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Extremal discs and holomorphic extension from convex hypersurfaces

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Abstract. Let D be a convex domain with smooth boundary in complex space and let f be a continuous function on the boundary of D. Suppose that f holomorphically extends to the extremal discs tangent to a convex subdomain of D. We prove that f holomorphically extends to D. The result partially answers a conjecture by Globevnik and Stout of 1991.

1. Introduction

Let $D\subset\mathbf{C}^n$ be a bounded domain with smooth boundary ∂D and let f be a continuous function on ∂D. Suppose that for every complex line L the restriction $f|_{L\cap\partial D}$ holomorphically extends into $L\cap D$. Then f extends to D as a holomorphic function of n variables (Stout [11]). The conclusion is still true if instead of the holomorphic extendibility of f into the sections $L\cap D$, we assume the weaker Morera condition

$$\int_{L\cap\partial D} f\alpha=0 \tag{1.1}$$

for every $(1,0)$-form α with constant coefficients and every complex line L (Globevnik and Stout [7]).

The condition of holomorphic extendibility into sections $L\cap D$ and even the Morera condition (1.1) for all lines L seem excessive, because it suffices to use only the lines close to the tangent lines to ∂D. Indeed, for simplicity assume $f\in C^1(\partial D)$. Then the Morera condition for L as L approaches a tangent line L_0 at $z_0\in\partial D$ implies that the $\bar\partial$ derivative of f at z_0 along L_0 equals zero. Then f holomorphically extends to D by the classical Hartogs–Bochner theorem. Therefore, of great interest are "small" families of lines, for which the result is still true. In particular, the family of lines should not contain the lines close to the tangent lines to ∂D.

Reducing the family of lines, Agranovsky and Semenov [1] show that if $D_2 \subset D_1$ are bounded domains in $\mathbf{C}^n$ and $f \in C(\partial D_1)$ holomorphically extends into sections $L \cap D_1$ by the lines that meet D_2, then f holomorphically extends to D_1. In the case of two concentric balls $D_2 \subset D_1$, Rudin [10] proves that the same conclusion is valid if one only assumes the extendibility into sections by the lines tangent to ∂D_2. Globevnik [5] observes that in Rudin's result one only needs the lines tangent to a sufficiently large open set in ∂D_2.

Globevnik and Stout [7] conjecture that Rudin's result is valid for every pair of bounded convex domains $D_2 \Subset D_1$, that is if $f \in C(\partial D_1)$ holomorphically extends into sections $L \cap D_1$ by the lines tangent to ∂D_2, then f holomorphically extends to D_1. They also observe in [7] (see also [2]) that for $n=2$ in Rudin's result one generally cannot replace the extendibility into the sections $L \cap D_1$ by the Morera condition (1.1), that is the latter suffices unless the ratio r_1/r_2 of the radii of the balls belongs to an exceptional countable set. For a counterexample in $\mathbf{C}^2$, take $r_1=1$, $r_2=\sqrt{1/3}$, and $f=z_1\bar{z}_2^2$. However if $n>2$, then Berenstein, Chang, Pascuas, and Zalcman [2] show that for the concentric balls the Morera condition for the tangent lines suffices without exceptions. Dinh [4] proves the conjecture of Globevnik and Stout assuming that the boundary ∂D_1 in some sense is very far from being real-analytic.

Further reduction of the family of lines is possible. Globevnik [6] shows that for the unit ball $D \subset \mathbf{C}^2$, the holomorphic extension property into sections by lines of certain two-parameter families suffices for the holomorphic extendibility into D. The set of lines in his result consists of two disjoint tori. The second author shows [13] that for every generating CR manifold $M \subset \mathbf{C}^n$ of dimension d there exists a $(d-1)$-parameter family of analytic discs attached to M so that if $f \in C(M)$ holomorphically extends to those discs, then f is a CR function on M.

Despite the large amount of work done on the subject, the conjecture of Globevnik and Stout has been open so far. In this paper we prove a version of the conjecture in which the complex lines are replaced by the complex geodesics of the Kobayashi metric for D_1 also known as extremal or stationary discs, whose theory was developed by Lempert [9]. We believe that the extremal discs for a general convex domain D_1 are more appropriate in the problem than the lines because they are intrinsically defined, invariant under biholomorphisms, and coincide with the lines for the ball. Hence, if D_1 is the ball and D_2 is an arbitrary strictly convex subdomain, then our result proves the conjecture of Globevnik and Stout for the lines as stated. As in Globevnik's result [5] cited above, we only need the extendibility into the extremal discs tangent to a sufficiently large open set in ∂D_2 (cf. Remark 3.6).

The authors of the results for the concentric balls use Fourier analysis and decomposition into spherical harmonics. This method does not seem to work for general convex domains. We employ the method of [14] according to which we add an extra variable, the fiber coordinate in the projectivized cotangent bundle. Then using the lifts of the extremal discs we lift the given function f to a CR function on the boundary of a wedge W whose edge is the projectivized conormal bundle of ∂D_1. Then using the theory of CR functions we extend it to a bounded holomorphic function in W. Finally since W contains "large" discs, we prove that the lifted function actually does not depend on the extra variable, which proves the result.

We feel that the method developed here has a wide scope, and we hope to use it in other occasions.

2. Extremal discs

We will collect here, and develop in some details, the main results of [9] which are needed for our discussion. Let D be a bounded domain of $\mathbf{C}^n$ with C^k-boundary; according to [9] we assume $k\geq 6$. We also assume that D is strongly convex in the sense that D has a global defining function with positive real Hessian. An analytic disc in $\mathbf{C}^n$ is a holomorphic mapping $\Delta\to\mathbf{C}^n$, smooth up to $\partial\Delta$, where Δ is the standard disc in $\mathbf{C}$. We denote by A the image set under ϕ. The disc A is said to be "attached" to ∂D when $\partial A\subset\partial D$.

