Ark. Mat., **48** (2010), 149–176
DOI: 10.1007/s11512-008-0087-8
© 2008 by Institut Mittag-Leffler. All rights reserved

General Hausdorff functions, and the notion of one-sided measure and dimension

Claude Tricot

Abstract. The main facts about Hausdorff and packing measures and dimensions of a Borel set E are revisited, using *determining* set functions $\phi_\alpha\colon \mathcal{B}_E \to (0,\infty)$, where $\mathcal{B}_E$ is the family of all balls centred on E and α is a real parameter. With mild assumptions on ϕ_α, we verify that the main density results hold, as well as the basic properties of the corresponding box dimension. Given a bounded open set V in $\mathbb{R}^D$, these notions are used to introduce the *interior* and *exterior* measures and dimensions of any Borel subset of ∂V. We stress that these dimensions depend on the choice of ϕ_α. Two determining functions are considered, $\phi_\alpha(B)=\mathrm{Vol}_D(B\cap V)\,\mathrm{diam}(B)^{\alpha-D}$ and $\phi_\alpha(B)=\mathrm{Vol}_D(B\cap V)^{\alpha/D}$, where Vol_D denotes the D-dimensional volume.

1. Introduction

Hausdorff and packing measures were first intended to generalize the notions of length, area, volume etc. In his 1919 paper [4], F. Hausdorff uses general *determining functions* (also called *Hausdorff*, *gauge* or *constituent functions*) $\phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ to define the Hausdorff measure in a metric space as

$$\mathrm{H}^\phi(E) = \lim_{\varepsilon\to 0} \inf\left\{\sum_{i\ge 0} \phi(\mathrm{diam}(E_i)) : E \subset \bigcup_{i\ge 0} E_i \text{ and } \mathrm{diam}(E_i) \le \varepsilon\right\}. \tag{1}$$

Let id be the identity function, so that $\mathrm{id}^\alpha(t)=t^\alpha$ for all $t\in\mathbb{R}$ and $\alpha>0$. The scale of functions $\Phi=\{\mathrm{id}^\alpha : \alpha>0\}$ is used to obtain a large family of Hausdorff measures $\mathrm{H}^{\mathrm{id}^\alpha}$, rather denoted by H^α for simplicity. Up to some constants, H^1 gives the proper definition for the notion of length, H^2 for the notion of area etc. For many references concerning the classical theory, see [2].

The Hausdorff dimension of a set E is introduced as a critical exponent: $\dim(E)=\inf\{\alpha : H^\alpha(E)=0\}$. The dimension is a real number in this context, but a different choice of Φ could lead to a different notion of dimension, as pointed out by Hausdorff himself in his seminal paper. In general the dimension is a Dedekind cut in a scale of functions.

The value of $\mathrm{H}^\alpha(E)$ is 0 when $\alpha > \dim(E)$, $+\infty$ when $\alpha < \dim(E)$. When $\mathrm{H}^\alpha(E)$ is non-zero and finite, its exact value is known only for very particular sets like curves of finite length or symmetric Cantor sets. As shown by A. S. Besicovitch, a finite measure μ such that $\mu(E)>0$ may help to get estimations of $\mathrm{H}^\alpha(E)$, by using the *μ-densities* $\mu(B(x,\varepsilon))/\varepsilon^\alpha$, where $B(x,\varepsilon)$ denotes the ball of centre x and radius ε. A typical result [6] stands as follows: If a and b are such that

$$a \leq \limsup_{\varepsilon \to 0} \frac{\mu(B(x,\varepsilon))}{(2\varepsilon)^\alpha} \leq b$$

for all $x \in E$, then $a \leq \mathrm{H}^\alpha(E) \leq 2b$. A better result may be obtained by changing somewhat the definition of H^α. If we consider only covers of E by centred balls, the *centred covering measure* $\mathrm{c}^\alpha(E)$ is obtained through a two-stages operation [9]. Then, with the same assumptions, the inequality $a \leq \mathrm{c}^\alpha(E) \leq b$ is always true.

The corresponding result for the dimension is often attributed to P. Billingsley [1]: Let μ be a finite measure such that $\mu(E)>0$. If there exists α and β such that

$$\alpha \leq \liminf_{\varepsilon \to 0} \frac{\log \mu(B(x,\varepsilon))}{\log \varepsilon} \leq \beta$$

for all $x \in E$, then $\dim(E) \in [\alpha, \beta]$.

Giving a sense to the above formulas after exchanging lim inf and lim sup is possible, at the condition of introducing new notions of measure and dimension. Using the scale $\Phi = \{\mathrm{id}^\alpha : \alpha > 0\}$, this was done in [12] for the dimension and in [11] and [9] for the measures, by introducing the *packing measures and dimension.* In the present paper we will give a detailed account of these notions, in a more general framework.

The above density results have been extended to functions $\phi\colon (0,\infty) \to (0,\infty)$ which are increasing, continuous at 0, and such that $\phi(2\varepsilon)/\phi(\varepsilon)$ is bounded [9]. More generally, $\phi(\mathrm{diam}(E_i))$ may be replaced by $\phi(E_i)$, where ϕ is a specific set function: See for example [7], where a multifractal analysis of measures uses $\phi(E_i) = \mathrm{diam}(E_i)^t \nu(E_i)^q$ for some prescribed measure ν. In [8], ϕ is a function defined on balls centred on the set E under analysis. This is also our point of view in the present paper.

Let $D \geq 1$ and Vol_D be the D-dimensional volume in $\mathbb{R}^D$. For any Borel set E, let $\mathcal{B}_E$ be the family of closed balls centred on E. In Section 2 we use functions $\phi\colon \mathcal{B}_E \to (0,\infty)$ to define covering and packing Borel measures H^ϕ and p^ϕ. We get corresponding results involving the μ-densities and prove the usual inequality

$\mathrm{H}^\phi \le \mathrm{p}^\phi$. For any Borel set E, replacing the measure μ by the restriction to E of the measures H^ϕ and p^ϕ allows us to state *local density results* in Section 3, giving rise to a notion of rectifiability. Until this point, there is no assumption whatsoever made on the determining function ϕ. We introduce parameterized scales of functions $\Phi=\{\phi_\alpha : \alpha\in\mathbb{R}\}$ in Section 4 to define the related covering and packing dimensions, $\dim_\Phi$ and Dim_Φ. Section 5 deals with the corresponding *box dimension.* With the help of mild assumptions on functions ϕ_α, we introduce a pre-measure and show that the corresponding critical index Δ_Φ verifies some of the fundamental properties of the usual box dimension, for example $\Delta_\Phi(E)=\Delta_\Phi(\overline{E})$. We investigate the particular case $\phi_\alpha(B)=\nu(B)\,\mathrm{diam}(B)^{\alpha-D}$, where ν is a finite, Borel measure. If $E(\varepsilon)$ is the *Minkowski sausage* (set of points at a distance $\le\varepsilon$ from E), we show that $\Delta_\Phi(E)=\limsup_{\varepsilon\to\infty}(D-\log\nu(E(\varepsilon)))/\log\varepsilon$.

Section 6 deals with the main purpose of this paper: defining *lateral* or *one-sided* measures and dimensions on both sides of a closed curve. This has already been done for the box-counting dimension [15] but for the Hausdorff dimension these notions seem to be new. More generally, being given an open set V and $E\subset\partial V$, we perform a fractal analysis of E *relative to* V, by using determining functions $\phi\colon \mathcal{B}_E\to(0,\infty)$ such that $\phi(B)$ depends on $B\cap V$. There are several ways to do this. We pay special attention to the function

$$\phi_\alpha(B)=\frac{\mathrm{Vol}_D(B\cap V)}{\mathrm{Vol}_D(B)}\,\mathrm{diam}(B)^\alpha.$$

It involves a notion of covering and packing interior measures (from the results of Section 2), and interior dimensions (from Section 4) denoted by $\dim_{\mathrm{int}}$ and $\mathrm{Dim}_{\mathrm{int}}$. Being given a finite measure on E such that $\mu(E)>0$, we can establish relationships between those dimensions and the quantities

$$\frac{\log\mu(B)}{\log(\mathrm{diam}(B))}+D-\frac{\log \mathrm{Vol}_D(B\cap V)}{\log(\mathrm{diam}(B))}.$$

It is worth pointing out that the following characteristic quantity

$$-D+\frac{\log \mathrm{Vol}_D(B\cap V)}{\log(\mathrm{diam}(B))}$$

has already been considered in [5] to perform a multifractal analysis of ∂V, seen from the interior of V.

Considering the interior of the complement of V, it is also possible to define *exterior dimensions* $\dim_{\mathrm{ext}}$ and $\mathrm{Dim}_{\mathrm{ext}}$. Relationships between these dimension are obtained, namely: If $\mathrm{Vol}_D(\partial V)=0$ and $E\subset\partial V$, then

(2)

$$\max\{\dim_{\mathrm{int}}(E),\dim_{\mathrm{ext}}(E)\}\le\dim(E)\le\mathrm{Dim}(E)=\max\{\mathrm{Dim}_{\mathrm{int}}(E),\mathrm{Dim}_{\mathrm{ext}}(E)\}.$$

Some examples are studied. An interesting feature of this determining function ϕ is that the interior dimension of a singleton depends on the shape of ∂V, and may take all values between $-\infty$ and 0. We also construct an open set V in $\mathbb{R}$, then in $\mathbb{R}^2$, such that the first inequality in (2) is strict. Finally we point out in Section 7 that this is not the only approach to a one-sided fractal analysis: other determining functions might be used, giving other results.

2. ϕ-measures

Being given a set E in $\mathbb{R}^D$, a *net* over E is a family $\mathcal{F}$ of sets such that every point in E belongs to a sequence E_n of $\mathcal{F}$ such that $\operatorname{diam}(E_n)\to 0$. In other words, $\mathcal{F}$ is a net if, for every $\varepsilon>0$, one can extract from $\mathcal{F}$ a cover of E by sets of diameter $\le\varepsilon$. It is sometimes called a *Vitali covering*, or a *fine covering*. The family of all closed centred balls $\mathcal{B}_E$ is a net.

A *centred net* over E will be a subfamily $\mathcal{F}$ of $\mathcal{B}_E$ such that every $x\in E$ is the centre of a sequence of balls in $\mathcal{F}$ whose diameters tend to 0.

For every set family $\mathcal{F}$, $\bigcup\mathcal{F}$ denotes the union of the sets in $\mathcal{F}$, as usual.

We will make use of the *Besicovitch lemma*, in the two following forms:

Lemma 2.1. *Let E be a bounded set, and $\mathcal{F}\subset\mathcal{B}_E$ be such that every $x\in E$ is the centre of some ball in $\mathcal{F}$. From $\mathcal{F}$ one can extract a number $K\le 6^D$ of families $\mathcal{R}_1,\dots,\mathcal{R}_K$ such that $\bigcup_{n=1}^K\mathcal{R}_n$ covers E and every $\mathcal{R}_n$ consists of disjoint centred balls.*

This result implies the following consequence:

Lemma 2.2. *Let E be a Borel set and μ be a Borel measure such that $\mu(E)<+\infty$. From every centred net $\mathcal{F}$ over E we can extract a family $\mathcal{R}$, consisting of disjoint balls, such that $\mu(E\setminus\bigcup\mathcal{R})=0$.*

Let us recall that a set function F is *subadditive* if $F(E_1\cup E_2)\le F(E_1)+F(E_2)$, and σ-subadditive if for any sequence $\{E_n\}_{n\ge 0}$, $F(\bigcup_{n\ge 0}E_n)\le\sum_{n\ge 0}F(E_n)$.

Let ϕ be an application defined on $\mathcal{B}_E$, with values in $(0,+\infty)$.

