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Singularities of functions of one and several bicomplex variables

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Abstract. In this paper we study the singularities of holomorphic functions of bicomplex variables introduced by G. B. Price (*An Introduction to Multicomplex Spaces and Functions*, Dekker, New York, 1991). In particular, we use computational algebra techniques to show that even in the case of one bicomplex variable, there cannot be compact singularities. The same techniques allow us to prove a duality theorem for such functions.

1. Introduction

The classical work [8] introduces the notion of bicomplex and, more generally, multicomplex numbers. Even though the author traces back the origin of these numbers to the Italian school of the early twentieth century (see [13]–[17]), other important contributions that need to be highlighted are the papers of Ryan ([11] and [12]). Price's book, however, is the first formal and thorough introduction to these concepts. On the basis of this work, Rochon, Shapiro, and others further developed the theory of bicomplex holomorphicity in [2], [9] and [10]. To avoid confusion with the complex case, in this paper we will refer to these functions as hyperholomorphic.

In a nutshell, one regards the space $\mathbb{C}$ of complex numbers as a real bidimensional algebra, and then complexifies the algebra itself. Thus, one obtains a four-dimensional algebra which is usually denoted by $\mathbb{C}_2$, or by $\mathbb{BC}$. The key point of the theory of functions on this algebra is that classical holomorphic functions can be extended from one complex variable to this algebra, and one can therefore develop a new theory of hyperholomorphic functions.

In [8] one can see how to define such a notion of hyperholomorphicity for bicomplex-valued functions defined on $\mathbb{BC}$ and its author shows how most of the classical properties of holomorphic functions of one complex variable can be extended to these new functions. For example, hyperholomorphic functions admit a representation in power series of a bicomplex variable, and the classical Cauchy theory (Cauchy theorem, Cauchy integral formula, etc.) can easily be obtained for these functions.

The last chapter of [8] highlights a few open problems in the theory, in particular the extension of this analysis to the case of several bicomplex variables.

In this paper we attempt two different tasks. First, in Section 2, we show how the space of hyperholomorphic functions on bicomplex numbers can be obtained in a new way; rather than looking at a generalization of complex numbers, we consider the space of quaternions, and we attempt to define a variation of the Cauchy–Fueter operator, whose solutions form a multiplicatively closed structure. We therefore discuss a variety of possible modifications of the Cauchy–Fueter system, and we show that some of those leads to a multiplicatively closed space of functions. As it turns out, this approach corresponds exactly to identifying $\mathbb{R}^4$ with $\mathbb{BC}$, and the systems one looks at are those who are necessary to define hyperholomorphicity in [8].

More important, however, we show that while Price has regarded the theory of hyperholomorphic functions as an extension of the theory of holomorphic functions of *one* complex variable, there are in fact important reasons why we should think of this as an extension of the theory of holomorphic functions of *several* complex variables. This was already understood by Spampinato ([15] and [16]), and we demonstrate it here by looking at the nature of singularities for hyperholomorphic functions.

Specifically, we will show that hyperholomorphic functions of a bicomplex variable can be seen as solutions of an overdetermined system of linear constant coefficient partial differential equations (unlike the case of holomorphic functions which are solutions of a determined system), and as such we are able to use some standard techniques from the theory of those systems to demonstrate, for example, that hyperholomorphic functions of a bicomplex variable do not admit compact singularities (in contrast with holomorphic functions of a complex variable).

Section 3 is therefore devoted to the introduction and the study of the differential operators which allow to define hyperholomorphicity in $\mathbb{BC}$. In Section 4, these operators are used to study the algebraic properties of hyperholomorphic functions in $\mathbb{BC}$, and they allow us to highlight the similarities between these functions and holomorphic functions of several complex variables. Finally, in Section 5 we show how this approach is useful to extend the theory to the case of several bicomplex variables. We will find, in this case, some interesting peculiarities, that give further interest to their study.

2. Generalized Cauchy–Fueter operators

Our motivation for this article started with the investigation of the failure for the space of solutions of the Cauchy–Fueter operator to be multiplicatively closed. In this section we use **1**, **I**, **J** and **K** to denote the usual quaternionic basis, in order to distinguish them from **1**, **i**, **j** and **k** which denote the bicomplex basis in the following section.

We denote the usual complex space with imaginary unit **I** by $\mathbb{C}_{\mathbf{I}}=\{z=a+\mathbf{I}b \mid a,b\in\mathbb{R}\}$, and, similarly, the complex space with imaginary unit **J** by $\mathbb{C}_{\mathbf{J}}$. Then the space of quaternions can be written as $\mathbb{H}=\mathbb{C}_{\mathbf{I}}\times_{\mathbb{C}_{\mathbf{J}}}\overline{\mathbb{C}}_{\mathbf{I}}$, where **I** and **J** anti-commute, $\mathbf{IJ}=\mathbf{K}$, and $\mathbf{I}^2=\mathbf{J}^2=\mathbf{K}^2=-\mathbf{1}$. Thus, a quaternion $q=x_0+\mathbf{I}x_1+\mathbf{J}x_2+\mathbf{K}x_3\in\mathbb{H}$ can be rewritten as $q=z+\mathbf{J}\overline{w}$, where $z=x_0+\mathbf{I}x_1$ and $w=x_2+\mathbf{I}x_3$ both belong to $\mathbb{C}_{\mathbf{I}}$.

We define the usual complex differential operators in both copies of $\mathbb{C}_{\mathbf{I}}$ as follows:

$$\partial_{\bar{z}}=\partial_{x_0}+\mathbf{I}\partial_{x_1}, \quad \partial_z=\partial_{x_0}-\mathbf{I}\partial_{x_1},$$
$$\partial_{\overline{w}}=\partial_{x_2}+\mathbf{I}\partial_{x_3}, \quad \partial_w=\partial_{x_2}-\mathbf{I}\partial_{x_3},$$

where (for every real or complex variable t) we use the shorthand ∂_t for $\partial/\partial t$.

