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On Nordlander's conjecture in the three-dimensional case

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Abstract. In the present paper we prove, that in the real normed space X, having at least three dimensions, the Nordlander's conjecture about the modulus of convexity of the space X is true, i.e. from the validity of Day's inequality for a fixed real number from the interval $(0,2)$, follows that X is an inner product space.

1. Introduction and notation

Let X be a real normed space and let $S=\{x\in X:\|x\|=1\}$. The modulus of convexity of X is the function $\delta_X\colon [0,2]\to[0,1]$, defined by

$$\delta_X(a)=\inf\bigl\{1-\tfrac{1}{2}\|x+y\| : x,y\in S \text{ and } \|x-y\|=a\bigr\}.$$

It follows from the Jordan–von Neumann parallelogram law, that if H is an inner product space then $\delta_H(a)=1-\frac{1}{2}\sqrt{4-a^2}$. Nordlander discovered [5], that for an arbitrary normed space X the following inequality is true

$$\delta_X(a)\le\delta_H(a) \quad \text{for all } a\in(0,2).$$

This inequality is called Nordlander's inequality and shows that inner product spaces are "the most uniformly convex" spaces in the class of normed spaces [7].

In Day [3, Theorem 4.1] it is proved that if

$$\delta_X(a)=\delta_H(a) \quad \text{for all } a\in(0,2),$$

then X is an inner product space. Nordlander conjectured [5], that the same conclusion should hold if the above equality takes place for some fixed $a\in(0,2)$. Alonso and Benitez [1], making use of real two-dimensional arguments, proved the validity of Nordlander's conjecture for $a\in(0,2)\setminus D$, where $D=\{2\cos(k\pi/2n):k=1,\dots,n-1;n=2,3,\dots\}$, and they gave counterexamples for $\dim X=2$ and $a\in D$. However, according to [2, p. 154], the validity of Nordlander's conjecture for any $a\in(0,2)$ is still open in the case $\dim X\geq 3$ and in the present paper we give an affirmative answer to it.

An immediate consequence of the parallelogram law is that X is an inner product space if and only if all of its two-dimensional (three-dimensional) subspaces are inner product spaces. Therefore we can divide this paper into two parts. The first one focuses on the results that can be obtained with two-dimensional arguments and the second one (the essential part of the paper) with results that need three-dimensional arguments.

2. Two-dimensional arguments

Let $\dim X=2$, i.e. X is the space $\mathbb{R}^2$ endowed with a norm with unit sphere S.

Lemma. (Nordlander) *For any $a\in(0,2)$, the area bounded by the curve $S_a=\{x+y:x,y\in S$ and $\|x-y\|=a\}$ is $(4-a^2)$ times the area bounded by S.*

Then if $\delta_X(a)\geq 1-\frac{1}{2}\sqrt{4-a^2}$, for some $a\in(0,2)$, we get the following consequences.

Corollary 1. *For $x,y\in S$, $\|x-y\|=a$ is equivalent to $\|x+y\|=\sqrt{4-a^2}$.*

Corollary 2. *For any $z\in S$ there exists a unique set $\{x,y\}$ of norm-one elements, such that $x+y=\sqrt{4-a^2}z$ and $\|x-y\|=a$.*

Proof. On the one hand, it is obvious that for any $0<t<1$ there exist $x_t,y_t\in S$, such that $x_t+y_t=tz$, and it follows from the above Corollary 1 that if $t=\sqrt{4-a^2}$, then $\|x_t-y_t\|=a$.

On the other hand, a simple drawing of $x_1,y_1,x_2,y_2\in S$, such that $x_1+y_1=x_2+y_2$ and $\|x_1-y_1\|=\|x_2-y_2\|$ shows that $\{x_1,y_1\}=\{x_2,y_2\}$. $\square$

Corollary 3. *If $u,v\in X$ are such that $\|u+v\|=\sqrt{4-a^2}$ and $\|u-v\|=a$, then $(\|u\|-1)(\|v\|-1)\leq 0$.*

Proof. Indeed, by Corollary 2 for all $z\in S$, there exists one pair of vectors u and v from S, such that $u+v=\sqrt{4-a^2}z$ and $\|u-v\|=a$. Now consider the line $l=\{tu+(1-t)v:t\in\mathbb{R}\}$. It is clear, that this line intersects the curve $\frac{1}{2}\sqrt{4-a^2}z+\frac{1}{2}aS$ only in the points u and v, and divides it into two parts: the first part consists of points having norm less than 1, while the second consists of points having norm greater than 1. If $\{u_1,v_1\}$ is different from $\{u,v\}$ and $u_1+v_1=\sqrt{4-a^2}z$, then u_1 and v_1 are located on different sides of the line l and, by the condition $\|u_1-v_1\|=a$, they belong to the curve $\frac{1}{2}\sqrt{4-a^2}z+\frac{1}{2}aS$. So, both of them cannot have norm greater (smaller) than one simultaneously. $\square$

3. Three-dimensional arguments. Main result

Theorem. *Let X be a real normed space with* $\dim X \geq 3$ *and let a be a fixed number from the interval* $(0,2)$. *Then X is an inner product space if and only if* $\delta_X(a) \geq 1-\frac{1}{2}\sqrt{4-a^2}$.