2.1. ϕ-covering measures

For every $\varepsilon>0$, let

$$\mathrm{C}^{\phi}_{\varepsilon}(E)=\inf\left\{\sum_{i\ge 0}\phi(B_i): B_i\in\mathcal{B}_E,\ E\subset\bigcup_{i\ge 0}B_i \text{ and } \operatorname{diam}(B_i)\le\varepsilon\right\}. \tag{3}$$

It is a decreasing function of ε, with values in $[0,+\infty]$. As a set function it is σ-subadditive. The quantity

$$\mathrm{C}^{\phi}(E)=\lim_{\varepsilon\to 0}\mathrm{C}^{\phi}_{\varepsilon}(E) \tag{4}$$

is defined like the Hausdorff measure. As a set function, C^{ϕ} is σ-subadditive, but not increasing. Hence it is not a measure. To change it into an increasing function, it is necessary to perform the following operation:

$$\mathrm{c}^{\phi}(E)=\sup\{\mathrm{C}^{\phi}(F): F \text{ is Borel and } F\subset E\}. \tag{5}$$

This new set function is increasing and σ-subadditive. Moreover we can set $\mathrm{c}^{\phi}(\varnothing)=0$. Then c^{ϕ} is an outer measure. Since it is also metric, c^{ϕ} is a Borel measure, called the *centred ϕ-covering measure* [9]. The next two results will be helpful to establish a density theorem for c^{ϕ}.

Proposition 2.3. *Let μ be a Borel measure and $\mathcal{F}$ be a centred net over E. Let us assume that*

$$\phi(B)\le\mu(B)\quad \textit{for all } B\in\mathcal{F}.$$

Then $\mathrm{c}^{\phi}(E)\le\mu(E)$.

Proof. Suppose that $\mu(E)<\infty$. Let $\varepsilon>0$, and for every open V, $\mathcal{G}(V,\varepsilon)=\{B\in\mathcal{F}:\operatorname{diam}(B)\le\varepsilon \text{ and } B\subset V\}$.

(a) Since E is Borel, there exists an open set V_E such that $E\subset V_E$ and $\mu(V_E)\le\mu(E)+\varepsilon$. The family $\mathcal{G}(V_E,\varepsilon)$ is a centred net over E. Using Lemma 2.2, we can extract from $\mathcal{G}(V_E,\varepsilon)$ a sequence $\{B_i\}_{i\ge 0}$ of disjoint balls such that $\mu(E\setminus\bigcup_{i\ge 0}B_i)=0$. Therefore

$$\sum_{i\ge 0}\phi(B_i)\le\mu(V_E)\le\mu(E)+\varepsilon.$$

(b) Let F be a Borel set in E. There exists an open set V_F such that $F\subset V_F$ and $\mu(V_F)\le\mu(F)+\varepsilon$. Using Lemma 2.1, one can extract from $\mathcal{G}(V_F,\varepsilon)$ a number K of families $\mathcal{R}_1,\dots,\mathcal{R}_K$ such that $\mathcal{R}_n$ is made up with disjoint balls and $F\subset\bigcup_{n=1}^{K}\mathcal{R}_n$. For every n,

$$\sum\{\phi(B): B\in\mathcal{R}_n\}\le\mu(V_F)\le\mu(F)+\varepsilon,$$

so that $\sum\{\phi(B):B\in\bigcup_{n=1}^{K}\mathcal{R}_n\}\le K(\mu(F)+\varepsilon)$. Applying this to $F=E\setminus\bigcup_{i\ge 0}B_i$, we get a cover $\bigcup_{n=1}^{K}\mathcal{R}_n$ of $E\setminus\bigcup_{i\ge 0}B_i$ such that

$$\sum\left\{\phi(B): B\in\bigcup_{n=1}^{K}\mathcal{R}_n\right\}\le K\varepsilon.$$

(c) Since $\{B_i\}_{i\geq 0}\cup(\bigcup_{n=1}^{K}\mathcal{R}_n)$ is an ε-covering of E,

$$\mathrm{C}_\varepsilon^\phi(E)\leq\sum_{i\geq 0}\phi(B_i)+\sum\left\{\phi(B):B\in\bigcup_{n=1}^{K}\mathcal{R}_n\right\}\leq\mu(E)+(K+1)\varepsilon.$$

Letting $\varepsilon\to 0$, we get $\mathrm{C}^\phi(E)\leq\mu(E)$. This result is true for any Borel subset G of E:

$$\mathrm{C}^\phi(G)\leq\mu(G)\leq\mu(E).$$

Taking the supremum over all $G\subset E$ gives $\mathrm{c}^\phi(E)\leq\mu(E)$. □

Here is a result in the other direction:

Proposition 2.4. *Let $\varepsilon_0>0$, $\mathcal{F}_{\varepsilon_0}$ be the family of all balls centred on E such that* $\mathrm{diam}(B)\leq\varepsilon_0$*, and μ be a Borel measure such that*

$$\phi(B)\geq\mu(B)\quad \textit{for all } B\in\mathcal{F}_{\varepsilon_0}.$$

Then $\mathrm{c}^\phi(E)\geq\mu(E)$.

Proof. Let $\varepsilon\in(0,\varepsilon_0]$. For every cover $\{B_i\}_{i\geq 0}$ of E by balls centred on E such that $\mathrm{diam}(B)\leq\varepsilon$,

$$\sum_{i\geq 0}\phi(B_i)\geq\sum_{i\geq 0}\mu(B_i)\geq\mu(E),$$

so that $\mathrm{C}_\varepsilon^\phi(E)\geq\mu(E)$. Letting $\varepsilon\to 0$, shows that $\mathrm{C}^\phi(E)\geq\mu(E)$. Since $\mathrm{c}^\phi\geq\mathrm{C}^\phi$ we get the desired result. □

Now we can set up the main density result. It allows us to give estimates of $\mathrm{c}^\phi(E)$ with the help of the μ-*densities* $\mu(B)/\phi(B)$.

The usual conventions hold: for every $a>0$, $0/a=0$ and $a/0=+\infty$. The cases of indetermination are $\frac{0}{0}$ and $+\infty/+\infty$.

Theorem 2.5. *Let E be a Borel set and μ be a Borel measure. Then*

$$\inf_{x\in E}\limsup_{\varepsilon\to 0}\frac{\mu(B(x,\varepsilon))}{\phi(B(x,\varepsilon))}\leq\frac{\mu(E)}{\mathrm{c}^\phi(E)}\leq\sup_{x\in E}\limsup_{\varepsilon\to 0}\frac{\mu(B(x,\varepsilon))}{\phi(B(x,\varepsilon))},\tag{6}$$

except in the cases of indetermination.

Proof. (a) Let D_1 be the left-hand side of (6). Assume that $D_1>0$. Let $0<A_1<D_1$, and $\mathcal{F}=\{B\in\mathcal{B}_E:A_1\phi(B)\leq\mu(B)\}$. The family $\mathcal{F}$ is a centred net over E. Let $\mu^*=(1/A_1)\mu$. Since $\phi(B)\leq\mu^*(B)$ for all $B\in\mathcal{F}$, Proposition 2.3 gives $\mathrm{c}^\phi(E)\leq\mu^*(E)$, so that $A_1\mathrm{c}^\phi(E)\leq\mu(E)$. Letting A_1 tend to D_1 gives $D_1\mathrm{c}^\phi(E)\leq\mu(E)$. This may be written as $D_1\leq\mu(E)/\mathrm{c}^\phi(E)$, except in the indetermination cases.

(b) Let D_2 be the right-hand side of (6). Assume that $D_2<\infty$. Let $A_2>D_2$, and

$$E_k = \left\{ x\in E : \varepsilon \le \frac{1}{k} \Rightarrow \mu(B(x,\varepsilon)) \le A_2\phi(B(x,\varepsilon)) \right\},$$

so that $E=\bigcup_{k\ge1} E_k$. Proposition 2.4 implies that $A_2 \mathrm{c}^\phi(E_k)\ge\mu(E_k)$. Since μ and c^ϕ are Borel measures and E_k is an increasing sequence of sets, $A_2\mathrm{c}^\phi(E)\ge\mu(E)$. To conclude, let A_2 tend to D_2. $\square$

2.2. ϕ-packing measures

Let E be a bounded Borel set. For every $\varepsilon>0$, $\{B_i\}_{i\ge0}$ is an *ε-packing of E* if $B_i\in\mathcal{B}_E$, the B_i are disjoint, and $\operatorname{diam}(B_i)\le\varepsilon$. Let

$$\mathrm{P}^\phi_\varepsilon(E) = \sup\left\{ \sum_{i\ge0} \phi(B_i) :\ \{B_i\}_{i\ge0} \text{ is an } \varepsilon\text{-packing of } E \right\}. \tag{7}$$

This set function is increasing and subadditive. As a function of ε it increases, and we can set the following *pre-measure*

$$\mathrm{P}^\phi(E) = \lim_{\varepsilon\to0} \mathrm{P}^\phi_\varepsilon(E). \tag{8}$$

It is also increasing and subadditive, but not σ-subadditive. We get the *ϕ-packing measure* through the standard operation [12], [10],

$$\mathrm{p}^\phi(E) = \inf\left\{ \sum_{n\ge0} \mathrm{P}^\phi(E_n) : E_n \text{ are Borel sets with } E = \bigcup_{n\ge0} E_n \right\}. \tag{9}$$

In particular, $\mathrm{p}^\phi\le\mathrm{P}^\phi$. Like H^ϕ and c^ϕ, p^ϕ is a metric, outer measure, and therefore a Borel measure. The main results relating the value of $\mathrm{p}^\phi(E)$ to $\mu(E)$, for some measure μ, are very similar to those already proved for c^ϕ. We begin with two propositions and get a density theorem.

In the following proofs, it is necessary to assume that E is bounded since the pre-measure P^ϕ is used; but all results, as well as (9), may be extended to unbounded sets E without change.

Proposition 2.6. *Let $\varepsilon_0>0$, $\mathcal{F}_{\varepsilon_0}$ be the family of all balls centred on E such that* $\operatorname{diam}(B)\le\varepsilon_0$*, and μ be a Borel measure such that*

$$\phi(B)\le\mu(B) \quad \textit{for all } B\in\mathcal{F}_{\varepsilon_0}.$$

Then $\mathrm{p}^\phi(E)\le\mu(E)$.

Proof. Assume that $\mu(E)<\infty$. Let V be an open set such that $E\subset V$. Let N be an integer such that $N\geq 1/\varepsilon_0$. For every $n\geq N$, let

$$E_n=\left\{x\in E: B\left(x,\frac{1}{n}\right)\subset V\right\}.$$

The sequence $\{E_n\}_{n\geq N}$ is increasing. If $\varepsilon\leq 1/n$ and $x\in E_n$, then $\phi(B(x,\varepsilon))\leq \mu(B(x,\varepsilon))$. For every ε-packing $\{B_i\}_{i\geq 0}$ of E_n, we get

$$\sum_{i\geq 0}\phi(B_i)\leq\sum_{i\geq 0}\mu(B_i)\leq\mu(V).$$

Therefore $\mathrm{P}^\phi_\varepsilon(E_n)\leq\mu(V)$. Letting $\varepsilon\to 0$ gives that $\mathrm{P}^\phi(E_n)\leq\mu(V)$. We deduce that $\mathrm{p}^\phi(E_n)\leq\mu(V)$ for all $n\geq N$, and since $E=\bigcup_{n\geq 0}E_n$, $\mathrm{p}^\phi(E)\leq\mu(V)$. Since E is Borel, we can choose V such that $\mu(V)$ is as close to $\mu(E)$ as we wish. Therefore $\mathrm{p}^\phi(E)\leq \mu(E)$. □

Proposition 2.7. *Let μ be a Borel measure and $\mathcal{F}$ be a centred net over E. Let us assume that*

$$\mu(B)\leq\phi(B)\quad \textit{for all } B\in\mathcal{F}.$$

Then $\mathrm{p}^\phi(E)\geq\mu(E)$.

