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Irreducible Harish-Chandra modules over extended Witt algebras

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Abstract. Let d be a positive integer, $A=\mathbb{C}[t_1^{\pm 1}, ..., t_d^{\pm 1}]$ be the Laurent polynomial algebra, and $W=\mathrm{Der}(A)$ be the derivation Lie algebra of A. Then we have the semidirect product Lie algebra $W\ltimes A$ which we call the extended Witt algebra of rank d. In this paper, we classify all irreducible Harish-Chandra modules over $W\ltimes A$ with nontrivial action of A.

1. Introduction

The Witt algebra W of rank d is the Lie algebra consisting of all derivations of the Laurent polynomial algebra $A=\mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, ..., t_d^{\pm 1}]$, where d is a fixed positive integer. This algebra is isomorphic to the algebra of diffeomorphisms of the d-dimensional torus. The algebra W is a natural higher-rank generalization of the centerless Virasoro algebra, which has wide applications in different branches of mathematics and physics (see [9] and [11]–[15]). Unlike the Virasoro algebra, when $d>1$, W is centrally closed and its representation theory is still under development.

Modules over Witt algebras were used by O. Mathieu [18] to model simple cuspidal weight modules with finite-dimensional weight spaces over simple finite-dimensional Lie algebras. Representations of Witt algebras are also closely connected with the representation theory of extended affine Lie algebras ([1]) and toroidal Lie algebras ([2], [7], and [8]).

Representation theory of Witt algebras has been intensively studied by many mathematicians and physicists, see [3], [5], [6], [10], [11]–[15], [19], and [22]. In particular, [19] asserted recently that any irreducible Harish-Chandra W-module is either dense (with uniformly bounded weight spaces) or punctured (with uniformly bounded weight spaces) or a simple quotient of some generalized Verma module. So far, the only known dense or punctured modules are subquotients of the modules $F^{\alpha}(\psi, e)$ (see Section 2.1) introduced and studied in [13] and [21], which are called

Shen modules or Larsson modules. Eswara Rao (see the introduction of [6]) said that it will be interesting to prove that any irreducible Harish-Chandra W-module is either a highest weight module or a Larsson module up to a twist of a $\mathrm{GL}(d,\mathbb{Z})$ action. This was formulated into a more precise conjecture in [10] as follows.

Conjecture. All nontrivial irreducible uniformly bounded modules over W are isomorphic to irreducible subquotients of $F^\alpha(\psi,e)$.

Assuming this conjecture and using the result in [4], we can deduce that the third class of modules mentioned above depend only on the first two classes of the modules over W. So it is crucial to classify dense modules and punctured modules over W.

As an attempt to obtain a proof of this conjecture, we consider representations of the Lie algebra $W\ltimes A$, which was started by Eswara Rao in [6]. The W-modules $F^\alpha(\psi,e)$ can be naturally made into $(W\ltimes A)$-modules. In [6], Eswara Rao proved that if the action of A is associative on an irreducible $(W\ltimes A)$-module V, then V must be isomorphic to $F^\alpha(\psi,e)$. In Section 3, we show that the condition of Eswara Rao is always satisfied for any irreducible uniformly bounded $(W\ltimes A)$-module with nontrivial action of A, up to a natural automorphism of $W\ltimes A$. As a result, any nontrivial irreducible uniformly bounded $(W\ltimes A)$-module is either an irreducible uniformly bounded W-module if the action of A is zero, or isomorphic to some $F^\alpha(\psi,e,c)$ (a deformation of $F^\alpha(\psi,e)$, see Section 2 for its definition) if the action of A is nonzero. At last in Section 4, we show that any nontrivial irreducible $(W\ltimes A)$-module which is not uniformly bounded must be isomorphic to the unique quotient module of some generalized Verma module induced from a nontrivial irreducible uniformly bounded module over some subalgebra isomorphic to $W_{d-1}\ltimes A_{d-1}$, where $A_{d-1}=\mathbb{C}[t_1^{\pm1},\dots,t_{d-1}^{\pm1}]$. If the conjecture mentioned above holds, we can give a complete classification of irreducible Harish-Chandra modules over $W\ltimes A$, using our results in Sections 3 and 4.

2. Notation and preliminaries

We denote by $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of all integers, nonnegative integers, positive integers, real numbers and complex numbers, respectively. For any positive integer k, denote by $M_k(\mathbb{C})$ the set of all $k\times k$ matrices over $\mathbb{C}$. For a matrix $B\in M_k(\mathbb{C})$, we denote its (i,j)-entry by $B(i,j)$. For a Lie algebra $\mathcal{L}$, we denote its enveloping algebra by $U(\mathcal{L})$.

2.1. Witt algebra W and the extended Witt algebra $W\ltimes A$

Throughout this paper, we fix a positive integer $d>1$. Let $\mathbb{C}^d$ be the vector space of $d\times 1$ matrices with the standard basis $\{e_1, e_2, ..., e_d\}$ and $\mathbb{Z}^d$ be the subset of $\mathbb{C}^d$ with entries in $\mathbb{Z}$. Let $(\,\cdot\,|\,\cdot\,)$ be the standard symmetric bilinear form such that $(u|v)=u^Tv\in\mathbb{C}$, where u^T is the matrix transpose of u.

Let $A=\mathbb{C}[t_1^{\pm 1}, ..., t_d^{\pm 1}]$ be the *Laurent polynomial algebra* over $\mathbb{C}$ and denote by W the algebra of all derivations of A, called the *Witt algebra*.