We know that if X is an inner product space then $\delta_X(a)=1-\frac{1}{2}\sqrt{4-a^2}$. To prove the converse we shall use three lemmas in which we suppose:

1. a is a fixed point of the interval $(0,2)$ and $a'=\frac{1}{2}\sqrt{4-a^2}$.
2. $\delta_X(a) \geq 1-a'$.
3. X is the space $\mathbb{R}^3$ endowed with a norm whose unit sphere and ball are S and B, respectively.
4. $\zeta \in S$ is such that there is a homogeneous plane P_ζ such that $\zeta+P_\zeta$ is tangent to S at ζ. Since B is a convex body this is true for almost every $\zeta \in S$. It is not restrictive to assume that $\zeta=(1,0,0)$ and $P_\zeta=\{(r,s,0): r,s \in \mathbb{R}\}$.
5. $\Gamma=S \cap \left(a'\zeta+\frac{1}{2}aS\right)$. By Corollary 2, in every section of S by a homogeneous plane containing ζ there is only a pair of points of Γ. Hence, Γ is a simple, closed, continuous curve which is symmetric with respect to the point $a'\zeta$.

Lemma 1. *There exists a parametrization* $(u(\alpha), v(\alpha), w(\alpha))$ *of the curve* Γ, *such that* $u(\alpha), v(\alpha)$, *and* $w(\alpha)$ *are absolutely continuous functions.*

Proof. We prove firstly that the orthogonal projection Γ_1 of Γ over P_ζ, which coincides with the orthogonal projection of $\Gamma'=-\Gamma$ over P_ζ, is a convex curve, i.e. that for every $x_1, y_1 \in \Gamma_1$ and $0<t<1$, $tx_1+(1-t)y_1$ is not outside of Γ_1.

Let $x, y \in \Gamma$ and $x', y' \in \Gamma'$ be such that $x_1=\Pr x=\Pr x'$ and $y_1=\Pr y=\Pr y'$.

On the one hand it is obvious that

$$tx_1+(1-t)y_1=\Pr[tx+(1-t)y]=\Pr[tx'+(1-t)y'],$$

and, on the other hand, there exist $z \in \Gamma$, $z' \in \Gamma'$, and $\rho>0$ such that

$$\rho[tx_1+(1-t)y_1]=\Pr z=\Pr z'.$$

Then, it suffices to see that $\rho \geq 1$. Indeed, if $\rho<1$ the line

$$l=\rho[tx_1+(1-t)y_1]+\{r\zeta : r \in \mathbb{R}\}$$

intersects the parallel segments $[tx+(1-t)y, a'\zeta]$ and $[tx'+(1-t)y', -a'\zeta]$ at interior points $\bar{z}$ and $\bar{z}'$ of B and since $z, z' \in l \cap S$ we have the contradiction $2a'=\|z-z'\|>\|\bar{z}-\bar{z}'\|=2a'$.

Consider now the stereographic projection $\Pr_s$ from $\xi=(0,0,2a'/(2-a))$ on the plane P_ζ and prove that $\Gamma_2=\Pr_s \Gamma$ is also a convex curve. (For any point

$\eta\neq\xi$, $\Pr_s\eta=\{t\eta+(1-t)\xi : t\in\mathbb{R}\}\cap P_\zeta$.) By Corollary 1 we have $\Gamma_2=\Pr_s\Gamma''$, where $\Gamma''=(2/a)(\Gamma-a'\zeta)$. As above, we need to show that for every $x_2,y_2\in\Gamma_2$ and $0<t<1$, $tx_2+(1-t)y_2$ is not outside of Γ_2.

Let $x,y\in\Gamma$ and $x'',y''\in\Gamma''$ be such that $x_2=\Pr_s x=\Pr_s x''$ and $y_2=\Pr_s y=\Pr_s y''$.

Also, it is obvious that

$$tx_2+(1-t)y_2=\Pr_s[tx+(1-t)y]=\Pr_s[tx''+(1-t)y''],$$

and, there exist $z\in\Gamma$, $z''\in\Gamma''$, and $\rho>0$ such that

$$\rho[tx_2+(1-t)y_2]=\Pr_s z=\Pr_s z''.$$

It suffices to show that $\rho\geq1$. Indeed, if $\rho<1$, the line

$$h=\{r\rho[tx_2+(1-t)y_2]+(1-r)\xi : r\in\mathbb{R}\}$$

intersects the parallel segments $[tx+(1-t)y,a'\zeta]$ and $[tx''+(1-t)y'',0]$ at interior points $\bar z$ and $\bar z''$ of B and since $z,z''\in h\cap S$, we have $z-a'\zeta=\lambda_1\bar z''+\mu_1\zeta$ and $z''=\lambda_2\bar z''-\mu_2\zeta$, where λ_1, λ_2, μ_1 and μ_2 are positive numbers. Hence, these vectors cannot be collinear. So we get that Γ_2 is convex.

