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On hypoellipticity of generators of Lévy processes

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Abstract. We give a sufficient condition on a Lévy measure μ which ensures that the generator L of the corresponding pure jump Lévy process is (locally) hypoelliptic, i.e., $\operatorname{sing\,supp} u \subseteq \operatorname{sing\,supp} Lu$ for all admissible u. In particular, we assume that $\mu|_{\mathbb{R}^d \setminus \{0\}} \in C^\infty(\mathbb{R}^d \setminus \{0\})$. We also show that this condition is necessary provided that $\operatorname{supp} \mu$ is compact.

1. Introduction

Hypoellipticity of elliptic partial differential operators or, more generally, pseudodifferential operators is one of the classical topics in the theory of partial differential equations. Briefly, an operator L is called *hypoelliptic* if the singular support of u is contained in the singular support of Lu for all u in the domain of L. In particular, hypoellipticity comprises the C^∞-regularity of functions on their domains of L-harmonicity where we call $u\colon \mathbb{R}^d \to \mathbb{R}$ *L-harmonic* on Ω if $Lu=0$ on Ω.

This analytic notion has, if L generates a strong Markov process $(X_t)_{t\geq 0}$ in an appropriate way, a probabilistic counterpart. More precisely, a bounded measurable function u is said to be harmonic on Ω with respect to $(X_t)_{t\geq 0}$ if $u(X_{t\wedge\tau_\Omega})$ is, for all $x\in\mathbb{R}^d$, a local $\mathbb{P}^x$-martingale. Here τ_Ω denotes the first exit time of Ω and $\mathbb{P}^x$ is the probability measure under which the process starts in x, i.e., $X_0=x$ a.s. If $(X_t)_{t\geq 0}$ is a Brownian motion, this yields the mean-value property of classical harmonic functions. In fact, harmonicity with respect to a reasonable Markov process can always be defined by a generalized mean-value property, see for example [6]. Functions harmonic in this sense play an important role in the potential theory of Markov processes. This motivates an increasing interest for example in questions of regularity of these operators by probabilists. Since, by the theorem of Courrège [7],

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generators of strong Markov processes are pseudodifferential operators with continuous negative definite symbols as described for example in [14] or [8], generally they do not fit in the framework of classical symbol classes, for example the Hörmander class $S^m_{1,0}$.

Regularity of functions which are harmonic with respect to jump processes has been an object of intense studies in the last years. Let us mention here for example [2], [4], [9], [11], [15], [16], [18] and [19]. Most of these papers deal with a-priori continuity estimates in the broader context of processes with space-dependent jump measures. They rely on a delicate interplay between lower and upper bounds on the jump measures, i.e., they deal with fixed order operators where the small jumps are in principle comparable to those of a rotationally symmetric α-stable Lévy process. The variable order case is far more difficult as it is for example emphasized by a counterexample in [1].

In the present paper we concentrate on stochastic processes with stationary and independent increments, so-called Lévy processes. Their generators are translation-invariant and map $C_0^\infty(\mathbb{R}^d)$ continuously to $C^\infty(\mathbb{R}^d)$. Hence they act as convolution with a distribution. We show that in this case essentially smoothness away from the origin and a lower bound on the Lévy measure are enough to yield smoothness of harmonic functions.

Let us give a precise formulation of our results: Let ν be a Lévy measure, i.e., a nonnegative Borel measure on $\mathbb{R}^d$ such that $\nu(\{0\})=0$ and $\int_{\mathbb{R}^d}\min(1,|h|^2)\,\nu(dh)<\infty$. Moreover, we assume that ν is absolutely continuous with respect to the Lebesgue measure with a density n satisfying the following assumptions:

(A1) $n\in C^\infty(\mathbb{R}^d\setminus\{0\})$ and $n|_{\mathbb{R}^d\setminus B_1(0)}\in H^\infty(\mathbb{R}^d\setminus B_1(0))$.

(A2) There exist $r_0, c>0$ and $\alpha\in(0,2)$ so that for all $\omega\in S^{d-1}=\{x\in\mathbb{R}^d:|x|=1\}$ and $0<r\le r_0$:

$$\int_{|h|\le r}|h\cdot\omega|^2\,\nu(dh)\ge cr^{2-\alpha}. \tag{1}$$

(A1) ensures that n is smooth on $\mathbb{R}^d\setminus\{0\}$ with all its derivatives square-integrable away from 0. Note also, that (A2) only assumes a lower bound on the growth of ν near 0. For example, (A2) holds if $n(h)\ge c|h|^{-d-\alpha}$ near 0. The generator of the associated Lévy process L is on $C_b^2(\mathbb{R}^d)$ given by

$$Lu(x)=\int_{\mathbb{R}^d}\left(u(x+h)-u(x)-\frac{h\cdot\nabla u(x)}{1+|x|^2}\right)\nu(dx). \tag{2}$$

It acts in the Fourier space as a multiplication operator with the continuous negative-definite function associated to ν by the Lévy–Khinchin formula, cf. (5) below. Moreover, it is of order 2 on certain weighted Sobolev spaces $H^{\psi,s}(\mathbb{R}^d)$, see Section 2 for precise definitions.

