Ark. Mat., **45** (2007), 123–139
DOI: 10.1007/s11512-006-0023-8

Characterization of Orlicz–Sobolev space

Heli Tuominen

Abstract. We give a new characterization of the Orlicz–Sobolev space $W^{1,\Psi}(\mathbf{R}^n)$ in terms of a pointwise inequality connected to the Young function Ψ. We also study different Poincaré inequalities in the metric measure space.

1. Introduction

Analysis in metric measure spaces, for example the theory of Sobolev type spaces, has been under active study during the past decade. In a general metric space we cannot speak about weak derivatives, which are an essential part of the definition of the classical Sobolev space. Hence there has been a need for characterizations of $W^{1,p}(\Omega)$ that do not involve derivatives. Recall here that, for a domain $\Omega\subset\mathbf{R}^n$, the Sobolev space $W^{1,p}(\Omega)$, $1\le p<\infty$, consists of the functions $u\in L^p(\Omega)$ whose all first order weak derivatives $\partial_j u$ belong to $L^p(\Omega)$. Pointwise inequalities for pairs of L^p-functions are used both as a definition of a Sobolev space in a metric space, and as a tool to show that different definitions give the same set of functions if the underlying metric space satisfies certain assumptions.

Let us recall the result that led Hajłasz to define the Sobolev space $M^{1,p}(X)$ on metric measure space in [9]. For $1<p<\infty$, the function $u\in L^p(\mathbf{R}^n)$ is in $W^{1,p}(\mathbf{R}^n)$ if and only if there is a function $0\le g\in L^p(\mathbf{R}^n)$ such that the pointwise inequality

$$|u(x)-u(y)|\le|x-y|(g(x)+g(y)) \tag{1.1}$$

holds for almost every $x,y\in\mathbf{R}^n$. The validity of (1.1) for $u\in W^{1,p}(\mathbf{R}^n)$ follows from the inequality

$$|u(x)-u(y)|\le C(n)|x-y|[\mathcal{M}_{2|x-y|}\,|\nabla u|(x)+\mathcal{M}_{2|x-y|}\,|\nabla u|(y)], \tag{1.2}$$

which holds for all $1\le p<\infty$, and the boundedness of the Hardy–Littlewood maximal function for $p>1$. The boundedness of $\mathcal{M}$ is essential; for a function $u\in W^{1,1}(\mathbf{R}^n)$

there is not necessarily any integrable function g such that inequality (1.1) holds, see [10]. In [10], Hajłasz gave the following new characterization of $W^{1,1}(\mathbf{R}^n)$ using a pointwise estimate with maximal functions on its right-hand side.

Theorem 1.1. ([10, Theorem 4]) *A function $u\in L^1(\mathbf{R}^n)$ is in $W^{1,1}(\mathbf{R}^n)$ if and only if there exists a function $0\le g\in L^1(\mathbf{R}^n)$, a constant $\sigma\ge 1$, and a set E with $|E|=0$ such that the pointwise inequality*

$$|u(x)-u(y)| \le |x-y|[\mathcal{M}_{\sigma|x-y|}\, g(x)+\mathcal{M}_{\sigma|x-y|}\, g(y)] \tag{1.3}$$

holds for all $x,y\in\mathbf{R}^n\setminus E$.

In this paper we study a generalization of Theorem 1.1 to Orlicz–Sobolev spaces. Recall that for a domain $\Omega\subset\mathbf{R}^n$ and a Young function Ψ, the Orlicz–Sobolev space $W^{1,\Psi}(\Omega)$ consists of the functions $u\in L^\Psi(\Omega)$ whose all first order weak derivatives $\partial_j u$ belong to the Orlicz space $L^\Psi(\Omega)$, see Section 2 for the definition of Young function and Orlicz space. The space $W^{1,\Psi}(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{W^{1,\Psi}(\Omega)} = \|u\|_{L^\Psi(\Omega)} + \||\nabla u|\|_{L^\Psi(\Omega)},$$

where $\|\cdot\|_{L^\Psi(\Omega)}$ is the Luxemburg norm and ∇u is the weak gradient of u.

In Theorem 1.2, we assume that $\Psi,\widetilde{\Psi}$ is a pair of complementary Young functions such that both functions are doubling. The easiest example of such a pair is $\Psi(t)=t^p/p$, $\widetilde{\Psi}(t)=t^{p'}/p'$, where $1/p+1/p'=1$. Note also that the function

$$\Psi(t) = t^p \log^\alpha(e+t), \tag{1.4}$$

where $p>1$ and $\alpha>0$, or $p>1-\alpha$ and $-1\le\alpha<0$, is doubling and has a doubling complementary function. This can be checked by standard tests for N-functions, (cf. [15, Chapter 4] and [17, Chapter 2.2.3]). Weakly differentiable functions with gradient in the Orlicz space $L^\Psi(\Omega)$, Ψ as in (1.4), are used in the theory of mappings of finite distortion, see for example [13], [14] and the references therein. Orlicz and Orlicz–Sobolev spaces with such Ψ are studied also in [2], [3], [6] and [8], the list not being exhaustive.

Theorem 1.2. *Assume that $\Psi,\widetilde{\Psi}$ is a complementary pair of doubling Young functions. A function $u\in L^\Psi(\mathbf{R}^n)$ is in $W^{1,\Psi}(\mathbf{R}^n)$ if and only if there exists a function $0\le g\in L^\Psi(\mathbf{R}^n)$, a constant $\sigma\ge 1$, and a set E with $|E|=0$ such that the pointwise inequality*

$$|u(x)-u(y)| \le C|x-y|[\Psi^{-1}(\mathcal{M}_{\sigma|x-y|}\, \Psi(g)(x))+\Psi^{-1}(\mathcal{M}_{\sigma|x-y|}\, \Psi(g)(y))], \tag{1.5}$$

holds for all $x,y\in\mathbf{R}^n\setminus E$.

As in the case of $W^{1,1}(\mathbf{R}^n)$, the proof consists of two parts which are interesting also as separate results. The first part, Theorem 3.2, together with earlier results, provides a close connection between the pointwise inequality and a Ψ-Poincaré inequality (1.6) below also in the metric space setting. In Theorem 3.3, we show that the validity of a Ψ-Poincaré inequality for a pair $u\in L^\Psi(\mathbf{R}^n)$, $0\le g\in L^\Psi(\mathbf{R}^n)$, guarantees that $u\in W^{1,\Psi}(\mathbf{R}^n)$.

Given a strictly increasing Young function Ψ, we say, as in [18], that a pair $u\in L^1_{\mathrm{loc}}(X)$ and a measurable function $g\ge 0$ *satisfies a Ψ-Poincaré inequality* if there exist constants $C>0$ and $\tau\ge 1$ such that

$$\fint_B |u-u_B|\,d\mu \le C_\Psi r\Psi^{-1}\biggl(\fint_{\tau B}\Psi(g)\,d\mu\biggr) \tag{1.6}$$

for each ball $B=B(x,r)$.