Definition 2.1. An analytic disc ϕ in D is said to be *stationary* when it is attached to ∂D and endowed with a meromorphic lift $\phi^*(\tau)\in(T^*\mathbf{C}^n)_{\phi(\tau)}$ for all $\tau\in\Delta$ with one simple pole in Δ such that $\phi^*(\tau)\in(T^*_{\partial D}\mathbf{C}^n)_{\phi(\tau)}$ when $|\tau|=1$. In other words, (ϕ,ϕ^*) is attached to the conormal bundle $T^*_{\partial D}\mathbf{C}^n$.

Definition 2.2. An analytic disc ϕ in D is said to be *extremal* when for any other disc ψ in D with $\psi(0)=\phi(0)$ and $\psi'(0)=\lambda\phi'(0)$, $\lambda\in\mathbf{C}$, we have $|\lambda|<1$.

It is shown in [9] that extremal and stationary discs coincide. Also, it is shown that they are stable under reparametrization. In particular, in Definition 2.2 we can replace 0 by any other value of $\tau\in\Delta$ which does not affect the stationarity or extremality of ϕ. It follows that the extremal discs are the geodesics of the Kobayashi metric in D; in particular they are embeddings of $\bar\Delta$ into $\mathbf{C}^n$. We recall some basic facts about the existence, uniqueness, and smooth dependence of the extremal discs on parameters (see [9], Proposition 11$'$):

(2.0) For any $z\in D$ and $v\in\dot{\mathbf{C}}^n:=\mathbf{C}^n\setminus\{0\}$, there exists a unique extremal disc $\phi=\phi_{z,v}$ such that $\phi(0)=z$ and $\phi'(0)=rv$ for $r\in\mathbf{R}^+$. Also, the mapping

$$D\times\dot{\mathbf{C}}^n\longrightarrow C^{2,1/2}(\bar\Delta),\quad (z,v)\longmapsto\phi_{z,v}$$

is of class C^{k-4}, where $C^{2,1/2}$ is the space of functions whose derivatives up to order 2 are $\frac{1}{2}$-Hölder-continuous.

If ϕ^* has its pole at τ_0, we multiply it by

$$\frac{(\tau-\tau_0)(1-\bar{\tau}_0\tau)}{\tau}, \quad \tau\in\Delta,$$

so that the pole is moved to 0. Next, we multiply ϕ^* by a real constant $\neq 0$ so that $\phi^*(1)$ is the outward unit conormal to D at $\phi(1)$. We will assume that ϕ^* is normalized by the two above conditions. It is essential for our discussion also to clarify the dependence of $\phi^*_{z,v}$ on the parameters z and v which is not explicitly stated in [9].

Proposition 2.3. *The mapping*

$$D\times\dot{\mathbf{C}}^n \longrightarrow C^{2,1/2}(\bar{\Delta}), \quad (z,v)\longmapsto \phi^*_{z,v}, \tag{2.1}$$

is C^{k-4}.

Proof. Our starting remark is that $\phi^*_{z,v}$ is explicitly described only over $\partial\Delta$ where it is in the form $g(\tau)\partial\rho(\phi_{z,v}(\tau))$ for a suitable function g, real on $\partial\Delta$. We can also assume that $g(1)=1$ and $\partial_{z_1}\rho(\phi_{z,v}(1))=1$. On the other hand, when evaluating $\phi^*_{z,v}$ at points $\tau\in\Delta$, we can use the Cauchy integral over $\partial\Delta$. Thus, if we are able to show that $(z,v)\mapsto\phi^*_{z,v}$ with values in $C^{2,\frac{1}{2}}(\partial\Delta)$ is C^{k-4}, the same will be true with values in $C^{2,\frac{1}{2}}(\bar{\Delta})$ since the Cauchy integral preserves fractional regularity. Now, $\partial\rho(\phi_{z,v})$ depends smoothly on z, v because of (2.0) and so what is needed is to prove that the same is true for $g_{z,v}$. Recall that $\partial\rho(\phi_{z,v}(\tau))$ extends meromorphically from $\partial\Delta$ to Δ with a simple pole. Thus, for one of the components of $\partial\rho$, e.g. for $\partial_{z_1}\rho$, we have that the index of the curve $\{\partial_{z_1}\rho(\phi_{z,v}(\tau)):\tau\in\partial\Delta\}$ around 0 is -1. It follows that the function $f(\tau)=\log(\tau\partial_{z_1}\rho(\phi_{z,v}(\tau)))$ is well defined. We now need a function $G=G_{z,v}$ holomorphic in Δ such that

$$\operatorname{Im} G_{z,v}(\tau)=\operatorname{Im}(f(\tau)), \quad \tau\in\partial\Delta. \tag{2.2}$$

We put $G=-T_1(\operatorname{Im}(f))+i\operatorname{Im}(f)$, where T_1 is the Hilbert transform normalized by the condition $T_1f(1)=0$. As T_1 preserves fractional regularity, $(z,v)\mapsto G_{z,v}\in C^{2,1/2}(\partial\Delta)$ is also C^{k-4}. We finally put

$$\nu_{z,v}(\tau):=\frac{e^{G(\tau)}}{\tau\partial_{z_1}\rho(\phi_{z,v})}.$$

We have

$$\begin{aligned}\nu_{z,v} &= \exp(\operatorname{Re} G + i \operatorname{Im} \log(\tau\partial_{z_1}\rho(\phi_{z,v})) - \log(\tau\partial_{z_1}\rho(\phi_{z,v}))) \\ &= \exp(\operatorname{Re} G - \operatorname{Re} \log(\tau\partial_{z_1}\rho)) \text{ is real,}\end{aligned} \tag{2.3}$$

$$\nu_{z,v}\tau\partial_{z_1}\rho(\phi_{z,v}) \text{ extends holomorphically from } \partial\Delta \text{ to } \bar{\Delta}. \tag{2.4}$$

