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Carleson measures for weighted holomorphic Besov spaces

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Abstract. We obtain characterizations of positive Borel measures μ on $\mathbf{B}^n$ so that some weighted holomorphic Besov spaces $B_s^p(\omega,\mathbf{B}^n)$ are embedded in $L^p(d\mu)$.

1. Introduction

The aim of this paper is to contribute to the theory of Carleson measures. Let us recall that a Carleson measure for a function space X^p (e.g. a Hardy space) is a positive Borel measure μ such that the space X^p is embedded in $L^p(d\mu)$. This class of measures for the Hardy spaces H^p were first introduced by Carleson in the study of the interpolation problem for Hardy spaces, and plays a crucial role, among others, in the context of characterizations of pointwise multipliers and the corona problem.

The Carleson measures for unweighted Hardy–Sobolev spaces $H_s^p(\mathbf{B}^n)$ have been thoroughly studied, although the characterization is still open for a whole range of s and p. If $n-sp<0$, the space $H_s^p(\mathbf{B}^n)$, consists of continuous functions on the unit ball $\overline{\mathbf{B}}^n$, and therefore, the Carleson measures are the finite measures. If $n-sp\geq 0$ and $n=1$, the characterization in terms of Riesz capacities is due to Stegenga [17]. But for $n>1$ and $s>0$, the characterization of the Carleson measures for $H_s^p(\mathbf{B}^n)$ still remains open. Close to the regular case, when $n-sp<1$, in [2], [6] and [7], extensions of the capacitary characterizations given in [17] have been obtained. The case $p=2$, and $s=(n-1)/2$, which corresponds to the Drury–Arveson space, has been considered by [3] and [19]. In the remaining cases there is not a complete characterization.

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The boundedness of both the restriction operator from the Hardy–Sobolev space $H^p_{s+1/p}(\mathbf{B}^{n+1})$ to the Besov space $B^p_s(\mathbf{B}^n)$, and of an extension operator from $B^p_s(\mathbf{B}^n)$ to $H^p_{s+1/p}(\mathbf{B}^{n+1})$, give as a consequence a characterization of the Carleson measures for the Besov spaces $B^p_s(\mathbf{B}^n)$ for the same ranges of s and p, as has been observed in [14] (see also, [3], [8] and the references therein).

In this paper we study Carleson measures for weighted holomorphic Besov spaces $B^p_s(\omega,\mathbf{B}^n)$, with $s\geq 0$, that is, the positive Borel measures μ on $\mathbf{B}^n$, for which the weighted holomorphic Besov space $B^p_s(\omega,\mathbf{B}^n)$ is embedded in $L^p(d\mu)$.

To be more precise, let us introduce some notation. Let ω be a weight in $\mathbf{B}^n$, the unit ball in $\mathbf{C}^n$, dv be the normalized Lebesgue measure on $\mathbf{B}^n$, and $\mathcal{R}$ be the radial derivative. We use the notation $z\overline{\eta}$ to indicate the complex inner product in $\mathbf{C}^n$ given by $z\overline{\eta}=\sum_{i=1}^n z_i\overline{\eta}_i$, if $z=(z_1,...,z_n)$ and $\eta=(\eta_1,...,\eta_n)$. We will consider the weighted Besov space $B^p_s(\omega,\mathbf{B}^n)$, for $1<p<+\infty$ and $s\in\mathbf{R}$, which consists of holomorphic functions on $\mathbf{B}^n$ such that

$$\|f\|^p_{B^p_s(\omega,\mathbf{B}^n)}=\int_{\mathbf{B}^n}|(I+\mathcal{R})^k f(y)|^p(1-|y|^2)^{(k-s)p-1}\omega(y)\,dv(y)<+\infty,$$

for some $k\in\mathbf{Z}_+$, $k>s$. As happens in the unweighted case, it can be shown that for adequate weights if the above integral is finite for some $k>s$, then it is also finite for any $k>s$ (see Section 3).

The weighted Hardy–Sobolev space $H^p_s(\omega,\mathbf{B}^n)$, $0\leq s<+\infty$, $1<p<+\infty$, consists of functions f holomorphic in $\mathbf{B}^n$ such that if $f(z)=\sum_{j=0}^{+\infty} f_j(z)$ is its homogeneous polynomial expansion, and $(I+\mathcal{R})^s f(z)=\sum_{j=0}^{+\infty}(1+j)^s f_j(z)$, we have that

$$\|f\|_{H^p_s(\omega,\mathbf{B}^n)}=\sup_{0<r<1}\|(I+\mathcal{R})^s f(r\zeta)\|_{L^p(\omega\,d\sigma)}<+\infty,$$

where $d\sigma$ is the Lebesgue measure on $\mathbf{S}^n$. When $\omega\equiv 1$, we obtain the classical Hardy–Sobolev spaces $H^p_s(\mathbf{B}^n)$.

The Carleson measures for certain weighted Hardy–Sobolev spaces $H^p_s(\omega,\mathbf{B}^n)$ have been studied in [5], extending the unweighted case. The weights considered there were in the class $A_p(\mathbf{S}^n)$ and satisfied a certain doubling condition.

The main purpose of this paper is to obtain characterizations of Carleson measures for certain weighted Besov spaces $B^p_s(\omega,\mathbf{B}^n)$, $1<p<+\infty$, $s\geq 0$, in dimension $n\geq 1$. The weights ω considered are in the class $\mathcal{A}_p(\mathbf{B}^n)$, with respect to balls associated with a pseudodistance in $\overline{\mathbf{B}}^n$, and are also in an appropriate class of doubling measures $\mathcal{D}_\tau(\mathbf{B}^n)$. We refer to Section 2 for the precise definitions.

Theorem 1.1. *Let* $1<p<+\infty$, $s\geq 0$ *and* $\omega\in\mathcal{A}_p(\mathbf{B}^n)\cap\mathcal{D}_\tau(\mathbf{B}^n)$, $\tau-p(s+1/p)<1$. *Let* μ *be a positive Borel measure on* $\mathbf{B}^n$. *We then have that the following statements are equivalent*:

(i) *There exists* $C>0$ *such that for any* $f\in B^p_s(\omega,\mathbf{B}^n)$,

$$\|f\|_{L^p(d\mu)}\le C\|f\|_{B^p_s(\omega,\mathbf{B}^n)};$$

(ii) *There exists* $C>0$ *such that for any* $f\in L^p(\omega\,dv)$,

$$\left\|\int_{\mathbf{B}^n}\frac{f(y)}{(1-z\overline{y})^{n+1-(s+1/p)}}\,dv(y)\right\|_{L^p(d\mu)}\le C\|f\|_{L^p(\omega\,dv)};$$

(iii) *There exists* $C>0$ *such that for any* $f\in L^p(\omega\,dv)$,

$$\left\|\int_{\mathbf{B}^n}\frac{f(y)}{|1-z\overline{y}|^{n+1-(s+1/p)}}\,dv(y)\right\|_{L^p(d\mu)}\le C\|f\|_{L^p(\omega\,dv)}.$$

We would like to remark that condition (iii) in the above theorem reduces the problem to the boundedness of an integral operator with positive kernel.

If $\omega\equiv 1$, then $\tau=n+1$, and the equivalence of (i) and (ii) in Theorem 1.1 for $n+1-(s+1/p)<1$ is immediate. The above theorem deals with the more general case $n+1-p(s+1/p)<1$.

In fact, we will extend the above theorem to a bigger class of weights, $\mathcal{B}_p(\mathbf{B}^n)\cap d_\tau(\mathbf{B}^n)$, where the balls involved in the definitions are just the ones that "intersect" the boundary. It can be proved that if $\tau-p(s+1/p)<0$, the space $B^p_s(w,\mathbf{B}^n)$ consists of continuous functions on $\overline{\mathbf{B}}^n$, and consequently, the Carleson measures are just the finite ones. We will just consider from now on the case $\tau-p(s+1/p)>0$.

Let us finally mention that if $s<0$, no derivative is necessarily involved in the definition of the norm of $B^p_s(\omega,\mathbf{B}^n)$, and it is in fact a weighted Bergman space. The corresponding Carleson measures for some classes of weights have been studied, among others, in [12] and [17] in dimension 1, and [9], [10] and [11] in dimension $n>1$. The object of this paper is the study of Carleson measures for weighted Besov spaces $B^p_s(\omega,\mathbf{B}^n)$ with $s\ge 0$.

The paper is organized as follows: In Section 2 we introduce the classes $\mathcal{A}_p(\mathbf{B}^n)$, $\mathcal{D}_\tau(\mathbf{B}^n)$ and the bigger classes $\mathcal{B}_p(\mathbf{B}^n)\cap d_\tau(\mathbf{B}^n)$ and state their main properties. In Section 3, we study the weighted Besov spaces, proving, among other results needed in forthcoming sections, the weighted theorems on extension and restriction. In Section 4, we give the proof of Theorem 1.1.

Finally, the usual remark on notation: we will adopt the convention of using the same letter for various absolute constants whose values may change in each occurrence, and we will write $A\preceq B$ if there exists an absolute constant M such that $A\le MB$. We will say that two quantities A and B are equivalent if both $A\preceq B$ and $B\preceq A$, and, in that case, we will write $A\simeq B$.

2. Weights in $\mathbf{B}^n$

2.1. A pseudodistance ρ in $\mathbf{B}^n$

Our approach to the study of Carleson measures for weighted Besov spaces in $\mathbf{B}^n$ uses the immersion of such spaces in holomorphic spaces defined in $\mathbf{B}^{n+1}$ via the natural projection $\Pi\colon \mathbf{S}^{n+1}\to\mathbf{B}^n$, given by $\Pi(z_1,...,z_{n+1})=(z_1,...,z_n)$. It is then convenient to consider a pseudodistance in $\overline{\mathbf{B}}^n$, $\rho(z,y)$, deduced from the hyperbolic pseudodistance in $\mathbf{S}^{n+1}$. We define $\rho(z,y)$ by the infimum of the Korányi pseudodistances of the antiimages by the mapping Π of the points z,y:

$$\rho(z,y)=\inf_{\varphi,\,\theta\in[0,2\pi)}\left|1-z\overline{y}-\sqrt{1-|z|^2}e^{i\varphi}\sqrt{1-|y|^2}e^{-i\theta}\right|$$

$$=\inf_{\theta\in[0,2\pi)}\left|1-z\overline{y}-\sqrt{1-|z|^2}\sqrt{1-|y|^2}e^{i\theta}\right|=|1-z\overline{y}|-\sqrt{1-|z|^2}\sqrt{1-|y|^2}.$$

Lemma 2.1. *ρ is a pseudodistance in $\overline{\mathbf{B}}^n$.*

Proof. Let $\theta\in[0,2\pi)$. If $z\in\mathbf{B}^n$, let $\Pi_\theta^{-1}(z)=\big(z,\sqrt{1-|z|^2}e^{i\theta}\big)$. We then have that for any $\theta_0\in[0,2\pi)$, $\rho(z,y)=\inf_\theta|1-\Pi_\theta^{-1}(z)\overline{\Pi_{\theta_0}^{-1}(y)}|$, an expression from which we easily obtain that ρ is a pseudodistance. □

We denote by $U_\rho(z,R)=\{y\in\overline{\mathbf{B}}^n;\rho(z,y)<R\}$ the ball of center z and radius R with respect to the pseudodistance ρ. In the following lemma we show that ρ is a pseudodistance whose balls centered at z of radius R are “equivalent”, in a sense that we will make precise, to polydisks of radius $R+R^{1/2}(1-|z|^2)^{1/2}$ in the complex normal direction and of radius $R^{1/2}$ in the complex-tangential directions. We will write $v(E)$ for the volume measure of a measurable subset E in $\overline{\mathbf{B}}^n$.