Now it is time to explain "the earlier results" mentioned above. As shown by Hajłasz and Koskela in [11, Theorem 3.2], a $(1,p)$-Poincaré inequality, that is, (1.6) with $\Psi(t)=t^p$, for a pair $u\in L^1_{\mathrm{loc}}(X)$ and a measurable function $g\ge 0$ implies a pointwise inequality of the same type as (1.3),

$$|u(x)-u(y)|\le C\,d(x,y)[(\mathcal{M}_{2\tau d(x,y)}\,g^p(x))^{1/p}+(\mathcal{M}_{2\tau d(x,y)}\,g^p(y))^{1/p}] \tag{1.7}$$

for μ-almost all $x,y\in X$.

A Ψ-Poincaré inequality implies a similar estimate for the oscillation of a function. Namely, if a pair $u\in L^1_{\mathrm{loc}}(X)$, $g\ge 0$ satisfies a Ψ-Poincaré inequality, then for μ-almost all $x,y\in X$,

$$|u(x)-u(y)|\le C\,d(x,y)[\Psi^{-1}(\mathcal{M}_{2\tau d(x,y)}\,\Psi(g)(x))+\Psi^{-1}(\mathcal{M}_{2\tau d(x,y)}\,\Psi(g)(y))], \tag{1.8}$$

where the constant $C>0$ depends only on the doubling constant C_μ of μ and on the constant C_Ψ of the Ψ-Poincaré inequality, [18, Lemma 5.15].

The paper is organized as follows. In Section 2 we introduce the notation and the standard assumptions used in the paper. Our main results, Theorem 1.2 together with the two steps of its proof, are proved in Section 3. Section 4 contains a discussion about Poincaré inequalities connected with different Young functions.

2. Notation and preliminaries

2.1. Basic assumptions

Although our main theorem is for Orlicz–Sobolev space in $\mathbf{R}^n$ with the Euclidean metric, part of our results hold also in the metric setting. Then we assume that $X=(X,d,\mu)$ is a metric measure space equipped with a metric d and a Borel regular, *doubling* outer measure μ. The doubling property means that there is a fixed constant $C_\mu>0$, called *a doubling constant of* μ, such that

$$\mu(B(x,2r))\leq C_\mu \mu(B(x,r)) \tag{2.1}$$

for each $x\in X$, and all $r>0$. Here $B(x,r)=\{y\in X: d(y,x)<r\}$ is the open ball of radius r centered at x. Given a ball $B=B(x,r)$ and $0<t<\infty$, we let $tB=B(x,tr)$. We also assume that the measure of every open set is positive and that the measure of each bounded set is finite.

As a special case of doubling spaces we consider Q-regular spaces, where the measure μ behaves almost as well as the Lebesgue measure in $\mathbf{R}^n$. More precisely, a metric space X with a Borel regular measure μ is (*Ahlfors*) *Q-regular*, $Q>1$, if there is a constant $C_Q\geq 1$ such that

$$C_Q^{-1} r^Q \leq \mu(B(x,r))\leq C_Q r^Q \tag{2.2}$$

for each $x\in X$, and for all $0<r\leq \operatorname{diam}(X)$. Here $\operatorname{diam}(X)$ is the diameter of X. We say that X is *a geodesic space* if every two points $x,y\in X$ can be joined by a curve whose length is equal to $d(x,y)$.

The mean value of a function $u\in L^1(A)$ over a μ-measurable set A with finite and positive measure is $u_A=⨍_A u\,d\mu=\mu(A)^{-1}\int_A u\,d\mu$. We say that a function u belongs to the local space $L^p_{\mathrm{loc}}(X)$ if it belongs to $L^p(B)$ for each ball $B\subset X$.

The *restricted Hardy–Littlewood maximal function* of a function $u\in L^1_{\mathrm{loc}}(X)$ is

$$\mathcal{M}_R\, u(x)=\sup_{0<r\leq R} ⨍_{B(x,r)} |u(y)|\,d\mu(y). \tag{2.3}$$

For $R=\infty$, $\mathcal{M}_\infty\, u$ is the *usual Hardy–Littlewood maximal function* $\mathcal{M}\, u$.

By ω_n, we denote the Lebesgue n-measure of the unit ball $B(0,1)\subset\mathbf{R}^n$, and by $|E|$, the Lebesgue n-measure of a measurable set $E\subset\mathbf{R}^n$. The characteristic function of a set $E\subset X$ is χ_E. In general, C will denote a positive constant whose value is not necessarily the same at each occurrence. By writing $C=C(K,\lambda)$ we indicate that the constant depends only on K and λ.

2.2. Review of Orlicz spaces

We will give a brief review to Orlicz spaces. Classical references to Young functions, Orlicz spaces, and Orlicz–Sobolev spaces $W^{1,\Psi}(\mathbf{R}^n)$ are [1], [16], [15], and [17]. For Orlicz–Sobolev spaces in metric space, see [18].

A function $\Psi\colon[0,\infty)\to[0,\infty]$ is *a Young function* if

$$\Psi(s)=\int_0^s \psi(t)\,dt, \tag{2.4}$$

where $\psi\colon [0,\infty)\to[0,\infty]$ is an increasing, left-continuous function which is neither identically zero nor identically infinity on $(0,\infty)$, and satisfies $\psi(0)=0$. A Young function Ψ is convex, increasing, left-continuous, $\Psi(0)=0$, and $\Psi(t)\to\infty$ as $t\to\infty$. A continuous Young function with the properties $\Psi(t)=0$ only if $t=0$, $\Psi(t)/t\to\infty$ as $t\to\infty$, and $\Psi(t)/t\to 0$ as $t\to 0$, is called *an N-function.*

Given a Young function Ψ, the function $\widetilde{\Psi}\colon [0,\infty)\to[0,\infty]$,

$$\widetilde{\Psi}(s)=\sup\{st-\Psi(t):t\geq 0\},$$

is *the complementary function*, and $\Psi^{-1}\colon [0,\infty]\to[0,\infty]$,

$$\Psi^{-1}(t)=\inf\{s:\Psi(s)>t\},$$

with $\inf\varnothing=\infty$, is the *generalized inverse* of Ψ. The function Ψ^{-1} is a right-continuous, increasing substitute for the inverse function. If a Young function is continuous and strictly increasing, then its usual inverse function coincides with the generalized inverse. The functions Ψ and Ψ^{-1} satisfy the inequality

$$\Psi(\Psi^{-1}(t))\leq t\leq \Psi^{-1}(\Psi(t)) \tag{2.5}$$

for all $t\geq 0$.

It follows easily from the convexity and the property $\Psi(0)=0$ that the function $t\mapsto\Psi(t)/t$ is increasing. This implies that if Ψ is strictly increasing, then the function $t\mapsto\Psi^{-1}(t)/t$ is decreasing.