Note that, if h is a quaternionic-valued function defined on $\mathbb{H}$, then

$$\partial_{\bar{z}}(\mathbf{J}h)=(\partial_{x_0}+\mathbf{I}\partial_{x_1})(\mathbf{J}h)=(\mathbf{J}\partial_{x_0}-\mathbf{JI}\partial_{x_1})h=\mathbf{J}\partial_z(h).$$

If $F\colon\mathbb{H}\to\mathbb{H}$ is the function $F(q)=f_0(q)+\mathbf{I}f_1(q)+\mathbf{J}f_2(q)+\mathbf{K}f_3(q)$, where $f_i\colon\mathbb{H}\to\mathbb{R}$, then F can be rewritten as $F=f+\mathbf{J}\bar{g}$, where $f=f_0+\mathbf{I}f_1$ and $g=f_2+\mathbf{I}f_3$, and $f,g\colon\mathbb{H}\to\mathbb{C}_{\mathbf{I}}$. From this point of view, the usual Cauchy–Fueter operator can be written (up to a factor of 4) as

$$\partial_{\bar{q}}=\partial_{x_0}+\mathbf{I}\partial_{x_1}+\mathbf{J}\partial_{x_2}+\mathbf{K}\partial_{x_3}=\partial_{\bar{z}}+\mathbf{J}\partial_w.$$

Let us then check if it is possible to modify the Cauchy–Fueter operator in such a way that its space of solutions becomes multiplicatively closed. We will consider the family of generalized Cauchy–Fueter operators of the type

$$\partial^{\varepsilon}_{\bar{q}}=\varepsilon_0\partial_{x_0}+\varepsilon_1\mathbf{I}\partial_{x_1}+\varepsilon_2\mathbf{J}\partial_{x_2}+\varepsilon_3\mathbf{K}\partial_{x_3}, \tag{1}$$

where $\varepsilon_i=\pm1$.

Note that the usual Cauchy–Fueter operator is obtained when $\varepsilon_i=1$ for all $i=0,...,3$. Similarly, the modified Cauchy–Fueter operator [5] is

$$\tilde{\partial}_{\bar{q}}=\partial_{x_0}+\mathbf{I}\partial_{x_1}+\mathbf{J}\partial_{x_2}-\mathbf{K}\partial_{x_3}=\partial_{\bar{z}}+\mathbf{J}\partial_{\overline{w}},$$

and the Cauchy–Fueter conjugate operator is given by

$$\partial_q = \partial_{x_0} - \mathbf{I}\partial_{x_1} - \mathbf{J}\partial_{x_2} - \mathbf{K}\partial_{x_3} = \partial_z - \mathbf{J}\partial_w .$$

A quaternionic function $F = f_0 + \mathbf{I}f_1 + \mathbf{J}f_2 + \mathbf{K}f_3$, is said to be *$\varepsilon$-regular* if

$$\partial^{\varepsilon}_{\bar{q}}(F) = 0,$$

or, alternatively, if its components satisfy the system

$$\begin{cases} \varepsilon_0\partial_{x_0}f_0 - \varepsilon_1\partial_{x_1}f_1 - \varepsilon_2\partial_{x_2}f_2 - \varepsilon_3\partial_{x_3}f_3 = 0, \\ \varepsilon_1\partial_{x_1}f_0 + \varepsilon_0\partial_{x_0}f_1 - \varepsilon_3\partial_{x_3}f_2 + \varepsilon_2\partial_{x_2}f_3 = 0, \\ \varepsilon_2\partial_{x_2}f_0 + \varepsilon_3\partial_{x_3}f_1 + \varepsilon_0\partial_{x_0}f_2 - \varepsilon_1\partial_{x_1}f_3 = 0, \\ \varepsilon_3\partial_{x_3}f_0 - \varepsilon_2\partial_{x_2}f_1 + \varepsilon_1\partial_{x_1}f_2 + \varepsilon_0\partial_{x_0}f_3 = 0. \end{cases}$$

A simple, though lengthy, computation shows that the following result holds.

Theorem 2.1. *For every choice of quaternions* q_1, q_2, q_3, *and* q_4 (*except for the trivial case in which all* q_i*'s coincide*), *the space of solutions to the generalized Cauchy–Fueter operator*

$$q_0\partial_{x_0} + q_1\mathbf{I}\partial_{x_1} + q_2\mathbf{J}\partial_{x_2} + q_3\mathbf{K}\partial_{x_3}$$

is not closed under multiplication.

Proof. It is easy to notice that it is sufficient to prove the statement of the theorem when $q_i = a_i$ are real numbers, instead of quaternions. Denoting by ∂^a_q the generalized Cauchy–Fueter operator, and by f_i and g_i, respectively, the components of two quaternion-valued functions F and G, the following relation holds:

$$\begin{aligned} &\partial^a_q(F\cdot G) \\ &\quad = \partial^a_q(F)\cdot G + F\cdot\partial^a_q(G) \\ &\qquad + \mathbf{1}(2a_1(-f_2\partial_1 g_3 + f_3\partial_1 g_2) + 2a_2(f_1\partial_2 g_3 - f_3\partial_2 g_1) + 2a_3(-f_1\partial_3 g_2 + f_2\partial_3 g_1)) \\ &\qquad + \mathbf{I}(2a_1(-f_2\partial_1 g_2 - f_3\partial_1 g_3) + 2a_3(f_1\partial_3 g_3 - f_2\partial_3 g_0) + 2a_2(f_3\partial_2 g_0 + f_1\partial_2 g_2)) \\ &\qquad + \mathbf{J}(2a_2(-f_1\partial_2 g_1 - f_3\partial_2 g_3) + 2a_3(f_2\partial_3 g_3 + f_1\partial_3 g_0) + 2a_1(f_2\partial_1 g_1 - f_3\partial_1 g_0)) \\ &\qquad + \mathbf{K}(2a_3(-f_1\partial_3 g_1 - f_2\partial_3 g_2) + 2a_2(-f_1\partial_2 g_0 + f_3\partial_2 g_2) + 2a_1(f_2\partial_1 g_0 + f_3\partial_1 g_1)). \end{aligned}$$

An easy analysis of the formula above leads to the proof of the theorem. □

The reason for this situation is, as is well known, due to the lack of commutativity in the skew field $\mathbb{H}$ of quaternions. It is therefore natural to think that maybe a different product could lead to a family of generalized Cauchy–Fueter operators for which the space of solutions would be closed under multiplication.