Set $t^n=t_1^{n_1}...t_d^{n_d}$ for $n=(n_1, n_2, ..., n_d)^T\in\mathbb{Z}^d$ and $\partial_i=t_i\partial/\partial t_i$ for $i\in\{1, ..., d\}$. Set $D(\gamma, m)=t^m\sum_{i=1}^d\gamma_i\partial_i$ for any $\gamma\in\mathbb{C}^d$ and $m\in\mathbb{Z}^d$. Then W is spanned by all $D(\gamma, m)$ with $\gamma\in\mathbb{C}^d$ and $m\in\mathbb{Z}^d$. The Lie bracket of W is given by

$$(1)\qquad [D(\gamma, m), D(\delta, n)] = D(\varepsilon, m+n) \quad \text{for } \gamma, \delta\in\mathbb{C}^d \text{ and } m, n\in\mathbb{Z}^d,$$

where $\varepsilon=(\gamma|n)\delta-(\delta|m)\gamma$. The extended Witt algebra $W\ltimes A=W\oplus A$ is a Lie algebra by extending the Lie structure of W in the following way:

$$(2)\quad [t^m, t^n]=0 \quad\text{and}\quad [D(\gamma, m), t^n]=(\gamma\,|\,n)t^{m+n} \text{ for any } \gamma\in\mathbb{C}^d \text{ and } m, n\in\mathbb{Z}^d.$$

Both W and $W\ltimes A$ are $\mathbb{Z}^d$-graded, and their homogeneous subspaces are defined by $(W)_m=\sum_{\gamma\in\mathbb{C}^d} D(\gamma, m)$ and $(W\ltimes A)_m=(W)_m\oplus\mathbb{C}t^m$ for any $m\in\mathbb{Z}^d$.

Take $\mathcal{L}=W$ or $\mathcal{L}=W\ltimes A$. Then $\mathcal{L}_0$ is the Cartan subalgebra of $\mathcal{L}$. An $\mathcal{L}$-module V is called a *weight module* if the action of $\mathcal{L}_0$ on V is diagonalizable. For any weight module V we have the decomposition $V=\bigoplus_{\lambda\in\mathcal{L}_0^*} V_\lambda$, where $\mathcal{L}_0^*=\mathrm{Hom}_{\mathbb{C}}(\mathcal{L}_0, \mathbb{C})$ and

$$V_\lambda=\{v\in V\mid \partial v=\lambda(\partial)v \text{ for all } \partial\in\mathcal{L}_0\}.$$

The space V_λ is called the *weight space* corresponding to the weight λ. The *support* of the weight module V, denoted by $\mathrm{Supp}(V)$, is the set of all weights λ with $V_\lambda\neq 0$. A weight module V is a *Harish-Chandra* module if $\dim V_\lambda<\infty$ for all $\lambda\in\mathrm{Supp}(V)$ and is called *uniformly bounded* if there is some $N\in\mathbb{N}$ such that $\dim V_\lambda\leq N$ for all $\lambda\in\mathrm{Supp}(V)$.

Let $\mathfrak{gl}_d$ be the Lie algebra of all $d\times d$ complex matrices and $\mathfrak{sl}_d$ be the subalgebra of $\mathfrak{gl}_d$ consisting of all traceless matrices. For $1\leq i, j\leq d$ we use E_{ij} to denote the matrix units, i.e., $E_{i,j}$ has entry 1 at (i, j) and 0 otherwise.

For any integral dominant weight ψ (on the Cartan subalgebra of $\mathfrak{sl}_d$ consisting of all diagonal matrices), let $V(\psi)$ be the irreducible finite-dimensional $\mathfrak{sl}_d$-module with highest weight ψ. We make $V(\psi)$ into a $\mathfrak{gl}_d$ module by defining the action of the identity matrix I as some scalar $e\in\mathbb{C}$. We denote the resulting module by $V(\psi, e)$.

For any $\alpha\in\mathbb{C}^d$, it is known that $F^\alpha(\psi,e)=V(\psi,e)\otimes A$ is a W-module by defining

$$D(\gamma,n)(v\otimes t^m)=(\gamma\,|\,m+\alpha)v\otimes t^{m+n}+((n\gamma^T)v)\otimes t^{m+n}$$

for any $m,n\in\mathbb{Z}^d$, $\gamma\in\mathbb{C}^d$ and $v\in V(\psi,e)$. Moreover, given any $c\in\mathbb{C}$, we can make $F^\alpha(\psi,e)$ into a $(W\ltimes A)$-module by defining

$$t^n(v\otimes t^m)=cv\otimes t^{m+n}$$

for any $m,n\in\mathbb{Z}^d$ and $v\in V(\psi,e)$. We will denote the resulted $(W\ltimes A)$-module by $F^\alpha(\psi,e,c)$. It is obvious that $F^\alpha(\psi,e)$ and $F^\alpha(\psi,e,c)$ are exp-polynomial modules in the sense of [4]. We now collect some known results on the modules $F^\alpha(\psi,e)$ and $F^\alpha(\psi,e,c)$ from [5], [6] and [10].

