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A note on the acyclicity of the Koszul complex of a module

José M. Giral and Francesc Planas-Vilanova

Abstract. We prove the vanishing of the Koszul homology group $H_\mu(\mathrm{Kos}(M)_\mu)$, where μ is the minimal number of generators of M. We give a counterexample that the Koszul complex of a module is not always acyclic and show its relationship with the homology of commutative rings.

1. Introduction

The acyclicity of the Koszul complex $\mathrm{Kos}(M)=\Lambda(M)\otimes_A S(M)$ of a module M over a commutative ring A, for a flat module M or a $\mathbb{Q}$-algebra A, has been known for many years (see e.g. [2] or [10]). Since then, and to our knowledge, no new general results have been shown. Recently, F. Sancho de Salas conjectured that the Koszul complex $\mathrm{Kos}(M)$ of a module M over a sheaf of rings $\mathcal{O}$ is always acyclic ([11, Conjecture 2.3]). The purpose of this note is to shed some light on this intriguing question. We first prove the vanishing of a specific Koszul homology group in full generality. Then we give a counterexample, minimal in some sense, to the conjecture of F. Sancho de Salas. Finally we discuss a particular case of flat dimension one and show its relationship with the André–Quillen homology theory.

Let A be a commutative ring, M be an A-module and $\mathrm{Kos}(M)=\Lambda(M)\otimes S(M)$ be its Koszul complex, where $\Lambda(M)$ and $S(M)$ stand for the exterior and symmetric algebras of M. Recall that $\mathrm{Kos}(M)=\bigoplus_{n\geq 0}\mathrm{Kos}(M)_n$ is a graded complex whose nth graded component $\mathrm{Kos}(M)_n$ is

$$0\longrightarrow\Lambda^n(M)\xrightarrow{\partial_{n,0}}\Lambda^{n-1}(M)\otimes M\xrightarrow{\partial_{n-1,1}}\dots\xrightarrow{\partial_{2,n-2}}M\otimes S^{n-1}(M)\xrightarrow{\partial_{1,n-1}}S^n(M)\longrightarrow 0.$$

Here $\Lambda^p(M)\otimes S^q(M)$ is the pth piece of the chain complex $\mathrm{Kos}(M)_n$, $p+q=n$. The differentials are defined as follows: if $x_1,\dots,x_p$ and $y_1,\dots,y_q$ are in M, then

$$\partial_{p,q}((x_1\wedge\dots\wedge x_p)\otimes(y_1\dots y_q))=\sum_{i=1}^{p}(-1)^{i+1}(x_1\wedge\dots\wedge x_{i-1}\wedge x_{i+1}\wedge\dots\wedge x_p)\otimes(x_iy_1\dots y_q).$$

Clearly, the 0th homology group $H_0(\mathrm{Kos}(M))$ is the base ring A. As said before, for $n\geq 1$, $\mathrm{Kos}(M)_n$ is acyclic provided that M is a flat A-module or n is invertible in A ([2, AX, Section 9.3, Example 1 and Proposition 3] and the solution to [2, AX, Section 9, Exercise 1] show that n invertible implies $\mathrm{Kos}(M)_n$ is acyclic). On the other hand, using general properties of the symmetric algebra, one can see that the first homology group $H_1(\mathrm{Kos}(M)_n)$ is zero for all $n\geq 1$ (see e.g. [9, Lemma 2.5]). In particular, if $H_p(\mathrm{Kos}(M)_n)$ were nonzero, then necessarily $p\geq 2$, $n\geq 2$ and the minimal number of generators $\mu=\mu(M)$ of M would have to be $\mu\geq 2$.

2. Vanishing of $H_\mu(\mathrm{Kos}(M)_\mu)$

Let $\mu\geq 2$ be the minimal number of generators of M. Then $H_\mu(\mathrm{Kos}(M)_\mu)=0$. Indeed, first consider the antisymmetrization homomorphism

$$a_\mu\colon \Lambda^\mu(M)\longrightarrow M^{\otimes\mu}$$

defined by

$$a_\mu(u_1\wedge\ldots\wedge u_\mu)=\sum_\sigma \varepsilon_\sigma u_{\sigma(1)}\otimes\ldots\otimes u_{\sigma(\mu)},$$

where $u_1,\ldots,u_\mu$ are in M, σ runs over the symmetric group of μ elements and ε_σ denotes the sign of σ (see e.g. [5, p. 3]). Writing $b_{\mu-1,0}=(-1)^{\mu+1}(a_{\mu-1}\otimes 1_M)$, one can see that a_μ is equal to the composition $a_\mu=b_{\mu-1,0}\circ\partial_{\mu,0}$,

$$\Lambda^\mu(M)\xrightarrow{\partial_{\mu,0}}\Lambda^{\mu-1}(M)\otimes M\xrightarrow{b_{\mu-1,0}} M^{\otimes(\mu-1)}\otimes M.$$