By reviewing the computations above, it is not difficult to see that this is the case if one defines, for example, the matrix product

$$F\cdot G=\begin{bmatrix} f_0g_0 & f_0g_1 & f_0g_2 & f_0g_3 \\ -f_1g_1 & f_1g_0 & -f_1g_3 & f_1g_2 \\ -f_2g_2 & -f_2g_3 & f_2g_0 & f_2g_1 \\ f_3g_3 & -f_3g_2 & -f_3g_1 & f_3g_0 \end{bmatrix}\begin{bmatrix} \mathbf{1} \\ \mathbf{I} \\ \mathbf{J} \\ \mathbf{K} \end{bmatrix}.$$

In this case one immediately notices that the product is commutative, and that the space of solutions to generalized Cauchy–Fueter operators are closed under this particular multiplication. But, obviously, this is not the quaternionic multiplication, and in fact one can easily show that the product above reflects exactly the structure of bicomplex numbers (which will be introduced formally in the next section). In other words, if we assume that F and G have values in the space of bicomplex numbers, then the product above is the natural product for those functions, and a theory of solutions of generalized Cauchy–Fueter operators can be reconstructed in the most natural way.

We recover, therefore, bicomplex numbers not as an extension of complex numbers, but rather as a new structure on $\mathbb{R}^4$ which maintains commutativity.

The next few sections will study in detail the properties of functions defined on bicomplex numbers (and with bicomplex values), which satisfy suitable systems (and combinations of systems) of differential equations.

3. Bicomplex differential operators

We summarize below the definition and properties of the space of bicomplex numbers, and we refer the reader to [8] and [10] for further details.

The *bicomplex space* $\mathbb{BC}$ is defined to be the product of two copies of the complex spaces $\mathbb{C}_{\mathbf{i}}$ over the space of complex numbers $\mathbb{C}_{\mathbf{j}}$, i.e. $\mathbb{BC}=\mathbb{C}_{\mathbf{i}}\times_{\mathbb{C}_{\mathbf{j}}}\mathbb{C}_{\mathbf{i}}$, where $\mathbf{i}$ and $\mathbf{j}$ commute, $\mathbf{ij}=\mathbf{ji}=\mathbf{k}$, $\mathbf{i}^2=\mathbf{j}^2=-1$ and $\mathbf{k}^2=1$.

We will write a bicomplex number as $Z=x_0+\mathbf{i}x_1+\mathbf{j}x_2+\mathbf{k}x_3=z+\mathbf{j}w$, where $z,w\in\mathbb{C}_{\mathbf{i}}$. A function $F\colon\mathbb{BC}\to\mathbb{BC}$ can be written as $F(Z)=f_0(Z)+\mathbf{i}f_1(Z)+\mathbf{j}f_2(Z)+\mathbf{k}f_3(Z)$, where $f_i\colon\mathbb{BC}\to\mathbb{R}$; we can also write $F=u+\mathbf{j}v$, where $u=f_0+\mathbf{i}f_1$ and $v=f_2+\mathbf{i}f_3$.

The algebra $\mathbb{BC}$ is not a division algebra since, for example, $(1+\mathbf{k})(1-\mathbf{k})=0$. From this point of view, it is useful to consider the following two elements of $\mathbb{BC}$:

$$\mathbf{e_1}=\frac{1+\mathbf{k}}{2}\quad\text{and}\quad\mathbf{e_2}=\frac{1-\mathbf{k}}{2}.$$

It is easy to verify that $\mathbf{e_1}$ and $\mathbf{e_2}$ are linearly independent over $\mathbb{C}$, and that $\mathbf{e_1}+\mathbf{e_2}=1$, $\mathbf{e_1}\cdot\mathbf{e_2}=0$, $\mathbf{e_1}^2=\mathbf{e_1}$, $\mathbf{e_2}^2=\mathbf{e_2}$ and $\mathbf{e_1}^2+\mathbf{e_2}^2=1$. If we denote by $I_1=\langle\mathbf{e_1}\rangle$ and $I_2=\langle\mathbf{e_2}\rangle$ the ideals they generate in $\mathbb{BC}$, we see that $I_1\cap I_2=\{0\}$, $I_1\cdot I_2=\{0\}$ and $\langle I_1+I_2\rangle=\mathbb{BC}$.

For all $Z=z+\mathbf{j}w\in\mathbb{BC}$, we can write:

$$Z=(z-\mathbf{i}w)\mathbf{e_1}+(z+\mathbf{i}w)\mathbf{e_2};$$

for obvious reasons, this is called the *idempotent representation* of a bicomplex number.

Let $Z=x_0+\mathbf{i}x_1+\mathbf{j}x_2+\mathbf{k}x_3=z+\mathbf{j}w\in\mathbb{BC}$. Unlike what happens for complex numbers, there are several natural ways to define a notion of conjugation in $\mathbb{BC}$. The various definitions are all interesting, and lead to different ways of regarding the algebra of bicomplex numbers.

Definition 3.1. The Z^*-conjugate is defined as

$$Z^*=\bar{z}-\mathbf{j}\overline{w},$$

where we took the conjugates in the two complex spaces $\mathbb{C}_\mathbf{i}$ corresponding to the variables z and w, respectively, and the conjugate in $\mathbb{C}_\mathbf{j}$ corresponding to $\mathbf{j}\mapsto-\mathbf{j}$.

The $\widetilde{Z}$-conjugate is defined as

$$\widetilde{Z}=\bar{z}+\mathbf{j}\overline{w},$$

where we took only the conjugates in the two complex spaces $\mathbb{C}_\mathbf{i}$ corresponding to the variables z and w, respectively.

Finally the $Z^\dagger$-conjugate is defined as follows

$$Z^\dagger=z-\mathbf{j}w,$$

where we took only the conjugate in $\mathbb{C}_\mathbf{j}$ corresponding to $\mathbf{j}\mapsto-\mathbf{j}$.

Remark 3.2. Note that

$$ZZ^*=|z|_\mathbf{i}^2+|w|_\mathbf{i}^2+2\mathbf{k}(x_0x_3-x_1x_2)$$

is an element of the real Clifford algebra $\mathbb{R}_{1,0}$, see [5] for its definition; this algebra is not a complex space because $\mathbf{k}^2=1$, and it is also known as the set of hyperbolic numbers or duplex numbers.

Similarly,

$$Z\widetilde{Z}=|z|_\mathbf{i}^2-|w|_\mathbf{i}^2+2\mathbf{j}(x_0x_2+x_1x_3)\in\mathbb{C}_\mathbf{j}$$

and

$$ZZ^\dagger=z^2+w^2\in\mathbb{C}_\mathbf{i}.$$

It is easy to notice that the combination of any two conjugates gives the third one,

$$\widetilde{Z^*} = Z^\dagger, \quad (Z^\dagger)^* = \widetilde{Z}, \quad \text{etc.}$$

Remark 3.3. A bicomplex number $Z = z + \mathbf{j}w$ is *nonsingular* (or *invertible*) if and only if

$$ZZ^\dagger = z^2 + w^2 \neq 0.$$

Since the nonzero complex number $z^2 + w^2 \in \mathbb{C}_\mathbf{i}$ has a natural complex inverse, the bicomplex number

$$Z^{-1} = \frac{Z^\dagger}{z^2 + w^2}$$

is easily seen to be the inverse of Z in $\mathbb{BC}$.