Theorem 2.1. (1) *$F^\alpha(\psi,e)$ is reducible as a W-module if and only if $(\psi,e)=(\delta_k,k)$; or $\psi=0$, $e\in\{0,d\}$ and $\alpha\in\mathbb{Z}^d$, where δ_k is the k-th fundamental weight of $\mathfrak{sl}_d$.*

(2) *$F^\alpha(\psi,e,c)$ is irreducible as a $(W\ltimes A)$-module if $c\neq 0$.*

(3) *Let V be an irreducible Harish-Chandra $(W\ltimes A)$-module satisfying $t^0v=v$ and $t^mt^nv=t^{m+n}v$ for any $m,n\in\mathbb{Z}^d$ and $v\in V$, then $V\cong F^\alpha(\psi,e,1)$ for suitable $\alpha\in\mathbb{C}^d$, $e\in\mathbb{C}$, and some ψ which is an integral dominant weight of $\mathfrak{sl}_d$.*

2.2. Heisenberg–Virasoro algebra

We first recall the definition of the Heisenberg–Virasoro algebra HVir over $\mathbb{C}$.

Definition 2.2. The *twisted Heisenberg–Virasoro algebra* HVir is a Lie algebra over $\mathbb{C}$ with the basis

$$\{x_i, I(i), C_D, C_{DI}, C_I\,|\,i\in\mathbb{Z}\}$$

and the Lie bracket given by

$$\begin{aligned}
[x_i,x_j]&=(j-i)x_{i+j}+\delta_{i,-j}\frac{i^3-i}{12}C_D,\\
[x_i,I(j)]&=jI(i+j)+\delta_{i,-j}(i^2+i)C_{DI},\\
[I(i),I(j)]&=i\delta_{i,-j}C_I,\\
[\mathrm{HVir},C_D]&=[\mathrm{HVir},C_{DI}]=[\mathrm{HVir},C_I]=0.
\end{aligned}$$

We can define weight modules over HVir by requiring that the action of the Cartan subalgebra $\mathbb{C}x_0 \oplus \mathbb{C}I(0) \oplus \mathbb{C}C_D \oplus \mathbb{C}C_{DI} \oplus \mathbb{C}C_I$ is diagonalizable. Then we have the concepts of Harish-Chandra modules and uniformly bounded modules similarly as we did for W and $W \ltimes A$. In particular, for any $a, b, c \in \mathbb{C}$ we have the module of intermediate series $V(a,b,c)$ over HVir which has a $\mathbb{C}$-basis $\{v_i \mid i \in \mathbb{Z}\}$ and the HVir-actions

$$\begin{aligned} x_j v_i &= (a+i+bj)v_{i+j}, \\ I(j)v_i &= cv_{i+j}, \\ C_D v_i &= C_I v_i = C_{DI} v_i = 0 \quad \text{for } i, j \in \mathbb{Z}. \end{aligned}$$

And $V(a,b,c)$ is reducible if and only if $a \in \mathbb{Z}$, $b \in \{0,1\}$ and $c=0$. The unique nontrivial irreducible subquotient of $V(a,b,c)$ is denoted by $V'(a,b,c)$.

In [17], Lu and Zhao gave a complete classification of irreducible Harish-Chandra modules over HVir. In particular, we have the following theorem.

Theorem 2.3. *Any nontrivial irreducible uniformly bounded module over* HVir *is isomorphic to* $V'(a,b,c)$ *for some* $a, b, c \in \mathbb{C}$.

We note that for any fixed $\gamma \in \mathbb{C}^d$ and $m \in \mathbb{Z}^d$ with $(\gamma \mid m) \neq 0$, the subalgebra of $W \ltimes A$ spanned by $\{D(\gamma, im), t^{im} \mid i \in \mathbb{Z}\}$, which we denote by $\mathrm{HVir}(\gamma, m)$, is isomorphic to $\mathrm{HVir}/(\mathbb{C}C_D \oplus \mathbb{C}C_{DI} \oplus \mathbb{C}C_I)$ by assigning $D(\gamma, im)/(\gamma \mid m)$ to x_i and $t^{im}/(\gamma \mid m)$ to $I(i)$.

3. Uniformly bounded modules over $W \ltimes A$

In this section, we give a description of irreducible uniformly bounded modules over $W \ltimes A$ with nontrivial action of A.

We first recall a result on $\mathbb{Z}^d$-graded A-modules. Here and later, we consider A as a $\mathbb{Z}^d$-graded commutative Lie algebra. Let V be a $\mathbb{Z}^d$-graded module over A, that is, $V = \bigoplus_{m \in \mathbb{Z}^d} V_m$ and $t^n V_m \subseteq V_{m+n}$ for any $m, n \in \mathbb{Z}^d$. A $\mathbb{Z}^d$-graded A-module is said to be *irreducible* if it does not contain any nonzero proper graded submodule.

Lemma 3.1. *Let* $V = \bigoplus_{n \in \mathbb{Z}^d} V_n$ *be an irreducible uniformly bounded* $\mathbb{Z}^d$*-graded* A*-module, then* $\dim V_n \leq 1$ *for all* $n \in \mathbb{Z}^d$.

Proof. For $n \in \mathbb{Z}^d$, let U_n be the nth homogeneous subspace of $U(A)$ with respect to the $\mathbb{Z}^d$-gradation on A. Then U_0 is a commutative associative algebra, and each nonzero homogeneous subspace V_n is a finite-dimensional irreducible

U_0-module. The finite-dimensional condition guarantees that the action of U_0 on V_n has common eigenvectors. Therefore $\dim V_n \leq 1$ for all $n \in \mathbb{Z}^d$. □

We will frequently use the following technical lemma later in this paper.