Proof. Let $\varepsilon>0$, and $\mathcal{G}_\varepsilon=\{B\in\mathcal{F}:\operatorname{diam}(B)\leq\varepsilon\}$. Using Lemma 2.2, from $\mathcal{G}_\varepsilon$ we can extract a disjoint family $\{B_i\}_{i\geq 0}$ such that $\mu(E)=\mu(E\cap(\bigcup_{i\geq 0}B_i))$. Therefore $\mu(E)=\sum_{i\geq 0}\mu(B_i)\leq\sum_{i\geq 0}\phi(B_i)\leq\mathrm{P}^\phi_\varepsilon(E)$. Letting $\varepsilon\to 0$ gives that $\mu(E)\leq\mathrm{P}^\phi(E)$.

This inequality is also true for any Borel subset of E. If $E=\bigcup_{k\geq 0}E_k$, then $\mu(E)\leq\sum_{k\geq 0}\mu(E_k)\leq\sum_{k\geq 0}\mathrm{P}^\phi(E_k)$. Taking the infimum over all Borel covers of E gives $\mu(E)\leq\mathrm{p}^\phi(E)$. □

Propositions 2.6 and 2.7 imply the following density theorem, symmetrical to Theorem 2.5:

Theorem 2.8. *Let E be a Borel set, μ be a Borel measure. Then*

$$\inf_{x\in E}\liminf_{\varepsilon\to 0}\frac{\mu(B(x,\varepsilon))}{\phi(B(x,\varepsilon))}\leq\frac{\mu(E)}{\mathrm{p}^\phi(E)}\leq\sup_{x\in E}\liminf_{\varepsilon\to 0}\frac{\mu(B(x,\varepsilon))}{\phi(B(x,\varepsilon))},\tag{10}$$

except in the cases of indetermination.

Proof. (a) Let D_1 be the left-hand side of (10). Assume that $D_1>0$. Let $0<A_1<D_1$, and

$$E_k=\left\{x\in E:\varepsilon\leq\frac{1}{k}\Rightarrow A_1\phi(B(x,\varepsilon))\leq\mu(B(x,\varepsilon))\right\}.$$

Proposition 2.6 implies that $A_1\mathrm{p}^\phi(E_k)\le\mu(E_k)$. Since $E=\bigcup_{k\ge1}E_k$ and E_k is increasing, $A_1\mathrm{p}^\phi(E)\le\mu(E)$. Then let A_1 tend to D_1.

(b) Let D_2 be the right-hand side of (10). Assume that $D_2<\infty$. Let $A_2>D_2$, and $\mathcal{F}=\{B\in\mathcal{B}_E:\mu(B)\le A_2\phi(B)\}$. The family $\mathcal{F}$ is a centred net over E. Proposition 2.7 gives $\mu(E)\le A_2\mathrm{p}^\phi(E)$. Then let A_2 tend to D_2. □

The relationships between the measures c^ϕ and p^ϕ are as usual:

Proposition 2.9. *For any Borel set E and $\phi\colon\mathcal{B}_E\to(0,\infty)$, the inequality $\mathrm{c}^\phi(E)\le\mathrm{p}^\phi(E)$ holds.*

Proof. Let $\varepsilon>0$, and $\mathcal{F}_\varepsilon=\{B\in\mathcal{B}_E:\mathrm{diam}(B)\le\varepsilon\}$. Since c^ϕ is a Borel measure, we may use Lemma 2.2 to extract a disjoint family $\{B_i\}_{i\ge0}$ from $\mathcal{F}_\varepsilon$ such that $\mathrm{c}^\phi(E\setminus\bigcup_{i\ge0}B_i)=0$. The set function $\mathrm{C}^\phi_\varepsilon$ is subadditive, so that

$$\mathrm{C}^\phi_\varepsilon(E)\le\mathrm{C}^\phi_\varepsilon\biggl(E\cap\bigcup_{i\ge0}B_i\biggr)+\mathrm{C}^\phi_\varepsilon\biggl(E\setminus\bigcup_{i\ge0}B_i\biggr).$$

From the general inequalities $\mathrm{C}^\phi_\varepsilon\le\mathrm{C}^\phi\le\mathrm{c}^\phi$, we deduce that $\mathrm{C}^\phi_\varepsilon(E\setminus\bigcup_{i\ge0}B_i)=0$. And $\mathrm{C}^\phi_\varepsilon(E\cap(\bigcup_{i\ge0}B_i))\le\sum_{i\ge0}\phi(B_i)\le\mathrm{P}^\phi_\varepsilon(E)$, so that

$$\mathrm{C}^\phi_\varepsilon(E)\le\mathrm{P}^\phi_\varepsilon(E).$$

Letting $\varepsilon\to0$ gives that $\mathrm{C}^\phi(E)\le\mathrm{P}^\phi(E)$.

For any $F\subset E$ we get $\mathrm{C}^\phi(F)\le\mathrm{P}^\phi(F)\le\mathrm{P}^\phi(E)$, so that $\mathrm{c}^\phi(E)\le\mathrm{P}^\phi(E)$. If $E=\bigcup_{n\ge0}E_n$, then $\mathrm{c}^\phi(E)\le\sum_{n\ge0}\mathrm{c}^\phi(E_n)\le\sum_{n\ge0}\mathrm{P}^\phi(E_n)$. Therefore $\mathrm{c}^\phi(E)\le\mathrm{p}^\phi(E)$. □

3. Local density inequalities

Local density inequalities are direct applications of the density theorems, where the measure μ is replaced by the Borel measures c^ϕ or p^ϕ restricted to E. Without looking for exhaustiveness we present here a few basic results. Again, our statements do not require any assumption on the determining function $\phi\colon\mathcal{B}_E\to(0,\infty)$.

Let E be a Borel set in $\mathbb{R}^D$. We make use of the classical notations

$$\underline{d}^\phi_\mathrm{c}(x,E)=\liminf_{\varepsilon\to0}\frac{\mathrm{c}^\phi(B(x,\varepsilon)\cap E)}{\phi(B(x,\varepsilon))},\quad \bar{d}^\phi_\mathrm{c}(x,E)=\limsup_{\varepsilon\to0}\frac{\mathrm{c}^\phi(B(x,\varepsilon)\cap E)}{\phi(B(x,\varepsilon))},$$
$$\underline{d}^\phi_\mathrm{p}(x,E)=\liminf_{\varepsilon\to0}\frac{\mathrm{p}^\phi(B(x,\varepsilon)\cap E)}{\phi(B(x,\varepsilon))},\quad \bar{d}^\phi_\mathrm{p}(x,E)=\limsup_{\varepsilon\to0}\frac{\mathrm{p}^\phi(B(x,\varepsilon)\cap E)}{\phi(B(x,\varepsilon))}.$$

Lemma 3.1. *Suppose that* $\mathrm{c}^\phi(E)<\infty$. *Let* F *be a Borel subset of* E.
If $\mathrm{c}^\phi(F)>0$, *then*

$$\inf_{x\in F} \bar{d}_{\mathrm{c}}^{\phi}(x,E)\leq 1\leq \sup_{x\in F} \bar{d}_{\mathrm{c}}^{\phi}(x,E). \tag{11}$$

If $\mathrm{p}^\phi(F)>0$, *then*

$$\inf_{x\in F} \underline{d}_{\mathrm{c}}^{\phi}(x,E)\leq \frac{\mathrm{c}^\phi(F)}{\mathrm{p}^\phi(F)}\leq \sup_{x\in F} \underline{d}_{\mathrm{c}}^{\phi}(x,E). \tag{12}$$

Proof. For any Borel set G, let $\mu(G)=\mathrm{c}^\phi(G\cap E)$. Then μ is a Borel measure, such that

$$\bar{d}_{\mathrm{c}}^{\phi}(x,E)=\limsup_{\varepsilon\to 0}\frac{\mu(B(x,\varepsilon))}{\phi(B(x,\varepsilon))},$$

and $\mu(F)=\mathrm{c}^\phi(F)$. A direct application of (6) gives (11). The condition $\mathrm{c}^\phi(F)>0$ ensures that the indetermination cases are avoided. Also,

$$\underline{d}_{\mathrm{c}}^{\phi}(x,E)=\liminf_{\varepsilon\to 0}\frac{\mu(B(x,\varepsilon))}{\phi(B(x,\varepsilon))},$$

and a direct application of (10) gives (12). □

Equations (6) and (10) may also be used to obtain the symmetric results:

Lemma 3.2. *Suppose that* $\mathrm{p}^\phi(E)<\infty$. *Let* F *be a Borel subset of* E *such that* $\mathrm{p}^\phi(F)>0$. *Then*

$$\inf_{x\in F} \bar{d}_{\mathrm{p}}^{\phi}(x,E)\leq \frac{\mathrm{p}^\phi(F)}{\mathrm{c}^\phi(F)}\leq \sup_{x\in F} \bar{d}_{\mathrm{p}}^{\phi}(x,E), \tag{13}$$

$$\inf_{x\in F} \underline{d}_{\mathrm{p}}^{\phi}(x,E)\leq 1\leq \sup_{x\in F} \underline{d}_{\mathrm{p}}^{\phi}(x,E). \tag{14}$$

Corollary 3.3. *If* $\mathrm{c}^\phi(E)<\infty$, *then* $\bar{d}_{\mathrm{c}}^{\phi}(x,E)=1$ c^ϕ*-almost everywhere, that is, for all* $x\in E$ *except on a subset* F *such that* $\mathrm{c}^\phi(F)=0$.

Proof. Let

$$F_k=\left\{x\in E:\bar{d}_{\mathrm{c}}^{\phi}(x,E)\leq 1-\frac{1}{k}\right\}\quad\text{and}\quad G_k=\left\{x\in E:\bar{d}_{\mathrm{c}}^{\phi}(x,E)\geq 1+\frac{1}{k}\right\}.$$

If $\mathrm{c}^\phi(F_k)>0$, then (11) implies that $1\leq 1-1/k$. This is impossible, so that $\mathrm{c}^\phi(F_k)=0$.

If $\mathrm{c}^\phi(G_k)>0$, then (11) implies that $1\geq 1+1/k$. This is impossible, so that $\mathrm{c}^\phi(G_k)=0$.

Taking countable unions of F_k and G_k, we deduce that

$$\mathrm{c}^{\phi}(\{x\in E:\bar{d}_{\mathrm{c}}^{\phi}(x,E)<1\})=\mathrm{c}^{\phi}(\{x\in E:\bar{d}_{\mathrm{c}}^{\phi}(x,E)>1\})=0.\quad\square$$

Similar arguments and (14) are used for the following consequence:

Corollary 3.4. *If* $\mathrm{p}^{\phi}(E)<\infty$*, then* $\underline{d}_{\mathrm{p}}^{\phi}(x,E)=1$ p^{ϕ}*-almost everywhere.*

The following results deal with the notion of *regularity*:

Corollary 3.5. *If* $\mathrm{p}^{\phi}(E)<\infty$*, then the following are equivalent*:
(i) $\mathrm{c}^{\phi}(E)=\mathrm{p}^{\phi}(E)$;
(ii) $\underline{d}_{\mathrm{c}}^{\phi}(x,E)=\bar{d}_{\mathrm{c}}^{\phi}(x,E)=1$ p^{ϕ}*-a.e.*;
(iii) $\underline{d}_{\mathrm{p}}^{\phi}(x,E)=\bar{d}_{\mathrm{p}}^{\phi}(x,E)=1$ p^{ϕ}*-a.e.*

Sets that fulfill these conditions may be called *strongly ϕ-regular.*

Proof. (i)$\Rightarrow$(ii) and (iii) For any Borel set $F\subset E$, $\mathrm{c}^{\phi}(F)\le\mathrm{p}^{\phi}(F)$. If $\mathrm{c}^{\phi}(E)=\mathrm{p}^{\phi}(E)$, then $\mathrm{c}^{\phi}(F)=\mathrm{p}^{\phi}(F)$. This implies that c^{ϕ}-a.e. is equivalent to p^{ϕ}-a.e. in this case.