Lemma 2.2. *Let $z\in\overline{\mathbf{B}}^n$, and $0<R<1$. Let $P(z,R)$ be the polydisk in $\overline{\mathbf{B}}^n$ centered at z, of radius $R+R^{1/2}(1-|z|^2)^{1/2}$ in the complex normal direction and of radius $R^{1/2}$ in the complex tangential directions. Then there exists $C>0$ such that $P(z,R/C)\subset U_\rho(z,R)\subset P(z,CR)$. In particular, $v(U_\rho(z,R))\simeq R^n(R+(1-|z|^2))$.*

Proof. Let $z\in\overline{\mathbf{B}}^n$ and $R>0$. A unitary change of variables gives that, without loss of generality, we may assume that $z=(r,0,...,0)$, $0\le r\le 1$. We begin showing that $P(z,R)\subset U_\rho(z,CR)$ for some fixed constant $C>0$. Let us consider first the case $R\le 1-r^2$. If $y=(y_1,...,y_n)\in P(z,R)$, the definition of the polydisk gives that

$|r-y_1|\le R+R^{1/2}(1-r^2)^{1/2}$, and $|y_i|\le R^{1/2}$, $i=2,\dots,n$. Then

$$\rho(z,y)=\frac{|1-ry_1|^2-(1-r^2)(1-|y|^2)}{|1-r\overline{y}_1|+\sqrt{1-|y|^2}\sqrt{1-r^2}}=\frac{|r-y_1|^2+(|y_2|^2+\dots+|y_n|^2)(1-r^2)}{|1-r\overline{y}_1|+\sqrt{1-|y|^2}\sqrt{1-r^2}}$$
$$\preceq\frac{R^2+R(1-r^2)}{1-r^2}\preceq R.$$

Assume now that $(1-r^2)\le R$. We have that

$$\rho(z,y)=|1-r\overline{y}_1|-\sqrt{1-|y|^2}\sqrt{1-r^2}\le|1-r\overline{y}_1|$$
$$\preceq(1-r^2)+r|r-y_1|\preceq(1-r^2)+R+R^{1/2}(1-r^2)^{1/2}\preceq R.$$

Hence in any case we have shown that $P(z,R)\subset U_\rho(z,CR)$.

Conversely, let $y\in\mathbf{B}^n$ be such that $\rho(z,y)<R$. The previous argument gives that

$$\rho(z,y)\simeq\frac{|r-y_1|^2+(|y_2|^2+\dots+|y_n|^2)(1-r^2)}{|1-r\overline{y}_1|}\preceq R.$$

In particular,

$$\frac{|r-y_1|^2}{(1-r^2)+|r-y_1|}\preceq R. \tag{1}$$

If $|r-y_1|\le(1-r^2)$, we have that $(1-r^2)+|r-y_1|\simeq(1-r^2)$, and we deduce from (1) that $|r-y_1|\preceq R^{1/2}(1-r^2)^{1/2}$. If on the other hand, $(1-r^2)\le|r-y_1|$, (1) gives that $|r-y_1|\preceq R$. Thus in any case we deduce that $|r-y_1|\preceq R+R^{1/2}(1-r^2)^{1/2}$. In order to finish we have to check that $|y_i|^2\preceq R$, $i=2,\dots,n$. It is clear that this is the case if $|r-y_1|\le 1-r^2$, since then $(|y_2|^2+\dots+|y_n|^2)(1-r^2)/|1-r\overline{y}_1|\preceq R$. So we may assume that $(1-r^2)\le|r-y_1|$. As we have shown that in this case $|r-y_1|\preceq R$, we obtain that

$$|y_2|^2+\dots+|y_n|^2\le 1-|y_1|^2\preceq|1-y_1|\preceq 1-r^2+|r-y_1|\preceq|r-y_1|\preceq R.$$

The assertion on the volume of the balls is obvious from the above. □

Observe that if $0<\varepsilon<1$ is small enough, then there exists a constant $C>0$ such that if $y\in U_\rho(z,\varepsilon(1-|z|))$, then we have that $(1-|z|^2)/C\le 1-|y|^2\le C(1-|z|^2)$. This is a consequence of the fact that by the above lemma, if $R=\varepsilon(1-|z|^2)$, the ball $U_\rho(z,\varepsilon(1-|z|))$ is contained in and contains a polydisk of radius $C\varepsilon(1-|z|^2)$ in the normal direction.

2.2. Weights in the classes $\mathcal{A}_p(\mathbf{B}^n)$ and $\mathcal{D}_\tau(\mathbf{B}^n)$

We next recall the definition of the class $A_p(\mathbf{S}^{n+1})$. We recall that $\sigma(F)$ will stand for the Lebesgue measure of a measurable subset F in $\mathbf{S}^{n+1}$. Observe that although we will denote by σ both the Lebesgue measure in $\mathbf{S}^n$ and in $\mathbf{S}^{n+1}$, it will be clear in any occasion if we are in dimension n or $n+1$.

Definition 2.3. A weight ϕ in $\mathbf{S}^{n+1}$ is in $A_p(\mathbf{S}^{n+1})$, $1<p<+\infty$, if there exists $C>0$ such that for any nonisotropic ball $B(\zeta,R)=\{\eta\in\mathbf{S}^{n+1};|1-\zeta\overline{\eta}|<R\}$, $\zeta\in\mathbf{S}^{n+1}$, $R>0$,

$$\left(\frac{1}{\sigma(B(\zeta,R))}\int_{B(\zeta,R)}\phi\,d\sigma\right)\left(\frac{1}{\sigma(B(\zeta,R))}\int_{B(\zeta,R)}\phi^{-(p'-1)}\,d\sigma\right)^{p-1}\leq C.$$

We also introduce the weights in the class $\mathcal{A}_p(\mathbf{B}^n)$ with respect to the pseudodistance ρ.

Definition 2.4. We say that a weight ω in $\overline{\mathbf{B}}^n$ is in $\mathcal{A}_p(\mathbf{B}^n)$, $1<p<+\infty$, if there exists $C>0$ such that for any ball $U_\rho=U_\rho(z,R)$ in $\overline{\mathbf{B}}^n$,

$$\left(\frac{1}{v(U_\rho)}\int_{U_\rho}\omega\,dv\right)\left(\frac{1}{v(U_\rho)}\int_{U_\rho}\omega^{-(p'-1)}\,dv\right)^{p-1}\leq C.$$

If in the above definition we consider just balls $U_\rho(z,r)$ that intersects the boundary, we obtain the class $\mathcal{B}_p(\mathbf{B}^n)$ defined in [4]. We will deal with this class of weights in Section 2.3 below.

The following lemma gives a characterization of weights in $\mathcal{A}_p(\mathbf{B}^n)$ in terms of their "lifting" to $\mathbf{S}^{n+1}$.

Definition 2.5. If ω is a weight in $\mathbf{B}^n$, we denote by ω_l the weight in $\mathbf{S}^{n+1}$ defined by $\omega_l(z_1,...,z_{n+1})=\omega(z_1,...,z_n)$. We will call ω_l the *lifted weight.*

Lemma 2.6. *Let* $1<p<+\infty$, $n\geq1$, *and let* ω *be a weight in* $\mathbf{B}^n$. *We then have that* $\omega\in\mathcal{A}_p(\mathbf{B}^n)$ *if and only if the lifted weight* $\omega_l\in A_p(\mathbf{S}^{n+1})$.

Proof. We begin proving that if $\omega\in\mathcal{A}_p(\mathbf{B}^n)$, then $\omega_l\in A^p(\mathbf{S}^{n+1})$.

We consider first the particular case where $z_{n+1}=0$, i.e. the center of the ball $B(z,R)$ lies in $\overline{\mathbf{B}}^n$. By a suitable change of variables we may assume that $z=(1,0,...,0)$. Then $\Pi^{-1}(U_\rho((1,0,...,0),R))=\{y\in\mathbf{S}^{n+1};|1-y_1|<R\}=B(z,R)$, and consequently,

$$\int_{B(z,R)}\omega_l\,d\sigma=\int_{\Pi^{-1}(U_\rho((1,0,\ldots,0),R))}\omega_l\,d\sigma=\int_{U_\rho((1,0,\ldots,0),R)}\omega\,dv,$$

and the same argument holds for $\omega_l^{-(p'-1)}$. These equalities, together with the fact that $\omega\in\mathcal{A}_p(\mathbf{B}^n)$ and $v(U_\rho(1,0,...,0),R)\simeq R^{n+1}$, give that

$$\left(\frac{1}{R^{n+1}}\int_{B(z,R)}\omega_l\,d\sigma\right)\left(\frac{1}{R^{n+1}}\int_{B(z,R)}\omega_l^{-(p'-1)}\,d\sigma\right)^{1/(p'-1)}\leq C.$$

In fact, this argument can be applied to nonisotropic balls $B(z,R)$ in $\mathbf{S}^{n+1}$ whose centers $z=(z',z_{n+1})$ satisfy $1-|z'|^2\leq R$. We just have to observe that in this case the ball $B(z,R)$ is included in a nonisotropic ball in $\mathbf{S}^{n+1}$ whose center lies in $\mathbf{S}^n$ and whose radius is comparable to R.

So we may assume that $R\leq 1-|z'|^2$. Without loss of generality we also may assume that $z=(r,\overset{n-1}{0,...,0},\sqrt{1-r^2}e^{i\theta})$, for some $0<r<1$ and $\theta\in[0,2\pi)$. The fact that $R\preceq 1-r^2$, gives that $v(U_\rho((r,0,...,0),R))\simeq R^n(1-r^2)$.

Next, it is immediate to check that the projection of the points in $\mathbf{S}^{n+1}$ at distance less than R from the set $\Pi^{-1}((r,0,...,0))$ (with respect to the pseudodistance in $\mathbf{S}^{n+1}$) is included in $U_\rho((r,0,...,0),R)$, and on the other hand, the ball $U_\rho((r,0,...,0),R)$ is included in the projection of the set of points in $\mathbf{S}^{n+1}$ at distance less than CR from the set $\Pi^{-1}((r,0,...,0))$. Moreover, the set of points of $\mathbf{S}^{n+1}$ at distance less than R from the set $\Pi^{-1}((r,0,...,0))$ is included in a union of nonisotropic balls of radius R in a number which is of the order of $(1-|r|^2)/R$, and includes the same number of disjoint balls of radius comparable to R. The integral of the lifted weight ω_l on each of these balls is equivalent since we transform one nonisotropic ball to another by a rotation that leaves ω_l invariant.

Altogether we obtain that

$$\begin{aligned}\frac{1-r^2}{R}\int_{B((r,0,...,0,\sqrt{1-r^2}e^{i\theta}),R)}\omega_l\,d\sigma &\simeq \int_{\Pi^{-1}(U_\rho((r,0,...,0),R))}\omega_l\,d\sigma\\ &=\int_{U_\rho((r,0,...,0),R)}\omega\,dv,\end{aligned}$$

and, since in the case we are now considering $v(U((r,0,...,0),R))\simeq R^n(1-r^2)$, we have that

$$\begin{aligned}\frac{1}{R^{n+1}}\int_{B((r,0,...,0,\sqrt{1-r^2}e^{i\theta}),R)}\omega_l\,d\sigma &\simeq \frac{1}{(1-|z|^2)R^n}\int_{U_\rho((r,0,...,0,),R)}\omega\,dv\\ &\simeq \frac{1}{v(U_\rho((r,0,...,0),R))}\int_{U((r,0,...,0,),R)}\omega\,dv,\end{aligned}$$

with a similar estimate for $\omega_l^{-(p'-1)}$. Hence, we have proved that if $\omega\in\mathcal{A}_p(\mathbf{B}^n)$, then $\omega_l\in A_p(\mathbf{S}^{n+1})$.

Let now ω be a weight in $\mathbf{B}^n$ satisfying $\omega_l \in A_p(\mathbf{S}^{n+1})$. The argument we have used above shows that if $z\in\bar{\mathbf{B}}^n$, $R>0$ and $U_\rho(z,R)$ is a ball in $\mathbf{B}^n$, such that $1-|z|^2\le R$, we can reduce ourselves to the case where the point z is in $\mathbf{S}^n$. Then $U_\rho(z,R)$ is just a tent centered at a point z in $\mathbf{S}^n$, and $U_\rho(z,R)=B((z,0),R)$. Consequently, ω satisfies the $\mathcal{A}_p(\mathbf{B}^n)$ condition for this class of balls.

