I)$ where I° is the interior of a closed interval I.

We now set out some basic properties of doubling measures which will be useful in proving the main results of the paper. By *adjacent* we will mean that $I\cup J$ is an interval of any kind and $I\cap J=\varnothing$.

Propositions 3.4 and 3.5 are well known. We refer the reader to [1].

Proposition 3.4. *The following are equivalent for a measure ν on $\mathbb{S}^1$ with $\nu(I)>0$ for non-empty intervals I:*

(1) *ν is doubling*;

(2) *$\nu(I)\sim\nu(J)$ for adjacent intervals I and J with $|I|=|J|$.*

The extension of Proposition 3.4 to measures which are doubling away from the origin is straightforward.

Proposition 3.5. *Let λ be a measure on $\mathbb{R}^+$. Then the following are equivalent*:

(1) *λ is doubling away from the origin*;

(2) *if $|I|\leq 10a$ then $\lambda(I)\sim\lambda(J)$, whenever I and J are adjacent intervals with $|I|=|J|$ and $a=\inf(J\cap J)$.*

Proposition 3.6. *Suppose ν is a non-doubling measure on $\mathbb{S}^1$ and λ is non-doubling away from the origin on $\mathbb{R}^+$. Then*

(1) *there exist adjacent intervals I and J in $\mathbb{S}^1$, whose union is open, with $|I|\geq n|J|$ and $\nu(J)/\nu(I)\geq n$ for any positive integer n*;

(2) *there exist adjacent intervals I and J in $\mathbb{R}^+$, with $10\inf(I\cup J)\geq|I|$, whose union is open, with $|I|\geq n|J|$ and $\lambda(J)/\lambda(I)\geq n$. Furthermore we may take I and J so that if $I\cup J=Q$ and $Q=(a,b)$ then $|I\cup J|/a\to 0$ as $n\to\infty$.*

Proof. (1) Let $a_0=1$. Define a sequence by $a_j=n\sum_{i=0}^{j-1}a_i$, and set $T=\sum_{j=1}^{n}a_j$. Now choose a half-open interval U and an adjacent open interval V, such that $\nu(U)/\nu(V)>T$, and $|U|=|V|$. This is possible since ν is not doubling. Divide U into n adjacent intervals $U_1,U_2,...,U_n$ of equal length, where U_1 is adjacent to V. At least one of these intervals satisfies $\nu(U_i)\geq a_i\nu(V)$. Let U_{i_0} denote the first such interval. Now set $J=V\cup\big(\bigcup_{i=1}^{i_0-1}U_i\big)$ and $I=U_{i_0}$. Then

$$\nu(J)\le\nu(V)+\sum_{i=1}^{i_0-1}a_i\nu(V)=\nu(V)\sum_{i=0}^{i_0-1}a_i=\frac{\nu(V)}{n}a_{i_0}\le\frac{\nu(U_{i_0})}{n}.$$

(2) The proof of the first part of (b) is analogous to that of (a). For the statement $|I\cup J|/a\to 0$ we suppose it is false. This implies that there exists an n_0 and $\frac{1}{20}>K>0$ such that if $n\ge n_0$, and I and J are adjacent intervals where the following hold:

(a) $|I|\ge n|J|$;

(b) $|I\cup J|/a\le K<\frac{1}{20}$, where $\overline{I\cup J}=[a,b]$;

then $\lambda(J)/\lambda(I)<n$.

Take an interval $A=(c,c+2r)$ with $r\le 10c$ and let $B=(c+r/2,c+3r/2)$. Let $M=\max\{100n_0,100/K\}$ and set $\Delta=|A|/M=2r/M$. Set for $1\le k\le M^2/4$,

$$B_k=\left(c+\frac{r}{2}-\frac{k\Delta}{M},c+\frac{r}{2}-\frac{(k-1)\Delta}{M}\right],$$
$$B'_k=\left(c+\frac{r}{2}-\frac{(k-1)\Delta}{M},c+\frac{r}{2}-\frac{(k-1)\Delta}{M}+\Delta\right).$$

Observe that

(a) $B'_1\subset B$;

(b) $|B_k|/|B'_k|=1/M\le 1/n_0$;

(c) $|B_k\cup B'_k|/(c+r/2-k\Delta/M)\le(\Delta+\Delta/M)/(r/10+r/2-k\Delta/M)\le 100/M\le K$;

and therefore

(d) $\lambda(B_k)\le n_0\lambda(B'_k)$.