Let $F_k=\{x\in E:\underline{d}_{\mathrm{c}}^{\phi}(x,E)\le 1-1/k\}$. If $\mathrm{p}^{\phi}(F_k)>0$, then (12) implies that $1\le 1-1/k$. This is impossible, so that $\mathrm{p}^{\phi}(F_k)=0$ for all k, or in other words $\underline{d}_{\mathrm{c}}^{\phi}(x,E)\ge 1$ p^{ϕ}-a.e.

Similarly, define $G_k=\{x\in E:\bar{d}_{\mathrm{p}}^{\phi}(x,E)\ge 1+1/k\}$ and use (13) to show that $\mathrm{p}^{\phi}(G_k)=0$ for all k, or in other words $\bar{d}_{\mathrm{p}}^{\phi}(x,E)\le 1$ p^{ϕ}-a.e.

With Corollaries 3.3 and 3.4, this suffices to prove (ii) and (iii).

(ii)$\Rightarrow$(i) Let $F=\{x\in E:\underline{d}_{\mathrm{c}}^{\phi}(x,E)\ge 1\}$. Equation (12) implies that $\mathrm{c}^{\phi}(F)\ge\mathrm{p}^{\phi}(F)$, hence $\mathrm{c}^{\phi}(F)=\mathrm{p}^{\phi}(F)$. By hypothesis, $\mathrm{p}^{\phi}(F)=\mathrm{p}^{\phi}(E)$, so that $\mathrm{p}^{\phi}(E\setminus F)=0$ and $\mathrm{c}^{\phi}(E\setminus F)=0$. Therefore $\mathrm{c}^{\phi}(F)=\mathrm{c}^{\phi}(E)$. We conclude that $\mathrm{c}^{\phi}(E)=\mathrm{p}^{\phi}(E)$.

(iii)$\Rightarrow$(i) This is shown by similar arguments, using (13). $\square$

Let us give an application, among many others, which deals with the Cartesian product of sets:

Proposition 3.6. *Let* $D_1\ge 1$ *and* $D_2\ge 1$*. We assume that* $\mathbb{R}^{D_1+D_2}$ *is endowed with a metric which is the Cartesian product of the metrics in* $\mathbb{R}^{D_1}$ *and* $\mathbb{R}^{D_2}$*, so that for all* $\varepsilon>0$*,* $x\in\mathbb{R}^{D_1}$ *and* $y\in\mathbb{R}^{D_2}$*, we have*:

$$B((x,y),\varepsilon)=B(x,\varepsilon)\times B(y,\varepsilon).$$

Let E be a Borel set of $\mathbb{R}^{D_1}$*, F be a Borel set of* $\mathbb{R}^{D_2}$*,* $\phi\colon\mathcal{B}_E\to(0,+\infty)$ *and* $\psi\colon\mathcal{B}_F\to(0,+\infty)$*.*

If $\mathrm{c}^{\phi}(E)<+\infty$ *and* $\mathrm{c}^{\psi}(F)<+\infty$*, then*

$$\mathrm{c}^{\phi}(E)\mathrm{c}^{\psi}(F)\le\mathrm{c}^{\phi\times\psi}(E\times F). \tag{15}$$

Notice that $\phi\times\psi$ is defined on $\mathcal{B}_{E\times F}$, and that

$$(\phi\times\psi)(B((x,y),\varepsilon))=\phi(B(x,\varepsilon))\psi(B(y,\varepsilon)).$$

Proof. Without loss of generality, we assume that $\mathrm{c}^\phi(E)>0$ and $\mathrm{c}^\psi(F)>0$. For all Borel sets $G_1\subset\mathbb{R}^{D_1}$ and $G_2\subset\mathbb{R}^{D_2}$, let $\mu(G_1)=\mathrm{c}^\phi(E\cap G_1)$ and $\nu(G_2)=\mathrm{c}^\psi(F\cap G_2)$. Then μ and ν are finite Borel measures. If $G_1\subset E$ and $G_2\subset F$, then $\mu(G_1)=\mathrm{c}^\phi(G_1)$, $\nu(G_2)=\mathrm{c}^\psi(G_2)$ and $(\mu\times\nu)(G_1\times G_2)=\mathrm{c}^\phi(G_1)\mathrm{c}^\psi(G_2)$.

The inequalities

$$\limsup_{\varepsilon\to0}\frac{(\mu\times\nu)(B((x,y),\varepsilon))}{(\phi\times\psi)(B((x,y),\varepsilon))}\le\limsup_{\varepsilon\to0}\frac{\mu(B(x,\varepsilon))}{\phi(B(x,\varepsilon))}\limsup_{\varepsilon\to0}\frac{\nu(B(y,\varepsilon))}{\psi(B(y,\varepsilon))}$$

are true for any $(x,y)\in E\times F$.

From Corollary 3.3 there exist $E'\subset E$ and $F'\subset F$ such that $\mu(E)=\mu(E')$, $\nu(F)=\nu(F')$ and for all $(x,y)\in E'\times F'$, $\bar{d}^\phi_{\mathrm{c}}(x,E)=\bar{d}^\psi_{\mathrm{c}}(y,F)=1$. Therefore for all such (x,y),

$$\limsup_{\varepsilon\to0}\frac{(\mu\times\nu)(B((x,y),\varepsilon))}{(\phi\times\psi)(B((x,y),\varepsilon))}\le\bar{d}^\phi_{\mathrm{c}}(x,E)\bar{d}^\psi_{\mathrm{c}}(y,F)\le1.$$

From (6) we deduce that $(\mu\times\nu)(E'\times F')\le\mathrm{c}^{\phi\times\psi}(E'\times F')$. Since $(\mu\times\nu)(E'\times F')=\mu(E)\nu(F)=\mathrm{c}^\phi(E)\mathrm{c}^\psi(F)$ and $\mathrm{c}^{\phi\times\psi}(E'\times F')\le\mathrm{c}^{\phi\times\psi}(E\times F)$ we obtain the inequality (15). □

4. Related dimensions

Let E be a bounded Borel set in $\mathbb{R}^D$. Let us call a *scale of functions* associated with E a family Φ of determining functions $\phi\colon\mathcal{B}_E\to(0,\infty)$, such that for any ϕ_1 and ϕ_2 in Φ, only the three following cases can occur:

(a) $\phi_1=\phi_2$;

(b) for any $x\in E$, $\lim_{\varepsilon\to0}(\phi_1/\phi_2)(B(x,\varepsilon))=0$;

(c) for any $x\in E$, $\lim_{\varepsilon\to0}(\phi_1/\phi_2)(B(x,\varepsilon))=+\infty$.

The measures $\mathrm{c}^\phi(E)$ and $\mathrm{p}^\phi(E)$ are significant only if ϕ belongs to such a scale. Then we may hope to associate a notion of *dimension* with Φ. We choose to consider families $\Phi=\{\phi_\alpha:\alpha\in\mathbb{R}\}$ parameterized by a real number α. To be more specific, let us consider throughout this paper the following type of determining functions:

Assumption 1. $\phi_\alpha(B)=p(B)q(B)^\alpha$, where $p\colon\mathcal{B}_E\to(0,\infty)$, $q\colon\mathcal{B}_E\to(0,\infty)$, and if

$$f(\varepsilon)=\sup\{q(B):B\in\mathcal{B}_E\text{ and }\operatorname{diam}(B)\le\varepsilon\},$$

then $\lim_{\varepsilon\to0}f(\varepsilon)=0$.

This property ensures that $\Phi=\{pq^\alpha:\alpha\in\mathbb{R}\}$ is a scale of functions.

Notation. We will write c^α instead of c^{ϕ_α} for simplicity. Similarly, C^α, P^α and p^α replace C^{ϕ_α}, P^{ϕ_α} and p^{ϕ_α}, respectively.

The classical theory deals with $p=1$ and $q=\mathrm{diam}$, but we will consider other scales Φ in the sequel.

Proposition 4.1. *Let* $\alpha<\beta$.
If $\mathrm{C}^\alpha(E)<\infty$, *then* $\mathrm{C}^\beta(E)=0$; *if* $\mathrm{C}^\beta(E)>0$, *then* $\mathrm{C}^\alpha(E)=\infty$.
The same results hold for c^α, P^α *and* p^α.

Proof. For every $\varepsilon>0$ and $B\in\mathcal{B}_E$ such that $\mathrm{diam}(B)\le\varepsilon$, we have $\phi_\alpha(B)\ge f(\varepsilon)^{\alpha-\beta}\phi_\beta(B)$. Therefore $\mathrm{C}^\alpha_\varepsilon(E)\ge f(\varepsilon)^{\alpha-\beta}\mathrm{C}^\beta_\varepsilon(E)$. Then let $\varepsilon\to0$. □

Dimensions. We may therefore define the following dimensions:

$$\begin{aligned}\dim_\Phi(E)&=\inf\{\alpha:\mathrm{c}^\alpha(E)=0\};\\ \Delta_\Phi(E)&=\inf\{\alpha:\mathrm{P}^\alpha(E)=0\};\\ \mathrm{Dim}_\Phi(E)&=\inf\{\alpha:\mathrm{p}^\alpha(E)=0\}.\end{aligned}$$

The properties of $\dim_\Phi$, Δ_Φ and Dim_Φ derive easily from those of the related measures or pre-measures. In particular we have the following facts:

Since c^α, P^α and p^α are increasing set functions, $\dim_\Phi$, Δ_Φ and Dim_Φ are also increasing set functions.

Since P^α is subadditive, Δ_Φ is *stable*, that is

$$\Delta_\Phi(E_1\cup E_2)=\max\{\Delta_\Phi(E_1),\Delta_\Phi(E_2)\}. \tag{16}$$

Since c^α and p^α are σ-subadditive, $\dim_\Phi$ and Dim_Φ are σ*-stable*, in other words

$$\dim_\Phi\Big(\bigcup_{n\ge0}E_n\Big)=\sup\{\dim_\Phi(E_n)\},\quad \mathrm{Dim}_\Phi\Big(\bigcup_{n\ge0}E_n\Big)=\sup\{\mathrm{Dim}_\Phi(E_n)\}. \tag{17}$$

This allows us to define $\dim_\Phi$ and Dim_ϕ on unbounded sets as well.

Since $\mathrm{c}^\alpha(E)\le\mathrm{p}^\alpha(E)\le\mathrm{P}^\alpha(E)$,

$$\dim_\Phi(E)\le\mathrm{Dim}_\Phi(E)\le\Delta_\Phi(E). \tag{18}$$

From Δ_Φ a direct definition of Dim_Φ may be derived:

Proposition 4.2. *For any Borel set E, let*

$$\hat\Delta_\Phi(E)=\inf\Big\{\sup\Delta_\Phi(E_n):E_n\ \textit{are bounded Borel sets, with}\ E=\bigcup_{n\ge0}E_n\Big\}.$$

Then

$$\hat{\Delta}_\Phi(E)=\mathrm{Dim}_\Phi(E). \tag{19}$$

Proof. Let E_n be a sequence of bounded sets covering E. Then

$$\mathrm{Dim}_\Phi(E)\le\sup_{n\ge0}\mathrm{Dim}_\Phi(E_n)\le\sup_{n\ge0}\Delta_\Phi(E_n).$$

This gives $\mathrm{Dim}_\Phi(E)\le\hat{\Delta}_\Phi(E)$.