If $R<1-|z|^2$, then $v(U_\rho(z,R))\simeq R^n(1-|z|^2)$, and again the argument used before gives then that for any θ,

$$\begin{aligned}\frac{1}{R^n(1-|z|^2)}\int_{U_\rho(z,R)}\omega\,dv &\simeq \frac{1}{R^n(1-|z|^2)}\int_{\Pi^{-1}(U_\rho(z,R))}\omega_l\,d\sigma\\ &\simeq \frac{1}{R^n(1-|z|^2)}\,\frac{1-|z|^2}{R}\int_{B((z,\sqrt{1-|z|^2}e^{i\theta}),CR)}\omega_l\,d\sigma\\ &= \frac{1}{R^{n+1}}\int_{B((z,\sqrt{1-|z|^2}e^{i\theta}),CR)}\omega_l\,d\sigma,\end{aligned}$$

with a similar relationship for $\omega^{-(p'-1)}$. Since $\omega_l\in A_p(\mathbf{S}^{n+1})$, we are done. □

It is easy to check that in a natural way, weights in $A_p(\mathbf{S}^n)$ give weights in $\mathcal{A}_p(\mathbf{B}^n)$.

Example 2.7. Assume that $w\in A_p(\mathbf{S}^n)$. Then the weight defined by

$$\widetilde{w}(z)=\frac{1}{(1-|z|^2)^n}\int_{I_z} w(\zeta)\,d\sigma(\zeta),$$

$z\in\mathbf{B}^n$, where $I_z=\{\zeta\in\mathbf{S}^n; |1-\frac{z}{|z|}\bar{\zeta}|\le c(1-|z|^2)\}$, $c>0$, is in $\mathcal{A}_p(\mathbf{B}^n)$.

Proof. We want to show that there exists $C>0$ such that if $a=(a_1,...,a_n)$, and $U=U(a,R)=\{\eta\in\mathbf{B}^n;\rho(\eta,a)<R\}$, then

$$\left(\frac{1}{v(U)}\int_U \widetilde{w}(z)\,dv(z)\right)\left(\frac{1}{v(U)}\int_U \widetilde{w}^{-(p'-1)}(z)\,dv(z)\right)^{-1/(p'-1)}\le C.$$

Assume first that $1-|a|\le R/\delta$, $\delta>0$ to be chosen. We can reduce this case to the one where $a=(a_1,0,...,0)\in\mathbf{S}^n$. We then have that if $z\in U$ and $\zeta\in I_z$, then $\zeta\in U(a,CR)$, for some fixed constant $C>0$. Next, Fubini's theorem gives that if

$D_\alpha(\zeta)=\{z\in\mathbf{B}^n;|1-z\overline{\zeta}|<\alpha(1-|z|^2)\}$, then

$$\begin{aligned}\frac{1}{v(U)}\int_U \widetilde{w}(z)\,dv(z) &= \frac{1}{v(U)}\int_U \frac{1}{(1-|z|)^n}\int_{I_z} w(\zeta)\,d\sigma(\zeta)\,dv(z)\\ &\preceq \frac{1}{R^{n+1}}\int_{B(a,CR)} w(\zeta)\,d\sigma(\zeta)\int_{U(a,R)\cap D_\alpha(\zeta)}\frac{dv(z)}{(1-|z|)^n}\\ &\simeq \frac{1}{R^n}\int_{B(a,R)} w(\zeta)\,d\sigma(\zeta),\end{aligned}$$

where we have used that if $\zeta\in I_z$, then $|1-z\overline{\zeta}|\preceq 1-|z|$. An analogous argument to the one we have used applied to $\widetilde{w}^{-1/(p-1)}$ finishes this case.

If $R\le\delta(1-|a|)$, we have that for any $z\in U$, $\widetilde{w}(z)\simeq\widetilde{w}(a)$, and consequently, the A_p condition in this case is obvious. □

A weight ω in $\mathbf{S}^{n+1}$ is a *doubling weight*, if there exists $A>0$ satisfying that for any $R>0$, $\omega(B(\zeta,2R))\le A\omega(B(\zeta,R))$. More precisely, we introduce the class $D_\tau(\mathbf{S}^{n+1})$ of doubling weights in the following definition.

Definition 2.8. A weight ω in $\mathbf{S}^{n+1}$ is in $D_\tau(\mathbf{S}^{n+1})$, $\tau>0$, if there exists $C>0$ such that for any nonisotropic ball $B(\zeta,R)=\{\eta\in\mathbf{S}^{n+1};|1-\eta\overline{\zeta}|<R\}$, $\zeta\in\mathbf{S}^{n+1}$, $R>0$, and any integer $j\ge1$, $w(B(\zeta,2^jR))\le C2^{j\tau}w(B(\zeta,R))$.

Obviously, any weight $\omega\in D_\tau(\mathbf{S}^{n+1})$ is in $D_{\tau_1}(\mathbf{S}^{n+1})$, for any $\tau_1\ge\tau$. It is well known that any weight $\omega\in A_p(\mathbf{S}^{n+1})$ is in $D_\tau(\mathbf{S}^n)$, with $\tau=p(n+1)$, see for instance Section 1.5 of Chapter 5 in [18].

In the proof of Lemma 2.6 we have seen in fact that if ω is a weight in $\mathbf{B}^n$ and ω_l is the corresponding lifted weight in $\mathbf{S}^{n+1}$, then

$$\omega_l(B(z,R))\simeq\frac{R}{R+(1-|z'|)}\omega(U_\rho(z',R)),$$

where $z=(z',z_{n+1})\in\mathbf{S}^{n+1}$. It is then natural to define the class $D_\tau(\mathbf{B}^n)$ of weights in $\mathbf{B}^n$ as follows.

Definition 2.9. We say that a weight ω in $\mathbf{B}^n$ is in $\mathcal{D}_\tau(\mathbf{B}^n)$ for some $\tau>0$, if there exists $C>0$, such that for any integer $j\ge1$, $z\in\mathbf{B}^n$ and $R>0$,

$$\omega(U_\rho(z,2^jR))\le C\frac{(1-|z|^2)+2^jR}{(1-|z|^2)+R}2^{j(\tau-1)}\omega(U_\rho(z,R)). \tag{2}$$

Since $v(U_\rho(z,R))\simeq R^n(R+(1-|z|^2))$, this condition can be rewritten as

$$\frac{\omega(U_\rho(z,2^jR))}{v(U_\rho(z,2^jR))}\le C\frac{2^{j\tau}}{2^{j(n+1)}}\frac{\omega(U_\rho(z,R))}{v(U_\rho(z,R))}.$$

Observe that for those balls $U_\rho(z,R)$ such that $1-|z|^2<R$, the $\mathcal{D}_\tau$ condition is just that $\omega(U_\rho(z,2^jR))\le C2^{j\tau}\omega(U_\rho(z,R))$.

We have the following lemma.

Lemma 2.10. *A weight ω is in $\mathcal{D}_\tau(\mathbf{B}^n)$, $\tau>0$, if and only if the lifted weight ω_l is in $D_\tau(\mathbf{S}^{n+1})$.*

Proof. Assume that $\omega\in\mathcal{D}_\tau(\mathbf{B}^n)$, and let $z=(z',z_{n+1})\in\mathbf{S}^{n+1}$, $R>0$ and $j\ge 1$. If $1-|z'|^2\le R$, then

$$\omega_l(B(z,2^jR))\simeq\omega(U_\rho(z',2^jR))\preceq 2^{j\tau}\omega(U_\rho(z',R))\simeq 2^{j\tau}\omega_l(B(z,R)).$$

If $1-|z'|^2>2^jR$, we have that (see the proof of Lemma 2.6) $\omega_l(B(z,2^jR))\simeq(2^jR/(1-|z'|^2))\omega(U_\rho(z',2^jR))$, and $\omega_l(B(z,R))\simeq(R/(1-|z'|^2))\omega(U_\rho(z',R))$. Hence,

$$\omega_l(B(z,2^jR))\simeq\frac{2^jR}{1-|z'|^2}\omega(U_\rho(z',2^jR))\preceq\frac{2^jR}{1-|z'|^2}2^{j\tau}\omega(U_\rho(z',R))\simeq 2^{j\tau}\omega_l(B(z,R)).$$

If $1-|z'|^2>R$ and $1-|z'|^2\le 2^jR$, we have that (see Lemma 2.6) $\omega_l(B(z,2^jR))\simeq\omega(U_\rho(z',2^jR))$, and $\omega_l(B(z,R))\simeq(R/(1-|z'|^2))\omega(U(z',R))$. Hence

$$\omega_l(B(z,2^jR))\simeq\omega(U_\rho(z',2^jR))\preceq\frac{2^jR}{1-|z'|^2}2^{j\tau}\omega(U_\rho(z',R))\preceq 2^{j\tau}\omega_l(B(z,R)).$$

So we have that $\omega_l\in D_\tau(\mathbf{S}^{n+1})$. The other implication is proved in a similar way. □

Observe that if $\omega\equiv 1$, then $\omega\in\mathcal{D}_{n+1}(\mathbf{B}^n)$. The following lemma is an immediate consequence of the differentiation theorem (see for instance Theorem 5.3.1 in [15]), and show that without loss of generality we always may assume that $\tau\ge n+1$.

Lemma 2.11. *If ω is a nonidentically zero weight in $L^1(\mathbf{B}^n)$ which is in $\mathcal{D}_\tau(\mathbf{B}^n)$, then $\tau\ge n+1$.*

Proof. Assume that $\tau<n+1$, and fix a ball $U_\rho=U_\rho(z,R)\subset\mathbf{B}^n$ that intersects $\mathbf{S}^n$. The fact that $\omega\in\mathcal{D}_\tau(\mathbf{B}^n)$, gives that there exists an integer j_0 such that for any integer $j\ge j_0$,

$$\omega(U_\rho)\preceq\frac{R}{2^{-j}R+(1-|z|^2)}2^{j(\tau-1)}\omega(U_\rho(z,2^{-j}R))\simeq\frac{R}{1-|z|^2}2^{j(\tau-1)}\omega(U_\rho(z,2^{-j}R)).$$

Since for $j\ge j_0$, $v(U_\rho(z,2^{-j}R))\simeq(2^{-j}R)^n(1-|z|^2)$, we have that

$$2^{j(n+1-\tau)}\omega(U_\rho)\preceq R^{n+1}\frac{\omega(U_\rho(z,2^{-j}R))}{v(U_\rho(z,2^{-j}R))}. \tag{3}$$

The differentiation theorem (see for instance Theorem 5.3.1 in [15]) gives that for almost every $z\in\mathbf{B}^n$,

$$\lim_{j\to+\infty}\frac{\omega(U_\rho(z,2^{-j}R))}{v(U_\rho(z,2^{-j}R))}\simeq\omega(z).$$

If $\tau<n+1$, the last estimate gives a contradiction to (3). □

2.3. Weights in $\mathcal{B}_p(\mathbf{B}^n)$ and in $d_\tau(\mathbf{B}^n)$

In this subsection we study the main properties of some bigger classes of weights $\mathcal{B}_p(\mathbf{B}^n)$ and $d_\tau(\mathbf{B}^n)$, which correspond to weights that satisfy the conditions $\mathcal{A}_p(\mathbf{B}^n)$ and $\mathcal{D}_\tau(\mathbf{B}^n)$, respectively, only for balls that intersect the unit sphere. More precisely we make the following definition.

Definition 2.12. We say that a weight ω is in $\mathcal{B}_p(\mathbf{B}^n)$ (see [4]) if there exists $C>0$ such that for any ball $U_\rho(z,R)$ that intersects $\mathbf{S}^n$, i.e., $\overline{U_\rho(z,R)}\cap\mathbf{S}^n\neq\varnothing$,

$$\left(\frac{1}{v(U(z,R))}\int_{U(z,R)}\omega\,dv\right)\left(\frac{1}{v(U(z,R))}\int_{U(z,R)}\omega^{-(p'-1)}\,dv\right)^{p-1}\le C.$$

Obviously, any $\mathcal{A}_p(\mathbf{B}^n)$ weight satisfies the condition $\mathcal{B}_p(\mathbf{B}^n)$.

We also have the following definition.