Lemma 3.2. *Suppose that V is an irreducible $(W \ltimes A)$-module and $g \in U(A)$. If $gv=0$ for some nonzero $v \in V$, then g is locally nilpotent on V.*

Proof. For any $\gamma \in \mathbb{C}^d$ and $s \in \mathbb{Z}^d$, we have

$$[[D(\gamma, s), g], g] = 0.$$

Therefore, we can show that

$$\begin{aligned} &g^{l+1} D(\gamma_1, s_1)...D(\gamma_l, s_l)v \\ &\quad = \sum_{i_1+...+i_l=l+1} \frac{(l+1)!}{i_1!...i_l!} ((\mathrm{ad} g)^{i_1} D(\gamma_1, s_1))...((\mathrm{ad} g)^{i_l} D(\gamma_l, s_l))v \\ &\quad = 0, \end{aligned}$$

where $\gamma_i \in \mathbb{C}^d$ and $s_i \in \mathbb{Z}^d$, $1 \leq i \leq l$. Since V is irreducible, we see that g is locally nilpotent on V. □

Theorem 3.3. *Let V be an irreducible uniformly bounded weight $(W \ltimes A)$-module. Let $t^0 v = cv$ for any $v \in V$, where $c \in \mathbb{C}$. Then the following are true*:

(1) *If $c \neq 0$, then $(t^m t^s - c t^{m+s})v = 0$ for any $m, s \in \mathbb{Z}^d$ and $v \in V$. In this case, $V \cong F^\alpha(\psi, e, c)$ for suitable $\alpha \in \mathbb{C}^d$, $e \in \mathbb{C}$, and ψ which is an integral dominant weight of $\mathfrak{sl}_d$.*

(2) *If $c=0$, then $AV=0$.*

Proof. There exists $\alpha \in \mathbb{C}^d$ such that $V = \bigoplus_{m \in \mathbb{Z}^d} V_m$, where

$$V_m = \{v \in V \mid D(\delta, 0)v = (\delta \mid \alpha + m)v \text{ for any } \delta \in \mathbb{C}^d\} \quad \text{for } m \in \mathbb{Z}^d.$$

(1) Suppose $c \neq 0$. We may assume that $c=1$ by replacing t^m with t^m/c for any $m \in \mathbb{Z}^d$ if necessary, which is equivalent to making a Lie algebra automorphism. Choose $\gamma \in \mathbb{C}^d$ such that $(\gamma | m) \neq 0$ for all nonzero $m \in \mathbb{Z}^d$.

Claim 1. *t^m acts injectively and $t^m t^{-m} - t^0$ acts locally nilpotently on V for any $m \in \mathbb{Z}^d$.*

For any $0\neq m\in\mathbb{Z}^d$, the subalgebra

$$\mathrm{HVir}(\gamma,m)=\Big(\bigoplus_{i\in\mathbb{Z}}\mathbb{C}D(\gamma,im)\Big)\ltimes\Big(\bigoplus_{i\in\mathbb{Z}}\mathbb{C}t^{im}\Big)$$

is isomorphic to the Heisenberg–Virasoro algebra modulo $\mathbb{C}C_D\oplus\mathbb{C}C_{DI}\oplus\mathbb{C}C_I$ and $V(m)=\sum_{i\in\mathbb{Z}}V_{im}$ is a uniformly bounded $\mathrm{HVir}(\gamma,m)$-module. There is a nontrivial irreducible $\mathrm{HVir}(\gamma,m)$-submodule X of $V(m)$. Then $(t^{im}t^{jm}-t^{(i+j)m})v=0$ for any $i,j\in\mathbb{Z}_+$ and $v\in X$, which implies by Lemma 3.2 that $t^{im}t^{jm}-t^{(i+j)m}$ is locally nilpotent on V. Claim 1 is proved.

Claim 2. *For any $m,n\in\mathbb{Z}^d$, there exists $\lambda_{m,n}\in\mathbb{C}^*$ such that $t^mt^n-\lambda_{m,n}t^{m+n}$ acts locally nilpotently on V.*

Since V is a uniformly bounded $\mathbb{Z}^d$-graded module over the Lie algebra A, by Lemma 3.1, we can see that V contains an irreducible $\mathbb{Z}^d$-graded A-submodule $X=\bigoplus_{n\in\mathbb{Z}^d}X_n$ such that $X_n\subseteq V_n$ and $\dim X_n\leq 1$. If there is some $n\in\mathbb{Z}^d$ such that $X_n=0$, then we can take any $0\neq v\in X_m$ for some $m\in\mathbb{Z}^d$ to deduce a contradiction to Claim 1, viz. $t^{n-m}v=0$.

Thus we have $\dim X_m=1$ for all $m\in\mathbb{Z}^d$ and there exist $\lambda_{m,n}\in\mathbb{C}^*$ satisfying

$$t^mt^nv=\lambda_{m,n}t^{m+n}\quad\text{for } m,n\in\mathbb{Z}^d \text{ and } v\in X.$$

Also by Lemma 3.2, $t^mt^n-\lambda_{m,n}t^{m+n}$ acts locally nilpotently on V. Claim 2 is proved.