Finally, since

$$\left.\frac{g}{\nu}\right|_{\partial\Delta} \in \mathbf{R}, \quad \frac{g}{\nu} \text{ extends holomorphically} \quad \text{and} \quad \frac{g}{\nu}(1) = 1,$$

we have $g \equiv \nu$. It follows that g, and hence φ^*, depends in C^{k-4} fashion on z and v. $\square$

The following statement is similar to (2.0) above: for any $z \in D$ and $\zeta \in \dot{\mathbf{C}}^n$ there is a unique stationary disc with lift $(\phi, \phi^*) = (\phi_{z,\zeta}, \phi^*_{z,\zeta})$ such that $\phi(0) = z$ and $\phi^*(0) = \zeta$, where $\phi^*(0)$ stands for the residue $\operatorname{Res}\phi^*(0)$.

We recall now some basics about the Lempert Riemann mapping. For any pair of points (z, w) in D, let $\phi_{z,w}$ be the (unique) stationary disc through z and w normalized by the condition $z = \phi_{z,w}(0)$, $w = \phi_{z,w}(\xi)$ for some $\xi \in (0, 1)$; we define

$$\Psi_z(w) := \xi \frac{\phi'_{z,w}(0)}{|\phi'_{z,w}(0)|}.$$

Let $\mathbf{B}^n$ denote the unit ball of $\mathbf{C}^n$ and put $\dot{\mathbf{B}}^n = \mathbf{B}^n \setminus \{0\}$. Consider the correspondence

$$(D \times D) \setminus \mathrm{Diag} \longrightarrow D \times \dot{\mathbf{B}}^n, \quad (z, w) \longmapsto (z, \Psi_z(w)), \tag{2.5}$$

where Diag denotes the diagonal. We have

- For fixed z, Ψ_z is a diffeomorphism of class C^{k-4} which extends as a diffeomorphism between the boundaries ∂D and $\partial\mathbf{B}^n$.
- (2.5) is differentiable of class C^{k-4}.

Write $v = v(z, w)$ for $\Psi_z(w)$. By the above statements, the smoothness of (2.0) and (2.1) are equivalent to those of

$$(z, w) \longmapsto \phi_{z,w} \quad \text{and} \quad (z, w) \longmapsto \phi^*_{z,w}. \tag{2.6}$$

Remark 2.4. Let z_ν and w_ν be sequences converging to z_0, and put $v_\nu := \phi'_{z_\nu,w_\nu}(0)$. If we define $v := \lim_{\nu\to\infty}(w_\nu - z_\nu)/|w_\nu - z_\nu|$, then $v = \lim_{\nu\to\infty} v_\nu/|v_\nu|$. Hence we have convergence (in the $C^{2,1/2}(\bar{\Delta})$ space):

$$\phi_{z_\nu,w_\nu} (= \phi_{z_\nu,v_\nu}) \longrightarrow \phi_{z_0,v}, \qquad \phi^*_{z_\nu,w_\nu} (= \phi^*_{z_\nu,v_\nu}) \longrightarrow \phi^*_{z_0,v}. \tag{2.7}$$

For our further needs it is convenient to state the following uniqueness theorem which is largely contained in former literature.

Theorem 2.5. *Let two stationary discs ϕ_j, $j=1,2$, be given in a strongly convex domain D and assume that for $\tau_j\in\Delta$, $j=1,2$, we have*

$$\begin{cases} \phi_1(\tau_1)=\phi_2(\tau_2), \\ \phi_1^*(\tau_1)=\lambda\phi_2^*(\tau_2) \quad \textit{for some } \lambda\in\mathbf{C}. \end{cases} \tag{2.8}$$

Then, after reparametrization of Δ, we have for a complex scalar function $\mu=\mu(\tau)$,

$$\phi_1=\phi_2 \quad \textit{and} \quad \phi_1^*=\mu\phi_2^*.$$

As before, if τ_j is a pole of ϕ_j^*, then $\phi_j^*(\tau_j)$ stands for $\operatorname{Res}\phi_j^*(\tau_j)$.

Proof. We compose each (ϕ_j,ϕ_j^*) with an automorphism of Δ which brings τ_j to 0. We are therefore reduced to

$$\begin{cases} \phi_1(0)=\phi_2(0), \\ \operatorname{Res}\phi_1^*(0)=\lambda\operatorname{Res}\phi_2^*(0), \end{cases} \tag{2.9}$$

for a new constant λ. We put $\lambda=re^{i\theta}$ and replace $(\phi_2(\tau),\phi_2^*(\tau))$ by $(\phi_2(e^{-i\theta}\tau), r\phi_2^*(e^{-i\theta}\tau))$. This transformation reduces (2.9) to $\lambda=1$. At this point we can prove that $\phi_1=\phi_2$. We reason by contradiction and suppose $\phi_1\neq\phi_2$. It follows that

$$\int_0^{2\pi}\operatorname{Re}\langle\phi_1^*(\tau)-\phi_2^*(\tau),\phi_2(\tau)-\phi_1(\tau)\rangle\,d\theta>0, \tag{2.10}$$

since the integrand is almost everywhere >0 on $\partial\Delta$ due to the strong convexity of the domain. On the other hand $d\theta=-i\,d\tau/\tau$; also, $(\phi_2-\phi_1)/\tau$ and $\phi_1^*-\phi_2^*$ are holomorphic. Hence the integrand in (2.10) is a $(1,0)$-form whose coefficient is the real part of a holomorphic function. Hence the integral (2.10) is 0, a contradiction. □

In particular in the situation of Theorem 2.5 we have coincidence of the image sets $\phi_1(\Delta)=\phi_2(\Delta)$.