Definition 2.13. We say that a weight ω in $\mathbf{B}^n$ is in $d_\tau(\mathbf{B}^n)$ for some $\tau>0$, if there exists $C>0$, such that for any integer $j\ge1$, $z\in\mathbf{B}^n$ and $R>0$ satisfying that $U_\rho(z,2^jR)$ intersects $\mathbf{S}^n$,

$$\omega(U_\rho(z,2^jR))\le C\frac{R}{(1-|z|^2)+R}2^{j\tau}\omega(U_\rho(z,R)). \tag{4}$$

As we have already observed, it is well known that if $\omega\in A_p(\mathbf{S}^{n+1})$, then $w\in D_{p(n+1)}(\mathbf{S}^{n+1})$ (see for instance [18]). An analogous argument shows that any weight $\omega\in\mathcal{B}_p(\mathbf{B}^n)$ is in $d_\tau(\mathbf{B}^n)$ for $\tau=p(n+1)$.

Lemma 2.14. *Let $1<p<+\infty$ and $\omega\in\mathcal{B}_p(\mathbf{B}^n)$. There exists $C>0$, such that for any integer $j\ge1$, $z\in\mathbf{B}^n$ and $R>0$ satisfying that $U_\rho(z,2^jR)$ intersects $\mathbf{S}^n$,*

$$\omega(U_\rho(z,2^jR))\le C\frac{R}{(1-|z|^2)+R}2^{j\tau}\omega(U_\rho(z,R)),$$

where $\tau=p(n+1)$.

As a consequence of Lemma 2.14, we have an equivalent definition of weights in $\mathcal{B}_p(\mathbf{B}^n)$ which coincides with the weights in the class $B_p(\mathbf{B}^n)$ introduced in [4]: a weight ω is in $B_p(\mathbf{B}^n)$ if there exists $C>0$ such that for any tent $T(B(\zeta,R))=\{z\in\mathbf{B}^n;|1-z\overline{\zeta}|\le R\}$, $\zeta\in\mathbf{S}^n$,

(5)
$$\left(\frac{1}{v(T(B(\zeta,R)))}\int_{T(B(\zeta,R))}\omega\,dv\right)\left(\frac{1}{v(T(B(\zeta,R)))}\int_{T(B(\zeta,R))}\omega^{-(p'-1)}\,dv\right)^{p-1}\le C.$$

This observation is a consequence of the fact that if a ball $U_\rho(\zeta,R)$ intersects $\mathbf{S}^n$, then it is included in a tent of radius comparable to R and, conversely, a tent of radius R is included in a ball that intersects $\mathbf{S}^n$ of comparable radius, and the "doubling" condition of those weights.

Let us give some examples of weights in $\mathcal{B}_p(\mathbf{B}^n)$ and weights in $d_\tau(\mathbf{B}^n)$.

Proposition 2.15. *Let* $1<p<+\infty$, *and let* $\varphi\colon(0,1]\to\mathbf{R}$ *be a nonnegative monotone function*, $C>0$, *and* $\alpha>0$, *satisfying one of the following alternative assumptions*:

(i) φ *is nondecreasing, and for any integer* $j\ge1$, $\varphi(2^jx)\le C2^{\alpha j}\varphi(x)$;

(ii) φ *is nonincreasing, and for any integer* $j\ge1$, $\varphi(x)\le C2^{\alpha j}\varphi(2^jx)$.

Let $\omega_\varphi(z)=\varphi(1-|z|)$. *We then have the following*:

(a) *The weight* ω_φ *is in* $\mathcal{B}_p(\mathbf{B}^n)$ *if and only if* $0<\alpha<p-1$ *if* φ *is as in case* (i) *or if and only if* $0<\alpha<1$ *if it is as in case* (ii).

(b) *The weight* ω_φ *is in* $d_\tau(\mathbf{B}^n)$ *if* φ *is as in case* (i) *for each* $\tau\ge n+\alpha+1$ *or* φ *is as in case* (ii) *for each* $\tau\ge n+1$.

Proof. We begin with the proof of (a). We have that $\omega_\varphi\in\mathcal{B}_p(\mathbf{B}^n)$ if and only if ω_φ satisfies (5). If $\zeta\in\mathbf{S}^n$, $R>0$ and $T(B(\zeta,R))$ is a tent, we have that $\omega_\varphi(T(B(\zeta,R)))\simeq R^n\int_0^R\varphi(t)\,dt$, and consequently, it is enough to show that

$$\left(\frac{1}{R}\int_0^R\varphi(t)\,dt\right)\left(\frac{1}{R}\int_0^R\varphi^{-(p'-1)}(t)\,dt\right)^{1/(p'-1)}\le C.$$

Assume first that φ satisfies the hypothesis in (i). Since φ is nondecreasing, we have that for any $R>0$,

$$\varphi\left(\frac{R}{2}\right)\frac{R}{2}\le\int_{R/2}^R\varphi(t)\,dt\le\int_0^R\varphi(t)\,dt\le R\varphi(R).$$

Since $\varphi(2x)\simeq\varphi(x)$, we obtain that $\int_0^R\varphi(t)\,dt\simeq R\varphi(R)$. Next,

$$\int_0^R\varphi^{-(p'-1)}(t)\,dt=\sum_{j=0}^{+\infty}\int_{R/2^{j+1}}^{R/2^j}\varphi^{-(p'-1)}(t)\,dt\simeq\sum_{j=0}^{+\infty}2^{-j}R\varphi\left(\frac{R}{2^j}\right)^{-(p'-1)}.$$

Thus, it suffices to check that $\sum_{j=0}^{+\infty} 2^{-j}\varphi(R/2^j)^{-(p'-1)} \preceq \varphi(R)^{-(p'-1)}$. The fact that φ satisfies a doubling condition can be restated as $\varphi(R)\preceq\varphi(R/2^j)((2/(1+\delta))^{p-1})^j$ for some $\delta>0$. And in consequence, the above estimate holds.

If φ satisfies the hypothesis in (ii), then $\varphi^{-(p'-1)}$ satisfies condition (i), and we deduce from the above argument that $\omega_{\varphi^{-(p'-1)}} \in \mathcal{B}_{p'}(\mathbf{B}^n)$. And that is equivalent to the fact that $\omega\in\mathcal{B}_p(\mathbf{B}^n)$. The remaining implications are proved in a similar way.

In order to prove (b), we will check that

$$\omega_\varphi(U_\rho(z,2^jR)) \le C\frac{R}{1-|z|^2+R}2^{j\tau}\omega_\varphi(U_\rho(z,R))$$

for any $U_\rho(z,R)$ and $j\ge0$ such that $U_\rho(z,2^jR)$ intersects $\mathbf{S}^n$.

We first recall that if $M\le x/2$, then we have that for any $t\in[x-M,x+M]$, $x/2\le t\le 3x/2$, and hence, $\int_{x-M}^{x+M}\varphi(t)\,dt\simeq\varphi(x)M$. If on the contrary, $M\ge x/2$, then $\int_{x-M}^{x+M}\varphi(t)\,dt\simeq M\varphi(M)$.

Let $z\in\mathbf{B}^n$, $R>0$ and let $j\ge1$ be an integer. If $(1-|z|^2)/2\le R$, the above considerations give easily that $\omega_\varphi(U_\rho(z,2^jR))\simeq(2^jR)^{n+1}\varphi(2^jR)$, and $\omega_\varphi(U_\rho(z,R))\simeq R^{n+1}\varphi(R)$. Thus the condition $d_\tau(\mathbf{B}^n)$ is fulfilled provided

$$(2^jR)^{(n+1)}\varphi(2^jR)\preceq 2^j2^{j(\tau-1)}R^{n+1}\varphi(R),$$

a condition which is in turn equivalent to

$$\varphi(2^jR)\preceq 2^{j(\tau-n-1)}\varphi(R). \tag{6}$$

Assume first that φ is as in case (i). The conditions on τ and φ give immediately that (6) is satisfied for this set of z's.

If $R\le(1-|z|^2)/2\le 2^jR$, an analogous argument gives now that $\omega_\varphi(U_\rho(z,R))\simeq R^n(1-|z|^2)\varphi(1-|z|)$. Hence it is enough to check in this case that

$$(2^jR)^{(n+1)}\varphi(2^jR)\preceq\frac{R}{1-|z|^2}2^{j\tau}R^n(1-|z|^2)\varphi(1-|z|),$$

i.e.,

$$\varphi(2^jR)\preceq 2^{j(\tau-n-1)}\varphi(1-|z|). \tag{7}$$

And this estimate is again a consequence of the fact that φ satisfies (i).

On the other hand, if φ is nonincreasing, estimates (6) and (7) are obvious. □

Corollary 2.16. *The weight $\omega_\alpha(z)=(1-|z|)^\alpha$, $-1<\alpha<p-1$, is in $\mathcal{B}_p(\mathbf{B}^n)$. If $0\le\alpha<p-1$, then $\omega_\alpha\in d_{n+\alpha+1}(\mathbf{B}^n)$, and if $-1<\alpha<0$, then $\omega_\alpha\in d_{n+1}(\mathbf{B}^n)$.*

The techniques we have applied in order to work with weighted holomorphic Besov spaces in $\mathbf{B}^n$ require that the weights are in $\mathcal{A}_p(\mathbf{B}^n)\cap\mathcal{D}_\tau(\mathbf{B}^n)$. As we will show in the following section, any weighted Besov space with a weight in $\mathcal{B}_p(\mathbf{B}^n)\cap d_\tau(\mathbf{B}^n)$ can, in fact, be defined in terms of a weight in the smaller class $\mathcal{A}_p(\mathbf{B}^n)\cap\mathcal{D}_\tau(\mathbf{B}^n)$. The way to achieve this is via the regularizations of the weights.

Definition 2.17. If ω is a weight in $\mathbf{B}^n$, $0<\varepsilon<1$, the regularized weight $\mathfrak{R}_\varepsilon\omega$ is defined as

$$\mathfrak{R}_\varepsilon\omega(z)=\frac{1}{v(U_\rho(z,\varepsilon(1-|z|^2)))}\int_{U_\rho(z,\varepsilon(1-|z|^2))}\omega(y)\,dv(y).$$

Remark 2.18. As an immediate consequence of Lemma 2.2, we have that if ω satisfies a doubling condition, then all its regularizations are equivalent, that is, if $\varepsilon,\varepsilon'>0$, then $\mathfrak{R}_\varepsilon\omega(z)\simeq\mathfrak{R}_{\varepsilon'}\omega(z)$, for any $z\in\mathbf{B}^n$, with constants that do not depend on z. We just have to observe that if $\varepsilon>0$ is fixed, there exists $C>\varepsilon$ such that for any $z\in\mathbf{B}^n$, $U_\rho(z,C(1-|z|))$ intersects $\mathbf{S}^n$. The fact that ω satisfies a doubling condition gives that $\omega(U_\rho(z,\varepsilon(1-|z|^2)))\simeq\omega(U_\rho(z,C(1-|z|^2)))$.

Observe that the regularization of a doubling weight ω satisfies that $\mathfrak{R}_\varepsilon(\mathfrak{R}_\varepsilon w)\simeq\mathfrak{R}_\varepsilon w$.

It is worthwhile to recall that analogous regularizations were already considered among others by [4] and [11], where the balls $U^d_\varepsilon(z)=\{y\in\mathbf{B}^n;d(z,y)<\varepsilon(1-|z|^2)\}$ were defined with respect to the pseudodistance $d(z,y)=\big||z|-|y|\big|+\big|1-z\overline{y}/|z|\,|y|\big|$.

Proposition 2.19. *Let* $1<p<+\infty$ *and assume that* ω *is a weight in* $\mathcal{B}_p(\mathbf{B}^n)$. *Then the weight* $\mathfrak{R}_\varepsilon\omega$ *is in* $\mathcal{A}_p(\mathbf{B}^n)$.

Proof. We want to show that there exists $C>0$ such that for any ball $U_\rho=U_\rho(a,R)$,

$$\left(\frac{1}{v(U_\rho)}\int_{U_\rho}\mathfrak{R}_\varepsilon w\,dv\right)\left(\frac{1}{v(U_\rho)}\int_{U_\rho}(\mathfrak{R}_\varepsilon w)^{-(p'-1)}\,dv\right)^{1/(p'-1)}\le C.$$

As we have already observed, without loss of generality we may assume that $\varepsilon>0$ is small enough, since for every $\varepsilon,\varepsilon'>0$, $\mathfrak{R}_\varepsilon\omega\simeq\mathfrak{R}_{\varepsilon'}\omega$.