We say that L is *locally hypoelliptic* with respect to $H{=}H^{-\infty}(\mathbb{R}^d)$ or $H{=}H^s(\mathbb{R}^d)$, $s\in\mathbb{R}$, if for any $f\in H$ and a distribution $u\in H^{-\infty}(\mathbb{R}^d)$ such that $Lu{=}f$ in $\mathbb{R}^d$, and $U\subset\mathbb{R}^d$ open we have

$$f|_U\in C^\infty(U) \quad\Longrightarrow\quad u|_U\in C^\infty(U).$$

The translation invariance of L implies that local hypoellipticity of L on $H^{s_0}(\mathbb{R}^d)$ for some $s_0\in\mathbb{R}$ entails local hypoellipticity on $H^{-\infty}(\mathbb{R}^d)$, cf. Lemma 2.4 below. Therefore we will sometimes simply call L locally hypoelliptic in this case.

Our main results now reads as follows.

Theorem 1.1. *Let ν be an absolutely continuous Lévy measure with density n that satisfies* (A1)–(A2). *Then the generator of the pure-jump Lévy process L given by* (2) *is locally hypoelliptic on $H^{-\infty}(\mathbb{R}^d)$.*

Moreover, in the case that ν is a compactly supported Lévy measure satisfying (1) it is also necessary that ν is smooth on $\mathbb{R}^d\setminus\{0\}$. More precisely, we have the following result.

Theorem 1.2. *Let ν be a compactly supported Lévy measure that satisfies* (A2). *Assume furthermore that L is locally hypoelliptic on $H^{-\infty}(\mathbb{R}^d)$. Then ν is absolutely continuous with a density which is smooth on $\mathbb{R}^d\setminus\{0\}$.*

Note that a compactly supported Lévy measure with smooth density on $\mathbb{R}^d\setminus\{0\}$ automatically satisfies (A1). Hence (A1) is also necessary in that case.

We want to finish this section by some examples.

Let $\alpha\in(0,2)$ and $f\colon S^{d-1}\to\mathbb{R}^+$ be a smooth function such that the support of f is not contained in any proper subspace of $\mathbb{R}^d$. We set $\nu(dh){=}|h|^{-d-\alpha}f(h/|h|)$. Then ν satisfies (A1) and (A2) and therefore the generator of the associated symmetric α-stable Lévy process is hypoelliptic.

It is also interesting to remark the following: In [3] there is given a counterexample of a Lévy process which does not admit a scale-invariant Harnack inequality. One can modify this construction in an obvious way such that our results apply. Hence in this example the related Lévy process still has a hypoelliptic generator.

2. Prerequisites

We start by recalling some basic concepts. Details can be found for example in [5], [12], [13], [14] and [17].

We denote by $\mathscr{S}(\mathbb{R}^d)$ the Schwartz space, by $\mathscr{D}'(\mathbb{R}^d)$ the space of distributions, i.e., the topological dual of $C_0^\infty(\mathbb{R}^d)$, by $\mathscr{E}'(\mathbb{R}^d)$ the space of compactly supported distributions, and by $\mathscr{S}'(\mathbb{R}^d)$ the space of tempered distributions, i.e., the dual of $\mathscr{S}(\mathbb{R}^d)$. Moreover, let $H^s(\mathbb{R}^d)$ be the usual L^2-Sobolev space of order $s\in\mathbb{R}$. Furthermore, we set $H^\infty(\mathbb{R}^d)=\bigcap_{s\in\mathbb{R}} H^s(\mathbb{R}^d)$ and $H^{-\infty}(\mathbb{R}^d)=\bigcup_{s\in\mathbb{R}} H^s(\mathbb{R}^d)$. We also write $\mathscr{F}$ for the Fourier transform and $\hat{u}=\mathscr{F}[u]$. Note that by the Paley–Wiener theorem [10, Theorem 7.3.1] $\mathscr{E}'(\mathbb{R}^d)\subset H^{-\infty}(\mathbb{R}^d)$.

A function $\phi\colon \mathbb{R}^d\to\mathbb{C}$ is called *negative definite* if the matrix $(\phi(\xi_i)+\overline{\phi(\xi_j)}-\phi(\xi_i-\xi_j))_{i,j=1}^k$ is positive definite for each choice of $k\in\mathbb{N}$ and $\xi_1,...,\xi_k\in\mathbb{R}^d$. Then ϕ satisfies for example $\phi(\xi)=\overline{\phi(-\xi)}$, $\operatorname{Re}\phi(\xi)\geq\phi(0)$ and the Peetre-type inequality

$$\frac{(1+|\phi(\xi)|)^s}{(1+|\phi(\eta)|)^s}\leq 2^{|s|}(1+|\phi(\xi-\eta)|)^{|s|}. \tag{3}$$

Note also the estimate

$$|\phi(\xi)-\phi(\eta)|\leq 4(1+|\phi(\xi-\eta)|)\big(1+\sqrt{\operatorname{Re}\phi(\xi)}\big). \tag{4}$$