A Young function $\Psi\colon [0,\infty)\to[0,\infty)$ is *doubling* (satisfies the Δ_2-condition) if there is a constant $C_2>0$ such that

$$\Psi(2t)\leq C_2\Psi(t)$$

for each $t\geq 0$. The smallest such C_2 is called *the doubling constant of* Ψ. The doubling condition implies that

$$\Psi(t)\leq C_2\left(\frac{t}{s}\right)^{\log_2 C_2}\Psi(s) \tag{2.6}$$

for all $0<s<t$. It also tells that for large t the growth of Ψ is dominated by the function Ct^p with some $p>1$ and $C>0$, see [17, the proof of Corollary II.2.3.5]. Hence the doubling condition excludes functions with exponential growth. Functions $\Psi_1(t)=at^p$, $a>0, p\geq 1$, and $\Psi_2(t)=(1+t)\log(1+t)-t$ are examples of doubling functions, whereas the complementary function of Ψ_2, $\widetilde{\Psi}_2(t)=e^t-t-1$ is not doubling.

Note that, as a convex function, a real-valued Young function Ψ is continuous on $(0,\infty)$. By the convexity and the property $\Psi(0)=0$, such a Ψ is continuous on $[0,\infty)$. In particular, a doubling or strictly increasing Young function is real-valued, and hence continuous. On the other hand, if Ψ is real-valued and $\Psi(t)=0$ only if $t=0$, then it is strictly increasing.

Given a Young function Ψ and an open set $\Omega\subset X$, *the Orlicz space* $L^\Psi(\Omega)$ consists of measurable functions $u\colon\Omega\to[-\infty,\infty]$ for which

$$\int_\Omega \Psi(\alpha|u|)\,d\mu<\infty$$

for some $\alpha>0$. If Ψ is doubling, then $L^\Psi(\Omega)$ coincides with the set of functions u for which $\int_\Omega \Psi(|u|)\,d\mu$ is finite. Equipped with *the Luxemburg norm*,

$$\|u\|_{L^\Psi(\Omega)}=\inf\biggl\{k>0:\int_\Omega \Psi\bigl(k^{-1}|u|\bigr)\,d\mu\le 1\biggr\}, \tag{2.7}$$

$L^\Psi(\Omega)$ is a Banach space. If $\Omega=X$, we let $\|u\|_\Psi=\|u\|_{L^\Psi(X)}$. We say that a function u is in the local space $L^\Psi_{\mathrm{loc}}(X)$ if it is in $L^\Psi(B)$ for each ball $B\subset X$. The Luxemburg norm is monotone in the sense that if the measurable functions u and v satisfy $|u|\le|v|$ μ-almost everywhere, then $\|u\|_\Psi\le\|v\|_\Psi$.

If Ψ, $\widetilde\Psi$ is a complementary pair of Young functions, then *the generalized Hölder inequality*

$$\int_\Omega |u(x)v(x)|\,d\mu\le 2\|u\|_{L^\Psi(\Omega)}\|v\|_{L^{\widetilde\Psi}(\Omega)} \tag{2.8}$$

holds for all $u\in L^\Psi(\Omega)$ and $v\in L^{\widetilde\Psi}(\Omega)$.

If Ψ is doubling and Ω a domain in $\mathbf{R}^n$, then the space $C_0^\infty(\Omega)$ of infinitely differentiable functions with compact support is dense in $L^\Psi(\Omega)$. The standard convolution approximations $J_\varepsilon*u$ of $u\in L^\Psi(\Omega)$ converge to u in norm as $\varepsilon\to0$. If, in addition, $\widetilde\Psi$ is doubling, then $L^\Psi(\Omega)$ is reflexive and $L^\Psi(\Omega)^*=L^{\widetilde\Psi}(\Omega)$ (see [1, Chapter 8], [16, Chapter 3] and [17, Chapter 4]). Moreover, a version of Riesz representation theorem holds for $L^\Psi(\Omega)$: for each functional $L\in L^\Psi(\Omega)^*$, there exists a unique $v\in L^{\widetilde\Psi}(\Omega)$ such that

$$L(u)=\int_\Omega uv\,d\mu \tag{2.9}$$

for each $u\in L^\Psi(\Omega)$. Furthermore, the operator norm of L is controlled by the norm of v,

$$\|v\|_{L^{\widetilde\Psi}(\Omega)}\le\|L\|\le2\|v\|_{L^{\widetilde\Psi}(\Omega)}.$$

In [18], we defined, in addition to a Ψ-Poincaré inequality (1.6), that the pair u,g *satisfies a* (Ψ,Ψ)*-Poincaré inequality in* Ω if

$$\fint_B \Psi\left(\frac{|u-u_B|}{Cr}\right)d\mu \le C_0 \fint_{\tau B}\Psi(g)\,d\mu \tag{2.10}$$

with some constants $C, C_0>0$ and $\tau\ge 1$.

We close this subsection by recalling the Jensen inequality, an important tool in the theory of Orlicz spaces. If $\Psi\colon \mathbf{R}\to\mathbf{R}$ is convex, $u\in L^1_{\mathrm{loc}}(X)$, and $A\subset X$ is of positive and finite measure, then

$$\Psi\left(\fint_A |u|\,d\mu\right)\le \fint_A \Psi(|u|)\,d\mu. \tag{2.11}$$

Using (2.11) and (2.5), we see that each function $u\in L^\Psi(X)$ with a real-valued Ψ is locally integrable. Indeed, if $\alpha>0$ is such that $\int_X \Psi(\alpha|u|)\,d\mu<\infty$ and B is a ball, then

$$\int_B |u|\,d\mu = \frac{\mu(B)}{\alpha}\fint_B \alpha|u|\,d\mu \le \frac{\mu(B)}{\alpha}\Psi^{-1}\left(\fint_B \Psi(\alpha|u|)\,d\mu\right).$$

3. Pointwise estimate, Poincaré inequality and Orlicz–Sobolev space

In this section, we will prove Theorem 1.2. We begin with a geometric lemma, and continue by showing that the result that a Ψ-Poincaré inequality implies a pointwise inequality (1.8) for the oscillation of the function, can be converted. The proof of Theorem 3.2 is a modification of the proof of Hajłasz [10, Lemma 5].

Lemma 3.1. *Suppose that* X *is a geodesic metric space. If* $B=B(x_0,R)$ *a ball in* X*,* $x\in B$*, and* $0<r\le 2R$*, then there is a ball of radius* $r/2$ *in* $B(x,r)\cap B$*.*

Proof. If $d(x,x_0)\ge r/2$, then the assumption that X is geodesic implies that there is a point z such that $d(z,x)=r/2$ and $d(z,x_0)=d(x,x_0)-r/2$, and hence $B(z,r/2)\subset B(x,r)\cap B$.