Claim 3. $\lambda_{m,n}=1$ *for all $m,n\in\mathbb{Z}^d$.*

Note that the claim is true for $n=-m$ by Claim 1. Set

$$X_{m,n}=t^mt^n-\lambda_{m,n}t^{m+n}.$$

Note that $\dim V_m=\dim V_n$ for all $m,n\in\mathbb{Z}^d$. Choose a basis $\{v_1,...,v_k\}$ of V_0. Then $\{t^mv_1,...,t^mv_k\}$ is a basis of V_m by Claim 1, and there exists $B_{m,n}\in M_k(\mathbb{C})$ such that

$$t^mt^n(v_1,...,v_k)=t^{m+n}(v_1,...,v_k)B_{m,n}.$$

From $t^rt^st^mt^n=t^mt^nt^rt^s$, we see that

$$B_{r,s}B_{m,n}=B_{m,n}B_{r,s}.$$

By Lie's theorem, there exists a matrix $S\in M_k(\mathbb{C})$ such that $C_{m,n}=S^{-1}B_{m,n}S$ are upper triangular matrices for all $m,n\in\mathbb{Z}^d$. Let

$$(w_1,...,w_k)=(v_1,...,v_k)S.$$

Then

$$t^m t^n(w_1,\dots,w_k)=t^{m+n}(w_1,\dots,w_k)C_{m,n}. \tag{3}$$

Consequently

$$X_{m,n}(w_1,\dots,w_k)=t^{m+n}(w_1,\dots,w_k)(C_{m,n}-\lambda_{m,n}I_k), \tag{4}$$

where I_k is the identity matrix in $M_k(\mathbb{C})$. By the fact that $X_{m,n}$ is locally nilpotent on V, we see that all the diagonal entries of $C_{m,n}$ are $\lambda_{m,n}$. We obtain that

$$X_{m,n}^k V_0=0 \quad \text{and} \quad X_{m,n}w_1=0. \tag{5}$$

We can compute

$$\begin{aligned} 0&=D(\gamma,-m-n)^k X_{m,n}^k(w_1,\dots,w_k)\\ &=(k![D(\gamma,-m-n),X_{m,n}]^k+X_{m,n}g_{-m-n})(w_1,\dots,w_k), \end{aligned} \tag{6}$$

where $g_{-m-n}\in U(W\ltimes A)$ is some element such that $g_{-m-n}V_0\subseteq V_{-m-n}$. From (3) we have that

$$\begin{aligned} &[D(\gamma,-m-n),X_{m,n}]^k(w_1,\dots,w_k)\\ &\qquad=((\gamma\,|\,m)t^{-n}t^n+(\gamma\,|\,n)t^{-m}t^m-(\gamma\,|\,m+n)\lambda_{m,n}t^0)^k(w_1,\dots,w_k)\\ &\qquad=(w_1,\dots,w_k)((\gamma\,|\,m)C_{-n,n}+(\gamma\,|\,n)C_{-m,m}-(\gamma\,|\,m+n)\lambda_{m,n}I_k)^k \end{aligned} \tag{7}$$

and from (3) and (4) that

$$\begin{aligned} X_{m,n}g_{-m-n}(w_1,\dots,w_k)&=t^{-m-n}X_{m,n}(w_1,\dots,w_k)G_{-m-n}\\ &=(w_1,\dots,w_k)(C_{m+n,-m-n}(C_{m,n}-\lambda_{m,n}I_k)G_{-m-n}), \end{aligned} \tag{8}$$

where $G_{-m-n}\in M_k(\mathbb{C})$ is given by

$$g_{-m-n}(w_1,\dots,w_k)=t^{-m-n}(w_1,\dots,w_k)G_{-m-n}.$$

Note that $C_{m,n}C_{r,s}=C_{r,s}C_{m,n}$. Then the kth row of the matrix

$$C_{m+n,-m-n}(C_{m,n}-\lambda_{m,n}I_k)G_{-m-n}=(C_{m,n}-\lambda_{m,n}I_k)C_{m+n,-m-n}G_{-m-n}$$

is zero and all the diagonal entries of the upper triangular matrix $C_{-n,n}$ are equal to 1 for all $n\in\mathbb{Z}^d$. Combining (5), (6) and (8) and considering the coefficient of w_k in $D(\gamma,-m-n)^k X_{m,n}^k w_k$, we have

$$(\gamma\,|\,m+n)^k(1-\lambda_{m,n})^k=0,$$

that is, $\lambda_{m,n}=1$ for all $m,n\in\mathbb{Z}^d$. Claim 3 is proved.

As a result of the formula (5) and Claim 3, we have $t^m t^n w_1 = t^{m+n} w_1$ for all $m, n \in \mathbb{Z}^d$. So the set

$$V' = \{v \in V \mid t^m t^n v = t^{m+n} v \text{ for all } m, n \in \mathbb{Z}^d\}$$

is nonzero. It is easy to check that $\mathrm{span}\{t^m t^n - t^{m+n} \mid m, n \in \mathbb{Z}^d\}$ is stable under the action of W, so V' is a $(W \ltimes A)$-submodule of V. By the irreducibility of the $(W \ltimes A)$-module V, we see that $V' = V$. Our result for this case follows from Theorem 2.1.