We also note that for $1\le k\le M^2/4$ we have $B'_k\subset B_{k-1}\cup B'_{k-1}$. From this it follows easily that there exists a constant $C_0>0$, independent of A and r, such that

$$\lambda(B_k)\le C_0\lambda(B'_1)\le C_0\lambda(B)\quad\text{for all } 1\le k\le\frac{M^2}{4}.$$

Hence

$$\lambda\left(\bigcup_{k=1}^{M^2/4}B_k\right)=\lambda\left(\left(c,c+\frac{r}{2}\right]\right)\le C_0\frac{M^2}{4}\lambda\left(\left(c+\frac{r}{2},c+\frac{3r}{2}\right)\right).$$

A similar argument gives

$$\lambda\left(\left[c+\frac{3r}{2},c+2r\right)\right)\le C_0\frac{M^2}{4}\lambda\left(\left(c+\frac{r}{2},c+\frac{3r}{2}\right)\right).$$

Therefore λ is doubling away from the origin which is a contradiction. □

4. A positive result

In this section we will establish a positive result. Our strategy will be similar to that in [4] in that we will show that $\mu(B)\sim\mu(S_B)$ and then divide the Euclidean balls into two groups to deduce the weak-type $(1,1)$ inequality. We assume that $\mu=\nu\times\lambda$ where ν is a doubling measure on $\mathbb{S}^1$ and λ is doubling away from the origin on $\mathbb{R}^+$.

Proposition 4.1. $\mu(B)\sim\mu(S_B)$.

Proof. Suppose N is chosen large enough to satisfy Proposition 2.5 and if $\theta\leq 2\pi/N$ then Propositions 2.4 and 2.6 are satisfied by θ. We examine three cases separately:

$$R<\frac{|x_0|}{4},\quad \frac{|x_0|}{4}\leq R<|x_0|\quad\text{and}\quad |x_0|\leq R.$$

Case 1. $R<|x_0|/4$. Write $B=B_1\cup B_2$ where B_1 is the half of B counterclockwise from the axis of B and B_2 is the other half. Let S_B^1 and S_B^2 be the corresponding halves of S_B. We will prove that $\mu(S_B^2)\sim\mu(B_2)$. That $\mu(S_B^1)\sim\mu(B_1)$ will follow by a similar argument.

Divide S_B^2 and B_2 along N sectors of equal angle $\arcsin(R/|x_0|)/N$ and label these sectors $S_1,S_2,\ldots,S_N$, where S_1 is bounded by the axis of B, and set $\beta=\arcsin(R/|x_0|)$. Let θ be an angle made by a ray passing through S^1 and the axis of B. Since $0\leq\theta\leq\arcsin(R/|x_0|)/N$, Proposition 2.5 implies that $r_2(\theta)-r_1(\theta)=r(\theta)\geq R$. This and the fact that $R<|x_0|/4$ gives

$$\lambda(|x_0|-R,|x_0|+R)\leq C\lambda(r_1(\theta),r_2(\theta)).$$

It follows that

$$\mu(B\cap S_1)\sim\mu(S_1).$$

Since ν is doubling, by repeatedly applying Proposition 3.4 we get that $\mu(S_i)\sim\mu(S_1)$ for $N\geq i\geq 1$, the similarity depending on N alone. Therefore

$$\begin{aligned}\mu(S_B^2)&\leq\mu(S_1)+\sum_{i=2}^{N}\mu(S_i)\leq C_1\mu(B\cap S_1)+(N-1)C_2\mu(B\cap S_1)\\&\leq C_1\mu(B_2)+(N-1)C_2\mu(B_2)\leq C_3\mu(B_2).\end{aligned}$$

Case 2. $|x_0|>R\geq|x_0|/4$. We note in this case that $\arcsin(R/|x_0|)=\beta\geq\arcsin\frac{1}{4}$. By parts (1) and (2) of Proposition 2.4 and the fact that λ is doubling away from the origin, we have for F_θ a ray passing through S_1, where S_1 is as before,

$$\lambda((|x_0|-R,|x_0|+R))=\lambda((r_1(\theta),r_2(\theta)))+\lambda((|x_0|-R,r_1(\theta)])+\lambda([r_2(\theta),|x_0|+R))$$

$$\begin{aligned}&\leq \lambda((r_1(\theta),r_2(\theta)))+\lambda((r_1(\theta)-g_1(\theta),r_1(\theta)+3g_1(\theta)))\\&\quad+\lambda((r_2(\theta)-3g_2(\theta),r_2(\theta)+g_2(\theta)))\\&\leq \lambda((r_1(\theta),r_2(\theta)))+C\lambda(r_1(\theta),r_1(\theta)+2g_1(\theta))\\&\quad+C\lambda((r_2(\theta)-2g_2(\theta),r_2(\theta))).\end{aligned}$$