This follows from the third inequality of Lemma 3.6.21 in [14] which implies

$$\begin{aligned}|\phi(\xi)-\phi(\eta)| &\leq |\phi(\xi)-\phi(\eta)+\phi(\eta-\xi)|+|\phi(\eta-\xi)|\\ &\leq 2\sqrt{\operatorname{Re}\phi(\eta-\xi)}\sqrt{\operatorname{Re}\phi(\xi)}+|\phi(\eta-\xi)|\big(1+\sqrt{\operatorname{Re}\phi(\xi)}\big).\end{aligned}$$

If ϕ is locally bounded we have in addition $|\phi(\xi)|\leq c(1+|\xi|^2)$. The set of continuous negative definite functions $\mathrm{CN}(\mathbb{R}^d)$ is a convex cone closed in the topology of uniform convergence on compact sets. Each $\phi\in\mathrm{CN}(\mathbb{R}^d)$ has the unique Lévy–Khinchin representation

$$\phi(\xi)=b+A\xi\cdot\xi+i\xi\cdot\gamma+\int_{\mathbb{R}^d}\left(1-e^{ih\cdot\xi}+\frac{ih\cdot\xi}{1+|h|^2}\right)\nu(dh). \tag{5}$$

Here, $b\geq0$, A is a positive definite matrix, $\gamma\in\mathbb{R}^d$ and ν is a Lévy measure, i.e., a nonnegative Borel measure on $\mathbb{R}^d$ with $\nu(\{0\})=0$ and $\int_{\mathbb{R}^d}(1\wedge|h|^2)\,\nu(dh)<\infty$.

By the theorem of Schönberg [5, Theorem 7.8], the elements $\phi\in\mathrm{CN}(\mathbb{R}^d)$ satisfying $\phi(0)=0$ are in one-to-one correspondence with Lévy processes $(X_t)_{t\geq0}$. More precisely, the Fourier transform of the distribution of X_t in $\mathbb{R}^d$ is given by $e^{-t\phi(\xi)}$, on the other hand the generator of $(X_t)_{t\geq0}$ is the pseudodifferential operator with symbol $-\phi(\xi)$,

$$-\phi(D_x)u(x)=-\mathscr{F}_{\xi\mapsto x}[\phi(\xi)\hat{u}(\xi)]=-\int_{\mathbb{R}^d}e^{ix\cdot\xi}\phi(\xi)\hat{u}(\xi)\,đ\xi,$$

where $đ\xi=(2\pi)^{-d}\,d\xi$. If $A=b=\gamma=0$, then $Lu=-\phi(D_x)u$, where Lu is as in (2).

An important example for continuous negative definite functions are the functions $\xi \mapsto |\xi|^\alpha$, where $\alpha \in (0,2]$. The corresponding Lévy processes are the rotationally invariant α-stable Lévy processes for $\alpha \in (0,2)$ and in particular a Brownian motion for $\alpha = 2$.

In our framework it is useful to introduce weighted (or anisotropic) Sobolev spaces tailored on the operators we consider here. We fix a continuous negative definite reference function $\psi \colon \mathbb{R}^d \to \mathbb{C}$ which satisfies $\lim_{|\xi|\to\infty} |\psi(\xi)| = \infty$ and set

$$H^{\psi,s}(\mathbb{R}^d) = \{u \in L^2(\mathbb{R}^d) : \|u\|_{\psi,s} := \|(1+|\psi(\,\cdot\,)|)^{s/2}\hat{u}\|_{L^2(\mathbb{R}^d)} < \infty\}.$$

Define also for an open set $\Omega \subset \mathbb{R}^d$,

$$H^{\psi,s}_{\mathrm{loc}}(\Omega) = \{u \colon \Omega \to \mathbb{R} : \chi u \in H^{\psi,s}(\mathbb{R}^d) \text{ for all } \chi \in C_0^\infty(\Omega)\}$$

and $H^s_{\mathrm{loc}}(\Omega) = H^{|\xi|^2,s}_{\mathrm{loc}}(\Omega)$. We have $H^s(\mathbb{R}^d) = H^{|\xi|^2,s}(\mathbb{R}^d)$ and $H^s(\mathbb{R}^d) \subset H^{\psi,s}(\mathbb{R}^d)$ due to $|\psi(s)| \le C(1+|\xi|^2)$. Note also, that $H^{\psi,s}(\mathbb{R}^d)^* \cong H^{\psi,-s}(\mathbb{R}^d)$: u is in $H^{\psi,s}(\mathbb{R}^d)$ if and only if $(u,v)_{L^2} \le c\|v\|_{\psi,-s}$ for all $v \in H^{\psi,-s}(\mathbb{R}^d)$.

Let $\phi \in \mathrm{CN}(\mathbb{R}^d)$ satisfy the following conditions:

(S1) There exists $\varkappa_1 > 0$ such that $|\phi(\xi)| \le \varkappa_1(1+|\psi(\xi)|)$.