In terms of the ideals I_1 and I_2, the set of nonsingular bicomplex numbers is $\mathbb{BC} \setminus (I_1 \cup I_2)$. A simple computation shows that, in terms of the $\{\mathbf{e_1}, \mathbf{e_2}\}$ idempotent representation, the inverse of a bicomplex number $Z = z + \mathbf{j}w \notin I_1 \cup I_2$ is given by

$$Z^{-1} = (z - \mathbf{i}w)^{-1} \mathbf{e_1} + (z + \mathbf{i}w)^{-1} \mathbf{e_2}.$$

Definition 3.4. Let us consider $z = x_0 + \mathbf{i}x_1$ and $w = x_2 + \mathbf{i}x_3$ in $\mathbb{C}_\mathbf{i}$. We define the $\ddagger$-*conjugate* of $\zeta = z + \mathbf{i}w = (x_0 - x_3) + \mathbf{i}(x_1 + x_2) \in \mathbb{C}_\mathbf{i}$ by analogy with the bicomplex conjugate $\dagger$, as follows:

$$\zeta^\ddagger = z - \mathbf{i}w = (x_0 + x_3) + \mathbf{i}(x_1 - x_2) \in \mathbb{C}_\mathbf{i}.$$

With this new notation, every bicomplex number Z can be written as $Z = \zeta^\ddagger \mathbf{e_1} + \zeta \mathbf{e_2}$, with $\zeta^\ddagger \cdot \zeta = ZZ^\dagger = z^2 + w^2 \in \mathbb{C}_\mathbf{i}$. Moreover we can now multiply invertible bicomplex numbers "term-by-term" as follows: if $Z = \zeta^\ddagger \mathbf{e_1} + \zeta \mathbf{e_2}$ and $U = \eta^\ddagger \mathbf{e_1} + \eta \mathbf{e_2}$, then:

$$\begin{aligned} ZU &= \zeta^\ddagger \eta^\ddagger \mathbf{e_1} + \zeta\eta \mathbf{e_2}, \\ ZU^{-1} &= \zeta^\ddagger (\eta^\ddagger)^{-1} \mathbf{e_1} + \zeta\eta^{-1} \mathbf{e_2}, \\ ZZ^{-1} &= 1\,\mathbf{e_1} + 1\,\mathbf{e_2} = 1, \\ Z^n &= (\zeta^\ddagger)^n \mathbf{e_1} + \zeta^n \mathbf{e_2}. \end{aligned}$$

In what follows we will indicate by A_1 and A_2 the two copies of $\mathbb{C}_\mathbf{i}$, determined by $\zeta^\ddagger$ and ζ.

For each one of the conjugates we introduced above, we can define corresponding differential operators. Since the usual complex derivatives in $\mathbb{C}_{\mathbf{i}}$ are given by

$$\partial_{\bar z}=\partial_{x_0}+\mathbf{i}\partial_{x_1},\quad \partial_z=\partial_{x_0}-\mathbf{i}\partial_{x_1},$$
$$\partial_{\overline{w}}=\partial_{x_2}+\mathbf{i}\partial_{x_3},\quad \partial_w=\partial_{x_2}-\mathbf{i}\partial_{x_3},$$

we can analogously define

$$\partial_{\zeta}=\partial_z+\mathbf{i}\partial_w,\quad \partial_{\zeta^{\ddagger}}=\partial_z-\mathbf{i}\partial_w,$$
$$\partial_{\bar\zeta}=\partial_{\bar z}+\mathbf{i}\partial_{\overline{w}},\quad \partial_{\overline{\zeta^{\ddagger}}}=\partial_{\bar z}-\mathbf{i}\partial_{\overline{w}}.$$

Consider now a bicomplex function $F\colon \mathbb{BC}\to\mathbb{BC}$, and write it in two equivalent ways as $F=u+\mathbf{j}v=\phi^{\ddagger}\,\mathbf{e_1}+\phi\,\mathbf{e_2}$, where $u,\ v\colon \mathbb{BC}\to\mathbb{C}_{\mathbf{i}}$, and $\phi^{\ddagger}=u-\mathbf{i}v$ and $\phi=u+\mathbf{i}v$ are functions from $\mathbb{BC}$ to A_1 and A_2, respectively. The following identities are immediate:

$$\partial_{\overline{\zeta^{\ddagger}}}\phi^{\ddagger}=(\partial_{\bar\zeta}\phi)^{\ddagger},\quad \partial_{\bar\zeta}\phi^{\ddagger}=(\partial_{\overline{\zeta^{\ddagger}}}\phi)^{\ddagger},$$
$$\partial_{\zeta^{\ddagger}}\phi^{\ddagger}=(\partial_{\zeta}\phi)^{\ddagger},\quad \partial_{\zeta}\phi^{\ddagger}=(\partial_{\zeta^{\ddagger}}\phi)^{\ddagger}. \tag{2}$$

This allows us to define the natural bicomplex differential operators, corresponding to the three conjugates:

$$\partial_{Z^*}=\partial_{x_0}+\mathbf{i}\partial_{x_1}+\mathbf{j}\partial_{x_2}+\mathbf{k}\partial_{x_3}=\partial_{\bar z}+\mathbf{j}\partial_{\overline{w}}=\partial_{\overline{\zeta^{\ddagger}}}\,\mathbf{e_1}+\partial_{\bar\zeta}\,\mathbf{e_2},$$
$$\partial_{\tilde Z}=\partial_{x_0}+\mathbf{i}\partial_{x_1}-\mathbf{j}\partial_{x_2}-\mathbf{k}\partial_{x_3}=\partial_{\bar z}-\mathbf{j}\partial_{\overline{w}}=\partial_{\bar\zeta}\,\mathbf{e_1}+\partial_{\overline{\zeta^{\ddagger}}}\,\mathbf{e_2},$$
$$\partial_{Z^{\dagger}}=\partial_{x_0}-\mathbf{i}\partial_{x_1}+\mathbf{j}\partial_{x_2}-\mathbf{k}\partial_{x_3}=\partial_z+\mathbf{j}\partial_w=\partial_{\zeta^{\ddagger}}\,\mathbf{e_1}+\partial_{\zeta}\,\mathbf{e_2},$$
$$\partial_Z=\partial_{x_0}-\mathbf{i}\partial_{x_1}-\mathbf{j}\partial_{x_2}+\mathbf{k}\partial_{x_3}=\partial_z-\mathbf{j}\partial_w=\partial_{\zeta}\,\mathbf{e_1}+\partial_{\zeta^{\ddagger}}\,\mathbf{e_2}.$$

We will now show that these operators act like derivatives, in the sense that they obey the Leibniz multiplication formula when applied to products of bicomplex functions.