Since these measures are all over segments in $B\cap F_\theta$ and in light of part (3) of Proposition 2.4, the claim follows as before.

Case 3. $R\geq|x_0|$. Using Proposition 2.6 for θ small enough we have,

$$\begin{aligned}\lambda((0,|x_0|+R)) &= \lambda((0,r(\theta)))+\lambda([r(\theta),r(\theta)+g(\theta)))\\&\leq \lambda((0,r(\theta)))+\lambda((r(\theta)-3g(\theta),r(\theta)+g(\theta)))\\&\leq \lambda((0,r(\theta)))+C\lambda((r(\theta)-2g(\theta),r(\theta)))\end{aligned}$$

and the conclusion follows as before. □

Proposition 4.2. *When* $R\leq|x_0|/4$, $\mu(S_B)\sim\mu(S_{2B})$.

Proof. Assume that the axis of B makes an angle α_0 with the x-axis. Then

$$\begin{aligned}\mu(S_{2B}) &= \nu\biggl(\alpha_0-\arcsin\frac{2R}{|x_0|},\alpha_0+\arcsin\frac{2R}{|x_0|}\biggr)\lambda((|x_0|-2R,|x_0|+2R))\\&\leq C_1C_2\nu\biggl(\alpha_0-\arcsin\frac{R}{|x_0|},\alpha_0+\arcsin\frac{R}{|x_0|}\biggr)\lambda(|x_0|-R,|x_0|+R)\\&= C_1C_2\mu(S_B)\end{aligned}$$

since $\arcsin(2R/|x_0|)/\arcsin(R/|x_0|)\leq C$, ν is doubling and λ is doubling away from the origin. □

Proposition 4.3. *If* $R\geq|x_0|/4$ *we have* $\mu(B)\sim\mu(A_B)$, *where*

$$A_B=\begin{cases}S_B, & \text{if } R\geq|x_0|,\\ \{x: |x_0|-R<|x|<|x_0|+R\}, & \text{if } R<|x_0|.\end{cases}$$

Proof. This follows immediately from the fact that $\mu(S_B)\leq C\mu(B)$, the assumption that ν is doubling, and that $\arcsin(R/|x_0|)\geq\arcsin\frac{1}{4}$ for balls with $|x_0|/4\leq R<|x_0|$. □

We are able to conclude the argument exactly as in [4].

Theorem 4.4. *If $\mu=\nu\times\lambda$ where ν is doubling and λ is doubling away from the origin, then M_μ is weak-type* $(1,1)$.

Proof. See [4], p. 15. □

5. Partial converses to Theorem 4.4

We continue to assume that $\mu=\nu\times\lambda$ is a Borel measure which is positive on open balls and finite on compact sets.

Definition 5.1. A measure σ defined on $\mathbb{R}^+$ or $\mathbb{S}^1$ will be called *uniform at a point* if there exists an $x\in\mathbb{R}^+$ or $x\in\mathbb{S}^1$, respectively, for which the following two properties hold:

(1) there exist $\rho_0>0$ and a constant $C>0$ such that for all $0<\rho\le\rho_0$,

$$\sigma((x-\rho,x+\rho))\le C\sigma((x-\rho/2,x+\rho/2));$$

(2) for any sequences of positive numbers $\varepsilon_n\to 0$ and $\rho_n\to 0$ with $\varepsilon_n/\rho_n\to 0$, and a collection of open intervals J_n with $J_n\subset(x-\rho_n,x+\rho_n)$ and $|J_n|=2\varepsilon_n$ we have

$$\frac{\sigma(J_n)}{\sigma((x-\rho_n,x+\rho_n))}\to 0.$$

Lemma 5.2. *Let σ be a Radon measure defined on $\mathbb{R}^+$ or $\mathbb{S}^1$ for which there exists a set of positive measure A in $\mathbb{R}^+$ or $\mathbb{S}^1$, respectively, on which*

$$0<\theta^s_*(\sigma,x)\le\theta^{*s}(\sigma,x)<\infty \quad \text{for all } x\in A.$$

Then σ is uniform at a point.