(S2) There exist $\varkappa_2, r_0 > 0$ such that $|\phi(\xi)| \ge \varkappa_2 |\psi(\xi)|$ if $|\xi| \ge r_0$.

Then, by (S1), $\phi(D_x)$ maps $H^{\psi,s+2}(\mathbb{R}^d)$ continuously to $H^{\psi,s}(\mathbb{R}^d)$.

Theorem 2.1. *Let ϕ satisfy* (S1) *and* (S2). *Let $t \in \mathbb{R}$ and $f \in H^{\psi,t}(\mathbb{R}^d)$. If $u \in H^{-\infty}(\mathbb{R}^d)$ is a solution of $\phi(D_x)u = f$ in $\mathscr{S}'(\mathbb{R}^d)$, then $u \in H^{\psi,t+2}(\mathbb{R}^d)$.*

Proof. Without loss of generality we may assume $f \in L^2(\mathbb{R}^d)$. Then $\phi\hat{u} \in L^2(\mathbb{R}^d)$, $\hat{u} \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$, and $\lim_{|\xi|\to\infty} |\phi(\xi)| = \infty$ imply $(1+|\phi|)\hat{u} \in L^2(\mathbb{R}^d)$. Thus (S2) implies the statement of the theorem. □

Moreover, the commutator $[\phi(D_x), \chi]$ of $\phi(D_x)$ and the multiplication with $\chi \in C_0^\infty(\mathbb{R}^d)$ acts with order 1 in $H^{\psi,-\infty}(\mathbb{R}^d)$, i.e.:

Lemma 2.2. *Let ϕ satisfy* (S1) *and* (S2), *$t \in \mathbb{R}$ and $\chi \in C_0^\infty(\mathbb{R}^d)$. Then for all $u \in H^{\psi,t+1}(\mathbb{R}^d)$ we have*

$$\|[\phi(D_x), \chi]u\|_{t,\psi} \le C\|u\|_{t+1,\psi},$$

where C is independent of u.

Proof. Let $u, v \in \mathscr{S}(\mathbb{R}^d)$. Then on one hand we have

$$\mathscr{F}([\phi(D), \chi]u)(\xi) = \int_{\mathbb{R}^d} \widehat{\chi}(\xi-\eta)(\phi(\xi)-\phi(\eta))\hat{u}(\eta)\, d\eta.$$

By the theorem of Plancherel, (S1), (4) and (3) we estimate:

$$\begin{aligned}
|(\phi(D_x),\chi]u,v)_{L^2(\Omega)}| &\leq \iint_{\mathbb{R}^{2d}} |\widehat{\chi}(\xi-\eta)|\,|\phi(\xi)-\phi(\eta)|\,|\hat{u}(\eta)|\,|\hat{v}(\xi)|\,d\eta\,d\xi \\
&\leq C\iint_{\mathbb{R}^{2d}} |\widehat{\chi}(\xi-\eta)|(1+|\xi-\eta|^2)(1+|\psi(\xi)|)^{1/2}\,|\hat{u}(\eta)|\,|\hat{v}(\xi)|\,d\eta\,d\xi \\
&= C\iint_{\mathbb{R}^{2d}} |\widehat{\chi}(\xi-\eta)|(1+|\xi-\eta|^2)\left(\frac{1+|\psi(\xi)|}{1+|\psi(\eta)|}\right)^{(t+1)/2} \\
&\quad\times(1+|\psi(\eta)|)^{(t+1)/2}|\hat{u}(\eta)|(1+|\psi(\xi)|)^{-t/2}|\hat{v}(\xi)|\,d\eta\,d\xi \\
&\leq C\iint_{\mathbb{R}^{2d}} |\widehat{\chi}(\xi-\eta)|(1+|\xi-\eta|^2)^{(|t|+3)/2}(1+|\psi(\eta)|)^{(t+1)/2}|\hat{u}(\eta)| \\
&\quad\times(1+|\psi(\xi)|)^{-t/2}|\hat{v}(\xi)|\,d\eta\,d\xi \\
&\leq C\|(1+|\xi|^2)^{(|t|+3)/2}\widehat{\chi}(\xi)\|_{L_1(\mathbb{R}^d)}\|u\|_{\psi,t+1}\|v\|_{\psi,-t} \\
&\leq C\|u\|_{\psi,t+1}\|v\|_{\psi,-t}.
\end{aligned}$$

The assertion now follows by continuity and the characterization of $H^{\psi,s}(\mathbb{R}^d)$ as dual of $H^{\psi,-s}(\mathbb{R}^d)$. □

A direct application of this commutator estimate yields local regularity of the following type.

Theorem 2.3. *Let ϕ satisfy* (S1) *and* (S2). *If $\chi\in C_0^\infty(\mathbb{R}^d)$, $t\in\mathbb{R}$, $f\in L^2(\mathbb{R}^d)$, with $\chi f\in H^{\psi,t}(\mathbb{R}^d)$, and $u\in H^{\psi,t+1}(\mathbb{R}^d)$ solves $\phi(D)u=f$, then $\chi u\in H^{\psi,t+2}(\mathbb{R}^d)$.*

Unfortunately, we cannot expect to iterate this result without additional assumptions as is illustrated by Theorem 1.2.