Suppose first that $\delta(1-|a|^2)\le R$, $\delta>0$ to be chosen later on. In this case, Lemma 2.1 gives that $v(U_\rho)\simeq R^{n+1}$. Since we also have that in that case $U_\rho(a,R)$ is included in a ball in $\overline{\mathbf{B}}^n$ centered at a point in $\mathbf{S}^n$ of radius comparable to R, we also may assume without loss of generality that $a\in\mathbf{S}^n$, and that $U=U_\rho(a,CR)$,

$C>0$. In particular we have that for any $x\in U_\rho(a,R)$, $1-|x|^2\preceq R$. Thus there exists $C_1>0$ such that if $z\in U_\rho(a,CR)$ and $y\in U_\rho(z,\varepsilon(1-|z|^2))$, $y\in U_\rho(a,CR)$, and Fubini's theorem gives that

$$\begin{aligned}\int_{U_\rho(a,CR)}\mathfrak{R}_\varepsilon\omega(z)\,dv(z) &\simeq \int_{U_\rho(a,CR)}\frac{1}{(1-|z|^2)^{n+1}}\int_{U_\rho(z,\varepsilon(1-|z|^2))}\omega(y)\,dv(y)\,dv(z)\\ &\preceq \int_{U_\rho(a,C_1R)}\omega(y)\frac{1}{(1-|y|^2)^{n+1}}\int_{U_\rho(y,\varepsilon'(1-|y|^2))}dv(z)\,dv(y)\\ &\preceq \int_{U_\rho(a,CR)}\omega(y)\,dv(y). \qquad (8)\end{aligned}$$

In order to estimate the integral involving $(\mathfrak{R}_\varepsilon\omega)^{-(p'-1)}$, we use the fact that if $\omega\in\mathcal{B}_p(\mathbf{B}^n)$, then $\omega^{-(p'-1)}\in\mathcal{B}_{p'}(\mathbf{B}^n)$ and both of them satisfy a doubling condition to get that,

$$\frac{1}{v(U_\varepsilon(z))}\int_{U_\rho(z,\varepsilon(1-|z|^2))}\omega\,dv\left(\frac{1}{v(U_\rho(z,\varepsilon(1-|z|^2)))}\int_{U_\rho(z,\varepsilon(1-|z|^2))}\omega^{-(p'-1)}\,dv\right)^{1/(p'-1)} \simeq 1.$$

Consequently $\mathfrak{R}_\varepsilon w\simeq(\mathfrak{R}_\varepsilon(w^{-(p'-1)}))^{-1/(p'-1)}$, and

$$\frac{1}{v(U_\rho(a,CR))}\int_{U_\rho(a,CR)}(\mathfrak{R}_\varepsilon\omega)^{-(p'-1)}\,dv\simeq\frac{1}{v(U_\rho(a,CR))}\int_{U_\rho(a,CR)}\mathfrak{R}_\varepsilon(\omega^{-(p'-1)})\,dv,$$

and the argument in (8) applied to $\mathfrak{R}_\varepsilon(\omega^{-(p'-1)})$ together with the fact that $\omega\in\mathcal{B}_p(\mathbf{B}^n)$, gives that in case $1-|a|^2\le R/\delta$, then

$$\left(\frac{1}{v(U_\rho(a,R))}\int_{U_\rho(a,R)}\mathfrak{R}_\varepsilon\omega\,dv\right)\left(\frac{1}{v(U_\rho(a,R))}\int_{U_\rho(a,R)}(\mathfrak{R}_\varepsilon\omega)^{-(p'-1)}\,dv\right)^{1/(p'-1)}\le C.$$

Assume next that $R\le\delta(1-|a|^2)$, and δ is small enough. In that case, for any $z\in U_\rho(a,R)$, $1-|z|^2\simeq 1-|a|$, and consequently $\mathfrak{R}_\varepsilon\omega(z)\simeq\mathfrak{R}_\varepsilon\omega(a)$ for any $z\in U_\rho(a,R)$. □

Lemma 2.20. *If ω is a doubling weight in $\mathbf{B}^n$ its regularization $\mathfrak{R}_\varepsilon\omega$ also satisfies a doubling condition.*

Proof. The proof follows the scheme of the previous proposition, and we just sketch it briefly. If $z\in\mathbf{B}^n$ and $0<R\le\delta(1-|z|^2)$, with $\delta>0$ small enough so that for any $y\in U_\rho(z,2R)$, $1-|y|^2\simeq 1-|z|^2$, we have that $\mathfrak{R}_\omega$ is "quasi"-constant on $U_\rho(z,2R)$. And the doubling condition is satisfied in this case.

If $\delta(1-|z|)\le R$, the desired doubling condition is obtained from Fubini's theorem and the fact that ω satisfies a doubling condition, using (8). □

Proposition 2.21. *Let $1<p<+\infty$, and assume that ω is a weight satisfying that $\omega\in d_\tau(\mathbf{B}^n)$. Then the weight $\mathfrak{R}_\varepsilon\omega$ is in $\mathcal{D}_\tau(\mathbf{B}^n)$.*

Proof. Our first observation is that the proof of Lemma 2.11 shows in fact that if $\omega\in d_\tau(\mathbf{B}^n)$, then $\tau\ge n+1$. The hypothesis on ω gives that for any $z\in\mathbf{B}^n$, any integer $j\ge 0$ and $R>0$, such that $U_\rho(z,2^jR)$ intersects $\mathbf{S}^n$,

$$\omega(U_\rho(z,2^jR))\preceq\frac{R}{(1-|z|^2)+R}2^{j\tau}\omega(U_\rho(z,R)).$$

In order to check that $\mathfrak{R}_\varepsilon\omega\in\mathcal{D}_\tau(\mathbf{B}^n)$, we fix $\delta>0$ to be chosen later on, and given $z\in\mathbf{B}^n$, an integer $j\ge 0$ and $R>0$, we will consider the following three possibilities:

(a) $2^{j-1}R\le\delta(1-|z|^2)$;

(b) $R<\delta(1-|z|^2)\le 2^{j-1}R$;

(c) $\delta(1-|z|^2)<R$.

We begin with case (a). In that case $\mathfrak{R}_\varepsilon\omega(y)\simeq\mathfrak{R}_\varepsilon\omega(z)$ for $y\in U_\rho(z,2^{j-1}R)$. Hence, by the preceding lemma,

$$\begin{aligned}\mathfrak{R}_\varepsilon\omega(U_\rho(z,2^jR))&\simeq\mathfrak{R}_\varepsilon\omega(U_\rho(z,2^{j-1}R))\simeq\mathfrak{R}_\varepsilon\omega(z)(2^jR)^n(1-|z|)\\&\simeq 2^{jn}\int_{U_\rho(z,R)}\mathfrak{R}_\varepsilon\omega(\eta)\,dv(\eta)\preceq 2^{j(\tau-1)}\mathfrak{R}_\varepsilon\omega(U_\rho(z,R)),\end{aligned}$$

where in the last inequality we have used that $\tau\ge n+1$. This shows case (a).

We consider now case (b). Let $j_0\ge 1$ be such that $2^{j_0-1}R\le\delta(1-|z|^2)<2^{j_0}R$.

Arguing as in (8), the fact that $\omega\in d_\tau(\mathbf{B}^n)$ together with $2^{j-j_0}R+(1-|z|^2)\le 2^jR+(1-|z|^2)\simeq 2^jR$, gives that

$$\int_{U_\rho(z,2^jR)}\mathfrak{R}_\varepsilon\omega\,dv\preceq\int_{U_\rho(z,C2^jR)}\omega\,dv\preceq\frac{R}{1-|z|^2}2^{(j-j_0)(\tau-1)+j}\int_{U_\rho(z,2^{j_0}R)}\omega\,dv. \tag{9}$$

Since $R\le\delta(1-|z|^2)<2^{j_0}R$, the argument established in case (a) gives that $\mathfrak{R}_\varepsilon w$ is "frozen" on $U_\rho(z,R/2)$. This observation, together with the fact that $\mathfrak{R}_\varepsilon w$ satisfies a doubling condition, gives that

$$\begin{aligned}\int_{U_\rho(z,R)} \mathfrak{R}_\varepsilon\omega(y)\,dv(y) &\simeq \int_{U_\rho(z,R/2)} \mathfrak{R}_\varepsilon\omega(y)\,dv(y)\\ &\simeq \mathfrak{R}_\varepsilon\omega(z)v\left(U_\rho\left(z,\frac{R}{2}\right)\right)\\ &\simeq \mathfrak{R}_\varepsilon\omega(z)R^n\left(\frac{R}{2}+(1-|z|^2)\right)\\ &\simeq \frac{R^n 2^{j_0}R}{(1-|z|^2)^{n+1}}\int_{U_\rho(z,2^{j_0}R)}\omega(y)\,d(y).\end{aligned}$$

Consequently, if we plug the above calculation in (9), we deduce that since $\tau\geq 1$,

$$\int_{U_\rho(z,2^jR)} \mathfrak{R}_\varepsilon\omega(y)\,dv(y) \preceq \frac{R}{1-|z|^2}2^{(j-j_0)(\tau-1)+j}\frac{(1-|z|^2)^n}{R^n}\int_{U_\rho(z,R)} \mathfrak{R}_\varepsilon\omega(y)\,dv(y).$$

In other words,

$$\frac{R}{1-|z|^2}2^{j_0(n+1-\tau)+j\tau}\int_{U_\rho(z,R)} \mathfrak{R}_\varepsilon\omega(y)\,dv(y) \preceq \frac{R}{1-|z|^2}2^{j\tau}\int_{U_\rho(z,R)} \mathfrak{R}_\varepsilon\omega(y)\,dv(y).$$

We finally have to deal with case (c), i.e. the case where $\delta(1-|z|^2)\leq R$. We have that if $y\in U_\rho(z,2^{j-1}R)$, and $x\in U_\rho(y,\delta(1-|z|^2))$, then $x\in U_\rho(z,C2^jR)$, and consequently Fubini's theorem gives that

$$\int_{U_\rho(z,2^jR)} \mathfrak{R}_\varepsilon\omega(y)\,dv(y) \preceq \int_{U_\rho(z,C2^jR)}\omega(x)\,dv(x) \preceq 2^{j\tau}\int_{U_\rho(z,R)}\omega(x)\,dv(x), \tag{10}$$

where we have used that $\omega\in d_\tau(\mathbf{B}^n)$.

On the other hand, $\mathfrak{R}_\varepsilon w$ satisfies a doubling condition. Thus, if $M>0$ is fixed, Fubini's theorem gives that

$$\begin{aligned}\int_{U_\rho(z,R)}\omega(y)\,dv(y) &\simeq \int_{U_\rho(z,\delta_1R)}\omega(y)\frac{1}{(1-|y|^2)^{n+1}}\int_{U_\rho(\eta,\delta(1-|\eta|^2))}dv(x)\,dv(y)\\ &\simeq \int_{U_\rho(z,MR)}R_\delta\omega(x)\,dv(x) \simeq \int_{U_\rho(z,R)}\mathfrak{R}_\varepsilon\omega(x)\,dv(x). \qquad \square\end{aligned}$$

Remark 2.22. Similar arguments can be used to show that if $\omega\in A_p(\mathbf{S}^n)\cap D_\tau(\mathbf{S}^n)$, then the weight $\widetilde{\omega}$ introduced in Example 2.7, is in $\mathcal{A}_p(\mathbf{B}^n)\cap\mathcal{D}_{\tau+1}(\mathbf{B}^n)$.

3. Weighted holomorphic Besov spaces

We now introduce the weighted holomorphic Besov spaces. Let ω be a $\mathcal{B}_p$-weight in $\mathbf{B}^n$, $1<p<+\infty$, and $s\geq 0$. The space $B^p_{s,k}(\omega,\mathbf{B}^n)$ is the space of holomorphic functions in $\mathbf{B}^n$ for which

$$\|f\|^p_{B^p_{s,k}(\omega,\mathbf{B}^n)}=\int_{\mathbf{B}^n}|(I+\mathcal{R})^k f(y)|^p(1-|y|^2)^{(k-s)p-1}\omega(y)\,dv(y)<+\infty,$$

where $k>s$ is an integer. In fact, the definition of the weighted holomorphic Besov spaces does not depend on the integer $k>s$. This is the object of the following result.