For the other direction let $\alpha>\mathrm{Dim}_\Phi(E)$. Then $\mathrm{p}^\alpha(E)=0$, and there exists $\{E_n\}_{n\ge0}$ such that $E=\bigcup_{n\ge0}E_n$ and $\sum_{n\ge0}\mathrm{P}^\alpha(E_n)\le1$. For every n, $\mathrm{P}^\alpha(E_n)\le1$, so that $\Delta_\Phi(E_n)\le\alpha$. Therefore $\hat{\Delta}_\Phi(E)\le\alpha$. □

As for the measures c^α and p^α, a Borel measure μ spread on E may help to evaluate the value of the dimensions:

Theorem 4.3. *Let μ be a finite or σ-finite Borel measure such that $\mu(E)>0$, and*

$$\alpha_\mu(B)=\frac{\log\mu(B)-\log p(B)}{\log q(B)}.$$

Then

$$\inf_{x\in E}\liminf_{\varepsilon\to0}\alpha_\mu(B(x,\varepsilon))\le\dim_\Phi(E)\le\sup_{x\in E}\liminf_{\varepsilon\to0}\alpha_\mu(B(x,\varepsilon)), \tag{20}$$

$$\inf_{x\in E}\limsup_{\varepsilon\to0}\alpha_\mu(B(x,\varepsilon))\le\mathrm{Dim}_\Phi(E)\le\sup_{x\in E}\limsup_{\varepsilon\to0}\alpha_\mu(B(x,\varepsilon)). \tag{21}$$

Proof. Without loss of generality, assume that $\mu(E)<\infty$.

For (20), we make use of (6). Let α_1 be the left-hand side of (20), and assume that $\alpha_1>0$. Let $\beta\in(0,\alpha_1)$. For any $x\in E$,

$$\liminf_{\varepsilon\to0}\alpha_\mu(B(x,\varepsilon))>\beta\quad\Longrightarrow\quad\limsup_{\varepsilon\to0}\frac{\mu(B(x,\varepsilon))}{\phi_\beta(B(x,\varepsilon))}\le1.$$

The right inequality of (6) gives $\mathrm{c}^\beta(E)\ge\mu(E)$. Therefore $\dim_\Phi(E)\ge\beta$, so that $\dim_\Phi(E)\ge\alpha_1$.

Similar arguments are used to obtain the right inequality of (20) from the left inequality of (6). In the same way, (21) derives from (10). □

For further application, let us deal with the sum of two determining functions:

Proposition 4.4. *Let p_1 and p_2 be two functions $:\mathcal{B}_E\to(0,\infty)$, and q be as above. We consider the function scales*

$$\Phi_1=\{p_1q^\alpha:\alpha\in\mathbb{R}\},\quad\Phi_2=\{p_2q^\alpha:\alpha\in\mathbb{R}\}\quad\textit{and}\quad\Phi_1=\{(p_1+p_2)q^\alpha:\alpha\in\mathbb{R}\}.$$

Then

(22)
$$\max\{\dim_{\Phi_1}(E),\dim_{\Phi_2}(E)\}\le\dim_\Phi(E)\le\mathrm{Dim}_\Phi(E)=\max\{\mathrm{Dim}_{\Phi_1}(E),\mathrm{Dim}_{\Phi_2}(E)\}.$$

Proof. It suffices to show that

$$\mathrm{Dim}_\Phi(E)\le\max\{\mathrm{Dim}_{\Phi_1}(E),\mathrm{Dim}_{\Phi_2}(E)\}.$$

For any determining functions ϕ and ψ, the inequality $\mathrm{P}^{\phi+\psi}\le\mathrm{P}^\phi+\mathrm{P}^\psi$ is obtained through elementary arguments. This implies that

$$\Delta_\Phi(E)\le\max\{\Delta_{\Phi_1}(E),\Delta_{\Phi_2}(E)\}.$$

Let $\alpha>\mathrm{Dim}_{\Phi_i}(E)$ for $i=1,2$. We can find a cover $\{E_i\}_{i\ge0}$ of E such that $\sup\Delta_{\Phi_1}(E_i)<\alpha$, and a cover $\{F_j\}_{j\ge0}$ of E such that $\sup\Delta_{\Phi_2}(F_j)<\alpha$. Let $G_{ij}=E_i\cap F_j$. Then $\{G_{ij}\}_{i,j\ge0}$ is a cover of E, and for all i,j, $\Delta_{\Phi_1}(G_{ij})<\alpha$ and $\Delta_{\Phi_2}(G_{ij})<\alpha$. Therefore $\sup_{i,j}\Delta_\Phi(G_{ij})\le\alpha$. This proves that $\mathrm{Dim}_\Phi(E)\le\alpha$, and gives the desired inequality. □

5. A special study of Δ_Φ

The set function Δ_Φ becomes the well-known *box dimension* when $\phi_\alpha=\mathrm{diam}^\alpha$ [12]. Among important features of this dimension are

(i) its definition using balls of equal diameter;

(ii) the equality $\Delta(E)=\Delta(\overline{E})$, where $\overline{E}$ is the closure of E.

Let $\phi_\alpha(B)=p(B)q(B)^\alpha$ as before. With the help of some mild assumptions on p and q we will be able to recover these properties.

The function $f(\varepsilon)$ has been introduced in the last section. The following strengthens Assumption 1:

Assumption 2. There exists two constants $c_0,\alpha_0>0$ such that for all $B\in\mathcal{B}_E$, $f(\varepsilon)\le c_0\varepsilon^{\alpha_0}$.

But we also need a few more conditions on p and q:

Assumption 3. Both functions p and q are *weakly increasing*, in the following sense: There exists real constants a_1 and a_2 such that for all balls B and B' of $\mathcal{B}_E$, $B'\subset B$, $\mathrm{diam}(B)\le2\,\mathrm{diam}(B')$,

$$p(B')\le a_1p(B)\quad\text{and}\quad q(B')\le a_2q(B).$$

With $\phi_\alpha=pq^\alpha$, this inequality implies that

$$\phi_\alpha(B')\le a_1a_2^\alpha\phi_\alpha(B). \tag{23}$$

From now on we will assume that Assumptions 2 and 3 are satisfied.

5.1. Packing with equal balls

In this section E is bounded. The following set function is defined like $\mathrm{P}^\alpha_\varepsilon$, with the only difference that the packings are made up with balls of equal diameter. Let

$$\mathrm{Q}^\alpha_\varepsilon(E)=\sup\Big\{\sum_{i\geq 0}\phi_\alpha(B_i):B_i\in\mathcal{B}_E \text{ are disjoint and } \mathrm{diam}(B_i)=\varepsilon\Big\}.$$

As a function of ε, Q^α does not need to be monotonous. Hence there is no limit as $\varepsilon\to 0$ in general. Let

$$\mathrm{Q}^\alpha(E)=\limsup_{\varepsilon\to 0}\mathrm{Q}^\alpha_\varepsilon(E).$$

The statement of Proposition 4.1 is still true for Q: If $\alpha<\beta$, then

$$\mathrm{Q}^\alpha(E)<\infty \implies \mathrm{Q}^\beta(E)=0 \quad\text{and}\quad \mathrm{Q}^\beta(E)>0 \implies \mathrm{Q}^\alpha(E)=\infty.$$

From that, we deduce the existence of a critical index, or dimension. We are willing to show that this dimension is precisely Δ_Φ.

The inequality $\mathrm{Q}^\alpha\leq\mathrm{P}^\alpha$ is easily verified. But these two set functions are not equivalent in general.

Proposition 5.1. *Let $\alpha<\beta$. If $\mathrm{P}^\beta(E)>0$, then $\mathrm{Q}^\alpha(E)=\infty$.*

We need the following elementary lemma:

Lemma 5.2. *Let $\{B(x_i,\varepsilon)\}_{i\geq 0}$ be a family of disjoint balls, and $a\geq 1$. There exists an integer $K_1\leq(2a+1)^D$ such that the family $\{B(x_i,a\varepsilon)\}_{i\geq 0}$ may be separated into K_1 families, each of them made up with disjoint balls.*

Proof. For $i\neq j$, $B(x_i,a\varepsilon)\cap B(x_j,a\varepsilon)\neq\varnothing\Leftrightarrow B(x_j,\varepsilon)\subset B(x_i,(2a+1)\varepsilon)$. The ball $B(x_i,(2a+1)\varepsilon)$ cannot contain more than $(2a+1)^D$ disjoint balls of radius ε. Thus $B(x_i,a\varepsilon)$ cannot meet more than $(2a+1)^D$ balls $B(x_j,a\varepsilon)$. A standard argument achieves the proof. □

Proof of Proposition 5.1. It suffices to show that $\mathrm{Q}^\alpha(E)>0$. Let $\eta=\beta-\alpha$, $\varepsilon\in(0,1)$, and $a\in(0,\mathrm{P}^\beta(E))$. There exists an ε-packing $\mathcal{R}=\{B_i\}_{i\geq 0}$ of E such that $\sum_{i\geq 0}\phi_\beta(B_i)\geq a$. For any non-negative integer k, let

$$\mathcal{R}_k=\{B\in\mathcal{R}:2^{-k-1}<\mathrm{diam}(B)\leq 2^{-k}\}.$$

When $\varepsilon \le 2^{-k-1}$, we have $\mathcal{R}_k = \varnothing$. The $\mathcal{R}_k$ are such that $\bigcup_{k\ge 0}\mathcal{R}_k=\mathcal{R}$, so that $\sum_{k\ge 0}\sum\{\phi_\beta(B){:}B\in\mathcal{R}_k\}\ge a$. Let c_0 and α_0 be as in Assumption 2. From the equality

$$a=\sum_{k\ge 0} a(1-2^{-\alpha_0\eta})2^{-k\alpha_0\eta}$$

we deduce the existence of an integer k_0 such that

$$\sum\{\phi_\beta(B): B\in\mathcal{R}_{k_0}\}\ge a(1-2^{-\alpha_0\eta})2^{-k_0\alpha_0\eta}. \tag{24}$$

Let $C_j=B(x_j,\varepsilon_j)$ be the balls of $\mathcal{R}_{k_0}$. Let $C_j^*=B(x_j,2^{-k_0-1})$ and $C_j^{**}=B(x_j,2\varepsilon_j)$. Since $2^{-k_0-1}<2\varepsilon_j\le 2^{-k_0}$, $C_j\subset C_j^*\subset C_j^{**}$. As the C_j are disjoint, Lemma 5.2 implies that for some integer $K_1\le 5^D$, the family $\{C_j^{**}\}_{j\ge 0}$ may be divided into K_1 families of disjoint balls. The same is true for the family $\{C_j^*\}_{j\ge 0}$, so that

$$\sum_{j\ge 0}\phi_\alpha(C_j^*)\le K_1 \mathrm{Q}^\alpha_{2^{-k_0}}(E). \tag{25}$$

For every j, $\phi_\beta(C_j^*)\le f(2^{-k_0})^\eta \phi_\alpha(C_j^*)$. Using Assumption 2,

$$\phi_\beta(C_j^*)\le c_0^\eta 2^{-k_0\alpha_0\eta}\,\phi_\alpha(C_j^*).$$

Since p and q are weakly increasing, (23) gives

$$\phi_\beta(C_j)\le c_1 2^{-k_0\alpha_0\eta}\phi_\alpha(C_j^*) \tag{26}$$

with $c_1=a_1a_2^\beta c_0^\eta$. Putting together (24) and (26) shows that

$$a(1-2^{-\alpha_0\eta})\le 2^{k_0\alpha_0\eta}\sum_{j\ge 0}\phi_\beta(C_j)\le c_1\sum_{j\ge 0}\phi_\alpha(C_j^*).$$

With (25) we obtain

$$\mathrm{Q}^\alpha_{2^{-k_0}}(E)\ge \frac{a(1-2^{-\alpha_0\eta})}{c_1K_1}.$$

The right-hand side is strictly positive and does not depend on ε. Since $2^{-k_0}<4\varepsilon$, we deduce that $\limsup_{\varepsilon\to 0}\mathrm{Q}^\alpha_\varepsilon(E)>0$. □

Corollary 5.3. $\Delta_\Phi(E)=\inf\{\alpha{:}\mathrm{Q}^\alpha(E)=0\}$.

Proof. If $\alpha>\Delta_\Phi(E)$, then $\mathrm{P}^\alpha(E)=0$, so that $\mathrm{Q}^\alpha(E)=0$.