Let $F\colon \mathbb{BC}\to\mathbb{BC}$ be the bicomplex function $F=u+\mathbf{j}v=\phi^{\ddagger}\,\mathbf{e_1}+\phi\,\mathbf{e_2}$, with $u,\ v\colon \mathbb{BC}\to\mathbb{C}_{\mathbf{i}}$, $\phi^{\ddagger}=u-\mathbf{i}v$ and $\phi=u+\mathbf{i}v$, and let $G\colon \mathbb{BC}\to\mathbb{BC}$ be the bicomplex function $G=s+\mathbf{j}t=\gamma^{\ddagger}\,\mathbf{e_1}+\gamma\,\mathbf{e_2}$, with $s,t\colon \mathbb{BC}\to\mathbb{C}_{\mathbf{i}}$, $\gamma^{\ddagger}=s-\mathbf{i}t$ and $\gamma=s+\mathbf{i}t$. Then the bicomplex product $F\cdot G$ is given by

$$F\cdot G=(us-vt)+\mathbf{j}(vs+ut)=\phi^{\ddagger}\gamma^{\ddagger}\,\mathbf{e_1}+\phi\gamma\mathbf{e_2}.$$

In real coordinates, if $F=f_0+\mathbf{i}f_1+\mathbf{j}f_2+\mathbf{k}f_3$ and $G=g_0+\mathbf{i}g_1+\mathbf{j}g_2+\mathbf{k}g_3$, then

$$F\cdot G=(f_0g_0-f_1g_1-f_2g_2+f_3g_3)+\mathbf{i}(f_0g_1+f_1g_0-f_2g_3-f_3g_2)$$
$$+\mathbf{j}(f_0g_2-f_1g_3+f_2g_0-f_3g_1)+\mathbf{k}(f_0g_3+f_1g_2+f_2g_1+f_3g_0).$$

Simple computations show that

$$\partial_{Z^*}(F)=(\partial_{\bar z}u-\partial_{\overline{w}}v)+\mathbf{j}(\partial_{\overline{w}}u+\partial_{\bar z}v)=\partial_{\overline{\zeta^\ddagger}}\phi^\ddagger\,\mathbf{e_1}+\partial_{\bar\zeta}\phi\,\mathbf{e_2}, \tag{3}$$

and that a similar formula holds for the other differential operators. As a consequence, we obtain that

$$\begin{aligned}\partial_{Z^*}(F\cdot G)&=\partial_{Z^*}(F)\cdot G+F\cdot\partial_{Z^*}(G),\\ \partial_{\tilde Z}(F\cdot G)&=\partial_{\tilde Z}(F)\cdot G+F\cdot\partial_{\tilde Z}(G),\\ \partial_{Z^\dagger}(F\cdot G)&=\partial_{Z^\dagger}(F)\cdot G+F\cdot\partial_{Z^\dagger}(G).\end{aligned}$$

Every one of these differential operators can be used to define a natural notion of holomorphicity (or regularity), by simply looking at its kernel. It turns out that the three theories that one obtains are very much similar, but what will be interesting will be to look at the intersection of the three kernels.

Definition 3.5. A function $F\colon \mathbb{BC}\to\mathbb{BC}$ will be called *bicomplex* Z^**-regular* if it satisfies the equation

$$\partial_{Z^*}(F)=0;$$

F will be called *bicomplex* $\widetilde{Z}$*-regular* if it satisfies the equation

$$\partial_{\tilde Z}(F)=0;$$

finally, F will be called *bicomplex* $Z^\dagger$*-regular* if it satisfies the equation

$$\partial_{Z^\dagger}(F)=0.$$

If we write F in terms of its complex coordinates as $F=u+\mathbf{j}v$, the regularity conditions just introduced are equivalent, respectively, to the following 2×2 complex systems (some sort of complex versions of the Cauchy–Riemann system):

$$\begin{cases}\partial_{\bar z}u-\partial_{\overline{w}}v=0,\\ \partial_{\overline{w}}u+\partial_{\bar z}v=0,\end{cases}\quad \begin{cases}\partial_{\bar z}u+\partial_{\overline{w}}v=0,\\ \partial_{\overline{w}}u-\partial_{\bar z}v=0,\end{cases}\quad\text{and}\quad \begin{cases}\partial_{z}u-\partial_{w}v=0,\\ \partial_{w}u+\partial_{z}v=0.\end{cases}$$

Finally, if we write $F=f_0+\mathbf{i}f_1+\mathbf{j}f_2+\mathbf{k}f_3$, the conditions of regularity can be interpreted in terms of 4×4 real systems that resemble the well-known Cauchy–Fueter system as follows (once again, the first system is for Z^* regularity, the second for $\widetilde{Z}$ regularity, and the third for $Z^\dagger$ regularity:

$$\begin{bmatrix}\partial_{x_0} & -\partial_{x_1} & -\partial_{x_2} & \partial_{x_3}\\ \partial_{x_1} & \partial_{x_0} & -\partial_{x_3} & -\partial_{x_2}\\ \partial_{x_2} & -\partial_{x_3} & \partial_{x_0} & -\partial_{x_1}\\ \partial_{x_3} & \partial_{x_2} & \partial_{x_1} & \partial_{x_0}\end{bmatrix}\begin{bmatrix}f_0\\ f_1\\ f_2\\ f_3\end{bmatrix}=\vec{0},$$

$$\begin{bmatrix} \partial_{x_0} & -\partial_{x_1} & \partial_{x_2} & -\partial_{x_3} \\ \partial_{x_1} & \partial_{x_0} & \partial_{x_3} & \partial_{x_2} \\ -\partial_{x_2} & \partial_{x_3} & \partial_{x_0} & -\partial_{x_1} \\ -\partial_{x_3} & -\partial_{x_2} & \partial_{x_1} & \partial_{x_0} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} = \vec{0},$$

$$\begin{bmatrix} \partial_{x_0} & \partial_{x_1} & -\partial_{x_2} & -\partial_{x_3} \\ -\partial_{x_1} & \partial_{x_0} & \partial_{x_3} & -\partial_{x_2} \\ \partial_{x_2} & \partial_{x_3} & \partial_{x_0} & \partial_{x_1} \\ -\partial_{x_3} & \partial_{x_2} & -\partial_{x_1} & \partial_{x_0} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} = \vec{0}.$$