Proof. We assume that σ is defined on $\mathbb{S}^1$, the argument for $\mathbb{R}^+$ being identical. Let r_0 be small enough so that there exist $C_2,C_1>0$ such that the set

$$A_0=\left\{x\in\mathbb{S}^1 : C_2>\frac{\sigma(B(x,r))}{(2r)^s}>C_1>0 \text{ for all } r\le r_0\right\}$$

has $\sigma(A_0)>0$. For σ-almost every $x\in A_0$ we have

$$\lim_{r\downarrow 0}\frac{\sigma(B(x,r)-A_0)}{(2r)^s}\le C_2\lim_{r\downarrow 0}\frac{\sigma(B(x,r)-A_0)}{\sigma(B(x,r))}=0.$$

See Corollary 2.14(1) on page 38 of [2] for the last equality.

Let

$$A_1=\left\{x\in A_0:\ \lim_{r\downarrow 0}\frac{\sigma(B(x,r)-A_0)}{(2r)^s}=0\right\}.$$

Choose any point x of A_1. Suppose there is a constant $K>0$, sequences $\varepsilon_n\to 0$ and $\rho_n\to 0$ with $\varepsilon_n/\rho_n\to 0$, and intervals $J_n\subset(x-\rho_n,x+\rho_n)$ with

$$\sigma(J_n)>K\sigma((x-\rho_n,x+\rho_n))\quad\text{and}\quad |J_n|=2\varepsilon_n.$$

Assume that for some subsequence of intervals $\{J_{n_k}\}_{k=1}^\infty$ there is a $z_k\in J_{n_k}\cap A_1$ for all k. Let $N=n_k$ for some large k, and write $J_N=[y-\varepsilon_N,y+\varepsilon_N]$. Then

$$\begin{aligned}KC_1(\rho_N)^s&\le K\sigma\Big(\Big[x-\frac{\rho_N}{2},x+\frac{\rho_N}{2}\Big]\Big)\le K\sigma((x-\rho_N,x+\rho_N))\\&\le\sigma(J_N)\le\sigma([z_k-2\varepsilon_N,z_k+2\varepsilon_N])\le C_2(4\varepsilon_N)^s.\end{aligned}$$

This implies that $[KC_1/C_2 4^s]^{1/s}\le\varepsilon_N/\rho_N$, a contradiction for k large enough. Hence for M large enough $A_1\cap J_m=\varnothing$ for all $m\ge M$.

Since $A_1\cap J_m=\varnothing$ for all $m\ge M$ we have

$$\begin{aligned}KC_1\rho_m^s&\le K\sigma((x-\rho_m,x+\rho_m))\le\sigma(J_m)\\&\le\sigma([x-\rho_m,x+\rho_m]-A_1)\le\frac{\sigma([x-\rho_m,x+\rho_m]-A_1)}{(2\rho_m)^s}(2\rho_m)^s.\end{aligned}$$

This implies that

$$\frac{KC_1}{2^s}\le\frac{\sigma([x-\rho_m,x+\rho_m]-A_1)}{(2\rho_m)^s}\to 0$$

as $m\to\infty$, a contradiction. Hence (2) of Definition 5.1 holds. That (1) of Definition 5.1 holds is trivial given that $x\in A_0$. □

Lemma 5.3. *If σ is a measure on $\mathbb{R}^+$ or $\mathcal{S}^1$ which is somewhere doubling, then σ is uniform at a point.*

Proof. Because somewhere doubling is a local property we give the proof assuming σ is defined on $\mathcal{S}^1$, the case of $\mathbb{R}^+$ being identical. Let I_0 be an interval for which there exists a $C>0$ such that $\sigma(2J)\le C\sigma(J)$ for every $J\subset I_0$ with $2J\subset I_0$. Let x denote the center of I_0 and write $I_0=(x-r,x+r)$ for some r. Let $\mathcal{U}$ denote the collection of intervals contained in $I_0/16$.

