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A long $\mathbb{C}^2$ which is not Stein

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Abstract. We construct a 2-dimensional complex manifold X which is the increasing union of proper subdomains that are biholomorphic to $\mathbb{C}^2$, but X is not Stein.

1. Introduction

We will address the following question (see for instance [3]): Is any "long" $\mathbb{C}^2$ biholomorphic to $\mathbb{C}^2$? (A complex manifold is a long $\mathbb{C}^2$ if it is the increasing union of proper subsets which are biholomorphic to $\mathbb{C}^2$). The answer is negative, and we will prove the stronger result:

Theorem 1.1. *A long $\mathbb{C}^2$ need not be biholomorphic to $\mathbb{C}^2$. In particular there exists a complex manifold X with the following two properties*:

(1) $X=\bigcup_{i=0}^{\infty} X_i$, $X_i\subset X_{i+1}$, $X_i\approx\mathbb{C}^2$;

(2) *X is not Stein.*

The theorem then also gives a negative answer to the *union problem* in dimension 2: *If X is an increasing limit of Stein manifolds, need X be Stein*? (Note that the X_j's also can be taken to be balls or polydisks, or whatever one can use to exhaust $\mathbb{C}^2$.) Fornæss [2] gave a negative answer to the latter question in dimension 3, and we will use the same idea of proof. Whereas in [2] a main ingredient was a construction by Wermer [4] of a non-Runge polydisk in $\mathbb{C}^3$ we will use the construction of a non-Runge Fatou–Bieberbach domain in $\mathbb{C}^2$, [6].

On the other hand if we assume a "Runge-pair hypothesis" we get the following result.

Theorem 1.2. *If $X=\bigcup_{i=0}^{\infty} X_i$ is a long $\mathbb{C}^2$ and each (X_i, X_{i+1}) is a Runge-pair, then X is biholomorphic to $\mathbb{C}^2$.*

Sketch of proof. The proof of this theorem is the same as that of Proposition 3 in [5], where we proved it in the case of X being contained in $\mathbb{C}^2$. The main point is to use Andersen–Lempert theory to inductively build a biholomorphism between X and $\mathbb{C}^2$: if $\varphi_j\colon X_j\to\mathbb{C}^2$ and $\varphi_{j+1}\colon X_{j+1}\to\mathbb{C}^2$ are biholomorphisms, then $\varphi_j\circ\varphi_{j+1}^{-1}$ can be approximated by an automorphism of $\mathbb{C}^2$, and so φ_{j+1} can be corrected to approximate φ_j, [1]. $\square$

Recall that, by definition, if X is a Stein manifold, then for all compact sets $K\subset X$ we have that the hull $\widehat{K}_{\mathcal{O}(X)}$ is compact in X, where

$$\widehat{K}_{\mathcal{O}(X)} := \{x\in X\,;\, |f(x)|\le \|f\|_K \text{ for all } f\in\mathcal{O}(X)\}.$$

(As usual we drop the subscript $\mathcal{O}(\mathbb{C}^2)$ if $X=\mathbb{C}^2$.)

2. Construction

Let us first recall a construction of an increasing sequence of complex manifolds. For each $i\in\mathbb{N}$ assume that we have a complex manifold X_i of dimension 2, and a holomorphic embedding $\varphi_i\colon X_i\hookrightarrow X_{i+1}$ such that $\varphi_i(X_i)$ is an open subset of X_{i+1}. In that case we can define a limiting manifold $\overrightarrow{X}$ as follows: Define an equivalence relation by $(x,X_i)\sim(y,X_k)$ if one of the following holds:

(a) $i=k$ and $x=y$,

(b) $k>i$ and $\varphi_{k-1}\circ\ldots\circ\varphi_i(x)=y$, or

(c) $i>k$ and $\varphi_{i-1}\circ\ldots\circ\varphi_k(y)=x$.