(2) Suppose now that $c=0$. Since V is uniformly bounded, there exists $N \in \mathbb{N}$ such that $\dim V_m < N$ for all $m \in \mathbb{Z}^d$. For $n \in \mathbb{Z}^d$, view $\bigoplus_{i \in \mathbb{Z}} V_{n+im}$ as a module over $\mathrm{HVir}(\gamma, m)$. The fact that $\bigoplus_{i \in \mathbb{Z}} V_{n+im}$ is uniformly bounded implies that $\bigoplus_{i \in \mathbb{Z}} V_{n+im}$ has an $\mathrm{HVir}(\gamma, m)$-modules composition series with the number of nontrivial composition factors not bigger than N. By Theorem 2.3, t^m acts trivially on each composition factor. We can also see that the total number of composition factors (trivial and nontrivial) in this composition series is not bigger than $2N$. So $(t^m)^{2N} V = 0$ for all $m \in \mathbb{Z}^d$. Let N_0 be the smallest integer such that $(t^m)^{N_0} V = 0$ for all $m \in \mathbb{Z}^d$. Clearly, $N_0 \geq 1$. Recall that $(\gamma | r) \neq 0$ for all $r \in \mathbb{Z}^d$. Then

$$0 = D(\gamma, r-m)(t^m)^{N_0} V = (\gamma \mid m) N_0 t^r (t^m)^{N_0 - 1} V \quad \text{for all } r, m \in \mathbb{Z}^d.$$

By the choice of N_0, there are some $v \in V$ and $0 \neq m \in \mathbb{Z}^d$ such that $u = (t^m)^{N_0-1} v \neq 0$. Thus, $t^r u = 0$ for all $r \in \mathbb{Z}^d$.

It is easy to check that the nonzero subspace

$$V' = \{u \in V \mid t^r u = 0 \text{ for all } r \in \mathbb{Z}^d\}$$

is a $(W \ltimes A)$-submodule of V. By the irreducibility of V, we have $V' = V$. Part (2) follows. □

4. Unbounded weight modules over $W \ltimes A$

To describe irreducible weight modules over $W \ltimes A$ which are not uniformly bounded, we need to introduce some new notation. Let $\mathcal{L} = \bigoplus_{m \in \mathbb{Z}^d} \mathcal{L}_m$ be a $\mathbb{Z}^d$-graded Lie algebra such that $\mathcal{L}_0$ is abelian and the gradation itself is the root space decomposition with respect to $\mathcal{L}_0$. We also assume that $[\mathcal{L}_m, \mathcal{L}_n] = \mathcal{L}_{m+n}$ for all distinct $m, n \in \mathbb{Z}^d$. It is clear that W, $W \ltimes A$ and (centerless) higher-rank or classical Virasoro algebras are examples of such algebras. We can define weight modules and Harish-Chandra modules and other similar concepts for $\mathcal{L}$ as we did for the algebras W and $W \ltimes A$.

For convenience, we denote by λ_n the weight of $\mathcal{L}_n$ in the adjoint representation of $\mathcal{L}$, that is, $\mathcal{L}_n=\{y\in\mathcal{L}\,|\,[x,y]=\lambda_n(x)y$ for all $x\in\mathcal{L}_0\}$, where $n\in\mathbb{Z}^d$. When considering weights with respect to $\mathcal{L}$, we identify $\{\lambda_n\,|\,n\in\mathbb{Z}^d\}$ with $\mathbb{Z}^d$ naturally.

Suppose that V is a weight $\mathcal{L}$-module. Then V is *dense* provided $\mathrm{Supp}(V)=\lambda+\mathbb{Z}^d$ for some $\lambda\in\mathcal{L}_0^*$, and is *cut* provided there are $\lambda\in\mathrm{Supp}(V)$, $\delta\in\mathbb{R}^d\setminus\{0\}$ and $m\in\mathbb{Z}^d$ such that $\mathrm{Supp}(V)\subseteq\lambda+m+\mathbb{Z}^{(\delta)}_{\leq 0}$, where $\mathbb{Z}^{(\delta)}_{\leq 0}=\{n\in\mathbb{Z}^d\,|\,(\delta|n)\leq 0\}$. An element $v\in V$ is called a *generalized highest weight vector* provided there exist a $\mathbb{Z}$-basis $\{s_1,\dots,s_d\}$ of $\mathbb{Z}^d$ and $N\in\mathbb{N}$ such that $\mathcal{L}_m v=0$ for all $m=\sum_{i=1}^d m_i s_i\in\mathbb{Z}^d$ with $m_i>N$, $i=1,\dots,d$.

Let us recall the following general result for $\mathcal{L}$-modules, see Theorem 4.1 in [19].

Theorem 4.1. *Let V be an irreducible weight $\mathcal{L}$-module, which is neither dense nor trivial. If V contains a generalized highest weight vector, then V is a cut module.*

Now we introduce a class of cut modules over $\mathcal{L}$. Choose some subgroup $H\subset\mathbb{Z}^d$ such that $\mathbb{Z}^d=H\oplus\mathbb{Z}\beta$ for some $\beta\in\mathbb{Z}^d\setminus\{0\}$. Define the following subalgebras of $\mathcal{L}$:

$$\mathcal{L}_H=\bigoplus_{g\in H}\mathcal{L}_g,\quad \mathcal{L}_H^+=\bigoplus_{\substack{g\in H\\ k\in\mathbb{N}}}\mathcal{L}_{g+k\beta}\quad\text{and}\quad \mathcal{L}_H^-=\bigoplus_{\substack{g\in H\\ k\in\mathbb{N}}}\mathcal{L}_{g-k\beta}.$$

Let X be an irreducible $\mathcal{L}_H$-module. We make X into an $(\mathcal{L}_H+\mathcal{L}_H^+)$-module by defining $\mathcal{L}_H^+X=0$. Then we define the *generalized Verma module* $M(H,\beta,X)$ over $\mathcal{L}$ as

$$M(H,\beta,X)=\mathrm{Ind}^{\mathcal{L}}_{\mathcal{L}_H^++\mathcal{L}_H}X=U(\mathcal{L})\otimes_{\mathcal{L}_H+\mathcal{L}_H^+}X.$$

It is easy to see that $M(H,\beta,X)$ has a unique irreducible quotient module, which we denote by $L^{\mathcal{L}}(H,\beta,X)$. If there are no ambiguities occurring, we simply denote it by $L(H,\beta,X)$.