Remark 2.6. Let $\dot{T}^*\mathbf{C}^n$ be the cotangent bundle to $\mathbf{C}^n$ with the 0-section removed, and let $\dot{T}^*\mathbf{C}^n/\dot{\mathbf{C}}\simeq\mathbf{C}^n\times\mathbf{P}^{n-1}_{\mathbf{C}}$ be the projectivization of its fibers. We denote by $(z,[\zeta])$ the variable in $\dot{T}^*\mathbf{C}^n/\dot{\mathbf{C}}$. We can rephrase Theorem 2.5 by saying that if two discs $(\phi_j,[\phi_j^*])$, $j=1,2$, have a common point, then, after reparametrization, they need to coincide. Also, it is useful to point out that, given a stationary disc $\phi(\Delta)$, its lift $[\phi^*(\Delta)]$ is unique. In fact, the different choices of ϕ obtained by reparametrization do not affect the class of ϕ^* in the projectivization of the cotangent bundle.

3. The main result

Let D_1 and D_2 be bounded domains of $\mathbf{C}^n$ with $D_2 \Subset D_1$. We assume that D_1 is strongly convex and with C^k boundary for $k \geq 6$ as is the setting of [9]. Let D_2 be defined by $\rho < 0$ for a real function ρ of class C^2 with $\partial\rho(z) \neq 0$ when $\rho(z) = 0$.

Definition 3.1. The domain D_2 is said to be *strongly convex with respect to the extremal discs of* D_1, if every such disc ϕ tangent to D_2 at $z_0 = \phi(0) \in \partial D_2$ has tangency of order 2, that is for some $c > 0$ we have $\rho(\phi(\tau)) \geq c|\tau|^2$ for all $\tau \in \Delta$, in particular $\overline{D}_2 \cap \phi(\Delta) = \{z_0\}$.

Here is the main result of our paper.

Theorem 3.2. *Let $D_2 \Subset D_1$ be bounded domains of $\mathbf{C}^n$ with D_1 strongly convex and of class C^k for $k \geq 6$, and D_2 strongly convex with respect to the extremal discs of D_1 and of class C^2. Let f be a continuous function which extends holomorphically along each extremal disc $\phi(\Delta)$ of D_1 which is tangent to ∂D_2. Then f extends holomorphically to D_1, continuous up to ∂D_1.*

Remark 3.3. We do not think that the assumption that D_2 is strongly convex with respect to the extremal discs is essential. We add it for the sake of simplicity and convenience of the proof.

Proof. We consider the cotangent (respectively tangent) bundle $\dot{T}^*\mathbf{C}^n/\dot{\mathbf{C}}$, resp. $\dot{T}\mathbf{C}^n/\dot{\mathbf{C}}$, with projectivized fibers $\mathbf{P}^{n-1}_{\mathbf{C}}$ and with coordinates $(z, [\zeta])$ and $(z, [v])$, respectively. The prefix $T^{\mathbf{C}}$ will be used to denote the complex tangent bundle. We fix a rule for selecting a "distinguished" representative v of $[v]$ and define a mapping

$$(\dot{T}^{\mathbf{C}}\partial D_2/\dot{\mathbf{C}}) \times \Delta \xrightarrow{\Phi} (\dot{T}^*\mathbf{C}^n/\dot{\mathbf{C}})|_{D_1 \setminus D_2}, \tag{3.1}$$

$$(z, [v], \tau) \overset{\Phi}{\longmapsto} (\phi_{z,v}(\tau), [\phi^*_{z,v}(\tau)]), \tag{3.2}$$

where $\phi_{z,v}$ is the unique stationary disc such that $\phi(0) = z$ and $\phi'(0) = rv$ for some $r \in \mathbf{R}^+$ and $\phi^*_{z,v}$ is its "lift" according to Section 2, (2.0). Note that by multiplying $\phi^*_{z,v}$ by $\nu(\tau)$, which is real on $\partial\Delta$, we can move the pole to $\tau = 0$.

We denote by $\mathcal{S}$ the image-set of Φ. We show that Φ is an injective smooth local parametrization of $\mathcal{S}$. First, it is injective: in fact, if $(z_1, [v_1], \tau_1)$ and $(z_2, [v_2], \tau_2)$ maps to the same image, then by Theorem 2.5 in Section 2, ϕ_{z_1,v_1} and ϕ_{z_2,v_2} coincide up to reparametrization. On the other hand, by the strong convexity of D_2 with respect to the stationary discs of D_1, we must have $z_2 = z_1$. Then we also have $v_1 = v_2$ by our rule of taking representatives and therefore the discs coincide (without need of reparametrization). Finally $\tau_2 = \tau_1$ because they are injective (cf. Section 2). As for the smoothness, we make a choice of our representative v smoothly depending on z, and point our attention to (2.0) and (2.1) of Section 2. If we then take evaluation of the discs and their lifts at $\tau\in\Delta$ we get the C^{k-4}-smoothness of (3.2). In the lines of what was remarked after Proposition 2.3, for any $(z,[\zeta])\in(\dot{T}^*\mathbf{C}^n/\dot{\mathbf{C}})|_{D_1\setminus D_2}$ there is a unique (ϕ,ϕ^*), up to reparametrization, such that $\phi(\tau)=z$, $[\phi^*(\tau)]=[\zeta]$ for some $\tau\in\Delta$. On the other hand, the class of stationary discs which are tangent to ∂D_2 divides the set of all stationary discs into two sets, the ones which are transversal to (resp. disjoint to) D_2. Accordingly, $\mathcal{S}$ divides $(\dot{T}^*\mathbf{C}^n/\dot{\mathbf{C}})|_{D_1\setminus\bar{D}_2}$ into two sets. We denote by $\mathcal{W}$ the first set and refer to it as to the "finite" side of $\mathcal{S}$ the complement being called a neighborhood of the "plane at infinity". The set $\mathcal{W}$ is a wedge type domain with boundary $\mathcal{S}$ and edge $\mathcal{E}:=\dot{T}^*_{\partial D_1}\mathbf{C}^n/\dot{\mathbf{C}}$.