We call the set of eqivalence classes $\overrightarrow{X}$. For each i we may define an injective map $\psi_i\colon X_i\to\overrightarrow{X}$ simply by $\psi_i(x)=[(x,X_i)]$, and we let $([X_i],\psi_i^{-1})$ be local charts on $\overrightarrow{X}$, where $[X_i]:=\psi_i(X_i)$. The following diagram commutes (for all $k>j$):

$$\begin{array}{ccc} [X_j] & \xhookrightarrow{\text{inclusion}} & [X_k] \\ \Big\downarrow{\scriptstyle \psi_j^{-1}} & & \Big\downarrow{\scriptstyle \psi_k^{-1}} \\ X_j & \xhookrightarrow{\varphi_{k-1}\circ\ldots\circ\varphi_j} & X_k. \end{array}$$

This shows that the local charts define a complex structure on $\overrightarrow{X}$. We have that $\overrightarrow{X}$ is Hausdorff and has a countable base for the topology.

The construction relies on the following fact, which is the content of [6].

Lemma 2.1. *There exists a compact set $Y\subset\mathbb{C}^2$ and a Fatou–Bieberbach domain Ω, $Y\subset\Omega\subset\mathbb{C}^*\times\mathbb{C}$, such that the following hold*:

(i) *The origin is contained in $\widehat{Y}$*;

(ii) *For any open set $U\subset\mathbb{C}^*\times\mathbb{C}$ there exists a $G\in\operatorname{Aut}_{\mathrm{hol}}(\mathbb{C}^*\times\mathbb{C})$ such that $Y\subset G(U)$.*

The construction of the set Y is in Section 2 of [6], and (ii) is Lemma 3.1 of [6]. The existence of Ω then follows since $\mathbb{C}^*\times\mathbb{C}$ admits Fatou–Bieberbach domains.

Proof of Theorem 1.1. We will define a limit of complex manifolds X_i as described above with each $X_i=\mathbb{C}^2$. Let $F\colon \mathbb{C}^2\to\Omega$ be a Fatou–Bieberbach map corresponding to the domain Ω in Lemma 2.1.

We will define the maps φ_i inductively, and we start by letting $X_0=\mathbb{C}^2$ and $\varphi_0\colon X_0\to X_1=\mathbb{C}^2$ be defined by $\varphi_0:=F$. Choose a compact set $K\subset X_0$ with interior such that $\varphi_0(K)\supset Y$.

The inductive assumption will be that we have choosen maps $\varphi_j\colon \mathbb{C}^2=X_j\to X_{j+1}=\mathbb{C}^2$ for $j=0,...,N$, $\varphi_j(X_j)\subset\mathbb{C}^*\times\mathbb{C}$, and

$$(*)\qquad Y\subset\varphi_k\circ\cdots\circ\varphi_0(K)$$

for all $k\le N$. It is clear that Lemma 2.1 allows us to pass from step N to step $N+1$; define $\varphi_{N+1}:=G\circ F$ for a suitable G from Lemma 2.1(ii).

Now let (X_i,φ_i) be a collection constructed inductively like this, i.e., we get $(*)$ for all k, and let $\overrightarrow{X}$ be the limiting manifold.

In local coordinate ψ_j^{-1} we have that

(A) $\psi_j^{-1}([X_j])=\mathbb{C}^2$, and

(B) $\psi_j^{-1}([X_{j-1}])\subset\mathbb{C}^*\times\mathbb{C}$.

Let $[K]\subset\overrightarrow{X}$ denote the set $\psi_0(K)$. Then $[K]$ is a compact subset of $\overrightarrow{X}$, since it is compact in the chart $[X_0]$. The final piece of information we need is that

(C) $\psi_j^{-1}([K])\supset Y$ for all $j\in\mathbb{N}$.

This is seen from the commutative diagram since we have $(*)$.

We can now show that $\overrightarrow{X}$ *is not holomorphically convex*. To show this we shall demonstrate that $\widehat{[K]}_{\mathcal{O}(X)}$ is not compact in $\overrightarrow{X}$. It is enough to show that

$$(**)\qquad \widehat{[K]}_{\mathcal{O}([X_j])}\cap([X_j]\setminus[X_{j-1}])\neq\varnothing$$

for all $j\ge1$.

To see this we use the local coordinate ψ_j^{-1}. By Lemma 2.1 we have that the origin, o, is contained in $\widehat{Y}$ which by (C) implies that o is contained in $\widehat{\psi_j^{-1}([K])}$. The claim $(**)$ then follows immediately from (B). $\square$

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