Given two adjacent intervals I_1 and I_2 of equal length in $I_0/16$, somewhere doubling implies that

$$\frac{1}{C^2}\sigma(I_1)\le\sigma(I_2)\le C^2\sigma(I_1).$$

Thus given an interval $I\subset I_0/16$ writing $I=I_1\cup I_2\cup I_3\cup I_4$, where the interval I_i is adjacent to I_{i+1} and $I/2=I_2\cup I_3$, we see that there is an $\eta>0$ such that

$$(1+\eta)\sigma\left(\frac{I}{2}\right)\leq\sigma(I)\quad\text{for all } I\in\mathcal{U}.$$

Set $\rho_0=r/16$, and take $\rho\leq\rho_0$. Let J be an interval in $(x-\rho,x+\rho)$ where $\rho/2^{n+1}\leq |J|\leq\rho/2^n$ for some $n\geq 10$. Then

$$\sigma(J)\leq\frac{1}{(1+\eta)^{n-1}}\sigma((x-2\rho,x+2\rho))\leq\frac{C}{(1+\eta)^{n-1}}\sigma((x-\rho,x+\rho)).$$

(2) in Definition 5.1 now follows easily, while the truth of (1) is trivial. □

We let

$$\Pi_1: \mathcal{S}^1\times\mathbb{R}^+\longrightarrow\mathcal{S}^1,$$
$$\Pi_2: \mathcal{S}^1\times\mathbb{R}^+\longrightarrow\mathbb{R}^+$$

denote the projections onto $\mathcal{S}^1$ and $\mathbb{R}^+$ respectively. Note that Π_2 is an isometry when restricted to rays passing through the origin.

Theorem 5.4. *Let $\mu=\nu\times\lambda$ where ν is uniform at a point. If λ is not doubling away from the origin, then the M_μ is not weak* $(1,1)$.

Proof. Assume λ is not doubling away from the origin. Using Proposition 3.6(2) take a sequence of intervals $U_n=(a_n,b_n)\subset\mathbb{R}^+$ with $U_n=U_1^n\cup U_2^n$, $n|U_1^n|\leq|U_2^n|$, $n\lambda(U_2^n)\leq\lambda(U_1^n)$ and $(b_n-a_n)/a_n\to 0$. Assume for convenience that U_1^n is to the left of U_2^n in $\mathbb{R}^+$. Let $x\in\mathcal{S}^1$ be a point given by Definition 5.1 and ρ_0 and C be the corresponding constants. Let B_1 and B_2 be two open balls of radius $r=(b_n-a_n)/4$, both of whose points closest to the origin are on the circle $|y|=a_n$, and which touch only at one point z, where $\Pi_1(z)=x$. Let $I_n=\Pi_1(B_1\cup B_2)$ and $J_n=\Pi_1(A_n\cap B_1)$ where $A_n=\mathcal{S}^1\times U_1^n$.

A routine calculation gives

$$|I_n|=4\arcsin\frac{b_n-a_n}{3a_n+b_n}\quad\text{and}\quad|J_n|\leq\frac{C}{\sqrt{n}}\frac{b_n-a_n}{3a_n+b_n}.$$

Thus for n large enough, since $(b_n-a_n)/a_n\to 0$, we have $|I_n|<\rho_0$, $|I_n|\to 0$, and $|J_n|/|I_n|\to 0$. Let $\mathcal{B}$ denote the collection of open balls with radius $(b_n-a_n)/4$ whose boundary points closest to the origin are on $I_n/2\times\{a_n\}$ and note that $z\in\bigcap_{B\in\mathcal{B}}B$.

Let $D=\bigcup_{B\in\mathcal{B}} B$. Let B be any ball in $\mathcal{B}$. Set $J=\Pi_1(B\cap A_n)$ and $U=(a_n, a_n+2r)$. Then $|J|=|J_n|$. Set

$$\mathcal{J}=\{J: J\subset I_n, J \text{ an open interval and } |J|=|J_n|\}$$

and $k_n=\sup_{J\in\mathcal{J}} \nu(J)/\nu(I_n)$. By (2) of Definition 5.1, $k_n\to 0$ as $n\to\infty$. For large enough n we have

$$\begin{aligned}\mu(B) &= \mu(B\cap A_n)+\mu(B-A_n)\\ &\leq \nu(J)\lambda(U)+\nu(I_n)\lambda(U_2^n)\\ &\leq k_n\nu(I_n)\lambda(U)+\nu(I_n)\frac{1}{n}\lambda(U_1^n)\\ &\leq \left(k_n+\frac{1}{n}\right)\nu(I_n)\lambda(U)\\ &\leq C\left(k_n+\frac{1}{n}\right)\nu\left(\frac{I_n}{2}\right)\lambda(U)\\ &\leq C\left(k_n+\frac{1}{n}\right)\mu(D).\end{aligned}$$