On the other hand, if $d(x,x_0)<r/2$, then $B(x_0,r/2)\subset B(x,r)\cap B$. □

Theorem 3.2. *Let* X *be a* Q*-regular, geodesic metric measure space, and* Ψ *a doubling Young function. If* $u\in L^1_{\mathrm{loc}}(X)$*,* $0\le g\in L^\Psi_{\mathrm{loc}}(X)$*, and* $\sigma\ge 1$*,* $C>0$ *are such that the inequality*

$$|u(x)-u(y)|\le C\,d(x,y)[\Psi^{-1}(\mathcal{M}_{\sigma d(x,y)}\Psi(g)(x))+\Psi^{-1}(\mathcal{M}_{\sigma d(x,y)}\Psi(g)(y))] \tag{3.1}$$

holds for μ*-almost all* $x,y\in X$*, then the pair* u,g *satisfies a* Ψ*-Poincaré inequality with* $\tau=3\sigma$*. The constant* $C_\Psi>0$ *of* (1.6) *will depend only on the constants of* μ*,* Ψ*, and of* C *of* (3.1)*.*

Proof. Let $B=B(x_0,R)$ be a ball in X. We begin the proof by checking what we can assume from u and g. Since neither the left hand side of (3.1) nor the Ψ-Poincaré inequality change if a constant is added to u, we may assume that $\operatorname{ess\,inf}_E |u|=0$ for a set $E\subset B$ with $\mu(E)>0$. We will choose the set E later. We define $\tau=3\sigma$, and $h=g\chi_{\tau B}$. The pointwise estimate (3.1) implies that, after a modification of u in a set of measure zero if necessary,

$$|u(x)-u(y)|\le C_0 d(x,y)[\Psi^{-1}(\mathcal{M}\,\Psi(h)(x))+\Psi^{-1}(\mathcal{M}\,\Psi(h)(y))] \tag{3.2}$$

for all $x,y\in B$.

We can assume that $h>0$ on a set of positive measure, since if $h=0$ almost everywhere in X, then u is constant in B, and the Ψ-Poincaré inequality (1.6) follows. By replacing h with the bigger function $\tilde{h}=h+\Psi^{-1}\big(⨍_{\tau B}\Psi(h)\,d\mu\big)$ if necessary, we may assume that on τB the function h satisfies

$$h\ge \frac{1}{C_2}\Psi^{-1}\biggl(⨍_{\tau B}\Psi(h)\,d\mu\biggr)>0. \tag{3.3}$$

Using the doubling property of Ψ and the fact that the function $t\mapsto\Psi^{-1}(t)/t$ is decreasing, we have that

$$\Psi^{-1}\biggl(⨍_{\tau B}\Psi(\tilde{h})\,d\mu\biggr)\le C_2\Psi^{-1}\biggl(⨍_{\tau B}\Psi(h)\,d\mu\biggr),$$

and so the change from h to $\tilde{h}$ will only increase the constant of the Ψ-Poincaré inequality. For each $k\in\mathbf{Z}$, we define

$$E_k=\{x\in B:\Psi^{-1}(\mathcal{M}\,\Psi(h)(x))\le 2^k\}\quad\text{and}\quad a_k=\sup_{E_k}|u(x)|.$$

Then $E_{k-1}\subset E_k$ and $a_{k-1}\le a_k$ for each k, and

$$\int_B|u-u_B|\,d\mu\le 2\int_B|u|\,d\mu\le 2\sum_{k=-\infty}^{\infty}a_k\mu(E_k\setminus E_{k-1}). \tag{3.4}$$

We will obtain an upper bound for the left-hand side of the Ψ-Poincaré inequality by estimating the value of $|u|$ in the sets E_k.

Our next goal is to find for each $x\in E_k$, a point $y\in E_{k-1}$ such that the distance from y to x is at most $C\mu(B\setminus E_{k-1})^{1/Q}$. By the pointwise estimate (3.2), the function u is $C_0 2^{k+1}$-Lipschitz in E_k. Hence, for each $x\in E_k$ and $y\in E_{k-1}$, we have that

$$|u(x)|\le|u(x)-u(y)|+|u(y)|\le C_0 2^{k+1}d(x,y)+a_{k-1}. \tag{3.5}$$

Fix $x\in E_k$. By Lemma 3.1, $B(x,r)\cap B$ contains a ball of radius $r/2$ if $0<r\le 2R$, and hence, by the Q-regularity of μ,

$$\mu(B(x,r)\cap B)\geq \frac{1}{C_Q}\Big(\frac{r}{2}\Big)^Q. \tag{3.6}$$

If $\mu(E_{k-1})>0$, we choose

$$r_k = 2^{1+1/Q}C_Q^{1/Q}\mu(B\setminus E_{k-1})^{1/Q}.$$

Then, by (3.6),

$$\mu(B(x,r_k)\cap B) > \mu(B\setminus E_{k-1}),$$

and hence there is $y\in B(x,r_k)\cap E_{k-1}$. The upper bound $r_k\le 2R$ for r_k holds by the Q-regularity if E_{k-1} is large enough, if

$$2C_Q^2\mu(B\setminus E_{k-1})\leq \mu(B). \tag{3.7}$$

Since Ψ is doubling, the function $\Psi(h)$ is in $L^1_{\mathrm{loc}}(X)$, and hence the weak-type estimate for the maximal function, [12, Theorem 2.2], implies that

$$\begin{aligned}\mu(B\setminus E_{k-1}) &= \mu(\{x\in B: \Psi^{-1}(\mathcal{M}\,\Psi(h)(x))>2^{k-1}\})\\ &\leq \mu(\{x\in B: \mathcal{M}\,\Psi(h)(x)>\Psi(2^{k-1})\})\\ &\leq \frac{C}{\Psi(2^{k-1})}\int_{\tau B}\Psi(h)\,d\mu.\end{aligned} \tag{3.8}$$

Now (3.5), the definition of r_k, and (3.8) imply that

$$a_k\leq a_{k-1}+C2^k\mu(B\setminus E_{k-1})^{1/Q}\leq a_{k-1}+C2^k\Psi(2^{k-1})^{-1/Q}\bigg(\int_{\tau B}\Psi(h)\,d\mu\bigg)^{1/Q}$$

whenever (3.7) holds for E_{k-1}.

Iterating the above estimate we have that if $\mu(E_{k_0})>0$ and (3.7) holds for k_0, that is, $2C_Q^2\mu(B\setminus E_{k_0})\le\mu(B)$, then

$$a_k\leq a_{k_0}+C\sum_{i=k_0+1}^{k}\frac{2^i}{\Psi(2^{i-1})^{1/Q}}\bigg(\int_{\tau B}\Psi(h)\,d\mu\bigg)^{1/Q} \tag{3.9}$$

for each $k>k_0$.