We now describe the fibers $\mathcal{S}_{z_0}=\pi^{-1}(z_0)\cap\mathcal{S}$ where $\pi:\dot{T}^*\mathbf{C}^n/\dot{\mathbf{C}}\to\mathbf{C}^n$ is the projection $\pi(z,[\zeta])=z$. Our plan is to use Ψ_{z_0}, interchange D_1 with $\mathbf{B}^n$ and z_0 with 0, analyze the situation in this new setting, and then bring back the conclusions to the former by $\Psi_{z_0}^{-1}$. Recall that Ψ_{z_0} interchanges the stationary discs through z_0 with the complex lines (the stationary discs of the ball) through 0. We first describe the set

(3.3)
$$\gamma_0=\{z\in\partial(\Psi_{z_0}(D_2))\colon \text{for some } v\in T^{\mathbf{C}}_z\partial(\Psi_{z_0}(D_2)) \text{ with } \phi_{z,v} \text{ passing through } 0\}.$$

If $\rho=0$ is an equation for $\partial(\Psi_{z_0}(D_2))$, γ_0 is defined by

$$\rho(z)=0 \quad\text{and}\quad \partial\rho(z)\cdot z=0.$$

This is a system of three real equations that we denote by $r=0$. We normalize our coordinates so that

$$\partial\rho(z)=(1,0,...) \quad\text{and}\quad z=(0,c,0,...).$$

We then have for the partial Jacobian

$$J_{z_1,\bar{z}_1,z_2,\bar{z}_2}r(z)=\begin{bmatrix}1 & 1 & 0 & 0\\ * & * & c\partial^2_{z_2,z_2}\rho & c\partial^2_{\bar{z}_2,z_2}\rho\\ * & * & c\partial^2_{z_2,\bar{z}_2}\rho & c\partial^2_{\bar{z}_2,\bar{z}_2}\rho\end{bmatrix}, \tag{3.4}$$

where the asterisks denote unimportant matrix coefficients. Let A be the 3×3 minor obtained by discarding the first column. We have

$$\det A=c^2\det\begin{bmatrix}\partial^2_{z_2,z_2}\rho & \partial^2_{z_2,\bar{z}_2}\rho\\ \partial^2_{\bar{z}_2,z_2}\rho & \partial^2_{\bar{z}_2,\bar{z}_2}\rho\end{bmatrix}=-c^2\det(\mathrm{Hess}(\rho)|_{\mathbf{C}z_2})<0, \tag{3.5}$$

where the real Hessian of ρ at z along the z_2-plane is positive because $\Psi_{z_0}(D_2)$ is strongly convex with respect to $\Psi_{z_0}(\phi_{z_0,z}(\Delta))$. In conclusion, $\operatorname{rank} J(r)=3$ and hence γ_0 is a regular real manifold of dimension $2n-3$, compact and without boundary. We now use the fact that Ψ_{z_0} is a diffeomorphism and conclude that $\gamma_{z_0}:=\Psi_{z_0}^{-1}(\gamma_0)$ is also a regular manifold of dimension $2n-3$ in ∂D_2, which enjoys the same properties as γ_0. It represents the set of points where the geodesics of D_1 through z_0 are tangent to ∂D_2. Let $\widetilde{\gamma}_{z_0}$ be the section $(z,[v(z)])$ of $(\dot{T}^{\mathbf{C}}\partial D_2/\dot{\mathbf{C}})|_{\gamma_{z_0}}$ where $[v(z)]$ is the direction tangent at z to the stationary disc connecting z_0 and z. We can parametrize the fiber $\mathcal{S}_{z_0}$ over $\widetilde{\gamma}_{z_0}\times\Delta$ by the same parametrization Φ as in (3.1). This being bijective, we conclude that $\mathcal{S}_{z_0}$ is a finite family of regular closed manifolds of dimension $2n-3$ without boundary, which do not intersect. We move now z from the fixed z_0 and describe the behavior of the fibers $\mathcal{S}_z$; they depend in a (C^{k-4}) fashion on z since the mapping in (2.5) is also C^{k-4}. As for their behavior at $z_0\in\partial D_2$, we consider the set Π_{z_0} defined by the diagram

$$\begin{array}{ccc} \dot{T}^{\mathbf{C}}_{z_0}\mathbf{C}^n/\dot{\mathbf{C}} & \xrightarrow[\sim]{} & \dot{T}^{*}_{z_0}\mathbf{C}^n/\dot{\mathbf{C}}\simeq\mathbf{P}^{n-1}_{\mathbf{C}} \\ \cup & & \cup \\ \dot{T}^{\mathbf{C}}_{z_0}\partial D_2/\dot{\mathbf{C}} & \xrightarrow[\sim]{} & \Pi_{z_0}, \end{array}$$

where the two horizontal arrows are given by the smooth injective mapping $v\mapsto[\phi^*_{z_0,v}(0)]$. Thus $\Pi_{z_0}:=\{[\phi^*(0)]\colon \phi \text{ is tangent to } \partial D_2 \text{ at } z_0\}$ is a 2-codimensional real submanifold of $\mathbf{P}^{n-1}_{\mathbf{C}}$ which reduces to a single point when $n=2$.