If $\mathrm{Q}^\alpha(E)=0$, then for all $\beta>\alpha$, $\mathrm{P}^\beta(E)=0$, so that $\beta\ge\Delta_\Phi(E)$. Therefore $\alpha\ge\Delta_\Phi(E)$. □

5.2. The dimension of $\overline{E}$

The inequality $\Delta_\Phi(E)\leq\Delta_\Phi(\overline{E})$ is always true. But when Assumption 3 is fulfilled, the converse inequality also holds. Let us first show a result on the pre-measures.

Lemma 5.4. *For every α, $\mathrm{Q}^\alpha(\overline{E})\leq a_1a_2^\alpha 9^D\mathrm{Q}^\alpha(E)$.*

Proof. Assume that $\mathrm{Q}^\alpha(\overline{E})>0$. Let $\varepsilon>0$ and $0<a<\mathrm{Q}^\alpha_{2\varepsilon}(\overline{E})$. Let $\{x_i\}_{i\geq 0}$ be a finite subset of $\overline{E}$ such that $d(x_i,x_j)>2\varepsilon$ for every $i\neq j$, and $\sum_{i\geq 0}\phi_\alpha(B(x_i,\varepsilon))>a$. Without loss of generality we assume that this sequence is maximal, in the sense that for any $x\in\overline{E}$, $d(x,\{x_i\}_{i\geq 0})\leq 2\varepsilon$. The balls $B(x_i,\varepsilon)$ are disjoint, and $\overline{E}\subset\bigcup_{i\geq 0}B(x_i,2\varepsilon)$.

For every i, let $y_i\in E$ be such that $d(x_i,y_i)\leq\varepsilon/2$, so that

$$B\Big(y_i,\frac{\varepsilon}{2}\Big)\subset B(x_i,\varepsilon)\subset B(y_i,2\varepsilon).$$

Letting $b=a_1a_2^\alpha$, (23) implies that

$$\phi_\alpha(B(x_i,\varepsilon))\leq b\phi_\alpha(B(y_i,2\varepsilon)).$$

Since the $B(y_i,\varepsilon/2)$ are disjoint, we make use of Lemma 5.2 to separate the family $\{B(y_i,2\varepsilon)\}_{i\geq 0}$ into $K\leq 9^D$ families made up with disjoint balls, so that

$$\sum_{i\geq 0}\phi_\alpha(B(y_i,2\varepsilon))\leq 9^D\mathrm{Q}^\alpha_{4\varepsilon}(E).$$

Gathering all intermediary results we obtain

$$a<\sum_{i\geq 0}\phi_\alpha(B(x_i,\varepsilon))\leq b\sum_{i\geq 0}\phi_\alpha(B(y_i,2\varepsilon))\leq b9^D\mathrm{Q}^\alpha_{4\varepsilon}(E).$$

Let a tend to $\mathrm{Q}^\alpha_{2\varepsilon}(\overline{E})$ and then $\varepsilon\to 0$ to get the result. □

This implies directly that $\Delta_\Phi(\overline{E})\leq\Delta_\Phi(E)$, hence

Proposition 5.5. *For any bounded Borel set E, $\Delta_\Phi(\overline{E})=\Delta_\Phi(E)$.*

As in [12], this result may be used to establish a relationship between $\Delta_\Phi(E)$ and $\mathrm{Dim}_\Phi(E)$ when E shows a homogeneous structure:

Corollary 5.6. *Let E be compact and $\alpha\in\mathbb{R}$ be such that for any open set V meeting E, $\Delta_\Phi(E\cap V)=\alpha$. Then*

$$\mathrm{Dim}_\Phi(E)=\Delta_\Phi(E)=\alpha.$$

Proof. It suffices to prove that $\hat{\Delta}_\Phi(E)\geq\alpha$. Let $\{E_n\}_{n\geq 0}$ be a sequence of Borel sets such that $E=\bigcup_{n\geq 0} E_n$. Baire's theorem implies that for some index N, E_N is somewhere dense in E: There exists an open set V such that $\overline{E}_N\cap V=E\cap V$. Then

$$\alpha=\Delta_\Phi(E\cap V)=\Delta_\Phi(\overline{E}_N\cap V)\leq\Delta_\Phi(\overline{E}_N).$$

Since $\Delta_\Phi(\overline{E}_N)=\Delta_\Phi(E_N)$, we finally get $\alpha\leq\sup_n \Delta_\Phi(E_n)$, and hence the required result. □

5.3. A special case: $\phi_\alpha(B)=\nu(B)\operatorname{diam}(B)^{\alpha-D}$

Let ν be a σ-finite Borel measure. Letting $p(B)=\nu(B)/\operatorname{diam}(B)^D$ and $q(B)=\operatorname{diam}(B)^\alpha$, it is easy to check that Assumptions 2 and 3 are verified. In particular, if $B'\subset B$ is such that $\operatorname{diam}(B)\leq 2\operatorname{diam}(B')$, then

$$p(B')=\frac{\nu(B')}{\operatorname{diam}(B')^D}\leq 2^D\frac{\nu(B)}{\operatorname{diam}(B)^D}=2^D p(B).$$

We may take $a_1=2^D$ and $a_2=1$.

An interesting case is $\nu=\mathrm{Vol}_D$, where $\phi_\alpha(B)\simeq\operatorname{diam}(B)^\alpha$. The corresponding dimension is the classical box dimension Δ. Let $E(\varepsilon)=\bigcup\{B(x,\varepsilon):x\in E\}$. We know [12] that Δ satisfies the equality

$$\Delta(E)=\limsup_{\varepsilon\to 0}\left(D-\frac{\log \mathrm{Vol}_D(E(\varepsilon))}{\log\varepsilon}\right).$$

Let us show that this result may be generalized.

Proposition 5.7. *Let $\phi_\alpha(B)=\nu(B)\operatorname{diam}(B)^{\alpha-D}$. Then*

$$\Delta_\Phi(E)=\limsup_{\varepsilon\to 0}\left(D-\frac{\log \nu(E(\varepsilon))}{\log\varepsilon}\right). \tag{27}$$

Proof. Denote by Δ' the right-hand side. It may be written as

$$\Delta'=\inf\{\alpha:\varepsilon^{\alpha-D}\nu(E(\varepsilon))\to 0\}.$$

(a) Let $\varepsilon>0$, and $\{x_i\}_i$ be a finite subset of E such that $d(x_i,x_j)>2\varepsilon$ if $i\neq j$. Since the balls $B(x_i,\varepsilon)$ are disjoint, $\sum_{i\geq 0}\nu(B(x_i,\varepsilon))\leq\nu(E(\varepsilon))$, so that $\sum_{i\geq 0}\phi_\alpha(B(x_i,\varepsilon))\leq\varepsilon^{\alpha-D}\nu(E(\varepsilon))$. Then $\mathrm{Q}^\alpha_{2\varepsilon}(E)\leq\varepsilon^{\alpha-D}\nu(E(\varepsilon))$. Letting $\varepsilon\to 0$ gives

$$\mathrm{Q}^\alpha(E)\leq\limsup_{\varepsilon\to 0}\varepsilon^{\alpha-D}\nu(E(\varepsilon)). \tag{28}$$

(b) On the other hand, it is possible to choose in E a subset $\{x_i\}_{i\geq 0}$ such that $d(x_i,x_j)>2\varepsilon$, if $i\neq j$, and for any $x\in E$, $d(x,\{x_i\}_{i\geq 0})\leq 2\varepsilon$. Let $B_i=B(x_i,3\varepsilon)$. Then $E(\varepsilon)\subset\bigcup_{i\geq 0}B_i$ and

$$\nu(E(\varepsilon))\leq\sum_{i\geq 0}\nu(B_i),$$

so that

$$\varepsilon^{\alpha-D}\nu(E(\varepsilon))\leq\sum_{i\geq 0}\varepsilon^{\alpha-D}\nu(B_i)=6^{D-\alpha}\sum_{i\geq 0}\phi_\alpha(B_i).$$

The balls $B(x_i,\varepsilon)$ are disjoint: Using Lemma 5.2, we can divide $\{B(x_i,3\varepsilon)\}_{i\geq 0}$ into $K\leq 7^D$ families, each made up of disjoint balls. Therefore

$$\sum_{i\geq 0}\phi^\alpha(B_i)\leq 7^D\mathrm{Q}^\alpha_{3\varepsilon}(E).$$

We deduce that $\varepsilon^{\alpha-D}\nu(E(\varepsilon))\leq 7^D 6^{D-\alpha}\mathrm{Q}^\alpha_{3\varepsilon}(E)$. Letting $\varepsilon\to 0$ shows that

$$\limsup_{\varepsilon\to 0}\varepsilon^{\alpha-D}\nu(E(\varepsilon))\leq 7^D 6^{D-\alpha}\mathrm{Q}^\alpha(E). \tag{29}$$

(c) Equations (28) and (29) prove that the set function $\limsup_{\varepsilon\to 0}\varepsilon^{\alpha-D}\nu(E(\varepsilon))$ is a *pre-measure* equivalent to Q^α. They have the same critical index. Therefore $\Delta'=\Delta_\Phi(E)$. □

Remark. The equality $\Delta_\Phi(E)=\Delta_\Phi(\overline{E})$ is much easier to prove in this case, by making use of Proposition 5.7. Indeed, it suffices to verify that $\overline{E}(\varepsilon)\subset E(2\varepsilon)$, so that

$$\varepsilon^{\alpha-D}\nu(\overline{E}(\varepsilon))\leq 2^{D-\alpha}(2\varepsilon)^{\alpha-D}\nu(E(2\varepsilon)).$$

This gives directly $\Delta_\Phi(\overline{E})\leq\Delta_\Phi(E)$.

6. One-sided measures and dimensions

Let V be a bounded open set in $\mathbb{R}^D$, with boundary ∂V such that $\mathrm{Vol}_D(\partial V)=0$. Let $W=\mathbb{R}^D\setminus\overline{V}$. From now on the notion of *inside* will refer to V; the notion of *outside* to the interior of the complement of V, which is W. Let us associate with V the Hausdorff function

$$\phi_\alpha(B)=\mathrm{Vol}_D(B\cap V)\,\mathrm{diam}(B)^{\alpha-D}\simeq\frac{\mathrm{Vol}_D(B\cap V)}{\mathrm{Vol}_D(B)}\,\mathrm{diam}(V)^\alpha$$

which belongs to the type studied in Section 5.3. It gives rise to a notion of *α-interior measure*. Theorems 2.5 and 2.8 are valid, as well as the local density results of Section 3. The family $\{\phi_\alpha\}_{\alpha\in\mathbb{R}}$ provides the *interior dimensions* $\dim_{\text{int}}$, Δ_{int}, and Dim_{int}. For E included in V they are equal to the classical dimensions. For E in W we get the dimensions of an empty set, which may be granted the value $-\infty$. The interesting case deals with the subsets of ∂V. Let us point out that the notion of interior box dimension Δ_{int} is known since a long time, see [3], [13], [14] and [15]. Following (27) it may be written as

$$\Delta_{\text{int}}(E)=\limsup_{\varepsilon\to 0}\left(D-\frac{\log \text{Vol}_D(E(\varepsilon)\cap V)}{\log\varepsilon}\right). \tag{30}$$

The inequalities

$$\dim_{\text{int}}(E)\le \text{Dim}_{\text{int}}(E)\le \Delta_{\text{int}}(E)$$

are always true.

We associate with W the Hausdorff function

$$\psi_\alpha(B)=\text{Vol}_D(B\cap W)\,\text{diam}(B)^{\alpha-D}\simeq\frac{\text{Vol}_D(B\cap W)}{\text{Vol}_D(B)}\,\text{diam}(V)^\alpha$$

which gives rise to the notion of the *exterior dimensions* $\dim_{\text{ext}}$, Δ_{ext}, and Dim_{ext}. The exterior box dimension may be written as

$$\Delta_{\text{ext}}(E)=\limsup_{\varepsilon\to 0}\left(D-\frac{\log \text{Vol}_D(E(\varepsilon)\cap W)}{\log\varepsilon}\right).$$

The relation $\phi_\alpha(B)+\psi_\alpha(B)\simeq\text{diam}(B)^\alpha$ implies that for any $E\subset\partial V$,

$$\Delta(E)=\max\{\Delta_{\text{int}}(E),\Delta_{\text{ext}}(E)\}, \tag{31}$$

but Δ_{int} and Δ_{ext} may take different values as shown in the quoted references. It is possible to construct an open set V such that $\Delta_{\text{int}}(\partial V)=1$ and $\Delta_{\text{ext}}(\partial V)=2$.