Theorem 3.1. *Let $1<p<+\infty$, $s\geq 0$, k_1,k_2 be integers such that $k_1>k_2>s$, and $\omega\in\mathcal{B}_p(\mathbf{B}^n)$. The following are equivalent:*

(i) $f\in B^p_{s,k_1}(\omega,\mathbf{B}^n)$;

(ii) $f\in B^p_{s,k_2}(\omega,\mathbf{B}^n)$.

Proof. We recall (see for instance Chapter 7 in [15]) that if h is a holomorphic function in $\mathbf{B}$, and $N>0$ is such that $(1-|z|^2)^N h\in L^1(dv)$, then there exists c_N such that the following reproducing formula holds for any $y\in\mathbf{B}^n$:

$$h(y)=c_N\int_{\mathbf{B}^n}\frac{(1-|z|^2)^N}{(1-\bar{z}y)^{n+N+1}}h(z)\,dv(z). \tag{11}$$

Assume that (i) holds. The fact that $f\in B^p_{s,k_1}(\omega,\mathbf{B}^n)$, means that

$$(I+\mathcal{R})^{k_1}f(y)(1-|y|^2)^{k_1-s-1/p}\in L^p(\omega\,dv).$$

In particular, there exists $p_1<p$ such that $(I+\mathcal{R})^{k_1}f(y)(1-|y|^2)^{k_1-s-1/p}$ is in $L^{p_1}(dv)$ (see [5], Lemma 2.1). Consequently, we may apply the reproducing formula (11) for $N>0$ big enough to the function $(I+\mathcal{R})^{k_1}f$ and obtain

$$(I+\mathcal{R})^{k_1}f(y)=c_N\int_{\mathbf{B}^n}(I+\mathcal{R})^{k_1}f(z)\frac{(1-|z|^2)^N}{(1-\bar{z}y)^{n+1+N}}\,dv(z).$$

Let us recall that the operator $(I+\mathcal{R})^m$ is invertible and that the operator $(I+\mathcal{R})^{-m}$ has the following integral representation (see for instance [13])

$$(I+\mathcal{R})^{-m}g(z)=\frac{1}{\Gamma(m)}\int_0^1\left(\log\frac{1}{r}\right)^{m-1}g(rz)\,dr.$$

We then have

$$\left|(I+\mathcal{R})^{k_2} f(y)\right| = \left|c_N \int_{\mathbf{B}^n} (I+\mathcal{R})^{k_1} f(z)\left(I+\sum_{i=1}^n y_i \frac{\partial}{\partial y_i}\right)^{k_2-k_1} \frac{(1-|z|^2)^N}{(1-\bar{z}y)^{n+1+N}}\, dv(z)\right|$$

$$\preceq \int_{\mathbf{B}^n} \frac{|(I+\mathcal{R})^{k_1} f(z)|(1-|z|^2)^N}{|1-\bar{z}y|^{n+1+N+k_2-k_1}}\, dv(z).$$

Using that by duality

$$\|f\|_{L^p(\omega\, dv)} = \sup_{\|\psi\|_{L^{p'}(w^{-(p'-1)}dv)}\le 1} \left|\int_{\mathbf{B}^n} f\psi\, dv\right|,$$

we have that

$$\|f\|_{B^p_{s,k_2}(\omega,\mathbf{B}^n)} = \sup_{\substack{\|\psi\|_{L^{p'}(\omega^{-(p'-1)}dv)}\le 1\\ \psi\ge 0}} \left|\int_{\mathbf{B}^n} (I+\mathcal{R})^{k_2} f(y)(1-|y|^2)^{k_2-s-1/p}\psi(y)\, dv(y)\right|$$

$$\preceq \sup_{\substack{\|\psi\|_{L^{p'}(\omega^{-(p'-1)}\,dv)}\le 1\\ \psi\ge 0}} \int_{\mathbf{B}^n}\int_{\mathbf{B}^n} \frac{|(I+\mathcal{R})^{k_1} f(z)|}{|1-\bar{z}y|^{n+1+N+k_2-k_1}}$$

$$\times (1-|z|^2)^N (1-|y|^2)^{k_2-s-1/p}\psi(y)\, dv(y)\, dv(z). \tag{12}$$

We now check that the mapping

$$\psi \longmapsto \int_{\mathbf{B}^n} \frac{(1-|z|^2)^{N-(k_1-s-1/p)}(1-|y|^2)^{k_2-s-1/p}}{|1-\bar{z}y|^{n+1+N+k_2-k_1}}\psi(y)\, dv(y),$$

is bounded from $L^{p'}(\omega^{-(p'-1)}dv)$ to itself.

Since

$$\frac{(1-|z|^2)^{N-(k_1-s-1/p)}(1-|y|^2)^{k_2-s-1/p}}{|1-\bar{z}y|^{n+1+N+k_2-k_1}} \preceq \frac{(1-|y|^2)^{k_2-s-1/p}}{|1-\bar{z}y|^{n+1+k_2-s-1/p}},$$

$k_2-s-1/p>-1$, and $\omega^{-(p'-1)}\in\mathcal{B}_{p'}(\mathbf{B}^n)$, this is a consequence of Proposition 2 in [4].

Consequently, from (12) and the above observations we have that

$$\|f\|_{B^p_{s,k_2}(\omega,\mathbf{B}^n)} \preceq \|f\|_{B^p_{s,k_1}(\omega,\mathbf{B}^n)} \sup_{\|\psi\|_{L^{p'}(\omega\, dv)}\le 1} \|\psi\|_{L^{p'}(\omega\, dv)} = \|f\|_{B^p_{s,k_1}(\omega,\mathbf{B}^n)}.$$

The other implication is proved in a similar way. □

By Theorem 3.1, the spaces $B^p_{s,k}(\omega,\mathbf{B}^n)$ do not depend on $k>s$, and from now on we will denote them simply by $B^p_s(\omega,\mathbf{B}^n)$.

Our next goal is to study the relations between the weighted Besov spaces in $\mathbf{B}^n$ and the weighted Hardy–Sobolev spaces with respect to the lifted weight in $\mathbf{S}^{n+1}$. These relations strongly rely on the boundedness of Bergman type operators on weighted L^p spaces obtained in [4]. We will show (see Corollary 3.7) that the restriction operator maps $H^p_{s+1/p}(\omega_l, \mathbf{B}^{n+1})$ onto $B^p_s(\omega, \mathbf{B}^n)$.

Definition 3.2. Let $1<p<+\infty$, $1\le q\le+\infty$, $s\ge 0$, and let ω be a weight in $\mathbf{S}^n$. The weighted holomorphic *Triebel–Lizorkin space* $HF^{pq}_s(\omega, \mathbf{B}^n)$ when $q<+\infty$, is the space of holomorphic functions f in $\mathbf{B}^n$ for which

$$\|f\|_{HF^{pq}_s(\omega,\mathbf{B}^n)} = \left(\int_{\mathbf{S}^n}\left(\int_0^1 |((I+\mathcal{R})^k f)(r\zeta)|^q(1-r^2)^{(k-s)q-1}\,dr\right)^{p/q}\omega(\zeta)\,d\sigma(\zeta)\right)^{1/p} < +\infty,$$

where $k\in(s,+\infty)$, whereas when $q=+\infty$,

$$\|f\|_{HF^{p\infty}_s(w,\mathbf{B}^n)} = \left(\int_{\mathbf{S}^n}\left(\sup_{0<r<1}|((I+\mathcal{R})^k f)(r\zeta)|(1-r^2)^{k-s}\right)^p\omega(\zeta)\,d\sigma(\zeta)\right)^{1/p} < +\infty.$$

As happens in the unweighted case, the definition does not depend on the integer $k>s$ (see [5]).

The proof of the following lemma can also be found in [5].

Lemma 3.3. *Let* $1<p<+\infty$ *and* $\theta\in A_p(\mathbf{S}^n)$. *We then have*:
(a) $HF^{p,2}_s(\theta, \mathbf{B}^n)=H^p_s(\theta, \mathbf{B}^n)$;
(b) *If* $q_0\le q_1\le+\infty$, *then* $HF^{pq_0}_s(\theta, \mathbf{B}^n)\subset HF^{pq_1}_s(\theta, \mathbf{B}^n)$.

We will also need the following lemma.

Lemma 3.4. *Let* $\omega\in\mathcal{A}_p(\mathbf{B}^n)$. *If* $M>0$, *let* P_M *be the operator given by*

$$P_M(\psi)(\zeta) = \int_{\mathbf{B}^n}\frac{\psi(y)(1-|y|^2)^M}{|1-\zeta\overline{y}|^{n+1+M}}\omega(y)\,dv(y)$$

for $\psi\in L^{p'}(\omega\,dv)$ *and* $\zeta\in\overline{\mathbf{B}}^{n+1}$. *We then have that* P_M *is bounded as an operator from* $L^{p'}(\omega\,dv)$ *in* $\mathbf{B}^n$ *to* $L^{p'}({\omega_l}^{-(p'-1)})$ *in* $\mathbf{S}^{n+1}$.

Proof. Observe that for every $\zeta'\in\mathbf{B}^n$, $P_M(\psi)$ is constant on $\Pi^{-1}(\zeta')$. Consequently,

$$\begin{aligned}\|P_M(\psi)\|^{p'}_{L^{p'}({\omega_l}^{-(p'-1)}d\sigma)} &= \int_{\mathbf{S}^{n+1}} P_M(\psi)^{p'}(\zeta){\omega_l}^{-(p'-1)}(\zeta)\,d\sigma(\zeta)\\ &= \int_{\mathbf{B}^n} P_M(\psi)^{p'}(\zeta')\omega^{-(p'-1)}(\zeta')\,dv(\zeta')\\ &= \|P_M(\psi)\|^{p'}_{L^{p'}(\omega^{-(p'-1)}dv)}.\end{aligned}$$

Hence, we just have to show that the operator defined by

$$Q_M(\psi_1)(z)=\int_{\mathbf{B}^n}\frac{\psi_1(y)(1-|y|^2)^M}{|1-z\overline{y}|^{n+1+M}}\,dv(y)$$

is bounded from $L^{p'}(\omega^{-(p'-1)}\,dv)$ to itself. And this estimate is a consequence of Proposition 3 in [4]. □

We next prove a weighted restriction theorem.

Theorem 3.5. *Let* $1<p<+\infty$, $1\leq q<+\infty$, $\omega\in\mathcal{A}_p(\mathbf{B}^n)$, *and* $s\geq 0$. *Then the restriction operator maps* $HF^{p,q}_{s+1/p}(\omega_l,\mathbf{B}^{n+1})$ *to* $B^p_s(\omega,\mathbf{B}^n)$.

Proof. Since by Lemmas 2.6 and 3.3 if $q_0\leq q_1\leq+\infty$,

$$HF^{pq_0}_{s+1/p}(\omega_l,\mathbf{B}^{n+1})\subset HF^{pq_1}_{s+1/p}(\omega_l,\mathbf{B}^{n+1}),$$

it is enough to show that $HF^{p\infty}_{s+1/p}(\omega_l,\mathbf{B}^{n+1})|_{\mathbf{B}^n}\subset B^p_s(\omega,\mathbf{B}^n)$.