We finish this section by showing that the notion of local hypoellipticity with respect to $H^s(\mathbb{R}^d)$ is independent of $s\in\mathbb{R}$.

Lemma 2.4. *Let $s_0\in\mathbb{R}$. If L is locally hypoelliptic with respect to $H^{s_0}(\mathbb{R}^d)$, then L is locally hypoelliptic with respect to $H^{-\infty}(\mathbb{R}^d)$.*

Proof. Let L be locally hypoelliptic with respect to $H^{s_0}(\mathbb{R}^d)$. In order to prove that L is locally hypoelliptic with respect to $H^s(\mathbb{R}^d)$, $s\in\mathbb{R}$, let $f\in H^s(\mathbb{R}^d)$, $u\in\mathscr{S}'(\mathbb{R}^d)$ with $Lu=f$ and $U\subset\mathbb{R}^d$ open be such that $f|_U\in C^\infty(U)$. Then $f'=\langle D_x\rangle^{s_0-s}f\in H^s(\mathbb{R}^d)$, where $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$ and $\langle D_x\rangle^s$ denotes the pseudodifferential operator with symbol $\langle\xi\rangle^s$. Moreover, $Lu'=f'$ with $u'=\langle D_x\rangle^{s_0-s}u$ since L

commutes with $\langle D_x\rangle^{s_0-s}$. Since $\langle\xi\rangle^{s_0-s}\in S^{s_0-s}_{1,0}(\mathbb{R}^d\times\mathbb{R}^d)$, $\langle D_x\rangle^{s_0-s}$ is pseudolocal, i.e., $\operatorname{sing\,supp} f'=\operatorname{sing\,supp}\langle D_x\rangle^{s_0-s}f\subseteq\operatorname{sing\,supp} f$, cf. e.g. [10, Theorem 18.1.16]. Hence $f'|_U\in C^\infty(U)$ and therefore $u'|_U\in C^\infty(U)$ due to the local hypoellipticity with respect to $H^{s_0}(\mathbb{R}^d)$. Finally, since $\langle D_x\rangle^{s-s_0}$ is pseudolocal too,

$$\operatorname{sing\,supp} u=\operatorname{sing\,supp}\langle D_x\rangle^{s-s_0}u'\subseteq\operatorname{sing\,supp} u'.$$

Thus $u|_U\in C^\infty(U)$. This shows that L is locally hypoelliptic with respect to $H^s(\mathbb{R}^d)$ for any $s\in\mathbb{R}$. Hence L is locally hypoelliptic with respect to $H^{-\infty}(\mathbb{R}^d)$. $\square$

3. Proof of Theorem 1.1

Let ψ be the continuous negative definite function associated by (5) to the pure-jump Lévy process with Lévy measure ν and let L be its generator. The real part of ψ is

$$\operatorname{Re}\psi(\xi)=\int_{\mathbb{R}^d}(1-\cos(h\cdot\xi))\,\nu(dh). \tag{6}$$

Lemma 3.1. *Assume* (A2). *Then there exists some $c>0$ such that $1+|\psi(\xi)|\geq c|\xi|^\alpha$.*

Proof. Using (A2) and the inequality $1-\cos x\geq x^2/4$ for $|x|\leq\frac{1}{2}$, we estimate for all $|\xi|\geq(2r_1)^{-1}$,

$$\begin{aligned}|\psi(\xi)|\geq\operatorname{Re}\psi(\xi)&\geq\int_{|h|\leq(2|\xi|)^{-1}}(1-\cos(h\cdot\xi))\,\nu(dh)\\&\geq\frac{|\xi|^2}{4}\int_{|h|\leq(2|\xi|)^{-1}}\left|h\cdot|\xi|^{-1}\xi\right|^2\nu(dh)\geq c|\xi|^\alpha.\end{aligned}\quad\square$$

Therefore for all $s>0$ the anisotropic Sobolev space $H^{\psi,s}(\mathbb{R}^d)$ is continuously embedded into $H^{\alpha s/2}(\mathbb{R}^d)$.

Observe that by (A2) the asymptotic behavior of $|\psi(\xi)|$ as $|\xi|\to\infty$ remains unchanged if one cuts off the large jumps of the Lévy process in the following sense: Fix for $r>0$ a function $\rho_r\in C_0^\infty(B_{2r}(0))$ with $0\leq\rho_r\leq1$ and $\rho_r\equiv1$ on $B_r(0)$. Then ψ can be decomposed as $\psi=\psi_{r,\mathrm{long}}+\psi_{r,\mathrm{short}}$, where

$$\begin{aligned}\psi_{r,\mathrm{long}}(\xi)&=\int_{\mathbb{R}^d}(1-e^{ih\cdot\xi})(1-\rho_r(h))\,\nu(dh),\\ \psi_{r,\mathrm{short}}(\xi)&=\int_{\mathbb{R}^d}\left(1-e^{ih\cdot\xi}+\frac{ih\cdot\xi}{1+|h|^2}\right)\rho_r(h)\,\nu(dh)+i\xi\cdot\int_{\mathbb{R}^d}\frac{h(1-\rho_r(h))}{1+|h|^2}\,\nu(dh).\end{aligned}$$