On the other hand, for a set of μ generators $u_1,\ldots,u_\mu$ of M, Flanders constructs a homomorphism

$$\Phi_\mu\colon M^{\otimes\mu}\longrightarrow\Lambda^\mu(M)$$

such that

$$\begin{aligned}\Phi_\mu(u_1\otimes\ldots\otimes u_\mu)&=u_1\wedge\ldots\wedge u_\mu,\\ \Phi_\mu(u_{i_1}\otimes\ldots\otimes u_{i_\mu})&=0, \qquad \text{if } (i_1,\ldots,i_\mu)\neq(1,\ldots,\mu),\end{aligned}$$

(see [5, proof of Theorem 5]). Since $\Lambda^\mu(M)$ is generated by the single element $u_1\wedge\ldots\wedge u_\mu$, then clearly $\Phi_\mu\circ a_\mu=1$. In particular, $\Phi_\mu\circ b_{\mu-1,0}\circ\partial_{\mu,0}=1$ and $\partial_{\mu,0}$ is injective. Thus $H_\mu(\mathrm{Kos}(M)_\mu)=0$.

3. A counterexample

Therefore, in order to find a module M minimally generated by $\mu\geq 2$ elements with $H_p(\mathrm{Kos}(M)_n)\neq 0$, being $p,n\geq 2$ as small as possible, we must consider $p=n=2$

and $\mu \geq 3$. Take a 3-dimensional noetherian local ring A containing a field of characteristic 2. Let x, y, z be a system of parameters of A and let $M=(x, y, z)$ be the ideal they generate. Consider the second homogeneous component $\mathrm{Kos}(M)_2$ of the Koszul complex $\mathrm{Kos}(M)$:

$$0 \longrightarrow \Lambda^2(M) \xrightarrow{\partial_{2,0}} M \otimes M \xrightarrow{\partial_{1,1}} S^2(M) \longrightarrow 0,$$

where, if u, v are in M, $\partial_{2,0}(u \wedge v)=v \otimes u-u \otimes v$ and $\partial_{1,1}(u \otimes v)=uv$. Take $u=x(y \wedge z)$ in $\Lambda^2(M)$. To see $u \neq 0$, consider the bilinear surjective map

$$f\colon M \times M \longrightarrow M^2/M^{[2]}$$

defined by $f(a, b)=ab+M^{[2]}$, where $M^{[2]}$ stands for the ideal generated by the squares of all elements of M. Since f vanishes over the elements (a, a), it extends to an epimorphism $f\colon \Lambda^2(M) \to M^2/M^{[2]}$. Remark that if 2 were invertible, $M^2=M^{[2]}$ and $f=0$. But since the characteristic is 2, $M^{[2]}=(x^2, y^2, z^2)$. Since x, y, z is a system of parameters of a noetherian local ring A containing a field, $f(u)=xyz+M^{[2]}= xyz+(x^2, y^2, z^2) \neq 0$ ([3, Theorem 9.2.1]). Hence $u \neq 0$. On the other hand,

$$x(y \otimes z)=(xy) \otimes z=y(x \otimes z)=x \otimes(yz)=z(x \otimes y)=(xz) \otimes y=x(z \otimes y).$$

Therefore, $\partial_{2,0}(u)=x(z \otimes y)-x(y \otimes z)=0$ and $H_2(\mathrm{Kos}(M)_2)=\ker(\partial_{2,0}) \neq 0$. In particular, $\mathrm{Kos}(M)$ is not a rigid complex.

We would like to point out that there is no straightforward extension of this example to characteristic p different from 2.

Notice that the example above also works if A is any (not necessarily noetherian) commutative ring having a regular sequence x, y, z (see e.g. [3, proof of Theorem 9.2.1]; see also [4, Exercises A2.6 and A2.14], for related results). Then, the ordinary Koszul complex of the regular sequence x, y, z defines a free resolution of length 2 of the ideal $M=(x, y, z)$. In other words, if M is flat or if M is generated by one element, then $\mathrm{Kos}(M)_n$ is acyclic for all $n \geq 1$, but if the flat dimension of M is $\mathrm{fd}_A(M)=2$ or if M is generated by three elements, then $\mathrm{Kos}(M)_n$ may not be acyclic for some $n \geq 2$. Thus, one should study the acyclicity of $\mathrm{Kos}(M)$ for modules M of flat dimension $\mathrm{fd}_A(M)=1$ or for modules M generated by two elements. We do not know the general answer to this question.