Let $f\in HF^{p\infty}_{s+1/p}(\omega_l,\mathbf{B}^{n+1})$, and $k>s+1/p$. We have that if $N>0$ is chosen big enough, the reproducing formula (11) gives that

$$(I+\mathcal{R})^k f(y)=C\int_{\mathbf{B}^{n+1}}(I+\mathcal{R})^k f(z)\frac{(1-|z|^2)^N}{(1-\bar{z}y)^{n+2+N}}\,dv(z).$$

Hence,

$$\|f\|^p_{B^p_s(\omega,\mathbf{B}^n)}$$
$$\preceq\int_{\mathbf{B}^n}\left|\int_{\mathbf{B}^{n+1}}(I+\mathcal{R})^k f(z)\frac{(1-|z|^2)^N}{|1-z\overline{y}|^{n+2+N}}\,dv(z)\right|^p(1-|y|^2)^{(k-s)p-1}\omega(y)\,dv(y).$$

Since by duality of $L^p(\omega\,dv)$,

$$\|f\|_{L^p(\omega\,dv)}=\sup_{\|\psi\|_{L^{p'}(\omega\,dv)}\leq 1}\left|\int_{\mathbf{B}^n}f\psi\omega\,dv\right|,$$

we have that

$$\|f\|^p_{B^p_s(\omega,\mathbf{B}^n)}\leq\sup_{\substack{\|\psi\|_{L^{p'}(\omega\,dv)}\leq 1\\ \psi\geq 0}}\int_{\mathbf{B}^n}\int_{\mathbf{B}^{n+1}}|(I+\mathcal{R})^k f(z)|\frac{(1-|z|^2)^N}{|1-z\overline{y}|^{n+2+N}}\,dv(z)$$
$$\times(1-|y|^2)^{k-s-1/p}\psi(y)\omega(y)\,dv(y).$$

Using polar coordinates in the z-variable, we obtain that

$$\|f\|^p_{B^p_s(\omega,\mathbf{B}^n)} \preceq \sup_{\substack{\|\psi\|_{L^{p'}(\omega\,dv)}\le 1\\ \psi\ge 0}} \int_{\mathbf{S}^{n+1}}\int_{\mathbf{B}^n} \sup_{r<1}(|(I+\mathcal{R})^k f(r\zeta)|(1-r^2)^{k-s-1/p})$$
$$\times\left(\int_0^1 \frac{(1-r^2)^{N+s+1/p-k}}{|1-r\zeta\overline{y}|^{n+2+N}}\,dr\right)\psi(y)(1-|y|^2)^{k-s-1/p}\omega(y)\,dv(y)\,d\sigma(\zeta).$$

Since we have that

$$\int_0^1 \frac{(1-r^2)^{N+s+1/p-k}}{|1-r\zeta\overline{y}|^{n+2+N}}\,dr \preceq \frac{1}{|1-\zeta\overline{y}|^{n+1+k-s-1/p}},$$

the above is bounded up to a constant by

$$\int_{\mathbf{S}^{n+1}} \sup_{r<1}(|(I+\mathcal{R})^k f(r\zeta)|(1-r^2)^{k-s-1/p})P_{k-s-1/p}(\psi)(\zeta)\,d\sigma(\zeta).$$

Applying Hölder's inequality, Lemma 3.4 gives that

$$\|f\|^p_{B^p_s(\omega,\mathbf{B}^n)}$$
$$\preceq \sup_{\substack{\|\psi\|_{L^{p'}(\omega\,dv)}\le 1\\ \psi\ge 0}} \int_{\mathbf{S}^{n+1}} \sup_{r<1}(|(I+\mathcal{R})^k f(r\zeta)|(1-r^2)^{k-s-1/p})$$
$$\times P_{k-s-1/p}(\psi)(\zeta)\omega_l^{1/p}(\zeta)\omega_l^{-1/p}(\zeta)\,d\sigma(\zeta)$$
$$\le \sup_{\substack{\|\psi\|_{L^{p'}(\omega\,dv)}\le 1\\ \psi\ge 0}} \left(\int_{\mathbf{S}^{n+1}} \sup_{r<1}(|(I+\mathcal{R})^k f(r\zeta)|(1-r^2)^{k-s-1/p})^p\omega_l(\zeta)\,d\sigma(\zeta)\right)^{1/p}$$
$$\times\left(\int_{\mathbf{S}^{n+1}} P_{k-s-1/p}(\psi)(\zeta)^{p'}\omega_l^{-(p'-1)}(\zeta)\,d\sigma(\zeta)\right)^{1/p'}$$
$$\preceq \|f\|_{HF^{p\infty}_{s+1/p}(\omega_l,\mathbf{B}^{n+1})}. \qquad \square$$

Now we prove an extension theorem for weighted holomorphic Besov spaces.

Theorem 3.6. *Let* $1<p<+\infty$, $s\ge 0$, *and* $\omega\in\mathcal{A}_p(\mathbf{B}^n)$. *We then have that the extension operator* $f\mapsto f_l$, *where* $f_l(z',z_{n+1})=f(z')$, *if* $(z',z_{n+1})\in\mathbf{S}^{n+1}$, $z'\in\mathbf{B}^n$, *maps* $B^p_s(\omega,\mathbf{B}^n)$ *boundedly to* $HF^{p,1}_{s+1/p}(\omega_l,\mathbf{B}^{n+1})$.

Proof. Let $f\in H(\mathbf{B}^n)$. If we denote by $\mathcal{R}_l$ the radial derivative in $\mathbf{B}^{n+1}$, then we have

$$(I+\mathcal{R}_l)f_l(z',z_{n+1})=\left(I+\sum_{i=1}^{n} z_i\frac{\partial}{\partial z_i}+z_{n+1}\frac{\partial}{\partial z_{n+1}}\right)f_l(z',z_{n+1})=(I+\mathcal{R})f(z'),$$

i.e., $(I+\mathcal{R}_l)f_l=((I+\mathcal{R})f)_l$. Consequently, if $k>s+1/p$, and N is chosen big enough, duality and the representation formula give

$$\begin{aligned}
&\|f_l\|_{HF^{p,1}_{s+1/p}(\omega_l,\mathbf{B}^{n+1})}\\
&\quad=\left(\int_{\mathbf{S}^{n+1}}\left|\int_0^1((I+\mathcal{R})^k f)_l(r\zeta)(1-r)^{k-s-1/p-1}\,dr\right|^p\omega_l(\zeta)\,d\sigma(\zeta)\right)^{1/p}\\
&\quad\simeq\left(\int_{\mathbf{B}^n}\left|\int_0^1(I+\mathcal{R})^k f(r\zeta')(1-r)^{k-s-1/p-1}\,dr\right|^p\omega(\zeta')\,dv(\zeta')\right)^{1/p}\\
&\quad=\sup_{\|\psi\|_{L^{p'}(\omega\,dv)}\le 1}\left|\int_{\mathbf{B}^n}\left|\int_0^1(I+\mathcal{R})^k f(r\zeta')(1-r)^{k-s-1/p-1}\,dr\right|\psi(\zeta')\omega(\zeta')\,dv(\zeta')\right|\\
&\quad\preceq\sup_{\|\psi\|_{L^{p'}(\omega\,dv)}\le 1}\int_{\mathbf{B}^n}\int_{\mathbf{B}^n}\int_0^1\frac{|(I+\mathcal{R})^k f(z)|}{|1-\bar{z}r\zeta'|^{n+1+N}}\\
&\qquad\times(1-|z|^2)^N(1-r)^{k-s-1/p-1}\,dr\,dv(z)|\psi(\zeta')|\omega(\zeta')\,dv(\zeta')\\
&\quad\preceq\sup_{\|\psi\|_{L^{p'}(\omega\,dv)}\preceq 1}\int_{\mathbf{B}^n}\int_{\mathbf{B}^n}\frac{|(I+\mathcal{R})^k f(z)|(1-|z|^2)^N}{|1-\bar{z}\zeta'|^{n+1+N-(k-s-1/p)}}\,dv(z)|\psi(\zeta')|\omega(\zeta')\,dv(\zeta').
\end{aligned}$$

The theorem finishes with an analogous argument to the one used in the restriction theorem. □

As an immediate consequence of the above two theorems and Lemma 3.3, we obtain the following corollary.

Corollary 3.7. *Let $1<p<+\infty$, $s\ge 0$, and $\omega\in\mathcal{A}_p(\mathbf{B}^n)$. Then the restriction operator from $H^p_{s+1/p}(\omega_l,\mathbf{B}^{n+1})$ to $B^p_s(\omega,\mathbf{B}^n)$ is onto.*

The above restriction and extension theorems permit us to reformulate the fact that a measure μ is Carleson for $B^p_s(\omega,\mathbf{B}^n)$ in terms of the lifted weight ω_l and the lifted measure μ_l.

Corollary 3.8. *Let $1<p<+\infty$, $s\ge 0$, and $\omega\in\mathcal{A}_p(\mathbf{B}^n)$. We then have that the following assertions are equivalent:*

(i) *There exists* $C>0$ *such that for any* $f\in B^p_s(\omega,\mathbf{B}^n)$,

$$\|f\|_{L^p(d\mu)}\le C\|f\|_{B^p_s(\omega,\mathbf{B}^n)};$$

(ii) *There exists* $C>0$ *such that for any* $f\in H^p_{s+1/p}(\omega_l,\mathbf{B}^{n+1})$,

$$\|f\|_{L^p(d\mu_l)}\le C\|f\|_{H^p_{s+1/p}(\omega_l,\mathbf{B}^{n+1})}, \tag{13}$$

where μ_l *is the measure on* $\mathbf{B}^{n+1}$ *defined by* $\int_{\mathbf{B}^{n+1}} f\,d\mu_l=\int_{\mathbf{B}^n} f(z',0)\,d\mu(z')$.

Proof. Let us show that (i) implies (13). If $f\in H^p_{s+1/p}(\omega_l,\mathbf{B}^{n+1})$, Theorem 3.5 gives that $f|_{B^n}\in B^p_s(\omega,\mathbf{B}^n)$, and that $\|f|_{B^n}\|_{B^p_s(\omega,\mathbf{B}^n)}\le C\|f\|_{H^p_{s+1/p}(\omega_l,\mathbf{B}^{n+1})}$. Since we are assuming that (i) holds, we then have that $\|f|_{B^n}\|_{L^p(d\mu)}\le C\|f|_{B^n}\|_{B^p_s(\omega,\mathbf{B}^n)}\le C\|f\|_{H^p_{s+1/p}(\omega_l,\mathbf{B}^{n+1})}$. Since $\|f|_{B^n}\|^p_{L^p(d\mu)}=\|f\|^p_{L^p(d\mu_l)}$, we are done.

If (13) holds and $f\in B^p_s(\omega,\mathbf{B}^n)$, the extension Theorem 3.6 gives that $f_l\in H^p_{s+1/p}(\omega_l,\mathbf{B}^{n+1})$, with $\|f_l\|_{H^p_{s+1/p}(\omega_l,\mathbf{B}^{n+1})}\le C\|f\|_{B^p_s(\omega,\mathbf{B}^n)}$. The hypothesis on ω gives then that

$$\|f_l\|_{L^p(d\mu_l)}\le C\|f_l\|_{H^p_{s+1/p}(\omega_l,\mathbf{B}^{n+1})}\le C\|f\|_{B^p_s(\omega,\mathbf{B}^n)}.$$

Since, $\|f_l\|^p_{L^p(d\mu_l)}=\|f\|_{L^p(d\mu)}$, we get (i). $\square$

We finish the subsection with a result that shows that the weighted Besov space associated with a weight in $B^p_s(\omega,\mathbf{B}^n)$ coincides with the corresponding weighted Besov space of its regularization. In particular, we deduce that in the definition of the spaces $B^p_s(\omega,\mathbf{B}^n)$ we can assume, without loss of generality, that $\omega\in\mathcal{A}_p(\mathbf{B}^n)$.