Because $\psi_{r,\text{long}}$ is bounded, Lemma 3.1 implies that $\psi_{r,\text{short}}$ satisfies (S1) and (S2). Note also that the operator associated to $\psi_{r,\text{short}}$ is $2r$-local in the sense that

$$\operatorname{supp}\psi_{r,\text{short}}(D_x)u \subset B_{2r}(\operatorname{supp} u) = \{x\in\mathbb{R}^d : \operatorname{dist}(x,\operatorname{supp} u) < 2r\}.$$

For the following we also assume that $\nu(dh)=n(h)\,dh$ and the density n satisfies (A1). The key step in our argument is the following regularity result.

Lemma 3.2. *Let $\Omega\subset\mathbb{R}^d$ be open. If $f\in L^2(\mathbb{R}^d)$ with $f|_\Omega\in H^{\psi,t}_{\text{loc}}(\Omega)$ and $u\in H^{\psi,1}(\mathbb{R}^d)$ with $u|_\Omega\in H^{\psi,t+1}_{\text{loc}}(\Omega)$ solve $\psi(D_x)u=f$ in $\mathscr{S}'(\mathbb{R}^d)$, then $u|_\Omega\in H^{\psi,t+2}_{\text{loc}}(\Omega)$.*

Proof. Let $\chi_1\in C_0^\infty(\Omega)$. We fix $r>0$ such that $4r<\operatorname{dist}(\mathbb{R}^d\setminus\Omega,\operatorname{supp}\chi_1)$ and choose a cut-off function $\chi_2\in C_0^\infty(\Omega)$ with $\chi_2\equiv 1$ on $B_{4r}(\operatorname{supp}\chi_1)$. If $\psi_{r,\text{short}}$ and $\psi_{r,\text{long}}$ are as above, then $\chi_2 u$ solves

$$\psi_{r,\text{short}}(D_x)(\chi_2 u) = f-\psi_{r,\text{long}}(D_x)u-\psi_{r,\text{short}}(D_x)((1-\chi_2)u)=\tilde f \quad \text{in } \mathscr{S}'(\mathbb{R}^d),$$

where $\tilde f$ is in $L^2(\mathbb{R}^d)$. By (A1), $\psi_{r,\text{long}}(D_x)u$ is the sum of a convolution of u with $(1-\rho_r)n\in H^\infty(\mathbb{R}^d)$—which is smooth—and a constant multiple of u. Since $\operatorname{supp}(1-\chi_2)u\subseteq\mathbb{R}^d\setminus B_{4r}(\operatorname{supp}\chi_1)$ the support of $\psi_{r,\text{short}}(D_x)((1-\chi_2)u)$ is contained in $\mathbb{R}^d\setminus\operatorname{supp}\chi_1$. Hence $\chi_1\tilde f\in H^{\psi,t}(\mathbb{R}^d)$ and Theorem 2.3 yields $\chi_1\chi_2 u=\chi_1 u\in H^{\psi,t+2}(\mathbb{R}^d)$. □

Proof of Theorem 1.1. Because of Lemma 2.4 it is sufficient to prove that L is locally hypoelliptic with respect to $L^2(\mathbb{R}^d)$. To this end let $\Omega\subset\mathbb{R}^d$ and let $u\in H^{-\infty}(\mathbb{R}^d)$ be a solution of $\psi(D_x)u=f$ in $\mathscr{S}'(\mathbb{R}^d)$ with $f\in L^2(\mathbb{R}^d)$ and $f|_\Omega\in C^\infty(\Omega)$. Since $f|_\Omega\in H^{\psi,\infty}_{\text{loc}}(\Omega)$, iterating Lemma 3.2 implies $u\in H^{\psi,\infty}_{\text{loc}}(\Omega)$. By Lemma 3.1 we have $u\in H^\infty_{\text{loc}}(\Omega)$ and therefore, by Sobolev embedding, $u\in C^\infty(\Omega)$. □

4. Proof of Theorem 1.2

The proof of Theorem 1.2 is based on the results of Chapter 16 in [10], which will be summarized below. Let us first fix some notation: If E is a set, then $\operatorname{ch} E$ denotes its convex hull. The supporting function of a convex, compact subset $E\subset\mathbb{R}^d$ is given by

$$H_E(x)=\sup_{y\in E} x\cdot y,$$

where $H_\varnothing\equiv-\infty$ by definition. This gives a one-to-one correspondence between convex compact subsets and the set $\mathcal{H}$ of convex, subadditive, positively homogeneous functions. For each $u\in\mathscr{E}'(\mathbb{R}^d)$ we denote by $\mathcal{H}(u)\subset\mathcal{H}$ the same set as defined in [10, Definition 16.3.2]. We omit the precise definition at this point since it is a bit involved and not needed for our purposes. In the following we will only use some of the properties of $\mathcal{H}(u)$, which we now summarize.