We will simply call $L(H,\beta,X)$ an *irreducible highest weight module*. It is easy to see that $L(H,\beta,X)$ is a cut module. It was shown in [4] that $L(H,\beta,X)$ is a Harish-Chandra module if X is a uniformly bounded exp-polynomial module.

Let $\gamma\in\mathbb{C}^d$ be such that $(\gamma|m)\neq 0$ for all $m\in\mathbb{Z}^d$. The subalgebra $\mathrm{Vir}(\gamma)=\sum_{m\in\mathbb{Z}^d}\mathbb{C}D(\gamma,m)$ of $W\ltimes A$ is a centerless rank-d Virasoro algebra (in the sense of [20]). We can see that $L^{\mathrm{Vir}(\gamma)}(H,\beta,X)$ is not uniformly bounded provided that X is nontrivial, by [16]. On the other hand, if X is trivial, so is $L^{\mathrm{Vir}(\gamma)}(H,\beta,X)$. Similar results hold for $W\ltimes A$ by considering $(W\ltimes A)$-modules as $\mathrm{Vir}(\gamma)$-modules.

Theorem 4.2. *Any irreducible Harish-Chandra* $\mathrm{Vir}(\gamma)$*-module which is not uniformly bounded is isomorphic to $L^{\mathrm{Vir}(\gamma)}(H,\beta,X)$ for some H, β and X, where $\beta\in\mathbb{Z}^d\setminus\{0\}$, H is a subgroup of $\mathbb{Z}^d$ such that $\mathbb{Z}^d=H\oplus\mathbb{Z}\beta$, and X is a nontrivial irreducible uniformly bounded* $\mathrm{Vir}(\gamma)_H$*-module.*

Using Theorems 4.1 and 4.2, we can give a description of irreducible Harish-Chandra $(W\ltimes A)$-modules which are not uniformly bounded.

Theorem 4.3. *Let V be an irreducible Harish-Chandra $(W\ltimes A)$-module which is not uniformly bounded. Then V is isomorphic to some $L(H,\beta,X)$, where $\beta\in\mathbb{Z}^d\setminus\{0\}$, H is a subgroup of $\mathbb{Z}^d$ such that $\mathbb{Z}^d=H\oplus\mathbb{Z}\beta$, and X is an irreducible uniformly bounded $(W\ltimes A)_H$-module.*

Proof. For convenience set $\mathcal{L}=W\ltimes A$. By the irreducibility of V, there is some $\lambda\in\mathcal{L}_0^*$ such that $\mathrm{Supp}(V)\subseteq\lambda+\mathbb{Z}^d$. Let $\{e_i\,|\,i=1,...,d\}$ be the canonical $\mathbb{Z}$-basis of $\mathbb{Z}^d$. Given any $n\in\mathbb{Z}^d$, we will denote by n_i the ith entry of n for $i=1,...,d$, that is, $n=\sum_{i=1}^d n_ie_i$.

Choose any $\gamma\in\mathbb{C}^d$ such that $(\gamma|m)\neq0$ for all $m\in\mathbb{Z}^d$. Then V can be viewed as a $\mathrm{Vir}(\gamma)=\sum_{n\in\mathbb{Z}^d}\mathbb{C}D(\gamma,n)$-module, which is not uniformly bounded. Moreover each $V(i)=\sum_{n\in\mathbb{Z}^d,n_1=0}V_{\lambda+ie_1+n}$ for $i\in\mathbb{Z}$ is a module over $\sum_{n\in\mathbb{Z}^d,n_1=0}\mathbb{C}D(\gamma,n)$, which is a centerless rank-$(d-1)$ Virasoro algebra for any $i\in\mathbb{Z}$, and $V'=\sum_{n\in\mathbb{Z}^d,n_2=0}V_{\lambda+n}$ is a module over $\sum_{n\in\mathbb{Z},n_2=0}\mathbb{C}D(\gamma,n)$, which is also a centerless rank-$(d-1)$ Virasoro algebra.

Note that any uniformly bounded module over classical or higher-rank Virasoro algebras has equal dimensions of weight spaces except for the weight 0. If all $V(i)$, $i\in\mathbb{Z}$, and V' are uniformly bounded, then there exists $N'\in\mathbb{N}$ such that $\dim V_{\lambda+n}\leq N'$ for all $n\in\mathbb{Z}^d$ with $n_2=0$. Set $N=\max\{N',\delta_{\lambda,\mathbb{Z}^d}\dim V_0\}$, where $\delta_{\lambda,\mathbb{Z}^d}=1$ if $\lambda\in\mathbb{Z}^d$ and $\delta_{\lambda,\mathbb{Z}^d}=0$ otherwise. Since $V(i)$ is uniformly bounded, we have

$$\dim V_{\lambda+ie_1+m}\leq\max\{\dim V_{\lambda+ie_1+m-m_2e_2},\delta_{\lambda,\mathbb{Z}^d}\dim V_0\}\leq N$$

for all $i\in\mathbb{Z}$ and $m\in\mathbb{Z}^d$ with $m_1=0$. In other words, $\dim V_{\lambda+m}\leq N$ for all $m\in\mathbb{Z}^d$, contradicting the fact that V is not uniformly bounded. Thus, either $V(i)$, for some $i\in\mathbb{Z}$, or V' is not uniformly bounded.