Lemma 3.4. *The sets $\mathcal{S}_{z_\nu}$ shrink to Π_{z_0} as $z_\nu\to z_0\in\partial D_2$; in particular, $\mathcal{S}_{z_\nu}$ consists of just one component when z_ν is close to z_0.*

Proof. By the strong convexity of ∂D_2, the manifolds γ_{z_ν} shrink to $\{z_0\}$ as $z_\nu\to z_0$. If we pick up any sequence $w_\nu\in\gamma_{z_\nu}$, we have

$$\frac{w_\nu-z_0}{|w_\nu-z_0|}\to v\in T^{\mathbf{C}}_{z_0}\partial D_2.$$

Let ϕ_{z_ν,w_ν} (resp. $\phi_{z_0,v}$) be the geodesic through z_ν and w_ν (resp. through z_0 with tangent v), normalized by the condition, $z_\nu=\phi_{z_\nu,w_\nu}(0)$ and $w_\nu=\phi_{z_\nu,w_\nu}(\xi_{w_\nu})$ for $\xi_{w_\nu}\in(0,1)$, (resp. $z_\nu=\phi_{z_0,v}(0)$ and $rv=\phi'_{z_0,v}(0)$ for $r\in\mathbf{R}^+$). Then

$$\phi_{z_\nu,w_\nu}\to\phi_{z_0,v} \quad\text{and}\quad \phi^*_{z_\nu,w_\nu}\longrightarrow\phi^*_{z_0,v},$$

with convergence in the $C^{2,1/2}(\bar{\Delta})$ norm. In particular, since

$$\mathcal{S}_{z_\nu}=\bigcup_{w_\nu\in\gamma_{z_\nu}}[\phi^*_{z_\nu,w_\nu}(\xi_{w_\nu})],$$

we have

$$\mathcal{S}_{z_\nu}\to\bigcup_{v\in\dot{T}^{\mathbf{C}}_{z_0}\partial D_2}[\phi^*_{z_0,v}(0)]. \quad\square$$

It follows that for the fibers $\mathcal{W}_{z_\nu}$, which are open domains of $\mathbf{P}^{n-1}_{\mathbf{C}}$ with boundary $\mathcal{S}_{z_\nu}$, we have merely by definition:

$$\mathcal{W}_{z_\nu} \to \mathbf{P}^{n-1}_{\mathbf{C}} \setminus \Pi_{z_0}, \quad \text{as } z_\nu \to z_0 \in \partial D_2.$$

If, instead, we move z_ν towards $z_0 \in \partial D_1$, then each $\mathcal{S}_{z_\nu}$ as well as their "finite" sides $\mathcal{W}_{z_\nu}$, shrink to the single point $(\dot{T}^*_{\partial D_1}\mathbf{C}^n/\dot{\mathbf{C}})|_{z_0}$.

Now we move z_0 all over $D_1 \setminus \overline{D}_2$. If we take a closer look on (3.4) and (3.5), we see that the set $\Psi_{z_0}(D_2)$ as well as its equation $\rho_{z_0}=0$, moves smoothly with respect to z_0 by the regularity properties of Ψ. It follows that the set defined by $\{(z_0, z) : z_0 \in D_1 \setminus \overline{D}_2 \text{ and } z \in \gamma_{z_0}\}$ is a $(4n-3)$-dimensional manifold. In particular, the set γ_{z_0} is a $(2n-3)$-dimensional manifold and it cannot turn from one to several components without passing through a singular point z_0. It follows that the set $\mathcal{S}_{z_0}$ also consists of one component.

By the preceding discussion and Sard's theorem, we can also say that $\mathcal{S}$ is a smooth regular manifold except possibly a closed subset of measure zero. Along with its natural foliation by the discs $(\phi_{z,v}, [\phi^*_{z,v}])$, we need to endow $\mathcal{S}$ with another foliation, locally on a neighborhood of each of its points, by CR manifolds $\mathcal{M}$ of dimension $2n$ and CR dimension 1 each one being still a union of discs. For this, we fix $z_0 \in \overline{D}_1 \setminus \overline{D}_2$, consider the submanifold γ_{z_0} of ∂D_2 with dimension $2n-3$ of points of tangency for the stationary discs through z_0, and denote by z the point which moves in γ_{z_0}. As above, we denote by ϕ_{z,z_0} the stationary disc through z and z_0, normalized by $\phi_{z,z_0}(0)=z$ and $\phi_{z,z_0}(\xi)=z_0$ for $\xi \in (0,1)$; we also write $\xi = \xi(z, z_0)$ and define $v(z, z_0) := \phi'_{z,z_0}(0)$. We set $\Gamma_{z_0} := \{(z, [v(z, z_0)], \xi(z, z_0)) : z \in \gamma_{z_0}\}$; then $\dim \Gamma_{z_0} = \dim \gamma_{z_0} = 2n-3$. Since $\Phi_1 := \pi \circ \Phi$ sends all points of Γ_{z_0} to the fixed z_0, then we have an inclusion $T\Gamma_{z_0} \subset \operatorname{Ker} \Phi'_1|_{\Gamma_{z_0}}$. But since the dimensions are the same, $T\Gamma_{z_0} = \operatorname{Ker} \Phi'_1|_{\Gamma_{z_0}}$. In particular, if p is the projection $(z, [v], \tau) \mapsto z$, then

$$p'(\operatorname{Ker} \Phi'_1|_{\Gamma_{z_0}}) \subset T\gamma_{z_0}. \tag{3.6}$$