Remark 3.6. If we write $F=\phi^\ddagger\,\mathbf{e_1}+\phi\,\mathbf{e_2}$, where $\phi^\ddagger=u-\mathbf{i}v$ and $\phi=u+\mathbf{i}v$, then F is Z^*-regular if and only if $\partial_{\overline{\zeta^\ddagger}}\phi^\ddagger=0$ (equivalently $\partial_{\bar\zeta}\phi=0$, see (2)). Similarly F is $\widetilde{Z}$-regular if and only if $\partial_{\bar\zeta}\phi^\ddagger=0$ (equivalently $\partial_{\overline{\zeta^\ddagger}}\phi=0$, see again (2)). Finally F is $Z^\dagger$-regular if and only if $\partial_{\zeta^\ddagger}\phi^\ddagger=0$ (equivalently $\partial_\zeta\phi=0$, see (2)).

Example 3.7. Easy computations show that the identity function $F(Z)=Z$ is bicomplex Z^*-regular, as well as $\widetilde{Z}$-regular, and $Z^\dagger$-regular. By the same token, every polynomial in Z is regular with respect to Z^*, $\widetilde{Z}$ and $Z^\dagger$. It can also be shown, for example, that $Z^\dagger$ is Z^*-regular, but cannot be written as a polynomial in Z. Thus, while polynomials in Z (and even converging power series in Z for which we assume the usual notions of uniform convergence on compact sets) are Z^*-regular, it is not true that all Z^*-regular functions can be written as converging power series in Z.

4. Hyperholomorphicity in one variable

In [8] and [10] the following notion of bicomplex derivative is introduced.

Definition 4.1. Let $U\subseteq\mathbb{BC}$ be open and $Z_0\in U$. A function $F\colon U\to\mathbb{BC}$ is called *bicomplex differentiable* at Z_0 with derivative equal to $F'(Z_0)\in\mathbb{BC}$, if

$$\lim_{\substack{Z\to Z_0 \\ Z-Z_0 \text{ invertible}}} (Z-Z_0)^{-1}(F(Z)-F(Z_0)) = F'(Z_0).$$

Functions which admit a bicomplex derivative are called *hyperholomorphic*, and it can be shown that this is equivalent to requiring that they admit a power series expansion in Z [8, Definition 15.2]. There is however a third equivalent notion which is more suitable for our purposes (see [10]).

Theorem 4.2. *Let U be open and $F\colon U\to\mathbb{BC}$ be such that $F=u+\mathbf{j}v\in C^1(U)$. Then F is hyperholomorphic if and only if*

(1) *u and v are complex holomorphic in z and w, and*

(2) *$\partial_z u=\partial_w v$ and $\partial_z v=-\partial_w u$ on U.*

Moreover, $F'=\frac{1}{2}\partial_Z F=\partial_z u+\mathbf{j}\partial_z v=\partial_w v-\mathbf{j}\partial_w u$ and $F'(Z)$ is invertible if and only if the Jacobian is nonzero.

As mentioned in [10], the condition $F\in C^1(U)$ can be dropped via Hartogs' theorem. Note here a major difference between quaternionic and bicomplex analysis: as is well known, in $\mathbb{H}$ the only functions who have quaternionic derivatives are quaternionic linear functions ([17] and [18]), while in the bicomplex setting the class of functions admitting derivatives is nontrivial and consists of functions admitting power series expansion.

In our interpretation, the conditions in Theorem 4.2 can be translated into the following: let $F=u+jv$ and set $\vec{F}=[u\ v]^t$; then F is a bicomplex holomorphic function if and only if

$$\begin{bmatrix} \partial_{\bar z} & 0 \\ \partial_{\bar w} & 0 \\ 0 & \partial_{\bar z} \\ 0 & \partial_{\bar w} \\ \partial_z & -\partial_w \\ \partial_w & \partial_z \end{bmatrix} \vec{F}=\vec{0}. \tag{4}$$

Note that the condition (4) can also be written as

$$\begin{bmatrix} \partial_{\bar z} & 0 \\ 0 & \partial_{\bar z} \\ \partial_{\bar w} & 0 \\ 0 & \partial_{\bar w} \\ \partial_z & -\partial_w \\ \partial_w & \partial_z \end{bmatrix} \vec{F}=\vec{0}. \tag{5}$$

Taking suitable linear combinations of the first four rows, the condition (4) becomes

$$P(D)\vec{F}=\begin{bmatrix} \partial_{\bar z} & -\partial_{\bar w} \\ \partial_{\bar w} & \partial_{\bar z} \\ \partial_{\bar z} & \partial_{\bar w} \\ \partial_{\bar w} & -\partial_{\bar z} \\ \partial_z & -\partial_w \\ \partial_w & \partial_z \end{bmatrix} \vec{F}=\vec{0} \tag{6}$$

which is nothing but the result in Lemma 1 from [9].

Theorem 4.3. *Let* $U\subseteq\mathbb{BC}$ *be open and* $F\colon U\to\mathbb{BC}$ *be such that* $F=u+\mathbf{j}v\in C^1(U)$*. Then* F *is hyperholomorphic if and only if* F *is* Z^*, $\widetilde{Z}$ *and* $Z^\dagger$*-regular.*

Note also that in the idempotent representation, a function F is bicomplex holomorphic if and only if

$$\begin{cases} \partial_{\bar{\zeta}}\phi=0,\\ \partial_{\overline{\zeta^\ddagger}}\phi=0,\\ \partial_{\zeta}\phi=0. \end{cases}$$

This system of equations is equivalent to system (5).

Remark 4.4. System (6) is elliptic; indeed any infinitely differentiable solution to the system can be expressed in a power series (see [8]).

We have the following result.