Claim. There is k_0 for which $2C_Q^2\mu(B\setminus E_{k_0})\le\mu(B)$ and a constant $C\ge 1$ such that

$$C^{-1}⨍_{\tau B}\Psi(h)\,d\mu\leq \Psi(2^{k_0})\leq C⨍_{\tau B}\Psi(h)\,d\mu. \tag{3.10}$$

Proof. Since the assumption (3.3) guarantees that E_k is empty for small k, and since $\mu(E_k)\to\mu(B)$ as $k\to\infty$, there is an index k_0 for which

$$\mu(E_{k_0-1}) < \Bigl(1-\frac{1}{2C_Q^2}\Bigr)\mu(B) \le \mu(E_{k_0}). \tag{3.11}$$

The function $\Psi(g)$ is in $L^1(B)$ because $g\in L^{\Psi}_{\mathrm{loc}}(X)$ and Ψ is doubling. Hence the definition of h and the Lebesgue differentiation theorem imply that $\mathcal{M}\,\Psi(h)\ge|\Psi(h)|$ almost everywhere in B. Then, by the assumption (3.3) on h, the doubling property of Ψ, and the selection of k_0 such that E_{k_0} is not empty, there is $x\in B$ and a constant $C=C(C_2)$, $0<C<1/C_2$, such that

$$C \fint_{\tau B} \Psi(h)\,d\mu \le \mathcal{M}\,\Psi(h)(x) \le \Psi(2^{k_0}). \tag{3.12}$$

To prove the opposite inequality of the claim, we use (3.11) and a weak type estimate as in (3.8) to obtain

$$\begin{aligned}(2C_Q^2)^{-1}\mu(B) &\le \mu(B\setminus E_{k_0-1}) = \mu(\{x\in B : \Psi^{-1}(\mathcal{M}\,\Psi(h)(x)) > 2^{k_0-1}\})\\ &\le \frac{C}{\Psi(2^{k_0-1})}\int_{\tau B}\Psi(h)\,d\mu.\end{aligned} \tag{3.13}$$

The claim (3.10) follows from estimates (3.12) and (3.13) using the doubling property of Ψ and the Q-regularity of μ. □

We select the set E discussed in the beginning of the proof to be E_{k_0}, and assume that $\operatorname{ess\,inf}_{E_{k_0}}|u|=0$. Then, by the 2^{k_0+1}-Lipschitz continuity of u in E_{k_0}, and (3.10), we have the following estimate for a_{k_0}:

$$a_{k_0} = \sup_{E_{k_0}}|u| \le 2^{k_0+1}\cdot 2R \le CR\,\Psi^{-1}\Bigl(\fint_{\tau B}\Psi(h)\,d\mu\Bigr). \tag{3.14}$$

By letting $A_k=E_k\setminus E_{k-1}$ and using (3.4) we have that

$$\frac12\int_B |u-u_B|\,d\mu \le \sum_{k=-\infty}^{\infty} a_k\mu(A_k),$$

which, by estimate (3.9) for a_k with $k>k_0$, is not larger than

$$\begin{aligned}&\sum_{k=-\infty}^{k_0} a_{k_0}\mu(A_k) + \sum_{k=k_0+1}^{\infty}\Bigl(a_{k_0} + C\sum_{i=k_0+1}^{k}\frac{2^i}{\Psi(2^{i-1})^{1/Q}}\Bigl(\int_{\tau B}\Psi(h)\,d\mu\Bigr)^{1/Q}\Bigr)\mu(A_k)\\ &\le \sum_{k=-\infty}^{\infty} a_{k_0}\mu(A_k) + C\Bigl(\int_{\tau B}\Psi(h)\,d\mu\Bigr)^{1/Q}\sum_{k=k_0+1}^{\infty}\sum_{i=k_0+1}^{k}\frac{2^i}{\Psi(2^{i-1})^{1/Q}}\mu(B\setminus E_{k-1}).\end{aligned} \tag{3.15}$$

By the weak-type estimate (3.8), the last term in (3.15) is at most

$$C\biggl(\int_{\tau B}\Psi(h)\,d\mu\biggr)^{1+1/Q}\sum_{k=k_0+1}^{\infty}\sum_{i=k_0+1}^{k}\frac{2^i}{\Psi(2^{i-1})^{1/Q}}\frac{1}{\Psi(2^{k-1})}. \tag{3.16}$$

Switching the order of summation and using the fact that $\Psi(2^j)/\Psi(2^{j+1})\le\frac{1}{2}$ for all j, which follows from the monotonicity of the function $t\mapsto\Psi(t)/t$, we have that the double sum in (3.16) is not larger than

$$\begin{aligned}\sum_{i=k_0+1}^{\infty}\frac{2^i}{\Psi(2^{i-1})^{1/Q}}\sum_{k=i}^{\infty}\frac{1}{\Psi(2^{k-1})}&\le\sum_{i=k_0+1}^{\infty}\frac{2^i}{\Psi(2^{i-1})^{1/Q}}\frac{1}{\Psi(2^{i-1})}\sum_{j=0}^{\infty}2^{-j}\\&\le\frac{2}{\Psi(2^{k_0})^{1+1/Q}}\sum_{i=k_0+1}^{\infty}\frac{2^i}{2^{(i-k_0-1)(1+1/Q)}}\\&\le\frac{C2^{k_0}}{\Psi(2^{k_0})^{1+1/Q}}.\end{aligned}\tag{3.17}$$

Now we use (3.14) for a_{k_0}, estimates (3.15)–(3.17), comparability of $\Psi(2^{k_0})$ and $⨍_{\tau B}\Psi(h)\,d\mu$, and the Q-regularity of μ, and obtain

$$\begin{aligned}\int_B|u-u_B|\,d\mu&\le CR\,\Psi^{-1}\biggl(⨍_{\tau B}\Psi(h)\,d\mu\biggr)\mu(B)+C\biggl(\int_{\tau B}\Psi(h)\,d\mu\biggr)^{1+1/Q}\frac{2^{k_0}}{\Psi(2^{k_0})^{1+1/Q}}\\&\le CR\,\Psi^{-1}\biggl(⨍_{\tau B}\Psi(h)\,d\mu\biggr)\mu(B)+C\Psi^{-1}\biggl(⨍_{\tau B}\Psi(h)\,d\mu\biggr)\mu(\tau B)^{1+1/Q}\\&\le CR\Psi^{-1}\biggl(⨍_{\tau B}\Psi(h)\,d\mu\biggr)\mu(B),\end{aligned}$$

which gives a Ψ-Poincaré inequality for u and h. By the definition of $h=g\chi_{\tau B}$, we also have a Ψ-Poincaré inequality for u and g. □

In the above proof, the assumption that X be a geodesic space was needed only for the use of Lemma 3.1.

Notice that Heisenberg and Carnot groups are Ahlfors Q-regular geodesic spaces for a suitable Q. However, the result above is not new in these spaces because they support a $(1,p)$-Poincaré inequality for all $p\ge1$ and a Ψ-Poincaré inequality follows from a $(1,1)$-Poincaré inequality, see Section 4.