Theorem 4.1. *Let* $u\in\mathcal{E}'(\mathbb{R}^d)$ *and let* H *be the supporting function of* $\operatorname{ch}\operatorname{sing}\operatorname{supp} u$. *Then*

$$H(x)=\sup_{h\in\mathcal{H}(u)} h(x).$$

The latter theorem coincides with [10, Theorem 16.3.4].

Theorem 4.2. *Let* $u\in\mathcal{E}'(\mathbb{R}^d)$ *and* $h\in\mathcal{H}(u)$. *Then there is some* $w\in\mathcal{E}'(\mathbb{R}^d)\cap C^0(\mathbb{R}^d)$ *satisfying* $\operatorname{sing}\operatorname{supp} w=\{0\}$ *such that* h *is the supporting function of*

$$\operatorname{ch}\operatorname{sing}\operatorname{supp} u*w.$$

The statement of the theorem is just the first statement of [10, Theorem 16.3.13] with the only difference that the statement is formulated with $w\in\mathcal{E}'(\mathbb{R}^d)$ only. That indeed there is some $w\in\mathcal{E}'(\mathbb{R}^d)\cap C^0(\mathbb{R}^d)$ with the stated properties is shown in the proof of [10, Theorem 16.3.13].

Finally, we note that u is called *invertible* if $-\infty\notin\mathcal{H}(u)$, cf. [10, Definition 16.3.12]. The following condition for u not to be invertible will be used several times.

Theorem 4.3. *Let* $\mu\in\mathcal{E}'(\mathbb{R}^d)$. *Then the following statements are equivalent*:

(1) $-\infty\in\mathcal{H}(\mu)$;

(2) *For every* $x\in\mathbb{R}^d$ *there is some* $w\in C^0(\mathbb{R}^d)\setminus C^1(\mathbb{R}^d)$ *such that* $\operatorname{sing}\operatorname{supp} w=\{x\}$ *and* $\mu*w\in C^\infty(\mathbb{R}^d)$;

(3) *There is some* $w\in\mathcal{E}'(\mathbb{R}^d)$ *such that* $\mu*w\in C^\infty(\mathbb{R}^d)$ *but* $w\notin C^\infty(\mathbb{R}^d)$.

The latter theorem coincides with [10, Theorem 16.3.9].

Theorem 4.4. *Let* $\mu\in\mathcal{E}'(\mathbb{R}^d)$. *Then the following statements are equivalent*:

(1) $u\in\mathcal{D}'(\mathbb{R}^d)$ *and* $\mu*u\in C^\infty(\mathbb{R}^d)$ *implies* $u\in C^\infty(\mathbb{R}^d)$;

(2) μ *is hypoelliptic in the sense of* [10], *i.e.*, μ *is invertible and*

$$\frac{|\operatorname{Im}\zeta|}{\log|\zeta|}\to\infty,\ \text{as } |\zeta|\to\infty,\quad \text{on } \{\zeta\in\mathbb{C}^d:\hat{\mu}(\zeta)=0\};$$

(3) *There is some* $E\in\mathcal{E}'(\mathbb{R}^d)$ *such that* $E*\mu-\delta\in C^\infty(\mathbb{R}^d)$ *and* $\operatorname{sing}\operatorname{supp} E=-\operatorname{sing}\operatorname{supp}\mu$.

Proof. The theorem follows directly from the equivalent conditions (i), (ii), and (v) of [10, Theorem 16.6.5], where we note that hypoellipticity is defined in Definition 16.6.4 of [10]. $\square$

Note the following: If ν is a Lévy measure, then the associated operator $L\colon C_0^\infty(\mathbb{R}^d)\to C^\infty(\mathbb{R}^d)$ is linear, translation invariant and continuous and can therefore be written as a convolution with a distribution $\mu\in\mathscr{D}'(\mathbb{R}^d)$, cf. [10, Theorem 4.2.1]. Moreover, for $u\in C_0^\infty(\mathbb{R}^d)$ and $x\notin\operatorname{supp} u$ it follows that

$$Lu(x)=\int_{\mathbb{R}^d}\left(u(x+h)-u(x)-\frac{h\cdot\nabla u(x)}{1+|x|^2}\right)\nu(dh)=\int_{\mathbb{R}^d}u(x+h)\,\nu(dh)=\tilde{\nu}*u,$$

where $\tilde{\nu}$ denotes the reflection of ν, i.e., $\langle\tilde{\nu},\varphi\rangle:=-\langle\nu,\widetilde{\varphi}\rangle$ and $\widetilde{\varphi}(x)=\varphi(-x)$ for all $\varphi\in C_0^\infty(\mathbb{R}^d)$. Thus μ and $\tilde{\nu}$ agree on $\mathbb{R}^d\setminus\{0\}$. As a consequence $\operatorname{supp}\mu=-\operatorname{supp}\nu$ is compact and $\operatorname{sing\,supp}\mu\setminus\{0\}=-\operatorname{sing\,supp}\nu\setminus\{0\}$. The following proposition relates local hypoellipticity for L as we have defined it above and to hypoellipticity of μ in the sense of Hörmander, cf. Theorem 4.4.