Proposition 4.5. *Bicomplex holomorphic functions form a sheaf* $\mathcal{H}$ *of rings.*

Proof. For any open set U the set $\mathcal{H}(U)$ of bicomplex holomorphic functions is a ring with respect to the usual sum and product of functions. Consider the sheaf $\mathcal{E}$ of infinitely differentiable functions and the system represented by $P(D)$ defined in (6). Since $\mathcal{H}$ is the kernel $\mathcal{E}^P$ of $P(D)$ in $\mathcal{E}$, it follows immediately that $\mathcal{H}$ is a sheaf of rings. □

In the sequel, we will derive cohomological properties of the sheaf of bicomplex holomorphic functions by using some standard tools described in detail in [5]. We will show that, even though the hyperholomorphic functions are functions in one (bicomplex) variable, they have properties which make them similar to functions in several variables. Specifically, most other theories of regularity in one variable (holomorphic functions in one complex variable, Cauchy–Fueter regular functions in one quaternionic variable, monogenic functions in one vector variable, and so on) can be algebraically characterized by noting that the resolution of the associated module has length one (in fancier terminology, the flabby dimension of the associated sheaves is one). In the case of hyperholomorphic functions on bicomplex variables, however, the resolution is not trivial, the flabby dimension is three, and the specific description of the resolution entails a number of significant analytic consequences.

In accordance with the process described in [5], and with some abuse of language, we will now construct the module associated with the system (6). To begin with, we consider the "Fourier transform" P of the matrix $P(D)$ in (6). The entries of P belong to the ring $R=\mathbb{C}[z,\bar{z},w,\overline{w}]$ and the cokernel of the map P^t, i.e. $M:=R^2/\langle P^t\rangle$, where $\langle P^t\rangle$ denotes the module generated by the columns of P^t, is the module associated with the system (6). We have the following result.

Theorem 4.6. *The minimal free resolution of the module* $M=R^2/\langle P^t\rangle$ *is*

$$0\longrightarrow R^2(-3)\xrightarrow{P_2^t}R^6(-2)\xrightarrow{P_1^t}R^6(-1)\xrightarrow{P^t}R^2\longrightarrow M\longrightarrow 0. \tag{7}$$

Moreover, $\operatorname{Ext}^i(M,R)=0$, $i=0,1,2$, *and* $\operatorname{Ext}^3(M,R)\neq 0$.

Proof. First of all let us rewrite the matrix P^t as

$$P^t=\begin{bmatrix}\bar{z} & \overline{w} & \bar{z} & \overline{w} & z & w\\ -\overline{w} & \bar{z} & \overline{w} & -\bar{z} & -w & z\end{bmatrix}=[Z^*\ \widetilde{Z}\ Z^\dagger]$$

with obvious meaning of the symbols. It is immediate to verify that any two elements among Z^*, $\widetilde{Z}$ and $Z^\dagger$ commute. The relations coming from these commutations are syzygies. There cannot be other relations among the columns of P^t, in fact the entries are the complex variables z, $\bar{z}$, w and $\overline{w}$ whose syzygies can only come from their commutation. Thus, the matrix of the first syzygies can be written as

$$P_1^t=\begin{bmatrix}-\widetilde{Z} & Z^\dagger & 0\\ Z^* & 0 & -Z^\dagger\\ 0 & -Z^* & \widetilde{Z}\end{bmatrix}$$

and the matrix which closes the complex is $P_2^t=[Z^\dagger\ \ \widetilde{Z}\ \ Z^*]^t$. The fact that $\operatorname{Ext}^0(M,R)=0$ follows from the fact that the matrix P^t has independent columns. Moreover, the greatest common divisor of the 2×2 minors of the matrix P^t, in the form (6), is one, and so Lemma 3.1 in [1] implies that $\operatorname{Ext}^1(M,R)=0$. The fact that $\operatorname{Ext}^2(M,R)=0$ can be proved directly by using the algebraic manipulation software CoCoA [3]. At the last spot of the complex we have a nontrivial cohomology, corresponding to the cokernel of the last map. □

Remark 4.7. An alternative way to prove Theorem 4.6 is to use the condition of regularity on the matrices representing Z^*, $\widetilde{Z}$ and $Z^\dagger$ given in [4].

An immediate corollary follows from the fact that the matrix $P_1(D)$ expresses the compatibility conditions on the solvability of the inhomogeneous system associated with the bicomplex hyperholomorphicity (see [5] and [6]).

Corollary 4.8. *Let $U \subseteq \mathbb{BC}$ be an open convex set and let g_i be infinitely differentiable functions on U. The inhomogeneous system*

$$\begin{cases} \partial_{Z^*} F = g_1, \\ \partial_{\tilde{Z}} F = g_2, \\ \partial_{Z^\dagger} F = g_3, \end{cases}$$

admits a solution F if and only if

$$\begin{cases} \partial_{Z^*} g_2 = \partial_{\tilde{Z}} g_1, \\ \partial_{Z^*} g_3 = \partial_{Z^\dagger} g_1, \\ \partial_{Z^\dagger} g_2 = \partial_{\tilde{Z}} g_3. \end{cases}$$

Two more corollaries are direct consequences of the vanishing of the Ext^i-modules [7, Chapter 8].

Corollary 4.9. *Let $K \subset \mathbb{BC}$ be a compact convex set and let U be an open neighborhood of K. Then the distribution solutions to the homogeneous system $P(D)f = 0$ on $U \setminus K$ can be uniquely extended to a solution of the system on U.*

Proof. This is a consequence of the vanishing of $\mathrm{Ext}^i(M, R)$ for $i = 0, 1$. □

Corollary 4.10. *The characteristic variety of M has dimension* 1.

As another consequence of Theorem 4.6 we can state the following duality result.

Proposition 4.11. *If K is a bounded convex set in $\mathbb{BC}$ then*

$$H^j_K(\mathbb{BC}, \mathcal{H}) = 0, \quad j = 1, 2.$$

If $K \subset \mathbb{BC}$ is a compact set, then

$$H^3_K(\mathbb{BC}, \mathcal{H}) \cong \mathcal{H}(K)'. \tag{8}$$

Proof. The vanishing of $\mathrm{Ext}^i(M, R)$ for $i = 1, 2$ immediately gives the first part of the statement. The duality (8) is a consequence of the topological duality

$$H^3_K(\mathbb{R}^4, (\mathcal{D}')^P) \cong H^0(K, \mathcal{E}^{P_2^t})'$$

which holds by the vanishing of $\mathrm{Ext}^i(M, R)$ for $i = 0, 1, 2$. The fact that $P_2^t = P$ and that the system is elliptic give that $(\mathcal{D}')^P = \mathcal{E}^P = \mathcal{H}$. □

5. Hyperholomorphicity for functions of several bicomplex variables

In this short section we will consider functions of several bicomplex variables. We will denote by $(Z_1, ..., Z_n)$ a variable in $\mathbb{BC}^n$. For each $Z_i = z_i + \mathbf{j}w_i$ we have the conjugates Z_i^*, $\widetilde{Z}_i$ and $Z_i^\dagger$. The notion of bicomplex hyperholomorphicity in several variables can be introduced by requiring bicomplex hyperholomorphicity in each variable as in the following definition.