Proposition 3.9. *Let* $1<p<+\infty$, $s\ge 0$, *and assume that* ω *is a weight in* $\mathcal{B}_p(\mathbf{B}^n)$. *Then the spaces* $B^p_s(\omega,\mathbf{B}^n)$ *and* $B^p_s(\mathfrak{R}_\varepsilon\omega,\mathbf{B}^n)$ *coincide.*

Proof. Assume that $\varepsilon<1$ and $z\in\mathbf{B}^n$, and let $k>s$. The fact that $(I+\mathcal{R})^k f$ is holomorphic in $\mathbf{B}^n$ gives easily that

$$|(I+\mathcal{R})^k f(y)|\preceq\frac{1}{|U_\varepsilon(y)|}\int_{U_\varepsilon(y)}|(I+\mathcal{R})^k f(z)|\,dv(z).$$

On the other hand, $1-|y|^2\simeq 1-|z|^2$ for any $y\in U_\varepsilon(z)$. Hence

$$(1-|y|^2)^{((k-s)p-1)/p}|(I+\mathcal{R})^k f(y)|$$
$$\preceq\frac{1}{|U_\varepsilon(y)|}\int_{U_\varepsilon(y)}(1-|z|^2)^{((k-s)p-1)/p}|(I+\mathcal{R})^k f(z)|\,dv(z).$$

Consequently,

$$\begin{aligned}\|f\|^p_{B^p_s(\mathfrak{R}_\varepsilon\omega)} &= \int_{\mathbf{B}^n} |(I+\mathcal{R})^k f(y)|^p (1-|y|^2)^{(k-s)p-1}\mathfrak{R}_\varepsilon\omega(y)\,dv(y)\\ &\preceq \int_{\mathbf{B}^n}\left[\frac{1}{(1-|y|^2)^{n+1}}\int_{U_\varepsilon(y)}(1-|z|^2)^{((k-s)p-1)/p}|(I+\mathcal{R})^k f(z)|\,dv(z)\right]^p\\ &\quad\times\mathfrak{R}_\varepsilon\omega(y)\,dv(y)\\ &\simeq \int_{\mathbf{B}^n}(\mathfrak{R}_\varepsilon((1-|z|^2)^{((k-s)p-1)/p}|(I+\mathcal{R})^k f(z)|)(y))^p\mathfrak{R}_\varepsilon\omega(y)\,dv(y).\end{aligned}$$

Since $\omega\in\mathcal{B}_p(\mathbf{B}^n)$ and $\int_{\mathbf{B}^n}(\mathfrak{R}_\varepsilon g)^p\mathfrak{R}_\varepsilon\omega\,dv\preceq\int_{\mathbf{B}^n} g^p\omega\,dv$ for any $g\geq 0$ (the proof is analogous to Lemma 9 in [4]), we deduce that the above is bounded by

$$\int_{\mathbf{B}^n}((1-|y|^2)^{((k-s)p-1)/p}|(I+\mathcal{R})^k f(y)|)^p\omega(y)\,dv(y)\simeq\|f\|^p_{B^p_s(w)}.$$

On the other hand, the fact that $|(I+\mathcal{R})^k f|^p$ is plurisubharmonic gives that

$$|(I+\mathcal{R})^k f(y)|^p\preceq\frac{1}{|U_\varepsilon(y)|}\int_{U_\varepsilon(y)}|(I+\mathcal{R})^k f(z)|^p\,dv(z).$$

Since $1-|y|^2\simeq 1-|z|^2$ in $U_\varepsilon(z)$, we have

$$\begin{aligned}\|f\|^p_{B^p_s(\omega,\mathbf{B}^n)} &= \int_{\mathbf{B}^n}|(I+\mathcal{R})^k f(y)|^p(1-|y|^2)^{(k-s)p-1}\omega(y)\,dv(y)\\ &\preceq\int_{\mathbf{B}^n}(\mathfrak{R}_\varepsilon((1-|y|^2)^{((k-s)p-1)/p}(I+\mathcal{R})^k f)^p(y))\omega(y)\,dv(y).\end{aligned}$$

Fubini's theorem gives that there exists $\varepsilon'>0$ such that for any $f,g\geq 0$,

$$\int_{\mathbf{B}^n} f\mathfrak{R}_\varepsilon g\,dv\preceq\int_{\mathbf{B}^n} gR_{\varepsilon'}f\,dv.$$

Hence, the above is bounded by

$$\int_{\mathbf{B}^n}(1-|y|^2)^{(k-s)p-1}|(I+\mathcal{R})^k f(y)|^p R_{\varepsilon'}\omega(y)\,dv(y)\simeq\|f\|^p_{B^p_s(\mathfrak{R}_\varepsilon\omega)}.\qquad\square$$

4. Carleson measures for $B_s^p(\omega,\mathrm{B}^n)$, $w\in\mathcal{A}_p(\mathrm{B}^n)\cap\mathcal{D}_\tau(\mathrm{B}^n)$

In [5], it is shown that the Carleson measures for a weighted Hardy–Sobolev space $H_s^p(\omega,\mathbf{B}^{n+1})$ for some range of s and for a class of weights ω in $\mathbf{S}^{n+1}$ coincide with the Carleson measures for a nonisotropic weighted potential space. This potential space is the image of the space $L^p(\omega)$ under the operator K_s defined by

$$K_s[f](z)=\int_{\mathbf{S}^{n+1}}\frac{f(\zeta)}{|1-z\overline{\zeta}|^{n+1-s}}\,d\sigma(\zeta).$$

This last problem has been thoroughly studied, and there are different types of characterizations (see for instance [1] and [16]).

The main result in this section shows that we can reduce the study of Carleson measures for weighted holomorphic Besov spaces to the study of the boundedness of an operator with positive kernel.

Theorem 4.1. *Let $1<p<+\infty$, $s\geq0$, ω be a weight in $\mathcal{A}_p(\mathbf{B}^n)\cap\mathcal{D}_\tau(\mathbf{B}^n)$, and $0<\tau-p(s+1/p)<1$. Let μ be a positive Borel measure on $\mathbf{B}^n$. We then have that the following assertions are equivalent*:

(i) *There exists $C>0$ such that for any $f\in B_s^p(\omega,\mathbf{B}^n)$,*

$$\|f\|_{L^p(d\mu)}\leq C\|f\|_{B_s^p(\omega,\mathbf{B}^n)};$$

(ii) *There exists $C>0$ such that for any $f\in L^p(\omega\,dv)$,*

$$\left\|\int_{\mathbf{B}^n}\frac{f(y)}{(1-z\overline{y})^{n+1-(s+1/p)}}\,dv(y)\right\|_{L^p(d\mu)}\leq C\|f\|_{L^p(\omega\,dv)};$$

(iii) *There exists $C>0$ such that for any $f\in L^p(\omega\,dv)$,*

$$\left\|\int_{\mathbf{B}^n}\frac{f(y)}{|1-z\overline{y}|^{n+1-(s+1/p)}}\,dv(y)\right\|_{L^p(d\mu)}\leq C\|f\|_{L^p(\omega\,dv)}.$$

Proof. We recall that we have already observed (Lemmas 2.6 and 2.10) that if $\omega\in\mathcal{A}_p(\mathbf{B}^n)\cap\mathcal{D}_\tau(\mathbf{B}^n)$, then the lifted weight w_l is in $A_p(\mathbf{S}^{n+1})\cap D_\tau(\mathbf{S}^{n+1})$. In particular, $\omega_l\in A_p(\mathbf{S}^{n+1})$, and by the observation given after Definition 2.13, $\tau\leq p(n+1)$. By hypothesis $0<\tau-p(s+1/p)$, thus we also have that $0<p(n+1)-p(s+1/p)$, and consequently that $n+1-(s+1/p)>0$. That is, the exponent that appears in conditions (ii) and (iii) is strictly bigger than zero.

First of all, observe that by Corollary 3.8, (i) is equivalent to

$$\|f\|_{L^p(d\mu_l)}\leq C\|f\|_{H^p_{s+1/p}(\omega_l,\mathbf{B}^{n+1})}.$$

Next, Theorem 2.13 in [5] gives that this can be rewritten as

$$\left\| \int_{\mathbf{S}^{n+1}} \frac{f(\zeta)}{(1-z\overline{\zeta})^{n+1-(s+1/p)}}\, d\sigma(\zeta) \right\|_{L^p(d\mu_l)} \leq C\|f\|_{L^p(\omega_l\, dv)}. \tag{14}$$

Let us check that (14) is equivalent to

$$\left\| \int_{\mathbf{B}^n} \frac{f(y)}{(1-z\overline{y})^{n+1-(s+1/p)}}\, dv(y) \right\|_{L^p(d\mu)} \leq C\|f\|_{L^p(\omega\, dv)} \tag{15}$$

for any $f\in L^p(\omega\, dv)$. Indeed, assume first that (14) holds. We then have that $f_l\in L^p(\omega_l\, dv)$, and $\|f_l\|_{L^p(\omega_l\, dv)}\simeq\|f\|_{L^p(\omega\, dv)}$. Moreover, if $z\in\mathbf{B}^n$, $z=(z',0)$, and if for any $\zeta\in\mathbf{S}^{n+1}$, $\zeta=(\zeta',\zeta_{n+1})$, $\zeta'\in\mathbf{B}^n$,

$$\int_{\mathbf{S}^{n+1}} \frac{f_l(\zeta)}{(1-z\overline{\zeta'})^{n+1-(s+1/p)}}\, d\sigma(\zeta) = C\int_{\mathbf{B}^n} \frac{f(\zeta')}{(1-z\overline{\zeta'})^{n+1-(s+1/p)}}\, dv(\zeta'),$$

and consequently, we obtain (15).

Assume now that (15) holds, and let $f\in L^p(\omega_l\, dv)$. Then the function

$$\tilde{f}(y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y, e^{i\theta}(1-|y|^2)^{1/2})\, d\theta \quad \text{for } y\in\mathbf{B}^n$$

is in $L^p(\omega\, dv)$ (just applying Hölder's inequality) and moreover,

$$\|\tilde{f}\|_{L^p(\omega\, dv)} \leq C\|f\|_{L^p(\omega_l\, dv)}.$$

In addition, if $z\in\mathbf{B}^{n+1}$, $z=(z',0)$,

$$\begin{aligned}\int_{\mathbf{B}^n} \frac{\tilde{f}(y)}{(1-z\overline{y})^{n+1-(s+1/p)}}\, dv(y) &= C\int_{\mathbf{B}^n} \frac{\int_{-\pi}^{\pi} f(y, e^{i\theta}(1-|y|^2)^{1/2})\, d\theta}{(1-z\overline{y})^{n+1-(s+1/p)}}\, dv(y) \\ &= C\int_{\mathbf{S}^{n+1}} \frac{f(\zeta)}{(1-z\overline{\zeta})^{n+1-(s+1/p)}}\, d\sigma(\zeta).\end{aligned}$$

And that proves that (i) is equivalent to (ii).

Next, the hypothesis give that $\tau-p(s+1/p)<1$. Since we have observed that the lifted weight $\omega_l\in D_\tau(\mathbf{S}^{n+1})$, Theorem 3.5 in [5] gives that (13) holds if and only if for any $f\in L^p(\omega_l\, dv)$, $f\geq 0$,

$$\left\| \int_{\mathbf{S}^{n+1}} \frac{f(\zeta)}{|1-z\overline{\zeta}|^{n+1-(s+1/p)}}\, d\sigma(\zeta) \right\|_{L^p(d\mu_l)} \leq C\|f\|_{L^p(\omega_l\, dv)}. \tag{16}$$

The same argument used for the holomorphic potential gives that (16) can be rewritten as (iii). □

Remark 4.2. The above theorem cannot be extended in general to the case where $\tau-p(s+1/p)>1$, even in the unweighted case, as is shown in [5].

Remark 4.3. If $0<n+1-(s+1/p)\leq 1$, and $\omega\in\mathcal{A}_p(\mathbf{B}^n)$, the theorem can be obtained directly. The equivalence of (i) and (ii) is proved as before, whereas the equivalence for (ii) and (iii) for the case $n+1-(s+1/p)<1$ is trivial. The equivalence of (ii) and (iii) in the case $n+1-(s+1/p)=1$, can be reduced to the previous one, using that $B^p_s(\omega,\mathbf{B}^n)=B_{s+\varepsilon/p}((1-|z|)^\varepsilon\omega)$, and that if $\omega\in\mathcal{A}_p(\mathbf{B}^n)$, there exists ε_0 such that for any $\varepsilon<\varepsilon_0$, $(1-|z|)^\varepsilon\omega\in\mathcal{A}_p(\mathbf{B}^n)$.

Finally, observe that Theorem 4.1 together with Propositions 2.19 and 2.21 permits one to extend the characterization of the Carleson measures obtained in Theorem 4.1 to weighted Besov spaces with respect to weights $\omega\in\mathcal{B}_p(\mathbf{B}^n)\cap d_\tau(\mathbf{B}^n)$, where $0<\tau-p(s+1/p)<1$, replacing in the statement of this theorem the weight ω by its regularization $\mathcal{R}_\varepsilon(\omega)$.

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