Let $(u_1(\alpha),v_1(\alpha))$ and $(u_2(\alpha),v_2(\alpha))$ be the parametric expressions of Γ_1 and Γ_2, parametrized by the angle α between the radius vectors of the curves and any axis which passes through the point 0. Since Γ_1 and Γ_2 are convex curves, the functions u_1, v_1, u_2 and v_2 are absolutely continuous on the segment $[0,2\pi]$. Denoting the parametric expression of Γ by $(u(\alpha),v(\alpha),w(\alpha))$, we get $u(\alpha)=u_1(\alpha)$, $v(\alpha)=v_1(\alpha)$ and $(2-a)w(\alpha)/2a'=(u_2(\alpha)-u_1(\alpha))/u_2(\alpha)=(v_2(\alpha)-v_1(\alpha))/v_2(\alpha)$. From this it follows, that $w(\alpha)$ is also absolutely continuous on $[0,2\pi]$. The proof of Lemma 1 is complete. $\square$

Lemma 2. *If there exists a tangent vector $T_x=(u'(\alpha_x),v'(\alpha_x),w'(\alpha_x))$ to Γ at a point $x\in\Gamma$, then $T_x\in P_\zeta$, i.e. $w'(\alpha_x)=0$.*

Proof. If the vector T_x is not in P_ζ then it is not tangent, at $a'\zeta$, to the convex curve $\overline{\Gamma}=a'S\cap\operatorname{span}\{\zeta,T_x\}$ and, therefore, there exists $h\in\overline{\Gamma}$ such that

$$h-a'\zeta=\lambda T_x+\mu\zeta,\quad \text{with } \lambda\neq0 \text{ and } \mu>0.$$

We shall prove that, hence, $\|x-a'\zeta+h\|>1$ and $\|x-a'\zeta-h\|>1$, which contradicts Corollary 3.

On the one hand, since T_x is tangent to Γ at x,

$$1 \leq \left\| x + \frac{a'\lambda}{a'+\mu} T_x \right\| \leq \left\| \frac{a'}{a'+\mu}(x - a'\zeta + h) \right\| + \left\| \frac{\mu}{a'+\mu}(x - a'\zeta) \right\|$$
$$\leq \left\| \frac{a'}{a'+\mu}(x - a'\zeta + h) \right\| + \frac{a\mu}{2(a'+\mu)},$$

from which it follows that

$$\|x - a'\zeta + h\| \geq 1 + \frac{(2-a)\mu}{a'} > 1.$$

On the other hand, since Γ is symmetric with respect to $a'\zeta$, the vector T_x is also tangent to it at the point $y = 2a'\zeta - x$ and, as above, we obtain $\|x - a'\zeta - h\| = \|y - a'\zeta + h\| > 1$.

Then, $w'(\alpha) = 0$ for almost all values of α and, since $w(\alpha)$ is absolutely continuous, it is constant (by Lebesgue's theorem). So, Γ is a plane curve parallel to P_ζ. $\square$

Lemma 3. *If $x, y \in S$ are so that either $x + y = 2a'\zeta$ or $x + y = a\zeta$, then $x - y \in P_\zeta$.*

Proof. It suffices to consider that Γ is a plane curve parallel to P_ζ. $\square$

Proof of the theorem. Suppose that $\zeta + P_\zeta$ and $\zeta' + P_{\zeta'}$ are tangent planes to S at ζ and ζ', respectively, and that $\zeta \in S$ is Birkhoff–James orthogonal to $\zeta' \in S$, (i.e. $\|\zeta\| \leq \|\zeta + t\zeta'\|$ for every $t \in \mathbb{R}$, which is equivalent to $\zeta' \in P_\zeta$).

By Lemma 3 we have that $\zeta \in P_{\zeta'}$, i.e. that $\zeta' \perp \zeta$. Then Birkhoff–James-orthogonality is symmetric and, since $\dim X \geq 3$, X is an inner product space [4]. The proof of the theorem is complete. $\square$

Corollary. *Let $a \in (0, 2)$. A real normed space of dimension ≥ 3 is an inner product space if and only if the set $C_a = \{\|x + y\| : \|x\| = \|y\| = 1$ and $\|x - y\| = a\}$ is a singleton.*

Proof. This is an immediate consequence of Corollary 1 and the above theorem. $\square$

Remark. A well-known result of Senechale [6] says that a real normed space X is an inner product space if and only if there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that $\|x + y\| = f(\|x - y\|)$ for any $x, y \in S$. An immediate consequence of the corollary is that for $\dim X \geq 3$, it suffices to consider, in Senechale's result, only such points x and y on the unit sphere for which $\|x - y\| = a$.

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