Theorem 3.3. *Let $\Psi,\widetilde{\Psi}$ be a complementary pair of doubling Young functions. If the pair $u,g\in L^\Psi(\mathbf{R}^n)$, where $g\ge0$, satisfies a Ψ-Poincaré inequality* (1.6)*, then $u\in W^{1,\Psi}(\mathbf{R}^n)$. Moreover, if $\int_{\mathbf{R}^n}\Psi(g)\,dx\ge1$, then $\||\nabla u|\|_\Psi\le C\int_{\mathbf{R}^n}\Psi(g(y))\,dy$, and if $\int_{\mathbf{R}^n}\Psi(g)\,dx<1$, then $\||\nabla u|\|_\Psi\le C(\int_{\mathbf{R}^n}\Psi(g(y))\,dy)^{1/\log_2 C_2}$, where C_2 is the doubling constant of Ψ.*

Proof. We have to show that u has weak partial derivatives that are in $L^{\Psi}(\mathbf{R}^n)$. For each $i=1,...,n$, there should exist an $L^{\Psi}(\mathbf{R}^n)$-function v_i such that the partial integration formula

$$-\int_{\mathbf{R}^n} u\partial_i\varphi\,dx=\int_{\mathbf{R}^n} v_i\varphi\,dx, \tag{3.18}$$

where $\partial_i\varphi$ is the i^{th} partial derivative of φ, holds for all $\varphi\in C_0^\infty(\mathbf{R}^n)$. Since both Ψ and its complementary function $\widetilde{\Psi}$ are doubling, by (2.9) and the density of $C_0^\infty(\mathbf{R}^n)$ in $L^{\widetilde{\Psi}}(\mathbf{R}^n)$, it suffices to show that the formula

$$(\partial_i u)(\varphi):=-\int_{\mathbf{R}^n} u\partial_i\varphi\,dx$$

determines a continuous functional on $C_0^\infty(\mathbf{R}^n)$ with respect to the $\|\cdot\|_{\widetilde{\Psi}}$-norm.

Fix $0<\varepsilon<1$. Let $\varphi\in C_0^\infty(\mathbf{R}^n)$, and let $J\in C_0^\infty(B(0,1))$ be the standard mollifier with $J\geq 0$ and $\int_{\mathbf{R}^n} J\,dx=1$, and let $J_\varepsilon(x)=\varepsilon^{-n}J(x/\varepsilon)$. Since Ψ is doubling and $u\in L^{\Psi}(\mathbf{R}^n)$, the convolution approximations

$$u_\varepsilon(x)=J_\varepsilon*u(x)=\int_{\mathbf{R}^n} J_\varepsilon(x-y)u(y)\,dy$$

converge to u in $L^{\Psi}(\mathbf{R}^n)$ and satisfy

$$-\int_{\mathbf{R}^n} u\partial_i\varphi\,dx=-\lim_{\varepsilon\to 0}\int_{\mathbf{R}^n} u_\varepsilon\partial_i\varphi\,dx=\lim_{\varepsilon\to 0}\int_{\mathbf{R}^n}(\partial_i J_\varepsilon*u)\varphi\,dx. \tag{3.19}$$

Since $\int_{\mathbf{R}^n}\partial_i J_\varepsilon\,dx=0$, we have that

$$(\partial_i J_\varepsilon*u)(x)=(\partial_i J_\varepsilon*(u-u_{B(x,\varepsilon)}))(x),$$

and, by the Ψ-Poincaré inequality

(3.20)

$$\left|\partial_i J_\varepsilon*u\right|(x)\leq C\varepsilon^{-n-1}\int_{B(x,\varepsilon)}|u(y)-u_{B(x,\varepsilon)}|\,dy\leq C\Psi^{-1}\biggl(\fint_{B(x,\tau\varepsilon)}\Psi(g(y))\,dy\biggr).$$

Using (3.19) and the Hölder inequality (2.8), we have that

(3.21)

$$|(\partial_i u)(\varphi)|\leq\liminf_{\varepsilon\to 0}\int_{\operatorname{supp}\varphi}|\partial_i J_\varepsilon*u||\varphi|\,dx\leq\liminf_{\varepsilon\to 0}2\|\varphi\|_{\widetilde{\Psi}}\|\partial_i J_\varepsilon*u\|_{L^{\Psi}(\operatorname{supp}\varphi)}.$$

By inequality (3.20) and the monotonicity of the Luxemburg norm, it suffices to estimate the norm of the function

$$h(x)=\Psi^{-1}\biggl(\fint_{B(x,\tau\varepsilon)}\Psi(g(y))\,dy\biggr)$$

to find an upper bound for $\|\partial_i J_\varepsilon*u\|_{L^{\Psi}(\operatorname{supp}\varphi)}$.

Let $K\subset\mathbf{R}^n$ be compact and $K_{\tau\varepsilon}=\{z\in\mathbf{R}^n : d(z,K)<\tau\varepsilon\}$. If $k\geq 1$, then we have that

$$\begin{aligned}\int_K \Psi\left(\frac{h(x)}{k}\right) dx &\leq k^{-1}\int_K \fint_{B(x,\tau\varepsilon)} \Psi(g(y))\,dy\,dx\\ &= k^{-1}\omega_n^{-1}(\tau\varepsilon)^{-n}\int_{K_{\tau\varepsilon}} \Psi(g(y))\int_K \chi_{B(y,\tau\varepsilon)}(x)\,dx\,dy\\ &\leq k^{-1}\int_{K_{\tau\varepsilon}} \Psi(g(y))\,dy.\end{aligned}$$

If $0<k<1$, we use (2.6) and a similar estimate for $\int_K \Psi(h)\,d\mu$ as above to obtain

$$\int_K \Psi\left(\frac{h(x)}{k}\right) dx \leq C_2 k^{-\log_2 C_2}\int_{K_{\tau\varepsilon}} \Psi(g(y))\,dy.$$

The norm estimate depends on $A=\int_K \Psi(g(y))\,dy$. By selecting $k=\int_{K_{\tau\varepsilon}} \Psi(g(y))\,dy$ if $A\geq 1$ and $\left(C_2\int_{K_{\tau\varepsilon}} \Psi(g(y))\,dy\right)^{1/\log_2 C_2}$ if $A<1$, we have that

$$\|h\|_{L^\Psi(K)} \leq \int_{K_{\tau\varepsilon}} \Psi(g(y))\,dy \leq \int_{\mathbf{R}^n} \Psi(g(y))\,dy \tag{3.22}$$

in the former, and

$$\|h\|_{L^\Psi(K)} \leq \left(C_2\int_{K_{\tau\varepsilon}} \Psi(g(y))\,dy\right)^{1/\log_2 C_2} \leq \left(C_2\int_{\mathbf{R}^n} \Psi(g(y))\,dy\right)^{1/\log_2 C_2} \tag{3.23}$$

in the latter case.