Proposition 4.5. *Let L be a Lévy operator that is locally hypoelliptic and satisfies* (A2). *Then for any $u\in\mathcal{D}'(\mathbb{R}^d)$ and $f\in C^\infty(\mathbb{R}^d)$ such that $Lu=f$ we have $u\in C^\infty(\mathbb{R}^d)$.*

Proof. Let $M>0$ be such that $\operatorname{supp}\mu\subseteq\overline{B_M(0)}$. In order to show that $u\in C^\infty(\mathbb{R}^d)$ it is sufficient to show that $u|_{B_R(0)}\in C^\infty(B_R(0))$ for any $R>0$. Therefore let $R>0$ be arbitrary and let $\eta\in C_0^\infty(\mathbb{R}^d)$ be such that $\eta\equiv1$ on $B_{R+M}(0)$. Then

$$Lu(x)=\mu*u(x)=\mu*(\eta u)(x)\quad\text{for all } x\in B_R(0),$$

where $\eta u\in\mathcal{E}'(\mathbb{R}^d)$. Now there is some $s\in\mathbb{R}$ such that $\eta u\in H^s(\mathbb{R}^d)$. Thus $L(\eta u)=f'\in H^{s-2}(\mathbb{R}^d)$ since $|\psi(\xi)|\le C(1+|\xi|)^2$ for every continuous negative definite function ψ. Moreover, $f'|_{B_R(0)}=Lu|_{B_R(0)}=f|_{B_R(0)}\in C^\infty(B_R(0))$, which implies that $\eta u|_{B_R(0)}\in C^\infty(B_R(0))$ because of the local hypoellipticity of L. Since $R>0$ was arbitrary we obtain $u\in C^\infty(\mathbb{R}^d)$. $\square$

Proof of Theorem 1.2. First of all, because of Proposition 4.5, the first statement of Theorem 4.4 is true. Therefore there is a parametrix $E\in\mathscr{E}'(\mathbb{R}^d)$ such that

$$k=E*\mu-\delta\in C^\infty(\mathbb{R}^d)\quad\text{ and }\quad\operatorname{sing\,supp}E=-\operatorname{sing\,supp}\mu.$$

Here even $k\in C_0^\infty(\mathbb{R}^d)$ since E and μ have compact support. As μ is in turn a parametrix for E, E is also hypoelliptic due to Theorem 4.4 again. In particular this implies that E is invertible, i.e., $-\infty\notin\mathcal{H}(E)$.

Next we show that $\mathcal{H}(E)=\{0\}$. To this end let $h\in\mathcal{H}(E)$. By Theorem 4.2 there is some $w\in\mathscr{E}'(\mathbb{R}^d)\cap C_0(\mathbb{R}^d)$ with $\operatorname{sing\,supp} w=\{0\}$ such that h is the supporting function of $\operatorname{ch\,sing\,supp} E*w$. In particular, $w\in L^2(\mathbb{R}^d)$. Next let

$$v:=\mathscr{F}^{-1}[\eta(\xi)\psi(\xi)^{-1}\hat{f}(\xi)],$$

where $\eta\in C^\infty(\mathbb{R}^d)$ such that $\eta(\xi)=1$ for $|\xi|\geq\rho+1$ and $\eta(\xi)=0$ for $|\xi|\leq\rho$, where ρ is as in (S2). Then $v\in H^{\psi,2}(\mathbb{R}^d)$ and

$$Lv=f+k', \quad \text{where } k'\in C^\infty(\mathbb{R}^d).$$

Now, if $u=E*w$, then $\mu*(u-v)=k*w-k'\in C_0^\infty(\mathbb{R}^d)$. Therefore $u-v\in H^\infty(\mathbb{R}^d)$ and therefore $u\in H^{\psi,2}(\mathbb{R}^d)$. Because convolution with μ is by assumption locally hypoelliptic, we have

$$\operatorname{sing\,supp} E*w\subseteq\operatorname{sing\,supp}\mu*E*w=\operatorname{sing\,supp}(w+k*w)=\operatorname{sing\,supp} w=\{0\}.$$

As noted above $-\infty\notin\mathcal{H}(E)$, and therefore h cannot be the supporting function of $\varnothing$. We conclude that $\operatorname{ch\,sing\,supp} E*w=\{0\}$ which implies $h\equiv0$. This shows that $\mathcal{H}(E)=\{0\}$.

Thus the supporting function of $\operatorname{ch\,sing\,supp} E$ is $H(x)=\sup_{h\in\mathcal{H}(E)}h(x)=0$ and finally

$$\{0\}=\operatorname{ch\,sing\,supp} E=\operatorname{sing\,supp} E=-\operatorname{sing\,supp}\mu.$$

This completes the proof. □

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