We define $\mathcal{M}$ locally at a point $(z_0, [\zeta]) \in \mathcal{S} \cup \mathcal{E}$; if $(z_0, [\zeta]) \in \mathcal{E}$, $\mathcal{M}$ will in fact be a manifold with boundary $\mathcal{E}$. Let $(z, [v], \tau)$ be the value of the parameter in $(T^{\mathbf{C}}\partial D_2) \times \Delta$ which corresponds to $(z_0, [\zeta])$ via Φ_1. Choose a germ of submanifold $\delta_{z_0} \subset \partial D_2$ transversal to γ_{z_0} at z with complementary dimension 2. By (3.6), we have

$$\operatorname{Ker}(\Phi'_1(z, [v], \tau)\big|_{T_z\delta_{z_0} \times \mathbf{P}^{n-1}_{\mathbf{C}} \times T_\tau \Delta}) = \{0\}. \tag{3.7}$$

Thus Φ_1 induces a diffeomorphism between a neighborhood $\Sigma = \Sigma_1 \times \Sigma_2$ of $(z, [v], \tau)$ in $(T^{\mathbf{C}}\partial D_2/\dot{\mathbf{C}})|_{\delta_{z_0}} \times \Delta$ and a neighborhood of z_0 in $\overline{D}_1$. We define $\mathcal{M} = \Phi(\Sigma)$ that is

$$\mathcal{M} = \bigcup_{(\phi, \phi^*)} (\phi, [\phi^*])(\Sigma_2) \tag{3.8}$$

for $(\phi(0),[\phi'(0)])\in\Sigma_1$. The map Φ is a diffeomorphic parametrization of $\mathcal{M}$ over Σ and hence $\mathcal{M}$ is a smooth manifold, in fact a graph over a neighborhood of z_0 in $\overline{D}_1$. This was not necessarily the case of $\mathcal{S}$ since Φ is a smooth and bijective parametrization of $\mathcal{S}$ but it might occur that Φ' is degenerate at some point. We define a function F on $\mathcal{S}$ by collecting all extensions $f_{\phi(\bar\Delta)}$ of the given f from $\phi(\partial\Delta)$ to $\phi(\bar\Delta)$. For $(z,[\zeta])\in\mathcal{S}$ we put

$$F(z,[\zeta])=f_{\phi(\bar\Delta)}(z) \quad \text{if } (\phi(\tau),[\phi^*(\tau)])=(z,[\zeta]) \text{ for some } \tau. \tag{3.9}$$

According to Theorem 2.5, F is well defined. We have the following result.

Proposition 3.5. *At every point of $\mathcal{S}\setminus\mathcal{E}$, the function F holomorphically extends to a one-sided neighborhood on the $\mathcal{W}$-side of $\mathcal{S}$.*

Proof. The ingredients of the proof are the foliation of $\mathcal{S}$ by manifolds with boundary $\mathcal{M}$, which are themselves union of discs, and the additional transversal foliation of $\mathcal{W}$ by the fibers $\mathcal{W}_z$. The starting remark is that F is holomorphic along each disc and therefore it is CR on each $\mathcal{M}$ since $\dim_{\mathrm{CR}}\mathcal{M}=1$.

(a) We approximate $F|_{\mathcal{E}}$ by a sequence of entire functions F_ν (cf. e.g. [3]). To this end it is important to notice, as was first pointed out by Webster, that since ∂D_1 is strongly convex, $\mathcal{E}$ is totally real. Then, in an identification $\mathcal{E}\simeq\mathbf{R}^{2n-1}$, these are defined by

$$\widehat{F}_\nu(\xi)=\Big(\frac{\nu}{\pi}\Big)^{(2n-1)/2}\int_{\mathbf{R}^{2n-1}}F(\eta)e^{-\nu(\eta-\xi)^2}\,dV, \tag{3.10}$$

(dV being the volume element in $\mathbf{R}^{2n-1}$). It is well known that $\widehat{F}_\nu\to F$ uniformly on compact subsets of $\mathcal{E}$. Also, F being CR on each $\mathcal{M}$, it is possible to deform the integration chain from $\mathcal{E}$ to another chain entering inside $\mathcal{M}$ and reaching any point of $\mathcal{M}$ in a neighborhood of $\mathcal{E}$. In other terms, the function F is approximated, over each $\mathcal{M}$ near $\mathcal{E}$, by the same sequence (3.10) of entire functions. Since the $\mathcal{M}$'s give a foliation of $\mathcal{S}$, it follows that the uniform approximation of F by the F_ν's holds on the whole $\mathcal{S}$ in a neighborhood of $\mathcal{E}$.

(b) By now using the foliation of $\mathcal{W}$ by the fibers $\mathcal{W}_z$, we can bring the approximation by entire functions from $\mathcal{S}$ to $\mathcal{W}$ in a neighborhood of $\mathcal{E}$: in fact, by the maximum principle, the sequence $\widehat{F}_\nu$ which is Cauchy over $\mathcal{S}_z$ will be Cauchy on the whole $\mathcal{W}_z$.

(c) We now use the theory of propagation of wedge extendibility along discs for each CR function $F|_{\mathcal{M}}$ by [12] which develops [8]. We put a suffix s in the notation of the disc Δ_s to specify its radius, and define

$$I=\Big\{r\in(0,1): F \text{ extends to the side } \mathcal{W} \text{ of } \mathcal{S} \text{ in } \bigcup_{(\phi,\phi^*)}(\phi,[\phi^*])(\Delta_1\setminus\bar\Delta_{1-r})\Big\}, \tag{3.11}$$

for all stationary discs ϕ tangent to ∂D_2 at $\tau=0$. (The last requirement is just a choice of the parametrization.) We have $I\neq\varnothing$ due to (b) above. We show now that we have indeed $I=(0,1)$ from which the proposition follows. We reason by contradiction, suppose $I\neq(0,1)$, and denote by r_0 the supremum of I; thus $r_0<1$. By propagation of wedge extendibility of $F|_{\mathcal{M}}$ for each $\mathcal{M}$, and since the wedge evolves continuously with the base point, then, on account also of a compactness argument, F would extend to the side $\mathcal{W}$ for a value of r bigger than r_0, a contradiction. $\square$

End of proof of Theorem 3.2.