Definition 5.1. Let U be an open set in $\mathbb{BC}^n$ and let F be a differentiable function from U to $\mathbb{BC}$. Then $F = u + \mathbf{j}v$ is bicomplex holomorphic if and only if

(1) u and v are complex holomorphic in z_i and w_i for every $i = 1, ..., n$, and
(2) $\partial_{z_i} u = \partial_{w_i} v$ and $\partial_{z_i} v = -\partial_{w_i} u$ on U.

We immediately have the following result.

Proposition 5.2. *Let $U \subseteq \mathbb{BC}^n$ be an open set and let $F = u + \mathbf{j}v$ be a differentiable function from U to $\mathbb{BC}$. Then F is bicomplex holomorphic in $(Z_1, ..., Z_n)$ if and only if F is Z_i^*, $\widetilde{Z}_i$ and $Z_i^\dagger$-regular for all $i = 1, ..., n$.*

Proof. By reasoning as in Section 4, we can write conditions (4) for each $i = 1, ..., n$ in the form (6) and the statement follows. □

To construct the module of the system associated with several bicomplex variables it is sufficient to consider the matrices $P_i(D)$ in (6) for each variable Z_i and then to form the matrix

$$Q(D) = \begin{bmatrix} P_1(D) \\ \vdots \\ P_n(D) \end{bmatrix},$$

whose "Fourier transform" is denoted by Q. The entries of Q belong to the ring $R = \mathbb{C}[z_1, \bar{z}_1, w_1, \overline{w}_1, ..., z_n, \bar{z}_n, w_n, \overline{w}_n]$ in $4n$ variables. We will consider the cokernel of the map Q^t i.e. $M := R^2/\langle Q^t \rangle$. We have the following result.

Theorem 5.3. *The minimal free resolution of the module $M = R^2/\langle Q^t \rangle$ is linear of length n:*

$$(9) \qquad 0 \longrightarrow R^2(-3n) \xrightarrow{Q^t_{3n-1}} ... \longrightarrow R^{2\binom{3n}{2}}(-2) \xrightarrow{Q_1^t} R^{6n}(-1) \xrightarrow{Q^t} R^2 \longrightarrow M \longrightarrow 0.$$

Moreover, $\mathrm{Ext}^i(M, R) = 0$, $i = 0, 1, ..., 3n-1$ *and* $\mathrm{Ext}^{3n}(M, R) \neq 0$.

Proof. We write the matrix Q^t in the form

$$Q^t = \begin{bmatrix} \bar{z}_1 & \bar{w}_1 & \bar{z}_1 & \bar{w}_1 & z_1 & w_1 & \dots & \bar{z}_n & \bar{w}_n & \bar{z}_n & \bar{w}_n & z_n & w_n \\ -\bar{w}_1 & \bar{z}_1 & \bar{w}_1 & -\bar{z}_1 & -w_1 & z_1 & \dots & -\bar{w}_n & \bar{z}_n & \bar{w}_n & -\bar{z}_n & -w_n & z_n \end{bmatrix}.$$

All its columns are independent and any two blocks representing a variable Z_i or one of its conjugates, commute. The matrix of the first syzygies has $2\binom{3n}{2}$ rows which come from the commutation relations among the 2×2 blocks. Also all the other matrices can be written using the Koszul-like relations and at the rth step they have dimension $2\binom{3n}{r+1}\times 2\binom{3n}{r}$. In particular, the matrix which closes the complex contains all the $3n$ building blocks in Q and thus defines the same system as Q. The fact that $\mathrm{Ext}^0(M,R)=0$ follows from the fact that the matrix Q^t has independent columns. All the other vanishing follow from the fact that the syzygies are of Koszul type. At the last spot of the complex we have a nontrivial cohomology, corresponding to the cokernel of the last map, and thus $\mathrm{Ext}^{3n}(M,R)\neq 0$. □

As in the previous section, we have the following immediate corollaries.

Corollary 5.4. *Let $U\subseteq\mathbb{BC}$ be an open convex set and let G be a $3n$-dimensional vector of infinitely differentiable functions on U. The inhomogeneous system*

$$Q(D)F = G$$

admits a solution F if and only if

$$Q_1(D)G = 0,$$

where $Q_1(D)$ is the anti-Fourier transform of the matrix Q_1.

Corollary 5.5. *Let $K\subset\mathbb{BC}^n$ be a compact convex set and let U be an open neighborhood of K. Then the distribution solutions to the homogeneous system $P(D)f=0$ on $U\setminus K$ can be uniquely extended to a solution of the system on U.*

Proof. This is a consequence of the vanishing of $\mathrm{Ext}^i(M,R)$ for $i=0,1$. □

Corollary 5.6. *The characteristic variety of M has dimension n.*

As another consequence of Theorem 5.3 (and its proof) we can state the following duality result whose proof is similar to the proof of Proposition 4.11, and where the crucial fact is the fact that the matrix which closes the complex in Theorem 5.3 is the same matrix that defines hyperholomorphicity in several bicomplex variables.

Proposition 5.7. *If K is a bounded convex set in $\mathbb{BC}^n$ then*

$$H^j_K(\mathbb{BC}^n, \mathcal{H}) = 0, \quad j = 1, ..., 3n-1.$$

If $K \subset \mathbb{BC}$ is a compact set, then

$$H^{3n}_K(\mathbb{BC}^n, \mathcal{H}) \cong \mathcal{H}(K)'. \tag{10}$$

Remark 5.8. This last result is perfectly parallel (with $3n$ replacing n) to the well known Köthe–Martineau–Grothendieck duality theorem for several complex variables; its beauty rests on the fact that the same sheaf of functions appears on both sides of the duality. This is not a feature that exists in the quaternionic case (at least not for $n>2$), as is shown for example in [1] and in [5]. This aspect, together with the fact that n is replaced by $3n$ in this case, may be an indication that an interesting hyperfunction theory can be defined in this setting.

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