Therefore

$$\mu\left(\left\{w\in \mathbb{S}^1\times\mathbb{R}^+ : \sup_{w\in B}\frac{\chi_B(z)}{\mu(B)} > \frac{1}{2C(k_n+1/n)\mu(D)}\right\}\right) \geq \mu(D)$$

while a weak-type $(1,1)$ inequality would imply that $\mu(D)\leq 2KC(k_n+1/n)\mu(D)$ for some $K>0$ (see [3]), which is a contradiction as $k_n\to 0$ as $n\to\infty$. $\square$

Theorem 5.5. *Let $\mu=\nu\times\lambda$. If λ is uniform at a point and ν is not doubling then M_μ is not weak-type* $(1,1)$.

Proof. Using Proposition 3.6(1) take a sequence of open intervals $U_n=(a_n, b_n)\subset \mathbb{S}^1$ with $U_n=U_1^n\cup U_2^n$, $n|U_1^n|\leq|U_2^n|$, $n\nu(U_2^n)\leq\nu(U_1^n)$ and the U_i^n adjacent. Assume for convenience that a_n is clockwise from b_n and U_1^n is counterclockwise from U_2^n. Set $\Delta_1^n=(b_n-a_n)/(n+1)$ and $\Delta_2^n=(b_n-a_n)/\sqrt{n}$, so $U_1^n=(b_n-\Delta_1^n, b_n)$. Let x be the point in $\mathbb{R}^+$ given by Definition 5.1 and ρ_0 and C be the corresponding constants. Let z be the point on $F_{b_n-\Delta_2^n}$ for which $\Pi_2(z)=x$. Let B_1 and B_2 be two balls both tangent to F_{b_n} whose boundaries touch only at z and whose radii are both equal to $d(z, F_{b_n})=|z|\sin\Delta_2^n=|x|\sin\Delta_2^n$, where $d(z, F_{b_n})$ denotes the distance of z to the ray F_{b_n}. Let p_1 and p_2 denote the points of tangency of B_1 and B_2 respectively to F_{b_n}. Let S denote the line segment with endpoints p_1 and p_2 and let $V_n=\Pi_2(S)$ and $I_n=2V_n$. Then it is obvious that $|I_n|=4|x|\sin\Delta_2^n$.

Let $\mathcal{B}$ denote the collection of balls tangent to F_{b_n} with radius $d(z, F_{b_n})$ whose point of tangency to F_{b_n}, call it p_B for $B\in\mathcal{B}$, satisfies $\Pi_2(p_b)\in I_n/2=V_n$. Then $z\in B$ for every $B\in\mathcal{B}$. For $B\in\mathcal{B}$ denote by T_B the segment on $F_{b_n-\Delta_1^n}$ that is contained in B. And let $J=\Pi_2(T_B)$. The following are seen to hold for sufficiently large n:

(1) $|I_n|\sim(b_n-a_n)/\sqrt{n}$,

(2) $|J|\le K_1(b_n-a_n)/n^{3/4}$,

(3) $|J|/|I_n|\le K_2/n^{1/4}$,

where the similarities, and constants depend only on $|x|$. Let $A=U_2^n\times\mathbb{R}^+$, $D=\bigcup_{B\in\mathcal{B}} B$, $\mathcal{J}=\{J\subset I_n\colon |J|\le K_1(b_n-a_n)/n^{3/4}\}$ and $k_n=\sup_{J\in\mathcal{J}}\lambda(J)/\lambda(I_n)$. Then for large enough n we have

$$\begin{aligned}
\mu(B) &= \mu(B\cap A)+\mu(B\setminus A)\\
&\le \lambda(I_n)\nu(U_2^n)+\lambda(J)\nu(U_1^n)\\
&\le \frac{1}{n}\lambda(I_n)\nu(U_1^n)+\lambda(J)\nu(U_1^n)\\
&\le \frac{1}{n}\lambda(I_n)\nu(U_1^n)+k_n\lambda(I_n)\nu(U_1^n)\\
&\le C\left(\frac{1}{n}+k_n\right)\lambda\left(\frac{I_n}{2}\right)\nu(U_1^n)\\
&\le C\left(\frac{1}{n}+k_n\right)\mu(D).
\end{aligned}$$

The conclusion follows as in the proof of Theorem 5.4. □

The following is now easily established.