Using (3.21) and the norm estimates (3.22) and (3.23), we have that, if $A\geq 1$,

$$|(\partial_i u)(\varphi)| \leq C\|\varphi\|_{\widetilde{\Psi}}\int_{\operatorname{supp}\varphi} \Psi(g(y))\,dy \leq C\|\varphi\|_{\widetilde{\Psi}}\int_{\mathbf{R}^n} \Psi(g(y))\,dy, \tag{3.24}$$

or if $A<1$,

$$\begin{aligned}|(\partial_i u)(\varphi)| &\leq C\|\varphi\|_{\widetilde{\Psi}}\left(\int_{\operatorname{supp}\varphi} \Psi(g(y))\,dy\right)^{1/\log_2 C_2}\\ &\leq C\|\varphi\|_{\widetilde{\Psi}}\left(\int_{\mathbf{R}^n} \Psi(g(y))\,dy\right)^{1/\log_2 C_2}.\end{aligned} \tag{3.25}$$

Hence $(\partial_i u)(\varphi)$ defines a continuous functional on $C_0^\infty(\mathbf{R}^n)$ and extends to an element of $L^{\widetilde{\Psi}}(\mathbf{R}^n)^*=L^\Psi(\mathbf{R}^n)$. By (2.9), there is a function $v_i\in L^\Psi(\mathbf{R}^n)$, that satisfies (3.18). Moreover,

$$\|v_i\|_\Psi \leq \|(\partial_i u)\| \leq C\int_{\mathbf{R}^n} \Psi(g(y))\,dy,$$

when $A\geq 1$ and

$$\|v_i\|_\Psi \leq \|\partial_i u\| \leq C\biggl(\int_{\mathbf{R}^n} \Psi(g(y))\,dy\biggr)^{1/\log_2 C_2}$$

if $A<1$, from which the theorem follows. □

Remark 3.4. By the above proof, we see that also a local version of Theorem 3.3 holds. Namely, if the pair $u,g\in L^\Psi_{\mathrm{loc}}(\mathbf{R}^n)$, $g\geq 0$, satisfies the Ψ-Poincaré inequality (1.6), then $u\in W^{1,\Psi}_{\mathrm{loc}}(\mathbf{R}^n)$. Moreover, if the pair u,g satisfies the (Ψ,Ψ)-Poincaré inequality (2.10), then it is enough to assume that $u\in L^1_{\mathrm{loc}}(\mathbf{R}^n)$, (cf. [18, Lemma 5.13]).

Proof of Theorem 1.2. If $u\in W^{1,\Psi}(\mathbf{R}^n)$, then u belongs to $W^{1,1}_{\mathrm{loc}}(\mathbf{R}^n)$. Inequality (1.2) holds for functions that are in the local Sobolev space $W^{1,1}_{\mathrm{loc}}(\mathbf{R}^n)$, see [5], [9] and [10]. Hence the pointwise inequality

$$|u(x)-u(y)| \leq C|x-y|[\Psi^{-1}(\mathcal{M}_{\sigma|x-y|}\,\Psi(|\nabla u|)(x))+\Psi^{-1}(\mathcal{M}_{\sigma|x-y|}\,\Psi(|\nabla u|)(y))]$$

follows from (1.2) using the Jensen inequality.

If there is a function $g\in L^\Psi(\mathbf{R}^n)$ such that (1.5) holds for u and g almost everywhere, then $u\in W^{1,\Psi}(\mathbf{R}^n)$ by Theorems 3.2 and 3.3. □

4. Connections between different Poincaré inequalities

In this section, we briefly study how a Ψ-Poincaré inequality depends on the Young function Ψ.

With the function $\Psi(t)=t^p$, (1.6) and (2.10) give a $(1,p)$- and a (p,p)-Poincaré inequality, where *a* (q,p)*-Poincaré inequality* is

$$\biggl(\fint_{B(x,r)} |u-u_B|^q\,d\mu\biggr)^{1/q} \leq Cr\biggl(\fint_{B(x,\tau r)} g^p\,d\mu\biggr)^{1/p}. \tag{4.1}$$

Recall that among the class of all $(1,p)$-Poincaré inequalities, the Hölder inequality shows that the $(1,1)$-Poincaré inequality is the strongest one. Moreover, a (q_1,p_1)-Poincaré inequality implies a (q_2,p_2)-Poincaré inequality for all $1\leq q_2\leq q_1$ and $p_2\geq p_1$. A deep result of Hajłasz and Koskela in [11] shows that if the measure μ is doubling, then a $(1,p)$-Poincaré inequality improves itself to a (q,p)-Poincaré inequality for some $q>p$, see also [7]. Concerning Poincaré inequalities with a Young function Ψ, we have the following results from [18]:

1. The Jensen inequality shows that a Ψ-Poincaré inequality follows from a $(1,1)$- and a (Ψ,Ψ)-Poincaré inequality for any Ψ.

2. If the function $\Psi_2\circ\Psi_1^{-1}$ is convex, then a Ψ_2-Poincaré inequality follows from a Ψ_1-Poincaré inequality by the Jensen inequality.

3. If the pair u,g satisfies a Ψ_1-Poincaré inequality, then it satisfies a $\Psi_2\circ\Psi_1$-Poincaré inequality for any Ψ_2, [18, Lemma 5.6]. This is a generalization of the above mentioned result that a $(1,q)$-Poincaré inequality follows from a $(1,p)$-Poincaré inequality for $1\le p<q$.

4. If Ψ is doubling, then a Ψ-Poincaré inequality implies a $(1,p)$-Poincaré inequality for $p\ge\log_2 C_2$, where C_2 is the doubling constant of Ψ, [18, Theorem 5.7].

Bhattacharya and Leonetti showed in [4, Lemma 1] that if $\Psi\colon[0,\infty)\to[0,\infty)$ is a continuous, convex function with $\Psi(0)=0$, $B\subset\mathbf{R}^n$ is a ball, and $u\in W^{1,1}(B)$, then inequality (2.10) holds for the pair $u,|\nabla u|$ with $C=2$, $C_0=C(n)$, and $\tau=1$. Hence, if Ψ is a continuous Young function and $u\in W^{1,\Psi}(\mathbf{R}^n)$, then the pair $u,|\nabla u|$ satisfies the (Ψ,Ψ)-Poincaré inequality. Note that such a u is in $W^{1,1}_{\mathrm{loc}}(\mathbf{R}^n)$ by the Jensen inequality and the continuity of Ψ.

In the next lemma, we generalize the Jensen inequality (2.11). Inequality (4.3) can also been seen as a generalization of the consequence $\bigl(\fint_A |u|^p\,d\mu\bigr)^{1/p}\le\bigl(\fint_A |u|^q\,d\mu\bigr)^{1/q}$ for $1\le p<q$ of the Hölder inequality.