4. Relationship with the homology of commutative rings

In the sequel, we study the following particular case: take a commutative (not necessarily noetherian) ring A and a 2-generated ideal M of A of flat dimension $\mathrm{fd}_A(M)=1$. Suppose that M is an ideal of linear type, that is, the symmetric algebra $S(M)$ of the ideal M is canonically isomorphic to the Rees algebra $\mathcal{R}(M)=\bigoplus_{n\geq 0} M^n$ of M. In particular, the nth graded component $\mathrm{Kos}(M)_n$ of the Koszul complex $\mathrm{Kos}(M)$ becomes

$$0 \longrightarrow \Lambda^2(M)\otimes M^{n-2} \xrightarrow{\partial_{2,n-2}} M\otimes M^{n-1} \xrightarrow{\partial_{1,n-1}} M^n \longrightarrow 0.$$

Then $\ker(\partial_{1,n-1})=\mathrm{Tor}_2^A(B, A/M^{n-1})$, where $B=A/M$. Since M is generated by two elements, $M\subseteq \mathrm{Ann}_A(\Lambda^2(M))$ and $\Lambda^2(M)\otimes M^{n-2}=\Lambda^2(G_1)\otimes G_{n-2}$, where G_n is the nth graded component of the associated graded ring $\mathcal{G}(M)=\bigoplus_{n\geq 0} M^n/M^{n+1}$ of M. For each $n\geq 2$, the short exact sequence $0\rightarrow G_{n-2}\rightarrow A/M^{n-1}\rightarrow A/M^{n-2}\rightarrow 0$ induces the long exact sequence of $\mathrm{Tor}^A(B,\,\cdot\,)$,

$$\ldots \longrightarrow \mathrm{Tor}_3^A(B, A/M^{n-2}) \longrightarrow \mathrm{Tor}_2^A(B, G_{n-2}) \xrightarrow{i_{2,n-2}} \mathrm{Tor}_2^A(B, A/M^{n-1}) \longrightarrow \ldots .$$

One can see ([8, Proposition 2.2 and Corollary 2.6]) that $\partial_{2,n-2}$ factorizes as

$$\Lambda^2(G_1)\otimes G_{n-2} \xrightarrow{\psi_{2,n-2}} \mathrm{Tor}_2^A(B, G_{n-2}) \xrightarrow{i_{2,n-2}} \mathrm{Tor}_2^A(B, A/M^{n-1}),$$

where $\psi_{2,n-2}$ is the morphism appearing in the five term exact sequence associated with the spectral sequence relating the André–Quillen homology groups $H_*(A,B,\,\cdot\,)$ to $\mathrm{Tor}_*^A(B,\,\cdot\,)$ (see [10, Theorem 6.16]):

$$\begin{aligned}\mathrm{Tor}_3^A(B, G_{n-2}) \longrightarrow H_3(A,B,G_{n-2}) \longrightarrow \Lambda^2(G_1)\otimes G_{n-2} \xrightarrow{\psi_{2,n-2}} \mathrm{Tor}_2^A(B, G_{n-2})&\\ \longrightarrow H_2(A,B,G_{n-2}) \longrightarrow 0.&\end{aligned}$$

Since $\mathrm{fd}_A(M)=1$, then $\mathrm{Tor}_3^A(B,\,\cdot\,)=0$, $\ker(\partial_{2,n-2})=\ker(\psi_{2,n-2})$ and

$$H_2(\mathrm{Kos}(M)_n)=H_3(A,B,G_{n-2}).$$

5. Noetherian case

We will now prove that if A is noetherian and M is a 2-generated ideal of A of finite projective dimension, then $H_2(\mathrm{Kos}(M)_n)=H_3(A,B,G_{n-2})=0$ for all $n\geq 2$.