6.1. Main results

The interior and exterior Hausdorff and packing dimensions satisfy the following relations (see (22)):

$$\max\{\dim_{\text{int}}(E),\dim_{\text{ext}}(E)\}\le\dim(E)\le\text{Dim}(E)=\max\{\text{Dim}_{\text{int}}(E),\text{Dim}_{\text{ext}}(E)\}. \tag{32}$$

Theorem 4.3 is rewritten as follows:

Theorem 6.1. *Let E be a Borel subset of V, μ be a finite or σ-finite Borel measure such that $\mu(E)>0$, and*

$$\alpha_\mu(B)=\frac{\log\mu(B)}{\log\operatorname{diam}(B)}+D-\frac{\log\operatorname{Vol}_D(B\cap V)}{\log\operatorname{diam}(B)}.$$

Then

$$\inf_{x\in E}\liminf_{\varepsilon\to 0}\alpha_\mu(B(x,\varepsilon))\le\dim_{\rm int}(E)\le\sup_{x\in E}\liminf_{\varepsilon\to 0}\alpha_\mu(B(x,\varepsilon)), \tag{33}$$

$$\inf_{x\in E}\limsup_{\varepsilon\to 0}\alpha_\mu(B(x,\varepsilon))\le\operatorname{Dim}_{\rm int}(E)\le\sup_{x\in E}\limsup_{\varepsilon\to 0}\alpha_\mu(B(x,\varepsilon)). \tag{34}$$

The same results are true for the exterior dimensions, if we replace $B\cap V$ by $B\cap W$ in the expression of $\alpha_\mu(B)$.

Let us mention that the quantity $\log\operatorname{Vol}_D(B\cap\Omega)/\log\operatorname{diam}(B)-D$ is used in [5] to perform a multifractal analysis along the boundary of a domain Ω.

Symmetry. When a bounded open set V is such that for every $x\in\partial V$ and $\varepsilon>0$,

$$\dim_{\rm int}(B(x,\varepsilon)\cap\partial V)=\dim_{\rm ext}(B(x,\varepsilon)\cap\partial V)=\dim(B(x,\varepsilon)\cap\partial V),$$

and the same holds for the dimensions Dim and Δ, then ∂V may be called *laterally symmetric*. This is the case when ∂V is a C^1 curve, or a *self-similar* curve in the sense that ∂V is a finite union of self-similar arcs (like the well-known snowflake curve). A sufficient condition for lateral symmetry is the following: There exists constants c_1 and c_2 in $(0,1)$ such that for every $x\in\partial V$ and $\varepsilon>0$,

$$c_1\operatorname{Vol}_D(B(x,\varepsilon)\setminus V)\le\operatorname{Vol}_D(B(x,\varepsilon)\cap V)\le c_2\operatorname{Vol}_D(B(x,\varepsilon)\setminus V).$$

6.2. The interior dimension may be negative

In the classical theory the Hausdorff dimension belongs to $[0,D]$. This is not so for the interior dimension. Depending on the shape of ∂V, the dimension of a set $E\subset\partial V$ may take any negative value, or even the value $-\infty$. Let us consider four examples.

Example 1. Our set E is the single point $x_0=(0,0)$ in $\mathbb{R}^2$. Let $\alpha<0$.

For all $\varepsilon\in[0,1]$, let $f(\varepsilon)=\varepsilon^{1/(1-\alpha)}$, which is continuous and strictly increasing. Let Γ_1 be the graph of f on $[0,1]$, Γ_2 be the polygonal curve with vertices $(1,1)$, $(0,1)$ and x_0, and Γ be the simple closed curve $\Gamma_1\cup\Gamma_2$.

The interior of Γ is V. Since $f(\varepsilon)\ge\varepsilon$, the area of $B(x_0,\varepsilon)\cap V$ is equivalent to

$$\int_0^\varepsilon f^{-1}(t)\,dt\simeq\varepsilon^{2-\alpha}.$$

If μ is the discrete measure of mass 1 over $\{x_0\}$, a straightforward application of Theorem 6.1 gives

$$\dim_{\mathrm{int}}(\{x_0\}) = \mathrm{Dim}_{\mathrm{int}}(\{x_0\}) = 2 - \frac{\log \varepsilon^{2-\alpha}}{\log \varepsilon} = \alpha.$$

Example 2. Now we choose α and β such that $\alpha \le \beta < 0$, and for $x_0 = (0,0)$, we construct a simple closed curve Γ such that $\Gamma = \partial V$, $\dim_{\mathrm{int}}(\{x_0\}) = \alpha$, and $\mathrm{Dim}_{\mathrm{int}}(\{x_0\}) = \beta$.

Let $\gamma = 2 - (\alpha+\beta)/2$, $\delta = (\beta-\alpha)/2$, $G(\varepsilon) = \exp((\gamma + \delta \sin\log\varepsilon)\log\varepsilon)$, and $g(\varepsilon) = G'(\varepsilon)$. One can check that there exists $\varepsilon_0 > 0$ such that g is continuous and strictly increasing on $(0, \varepsilon_0]$, and $\lim_{\varepsilon\to 0} g(\varepsilon) = 0$. We take $f = g^{-1}$ on $(0, g(\varepsilon_0)]$, with $f(0) = 0$. Let Γ_1 be the graph of f on $[0, g(\varepsilon_0)]$, Γ_2 be the polygonal curve with vertices $(g(\varepsilon_0), \varepsilon_0)$, $(0, \varepsilon_0)$, and x_0, and Γ be the simple closed curve $\Gamma_1 \cup \Gamma_2$.

As in Example 1, the interior of Γ is V. Since $\beta < 0$, we have $f(\varepsilon) \ge \varepsilon$. The area of $B(x_0, \varepsilon) \cap V$ is equivalent to

$$\int_0^{\varepsilon} f^{-1}(t)\, dt = G(\varepsilon).$$

With the help of the discrete measure of mass 1 over $\{x_0\}$, we obtain

$$\dim_{\mathrm{int}}(\{x_0\}) = \liminf_{\varepsilon\to 0} \left(2 - \frac{\log G(\varepsilon)}{\log \varepsilon}\right) = 2 - \gamma - \delta = \alpha,$$

$$\mathrm{Dim}_{\mathrm{int}}(\{x_0\}) = \limsup_{\varepsilon\to 0} \left(2 - \frac{\log G(\varepsilon)}{\log \varepsilon}\right) = 2 - \gamma + \delta = \beta.$$

Example 3. In this example, we construct an open set V in $\mathbb{R}$, with a boundary K such that $\dim(K) = \log 2/\log 3$, $\dim_{\mathrm{int}}(K) \le 0$, and $\dim_{\mathrm{ext}}(K) \le 0$. This shows that the first inequality in (32) may be strict.

We denote the length by L (instead of Vol_1).

Let K be the triadic Cantor set in $[0,1]$.

Let $\mathcal{F}$ be the family of the complementary intervals of K: For all $n \ge 1$ there are 2^{n-1} open intervals of length 3^{-n} in $\mathcal{F}$. For every $I \in \mathcal{F}$, $\mathrm{rk}(I)$ is the *rank* of I in the triadic net, that is the integer such that $L(I) = 3^{-\mathrm{rk}(I)}$.

Let K_n be the set obtained by removing the first $2^n - 1$ intervals of $\mathcal{F}$ from $[0,1]$: K_n is the union of 2^n closed intervals of length 3^{-n}, $K_n \supset K_{n+1}$, and $\bigcap_{n\ge 1} K_n = K$.

Let $\{a_n\}_{n\ge 1}$ and $\{b_n\}_{n\ge 1}$ be two sequences of $\mathbb{N}$ such that $n \le a_n < b_n$, $a_n/b_n \to 0$, and $b_n/a_{n+1} \to 0$ (for example, take $a_n = 2^{n^2}$ and $b_n = 2^{n^2+n}$). Let $a_0 = 0$.

Construction of the subset $K' \subset K$. The set K_{a_n} is made up of a finite number of intervals. We can consider all points in K_{a_n} which are at distance $\ge \varepsilon_n$ from the boundary, for ε_n small enough. We choose $\varepsilon_n = 3^{-b_n}$ and construct the closed set

$$E_n = \{x \in K : B(x, \varepsilon_n) \subset K_{a_n}\}.$$

It is made up of 2^{a_n} intervals of length $3^{-a_n}-2\varepsilon_n$. Let

$$K'=\bigcup_{n\geq 1}\bigcap_{k\geq n}E_k=\{x\in K:\text{there exists an } n \text{ such that } k\geq n\Rightarrow B(x,\varepsilon_k)\subset K_{a_k}\}.$$

Showing that $\dim(K\setminus K')=0$. Let $\alpha>0$. The set $K\setminus E_k$ is a subset of $K_{a_k}\setminus E_k$, which itself is covered by 2^{a_k+1} intervals of length ε_k. Since $K\setminus K'\subset\bigcup_{k\geq n}(K\setminus E_k)$ for all n,

$$\mathrm{H}^{\alpha}_{2\varepsilon_n}(K\setminus K')\leq \mathrm{H}^{\alpha}_{2\varepsilon_n}\left(\bigcup_{k\geq n}(K\setminus E_k)\right)\leq\sum_{k\geq n}2^{a_k+1}\varepsilon_k^{\alpha}.$$

Since $b_k\geq k$ and $a_k/b_k\to 0$, the series $\sum_{k\geq 1}2^{a_k}\varepsilon_k^{\alpha}$ converges, so that $\mathrm{H}^{\alpha}_{2\varepsilon_n}(K\setminus K')\to 0$ as $n\to\infty$. This proves that $\mathrm{H}^{\alpha}(K\setminus K')=0$, and $\dim(K\setminus K')\leq 0$.

Construction of the open set V. Let

$$\begin{aligned}F_1(k)&=\bigcup\{I\in\mathcal{F}:a_{2k}\leq \mathrm{rk}(I)<a_{2k+1}\};\\ F_2(k)&=\bigcup\{I\in\mathcal{F}:a_{2k+1}\leq \mathrm{rk}(I)<a_{2k+2}\};\\ G(n)&=\bigcup\{I\in\mathcal{F}:\mathrm{rk}(I)\geq n\}.\end{aligned}$$

We define $V=\bigcup_{k\geq 0}F_1(k)$ and $W=\bigcup_{k\geq 0}F_2(k)$. These are two disjoint open sets such that $K=\partial V=\partial W$ and $K\cup V\cup W=[0,1]$. For all n, $K_n=K\cup G(n+1)$ and $L(G(n))=(2/3)^{n-1}$.

The interior and exterior dimensions of K'. Let $x\in K'$. There exists n such that $k\geq n\Rightarrow B(x,\varepsilon_k)\subset K_{a_k}$. Let $k\geq n$. Then the following are true:

(i) $B(x,\varepsilon_{2k})\subset K\cup G(a_{2k}+1)$, so that $B(x,\varepsilon_{2k})\cap W\subset\bigcup_{i\geq k}F_2(i)\subset G(a_{2k+1})$. Thus $L(B(x,\varepsilon_{2k})\cap W)\leq(2/3)^{a_{2k+1}-1}$.

(ii) $B(x,\varepsilon_{2k+1})\subset K\cup G(a_{2k+1}+1)$, so that $B(x,\varepsilon_{2k+1})\cap V\subset\bigcup_{i\geq k+1}F_1(i)\subset G(a_{2k+2})$. Thus $L(B(x,\varepsilon_{2k+1})\cap V)\leq(2/3)^{a_{2k+2}-1}$.