Lemma 4.1. *Let $\Psi_1,\Psi_2\colon[0,\infty)\to[0,\infty)$ be strictly increasing Young functions, $A\subset X$ be of positive and finite measure, and let $u\in L^1(A)$. If there are constants $C_1,C_2>0$ such that*

$$\frac{\Psi_1(t)}{\Psi_2(C_2t)}\le C_1\frac{\Psi_1(s)}{\Psi_2(C_2s)} \tag{4.2}$$

for all $0<s\le t$, then there is $C=C(C_1,C_2)$ such that

$$\Psi_1^{-1}\biggl(\fint_A\Psi_1(|u|)\,d\mu\biggr)\le C\Psi_2^{-1}\biggl(\fint_A\Psi_2(C_2|u|)\,d\mu\biggr). \tag{4.3}$$

Proof. Let $\lambda>0$, and $A_\lambda=\{x\in A:|u(x)|>\lambda\}$. By the assumption (4.2), we have that

$$\begin{aligned}\int_A\Psi_1(|u|)\,d\mu&=\int_{A\setminus A_\lambda}\Psi_1(|u|)\,d\mu+\int_{A_\lambda}\Psi_1(|u|)\,d\mu\\&\le\Psi_1(\lambda)\mu(A)+C_1\int_{A_\lambda}\Psi_2(C_2|u|)\frac{\Psi_1(\lambda)}{\Psi_2(C_2\lambda)}\,d\mu\\&\le\Psi_1(\lambda)\biggl(\mu(A)+\frac{C_1}{\Psi_2(C_2\lambda)}\int_A\Psi_2(C_2|u|)\,d\mu\biggr).\end{aligned}$$

For $\lambda = C_2^{-1}\Psi_2^{-1}\big(\fint_A \Psi_2(C_2|u|)\,d\mu\big)$, the above inequality implies that

$$
\begin{aligned}
(4.4)\qquad \int_A \Psi_1(|u|)\,d\mu &\le \Psi_1(\lambda)\left(\mu(A)+C_1\frac{\int_A \Psi_2(C_2|u|)\,d\mu}{\fint_A \Psi_2(C_2|u|)\,d\mu}\right)\\
&= \mu(A)(1+C_1)\Psi_1\left(C_2^{-1}\Psi_2^{-1}\left(\fint_A \Psi_2(C_2|u|)\,d\mu\right)\right).
\end{aligned}
$$

The claim follows from inequality (4.4) with $C=(1+C_1)C_2^{-1}$ because the function $t\mapsto \Psi_1^{-1}(t)/t$ is decreasing. □

Using the lemmas above, we obtain a connection between Ψ_1- and Ψ_2-Poincaré inequalities.

Corollary 4.2. *Let $\Psi_1, \Psi_2\colon [0,\infty)\to[0,\infty)$ be strictly increasing Young functions such that* (4.2) *holds for Ψ_1 and Ψ_2. If a pair u, g satisfies a Ψ_1-Poincaré inequality, then it also satisfies a Ψ_2-Poincaré inequality.*

Example 4.3. Calculations using the derivative of the function $f(t)=\Psi_1(t)/\Psi_2(t)$ show that the following pairs of functions satisfy (4.2) with $C_1=C_2=1$:

1. the complementary pair $\Psi_1(t)=(1+t)\log(t+1)-t$, $\Psi_2(t)=e^t-t-1$;
2. $\Psi_1(t)=(1+t)\log(t+1)-t$ and $\Psi_2(t)=t^p$, $p\ge 2$;
3. $\Psi_1(t)=t^p$, $1\le p\le 2$, and $\Psi_2(t)=e^t-t-1$;
4. $\Psi_1(t)=t^p$ and $\Psi_2(t)=t^q\log^\alpha(t+e)$, $1\le p\le q$, $\alpha\ge 0$;
5. $\Psi_1(t)=t^p$ and $\Psi_2(t)=t^q\log^\alpha(t+e)$, $q\ge p+1$, $\alpha\ge -1$.

Acknowledgements. I wish to thank Piotr Hajłasz for his comments and encouragement. I thank the referee for carefully reading the paper, helpful comments and for his/her help for avoiding an unnecessary technical assumption in the main theorem. This research was supported by the Centre of Excellence Geometric Analysis and Mathematical Physics of the Academy of Finland.

References

1. Adams, R. A., Sobolev spaces, *Pure and Applied Mathematics*, **65**, Academic Press, New York–London, 1975.
2. Adams, D. R. and Hurri-Syrjänen, R., Capacity estimates, *Proc. Amer. Math. Soc.* **131** (2003), 1159–1167.
3. Adams, D. R. and Hurri-Syrjänen, R., Vanishing exponential integrability for functions whose gradients belong to $L^n(\log(e+L))^\alpha$, *J. Funct. Anal.* **197** (2003), 162–178.

4. Bhattacharya, T. and Leonetti, F., A new Poincaré inequality and its application to the regularity of minimizers of integral functionals with nonstandard growth, *Nonlinear Anal.* **17** (1991), 833–839.
5. Bojarski, B. and Hajłasz, P., Pointwise inequalities for Sobolev functions and some applications, *Studia Math.* **106** (1993), 77–92.
6. Edmunds, D. E., Gurka, P. and Opic, B., Double exponential integrability of convolution operators in generalized Lorentz-Zygmund spaces, *Indiana Univ. Math. J.* **44** (1995), 19–43.
7. Franchi, B., Hajłasz, P. and Koskela, P., Definitions of Sobolev classes on metric spaces, *Ann. Inst. Fourier* (*Grenoble*) **49** (1999), 1903–1924.
8. Fusco, N., Lions, P.-L. and Sbordone, C., Sobolev imbedding theorems in borderline cases, *Proc. Amer. Math. Soc.* **124** (1996), 561–565.
9. Hajłasz, P., Sobolev spaces on an arbitrary metric space, *Potential Anal.* **5** (1996), 403–415.
10. Hajłasz, P., A new characterization of the Sobolev space, *Studia Math.* **159** (2003), 263–275.
11. Hajłasz, P. and Koskela, P., Sobolev met Poincaré, *Mem. Amer. Math. Soc.* **145** (2000), 1–101.
12. Heinonen, J., *Lectures on Analysis on Metric Spaces*, Universitext, Springer, New York, 2001.
13. Iwaniec, T., Koskela, P. and Onninen, J., Mappings of finite distortion: monotonicity and continuity, *Invent. Math.* **144** (2001), 507–531.
14. Kauhanen, J., Koskela, P., Malý, J., Onninen, J. and Zhong, X., Mappings of finite distortion: sharp Orlicz-conditions, *Rev. Mat. Iberoamericana* **19** (2003), 857–872.
15. Krasnosel'skiĭ, M. A. and Rutickiĭ, J. B., *Convex Functions and Orlicz Spaces*, Noordhoff, Groningen, 1961.
16. Kufner, A., John, O. and Fučík, S., *Function Spaces*, Noordhoff, Leyden; Academia, Prague, 1977.
17. Rao, M. M. and Ren, Z. D., *Theory of Orlicz Spaces*, Monographs and Textbooks in Pure and Applied Mathematics, **146**, Marcel Dekker, New York, 1991.
18. Tuominen, H., Orlicz–Sobolev spaces on metric measure spaces, *Ann. Acad. Sci. Fenn. Math. Diss.* **135** (2004).

Heli Tuominen
Department of Mathematics and Statistics
P.O. Box 35 (MaD)
FI-40014 University of Jyväskylä
tuheli@maths.jyu.fi

Received September 19, 2005
in revised form December 21, 2005
published online February 10, 2007