Theorem 5.6. *If ν and λ are Radon measures and*

(1) *there is a set $A\subset\mathcal{S}^1$, $\nu(A)>0$, with $0<\theta_*^s(x,\nu)\le\theta^{*s}(x,\nu)<\infty$ for $x\in A$; or*

(2) *there is a set $A\subset\mathbb{R}^+$, $\lambda(A)>0$, with $0<\theta_*^s(x,\lambda)\le\theta^{*s}(x,\lambda)<\infty$ for $x\in A$;*

then M_μ is weak-type $(1,1)$ if and only if ν is doubling and λ is doubling away from the origin.

Proof. The converse here is Theorem 4.4. Assume that M_μ is weak-type $(1,1)$ and (1) holds. Then Lemma 5.2 and Theorem 5.4 imply that λ is doubling away from the origin. Lemma 5.3 and Theorem 5.5 then imply that ν is doubling. A similar argument may be applied under the assumption of (2). □

When $s=1$ we have the following as a consequence of the above and the Lebesgue density theorem.

Corollary 5.7. *If the Lebesgue decomposition of either ν or λ has a non-zero absolutely continuous part, then M_μ is weak-type $(1,1)$ if and only if ν is doubling and λ is doubling away from the origin.*

A similar argument to that used in the proof of Theorem 5.6 can be used to give the following result.

Theorem 5.8. *If either ν or λ is somewhere doubling, then M_μ is weak-type $(1,1)$ if and only if ν is doubling and λ is doubling away from the origin.*

We end with an example showing that the converse to Theorem 4.4 is not true in general. In the following μ will be an atomic measure.

Example 5.9. Let $S=\{q_n\}_{n=1}^\infty$ be a dense set in $\mathbb{R}^2$ and μ be a measure satisfying $\mu(\mathbb{R}^2\setminus S)=0$ for which there exists a constant $C>0$ such that $C\mu(q_n)\geq\sum_{m>n}\mu(q_m)$ for all n. It is clear that

$$\mu(\{q_i : M_\mu f(q_i)>\lambda\})\leq C'\mu(q_{n(\lambda)}),$$

where $n(\lambda)$ is the smallest index of the points in this set. Since for some B with $q_{n(\lambda)}\in B$,

$$\lambda<\frac{1}{\mu(B)}\int_B|f|\,d\mu\leq\frac{1}{\mu(q_{n(\lambda)})}\int_B|f|\,d\mu$$

we have $\mu(q_{n(\lambda)})<(1/\lambda)\int_B|f|\,d\mu$ and M_μ is weak-type $(1,1)$.

A product measure $\mu=\nu\times\lambda$ satisfying the above may be found as follows. Let $\{r_n\}_{n=1}^\infty$ and $\{s_n\}_{n=1}^\infty$ be countable dense sets in $\mathcal{S}^1$ and $\mathbb{R}^+$ respectively. Let $\nu(r_n)=2^{-2^n}$ and $\lambda(s_n)=2^{-2^n}$. Enumerate the q_n by setting q_1 to be the pair (r_i,s_j) with greatest μ-measure, q_2 to be the pair with second greatest μ-measure to be and so on. Only two points may have the same measure, so in the event of a tie choose arbitrarily. Observe that the pairs (n,m) and (i,j) satisfy $2^{-2^m}2^{-2^n}\geq 2^{-2^i}2^{-2^j}$ if and only if $(\max\{n,m\},\min\{n,m\})\leq(\max\{i,j\},\min\{i,j\})$ lexicographically. Hence if $q_n=(r_p,r_t)$, with $p\leq t$, we have

$$\sum_{m>n}\mu(q_m)\leq 2\biggl(\sum_{i=t+1}^\infty\sum_{j=1}^\infty\frac{1}{2^{2^i}}\frac{1}{2^{2^j}}+\frac{1}{2^{2^t}}\sum_{j=p}\frac{1}{2^{2^j}}\biggr)<C_1\frac{1}{2^{2^{t+1}}}+C_2\frac{1}{2^{2^t}}\frac{1}{2^{2^p}}\leq C\mu(q_n).$$

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