• First, recall that for z moving from ∂D_1 to $z_0\in\partial D_2$, the fibers $\mathcal{W}_z$ grow from a single point to $\mathcal{W}_{z_0}=\mathbf{P}^{n-1}_{\mathbf{C}}\setminus\Pi_{z_0}$. Also, recall that by approximation, F extends holomorphically from $\mathcal{S}_z$ to $\mathcal{W}_z$ when z is close to ∂D_1, and, by propagation, to a neighborhood of $\mathcal{S}_z$ in $\mathcal{W}$ when z is no longer close to ∂D_1. Then F extends to the whole set $\mathcal{W}$ by the Hartogs continuity principle. For $n>2$ the same conclusion also follows by the Hartogs extension theorem.

• The boundary values of F on $\pi^{-1}(\partial D_2)\cap\overline{\mathcal{W}}\subset\partial\mathcal{W}$ are constant on the fibers $\mathcal{W}_{z_0}$, $z_0\in\partial D_2$. Indeed, $F|_{\mathcal{W}_{z_0}}$ holomorphically extends to the whole projective space $\mathbf{P}^{n-1}_{\mathbf{C}}$ because the set Π_{z_0} of codimension 2 is removable, hence it is constant. Now since F is constant on the fibers of $\overline{\mathcal{W}}$ on an open set of the boundary of $\mathcal{W}$, then F is constant on the fibers of $\mathcal{W}$ everywhere in $\mathcal{W}$. Then $\tilde{f}(z):=F(z,[\zeta])$, $(z,[\zeta])\in\mathcal{W}$, is a well-defined holomorphic extension of the original function f to $D_1\setminus\overline{D}_2$. Then f further extends to D_2 by the Hartogs theorem. The proof is now complete. $\square$

Remark 3.6. Take a line segment I connecting a pair of points z_1 and z_2 of ∂D_1 and ∂D_2, resp., fix a neighborhood $U\supset I$ in $\mathbf{C}^n$, denote by $\mathcal{I}$ the family of discs tangent to ∂D_2 and passing through $U\cap D_1$ and denote by $\mathcal{J}$ the family of the discs tangent to ∂D_2 which have some common boundary point with those of $\mathcal{I}$. Set $V_1:=\bigcup_{\phi\in\mathcal{I}}\phi(\partial\Delta)$ and $V_2:=\bigcup_{\phi\in\mathcal{J}}\phi(\partial\Delta)$. Assume that f is defined and continuous in V_2 and extends holomorphically to the discs which belong to $\mathcal{J}$; then f extends holomorphically to a one-sided neighborhood of V_1 in D_1. In fact, by moving z from z_1 to z_2 along I we will have the same conclusions for the fibers $\mathcal{S}_z$ and $\mathcal{W}_z$ as in the proof of Theorem 3.2. In particular we will conclude that F is independent of $[\zeta]$ in a neighborhood of z_1. But then F is independent of $[\zeta]$ wherever it is defined, in particular in a one-sided neighborhood of V_1.

References

[1] Agranovsky, M. L. and Semenov, A. M., Boundary analogues of the Hartogs theorem, *Sibirsk. Mat. Zh.* **32** (1991), 168–170, 222 (Russian). English transl: *Siberian Math. J.* **32** (1991), 137–139.

[2] Berenstein, C., Chang, D.-C., Pascuas, D. and Zalcman, L., Variations on the theorem of Morera, in *The Madison Symposium on Complex Analysis* (*Madison, WI, 1991*), Contemp. Math., **137**, pp. 63–78, Amer. Math. Soc., Providence, RI, 1992.

[3] Boggess, A., *CR manifolds and the tangential Cauchy–Riemann complex*, CRC Press, Boca Raton, FL, 1991.

[4] Dinh, T.-C., Conjecture de Globevnik–Stout et théorème de Morera pur une châine holomorphe, *Ann. Fac. Sci. Toulouse Math.* **8** (1999), 235–257.

[5] Globevnik, J., On holomorphic extensions from spheres in C^2, *Proc. Roy. Soc. Edinburgh Sect. A* **94** (1983), 113–120.

[6] Globevnik, J., A family of lines for testing holomorphy in the ball of C^2, *Indiana Univ. Math. J.* **36** (1987), 639–644.

[7] Globevnik, J. and Stout, E. L., Boundary Morera theorems for holomorphic functions of several complex variables, *Duke Math. J.* **64** (1991), 571–615.

[8] Hanges, N. and Trèves, F., Propagation of holomorphic extendability of CR functions, *Math. Ann.* **263** (1983), 157–177.

[9] Lempert, L., La métrique de Kobayashi et la représentation des domaines sur la boule, *Bull. Soc. Math. France* **109** (1981), 427–474.

[10] Rudin, W., *Function theory in the unit ball of* C^n, *Grundlehren der Mathematischen Wissenschaften*, **241**, Springer, New York–Berlin, 1980.

[11] Stout, E. L., The boundary values of holomorphic functions of several complex variables, *Duke Math. J.* **44** (1977), 105–108.

[12] Tumanov, A., Connections and propagation of analyticity for CR functions, *Duke Math. J.* **73** (1994), 1–24.

[13] Tumanov, A., Thin discs and a Morera theorem for CR functions, *Math. Z.* **226** (1997), 327–334.

[14] Tumanov, A., A Morera type theorem in the strip, *Math. Res. Lett.* **11** (2004), 23–29.

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