Clearly, to prove it one can suppose that A is local and M is minimally 2-generated. Since M is of finite projective dimension, then the projective dimension of M is $\mathrm{pd}_A(M)=1$ and M contains a nonzero divisor (see e.g. [7, Theorem 17.1], and [3, Corollary 1.4.6]). By prime avoidance, one can assume that M is generated by x and y, where x is regular. Then there exist an exact sequence

$$0 \longrightarrow A \xrightarrow{g} A^2 \xrightarrow{f} M \longrightarrow 0,$$

where $f(a,b)=ax+by$. Let $(z,t)\in A^2$ be a generator of $Z_1=\ker(f)=\operatorname{im}(g)$, the first module of syzygies of M. In particular, since $(y,-x)\in Z_1$, there exists $d\in A$ such that $y=dz$ and $x=-dt$, so d is not a zero divisor. Thus $M=dN$, where N is the ideal of A generated by z and t. By the Hilbert–Burch theorem (see e.g. [3, Theorem 1.4.16]), N is a perfect ideal of grade 2. Thus M is isomorphic to an ideal generated by a regular sequence and hence of linear type. Since M is 2-generated and of linear type, then for $n\geq 2$, $\operatorname{Kos}(M)_n$ is

$$0\longrightarrow \Lambda^2(M)\otimes M^{n-2}\xrightarrow{\partial_{2,n-2}} M\otimes M^{n-1}\xrightarrow{\partial_{1,n-1}} M^n\longrightarrow 0,$$

where $\Lambda^2(M)\otimes M^{n-2}=M^{n-2}/JM^{n-2}$ and $J=\operatorname{Ann}_A(\Lambda^2(M))$ is the annihilator of the A-module $\Lambda^2(M)$. On the other hand, since d is not a zero divisor, the localization morphism $\psi\colon A\to A_d$, $\psi(u)=u/1$, is injective. Moreover,

$$\ker(\partial_{1,n-1})=\operatorname{Tor}_1^A(M,A/M^{n-1})=\frac{Z_1\cap M^{n-1}A^2}{M^{n-1}Z_1}.$$

Through these isomorphisms, the morphism $\partial_{2,n-2}$ is defined by sending $u+JM^{n-2}$ to $u(y,-x)+M^{n-1}Z_1$. If $u+JM^{n-2}\in\ker(\partial_{2,n-2})$, then $u(y,-x)=v(z,t)$ with $v=xp+yq$, $p,q\in M^{n-2}$. Thus $v(z,t)=u(y,-x)=ud(z,t)$ and $v-ud\in\ker(g)=0$. So $v=ud$ and

$$\psi(u)=u/1=ud/d=v/d=(x/d)p+(y/d)q=(-t/1)p+(z/1)q=\psi(-tp+zq).$$

Since ψ is injective, $u=-tp+zq$. Since $xy=xdz=-dty$ and d is a nonzero divisor, $xz=-ty$, hence z and t are in J, and $u\in JM^{n-2}$. Therefore $\operatorname{Kos}(M)_n$ is acyclic for all $n\geq 1$ and, by the former point, $H_3(A,B,G_{n-2})=0$ for all $n\geq 2$.

For instance, if M is locally generated by a regular sequence of length 2, then its projective dimension is $\operatorname{pd}_A(M)=1$ and M is of linear type ([6, Théorème 1]). Thus $H_2(\operatorname{Kos}(M)_n)=H_3(A,B,G_{n-2})=0$. Remark that if M is locally generated by a regular sequence, not only does $H_3(A,B,G_{n-2})$ vanish, but the whole homology functor $H_3(A,B,\cdot)$ is zero ([1, Théorème 6.25]). In fact, under the hypothesis that A is noetherian, M being locally generated by a regular sequence is equivalent to M being of finite projective dimension and $H_3(A,B,\cdot)$ being zero (see [1, Théorème 17.11]).

Remark that there exist examples of 2-generated ideals M of projective dimension $\operatorname{pd}_A(M)=1$ and of linear type which are not locally generated by a regular sequence. For example, take a factorial domain A and an ideal M of A minimally generated by two elements. Extracting the greatest common divisor of the two generators, one sees that M is isomorphic to an ideal generated by a regular sequence of length 2. Therefore, M is of projective dimension 1 and of linear type although it is not necessarily locally generated by a regular sequence.

We do not know an example of a module M of flat dimension $\mathrm{fd}_A(M)=1$ or minimally generated by two elements such that $\mathrm{Kos}(M)_n$ is not acyclic for some $n\geq 3$.

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José M. Giral
Departament d'Àlgebra i Geometria
Universitat de Barcelona
Gran Via de les Corts Catalanes 585
ES-08007 Barcelona
Spain
giral@ub.edu

Francesc Planas-Vilanova
Departament de Matemàtica Aplicada 1
Universitat Politècnica de Catalunya
Diagonal 647, ETSEIB
ES-08028 Barcelona
Spain
francesc.planas@upc.edu

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