Let ν be the canonical measure on K: For any $x\in K$, $\nu(B(x,\varepsilon))\simeq\varepsilon^{\log 2/\log 3}$, so that $\nu(B(x,\varepsilon_n))\geq c2^{-b_n}$ for some constant c. Since $a_{2k+1}/b_{2k}\to\infty$, we deduce that

$$\begin{aligned}&\limsup_{k\to\infty}\left(\frac{\log\nu(B(x,\varepsilon_{2k}))}{\log\varepsilon_{2k}}+1-\frac{\log L(B(x,\varepsilon_{2k})\cap W)}{\log\varepsilon_{2k}}\right)\\ &\qquad\leq\lim_{k\to\infty}\left(\frac{1}{-b_{2k}\log 3}\left(-b_{2k}\log 2-a_{2k+1}\log\tfrac{2}{3}\right)\right)+1=-\infty.\end{aligned}$$

Since this is true for any $x\in K'$, $\dim_{\mathrm{ext}}(K')=-\infty$.

Similarly,

$$\limsup_{k\to\infty}\left(\frac{\log\nu(B(x,\varepsilon_{2k+1}))}{\log\varepsilon_{2k+1}}+1-\frac{\log L(B(x,\varepsilon_{2k+1})\cap V)}{\log\varepsilon_{2k+1}}\right)$$
$$\leq\lim_{k\to\infty}\left(\frac{1}{-b_{2k+1}\log 3}\left(-b_{2k+1}\log 2-a_{2k+2}\log\tfrac{2}{3}\right)\right)+1=-\infty,$$

so that $\dim_{\mathrm{int}}(K')=-\infty$.

Conclusion. Using the stability of $\dim_{\mathrm{int}}$,

$$\dim_{\mathrm{int}}(K)=\max\{\dim_{\mathrm{int}}(K'),\dim_{\mathrm{int}}(K\setminus K')\}\leq\max\{\dim_{\mathrm{int}}(K'),\dim(K\setminus K')\}=0.$$

Similar methods give $\dim_{\mathrm{ext}}(K)\leq 0$.

Example 4. From Example 3, we can derive an example in $\mathbb{R}^2$ with the help of Cartesian products. Let us construct an open set V such that

$$\dim_{\mathrm{int}}(\partial V)=\dim_{\mathrm{ext}}(\partial V)=1\quad\text{and}\quad\dim(\partial V)=1+\frac{\log 2}{\log 3}.$$

Let K, K', $F_1(k)$, and $F_2(k)$ in $[0,1]$ as before. Let us define

$$E=K\times[0,1],\quad\widetilde{F}_1=\bigcup_{k\geq 0}F_1(k)\times\{0\}\quad\text{and}\quad\widetilde{F}_2=\bigcup_{k\geq 0}F_2(k)\times\{1\}.$$

The sets $\widetilde{F}_1$ and $\widetilde{F}_2$ are unions of horizontal segments making "bridges" between the vertical segments of E, at levels 0 and 1, respectively.

We complete the closed set $E\cup\widetilde{F}_1\cup\widetilde{F}_2$ by some arc with endpoints $(0,0)$ and $(1,0)$, so as to form a closed curve of interior V which looks like a "comb" with irregular teeth (Figure 1). The open set W is $\mathbb{R}^2\setminus\overline{V}$.

Let ν_1 be the canonical measure on K, ν_2 be the Lebesgue measure on $[0,1]$, and $\nu=\nu_1\times\nu_2$. Then ν has support E and for all $x\in E$, $\nu(B(x,\varepsilon))\simeq\varepsilon^{1+\log 2/\log 3}$. Using ν and Theorem 6.1, calculations give

$$\dim((K\setminus K')\times[0,1])=1\quad\text{and}\quad\dim_{\mathrm{int}}(K'\times[0,1])=\dim_{\mathrm{ext}}(K'\times[0,1])=-\infty,$$

so that

$$\dim_{\mathrm{int}}(E)=\dim_{\mathrm{ext}}(E)\leq 1,$$

whilst $\dim(E)=1+\log 2/\log 3$. The same results are true for ∂V.

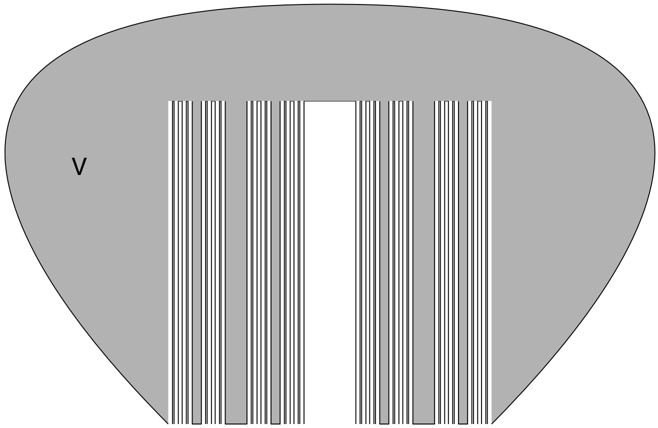

Figure 1. A fractal comb, enclosed by a closed curve $\Gamma = \partial V$. The one-sided dimensions $\dim_{\rm int}(\Gamma)$ and $\dim_{\rm ext}(\Gamma)$ are 1, whilst $\dim(\Gamma) = 1 + \log 2/|\log a|$. Here we chose $a > 1/3$ to make the picture more legible.

7. Using other determining functions

In the last section we used the determining function ϕ_α defined as $\phi_\alpha(B) = \mathrm{Vol}_D(B\cap V)\,\mathrm{diam}(B)^{\alpha-D}$ to obtain one-sided measures and dimensions. This is not the only possible choice. For example, let us take $p(B)=1$ and $q(B)=\mathrm{Vol}_D(B\cap V)^{1/D}$, so that

$$\phi_\alpha(B) = \mathrm{Vol}_D(B\cap V)^{\alpha/D}.$$

For any $E\subset V$, the family $\{\phi_\alpha\}_{\alpha\in\mathbb{R}}$ will provide measures and dimensions equivalent to the classical ones. It fulfills Assumption 2, with $\alpha_0=1$. Since p and q are increasing ($a_1=a_2=1$), Assumption 3 is also satisfied, which means that all the previous results and definitions remain valid, up to and including Section 5.2. In particular, the set function α_μ in Theorem 4.3 may be written as

$$\alpha_\mu(B) = D\frac{\log \mu(B)}{\log \mathrm{Vol}_D(B\cap V)}.$$

New notions of interior and exterior dimensions in ∂V stem from this determining function. As could be expected the dimension values are not the same as in Section 6. A major difference lies in the following result:

Proposition 7.1. *With V and ϕ_α as above,*

$$0 \le \dim_{\rm int}(E) \le \mathrm{Dim}_{\rm int}(E) \le \Delta_{\rm int}(E) \le D$$

for any $E\neq\varnothing$, $E\subset\partial V$.

Proof. Since $\mathrm{Vol}_D(B\cap V)\le\mathrm{Vol}_D(B)$, we find easily that $\Delta_{\mathrm{int}}(E)\le D$. Therefore it suffices to show that $\dim_{\mathrm{int}}(E)\ge 0$. Since $\dim_{\mathrm{int}}$ is increasing, all we have to do is to prove this inequality when $E=\{x_0\}$, $x_0\in\partial V$.

Indeed, for such x_0 we have $\mathrm{Vol}_D(B(x_0,\varepsilon)\cap V)>0$ for any $\varepsilon>0$, so that $\mathrm{C}^0(\{x_0\})=1$. This implies that $\mathrm{c}^0(\{x_0\})=1$, and $\dim_{\mathrm{int}}(\{x_0\})=0$. □

From the last result we could find this new determining function more "natural" than the one studied in Section 6. One drawback is that there is no such simple formula as (30) for Δ_{int}.

For a set E consisting of one point, $\dim_{\mathrm{int}}(E)=\mathrm{Dim}_{\mathrm{int}}(E)=0$, and the same is true symmetrically for $\dim_{\mathrm{ext}}$ and $\mathrm{Dim}_{\mathrm{ext}}$. For the Cantor set K of Example 3 (Section 6), we may verify that $\dim_{\mathrm{int}}(K)=\dim_{\mathrm{ext}}(K)=0$ and $\dim(K)=\log 2/\log 3$. For the fractal comb of Example 4, $\dim_{\mathrm{int}}(\partial V)=\dim_{\mathrm{ext}}(\partial V)=1$ and $\dim(\partial V)=1+\log 2/\log 3$.

As a final remark, if two families of determining functions ϕ_α and ψ_α satisfy Assumptions 2 and 3, then the same is true for any product of the kind $\phi_\alpha^r\psi_\alpha^s$, where $r>0$ and $s>0$. In theory we could find many suitable types of determining function, each of them giving rise to a notion of one-sided dimension. But let us keep in mind that a determining function ϕ_α has an interest only when it is a generalization of the classical one, more precisely when the following condition is verified:

There exists constants $0<c_1\le c_2$ such that for any ball $B\subset V$,

$$c_1\,\mathrm{diam}(B)^\alpha\le\phi_\alpha(B)\le c_2\,\mathrm{diam}(B)^\alpha.$$

We could call a family of such Hausdorff functions *normal*. Both families tackled in Sections 6 and 7 are normal. If ϕ_α and ψ_α are normal, then $\phi_\alpha^r\psi_\alpha^s$ is normal only if $r+s=1$.

References

1. Billingsley, P., *Ergodic Theory and Information*, Wiley, New York, 1965.
2. Falconer, K. J., *The Geometry of Fractal Sets*, Cambridge Tracts in Mathematics **85**, Cambridge University Press, Cambridge, 1986.
3. Grebogi, C., McDonald, S. W., Ott, E. and Yorke, J. A., Exterior dimension of fat fractals, *Phys. Lett. A* **110** (1985), 1–4; correction in *Phys. Lett. A* **113** (1986), 495.
4. Hausdorff, F., Dimension und äußeres Maß, *Math. Ann.* **79** (1918), 157–179.
5. Heurteaux, Y. and Jaffard, S., Multifractal Analysis of Images: New connexions between Analysis and Geometry, in *Proc. of the NATO-ASI Conference on Imaging for Detection and Identification*, pp. 169–194, Springer, Berlin–Heidelberg, 2008.

6. Morse, A. P. and Randolph, J. F., The ϕ rectifiable subsets of the plane, *Trans. Amer. Math. Soc.* **55** (1944), 236–305.
7. Olsen, L., A multifractal formalism, *Adv. Math.* **116** (1995), 82–196.
8. Peyrière, J., A vectorial multifractal formalism, in *Fractal Geometry and Applications: a Jubilee of Benoît Mandelbrot*, Proc. Sympos. Pure Math. **72**, pp. 217–230, Amer. Math. Soc., Providence, RI, 2004.
9. Saint Raymond, X. and Tricot, C., Packing regularity of sets in n-space, *Math. Proc. Cambridge Philos. Soc.* **103** (1988), 133–145.
10. Taylor, S. J. and Tricot, C., Packing measure, and its evaluation for a Brownian path, *Trans. Amer. Math. Soc.* **288** (1985), 679–699.
11. Taylor, S. J. and Tricot, C., The packing measure of rectifiable subsets of the plane, *Math. Proc. Cambridge Philos. Soc.* **99** (1986), 285–296.
12. Tricot, C., Two definitions of fractional dimension, *Math. Proc. Cambridge Philos. Soc.* **91** (1982), 57–74.
13. Tricot, C., The geometry of the complement of a fractal set, *Phys. Lett. A* **114** (1986), 430–434.
14. Tricot, C., Dimensions aux bords d'un ouvert, *Ann. Sci. Math. Québec* **11** (1987), 205–235.
15. Tricot, C., *Curves and Fractal Dimension*, Springer, New York, 1995.

Claude Tricot
Département de Mathématiques
Université Clermont-Ferrand II
FR-63177 Aubière
France
claude.tricot@math.univ-bpclermont.fr

Received January 21, 2008
published online September 26, 2008