Without loss of generality, we may assume that $V(0)$ is not uniformly bounded. So there exists some $n\in\mathbb{Z}^d$ with $n_1=0$ such that

$$\dim V_{\lambda-n}>(d+1)\left(\dim V_{\lambda+e_1}+\sum_{i=2}^d\dim V_{\lambda+e_1+e_i}\right).\tag{9}$$

We set $s_1=n+e_1$ and $s_i=n+e_1+e_i$ for $i=2,...,d$. It is easy to check that the set $\{s_i\,|\,i=1,...,d\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^d$ since $n=\sum_{i=2}^d n_ie_i$. By (9), there exists some nonzero element $v\in V_{\lambda-n}$ such that $D(e_i,s_j)v=0$ and $t^{s_j}v=0$ for any $i,j=1,...,d$. Thus, v is a generalized highest weight vector relative to the basis $\{s_i\,|\,i=1,...,d\}$ of $\mathbb{Z}^d$. By the irreducibility of the $\mathcal{L}$-module V and the Poincaré–Birkhoff–Witt theorem, we see that $\sum_{j=1}^d\mathbb{N}s_j\notin\mathrm{Supp}(V)$. Clearly, V is neither dense nor trivial. By Theorem 4.1, there exists some $m\in\mathbb{Z}^d$ and $\delta\in\mathbb{R}^d\setminus\{0\}$ such that $\mathrm{Supp}(V)\subseteq\lambda+m+\mathbb{Z}^{(\delta)}_{\leq 0}$. Set $G_\delta=\{n\in\mathbb{Z}^d\,|\,(\delta,n)=0\}$.

Now consider V as a $\mathrm{Vir}(\gamma)$-module. Then V has an irreducible nontrivial $\mathrm{Vir}(\gamma)$-subquotient, say Y, which is not uniformly bounded. By the representation of higher-rank Virasoro algebras, we know that $Y\cong L^{\mathrm{Vir}(\gamma)}(H,\beta,Y')$ for some $\beta\in\mathbb{Z}^d\setminus\{0\}$, a subgroup H of $\mathbb{Z}^d$ with $\mathbb{Z}^d\cong H\oplus\mathbb{Z}\beta$, and Y' being an irreducible uniformly bounded module over $\mathcal{L}_H\cap\mathrm{Vir}(\gamma)$ which is a rank-$(d-1)$ centerless Virasoro algebra. In particular,

$$\lambda-i\beta+H\subseteq\mathrm{Supp}(Y)\subseteq\mathrm{Supp}(V)\subseteq\lambda+m+\mathbb{Z}^{(\delta)}_{\leq 0}$$

for sufficiently large $i\in\mathbb{N}$. It follows that

$$(\delta\,|\,n)\leq(\delta\,|\,m+i\beta)\quad\text{for } n\in H \text{ and sufficiently large } i\in\mathbb{N},$$

yielding that $H=G_\delta$ and $(\delta|\beta)>0$.

Let $i_0\in\mathbb{Z}$ be the maximal number such that $X=V_{\lambda+i_0\beta+H}\neq 0$. Thus, $\mathrm{Supp}(V)\subseteq\sum_{i\leq i_0}(\lambda+i\beta+H)$ and $\mathcal{L}_H^+X=0$. Note that X is an $\mathcal{L}_H$-module. Then the irreducibility of V over $\mathcal{L}$ implies the irreducibility of X over $\mathcal{L}_H$ and that $V=U(\mathcal{L}_H^-)X$.

Since V is nontrivial over $\mathcal{L}$, we know that X is nontrivial over $\mathcal{L}_H$. If X is not a uniformly bounded $\mathcal{L}_H$-module, then X has an irreducible $\mathcal{L}_H\cap\mathrm{Vir}(\gamma)$-subquotient X' which is not uniformly bounded. The unique irreducible $\mathrm{Vir}(\gamma)$-quotient module of the induced $\mathrm{Vir}(\gamma)$-module

$$U(\mathrm{Vir}(\gamma))\otimes_{(\mathcal{L}_H+\mathcal{L}_H^+)\cap\mathrm{Vir}(\gamma)}X'$$

is not a Harish-Chandra module by Theorem 2.3(2) of [16], contradicting the fact that V is a Harish-Chandra $\mathcal{L}$-module. Thus we must have that X is uniformly bounded over $\mathcal{L}_H$. This completes the proof. □

Using Theorems 3.3 and 4.3, we can obtain a classification of irreducible Harish-Chandra modules over $W\ltimes A$ with nontrivial action of A.

Theorem 4.4. *Suppose that V is a nontrivial irreducible Harish-Chandra $(W\ltimes A)$-module with nonzero action of A.*

(1) If V is uniformly bounded, then V is isomorphic to some $F^\alpha(\psi,e,c)$ for suitable $\alpha\in\mathbb{C}^d$, $e,c\in\mathbb{C}$, and ψ which is an integral dominant weight of $\mathfrak{sl}_d$.

(2) If V is not uniformly bounded, then V is isomorphic to some $L(H,\beta,X)$, where $\beta\in\mathbb{Z}^d\setminus\{0\}$, H is a subgroup of $\mathbb{Z}^d$ such that $\mathbb{Z}^d=H\oplus\mathbb{Z}\beta$, and X is a nontrivial irreducible uniformly bounded $(W\ltimes A)_H$-module.

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