K-homology and index theory
on contact manifolds

by

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With admiration and affection we dedicate this paper
to Sir Michael Atiyah on the occasion of his 85th birthday.

Contents

1. Introduction

This paper applies K-homology to solve the index problem for a class of hypoelliptic (but not elliptic) operators on contact manifolds. K-homology is the dual theory to K-theory. We explicitly calculate the K-cycle (i.e., the element in geometric K-homology) determined by any hypoelliptic Fredholm operator in the Heisenberg calculus.

The index theorem of this paper precisely indicates how the analytic versus geometric K-homology setting provides an effective framework for extending formulas of Atiyah–Singer type to non-elliptic Fredholm operators. Given an index problem, the K-homology framework provides a guide and hint as to what the solution of that index problem might be.

A differential operator P on a closed contact manifold X of dimension $2n+1$,

$$P \colon C^\infty(X, F^0) \longrightarrow C^\infty(X, F^1),$$

is *Heisenberg-elliptic* if its principal symbol in the Heisenberg calculus is invertible. Any Heisenberg-elliptic P is Fredholm and hypoelliptic, but not elliptic. Invertibility of the principal Heisenberg symbol does not imply invertibility of the usual principal symbol used by Atiyah and Singer. However, the analytic properties of a Heisenberg-elliptic P are such that P determines an element in Kasparov (i.e., analytic) K-homology

$$[P] \in \mathrm{KK}^0(C(X), \mathbb{C}).$$

The main theorem of this paper is a topological formula for this analytical K-homology element $[P]$. Our formula is the following.

THEOREM.

$$[P] = [\sigma_H^+(P)] \cap [X^+] + [\sigma_H^-(P)] \cap [X^-]. \tag{1.1}$$

Here $[X^+]$ and $[X^-]$ are the fundamental cycles in the K-homology group $K_1(X)$ given by the two natural Spin^c structures of X, one for each co-orientation of the contact structure. In (1.1), $\sigma_H^+(P)$ and $\sigma_H^-(P)$ are elements in the K-theory group $K^1(X)$ extracted from the *principal Heisenberg symbol* of P. More specifically, $\sigma_H^+(P)$ and $\sigma_H^-(P)$ are explicit automorphisms of two $\mathbb{C}$-vector bundles on X intrinsically associated with the contact structure of X. The cap product $\cap$ in (1.1) is the $K^\bullet(X)$-module structure of $K_\bullet(X)$.

Applying the Chern character to (1.1) gives a characteristic class formula in rational homology $H_\bullet(X, \mathbb{Q})$ for the index of P. Our proof of (1.1) is sufficiently canonical so that the same formula is valid for the equivariant index problem and the families index problem for the Heisenberg calculus.

The index problem for Heisenberg-elliptic operators on contact manifolds has been studied, and special cases were solved by C. Epstein and R. Melrose [20], [13], [14] and E. van Erp [15], [16]. Our work builds on these partial results. See also the related work by A. Connes and H. Moscovici [11], [12] in which they solve the index problem for their Heisenberg-elliptic signature operator associated with the (non-commutative) "space of leaves" of any foliation. From our point of view, equation (1.1) solves the index problem (including the equivariant and families problems) for Heisenberg-elliptic operators on contact manifolds.

Acknowledgment

2. The problem and the solution

2.1. K-homology

The point of view of [3] is that index theory, in general, is based on the equivalence between geometric K-homology and analytic K-homology. For a topological space X, a geometric K-cycle is a triple (M, E, φ) consisting of a closed Spin^c manifold M, a complex vector bundle E on M, and a continuous map $\varphi \colon M \to X$. The collection of (M, E, φ)-cycles, subject to a certain equivalence relation, forms an abelian group under disjoint union. We denote this group by $K_*^{\mathrm{top}}(X)$.

There is a natural map

$$\mu \colon K_j^{\mathrm{top}}(X) \longrightarrow \mathrm{KK}^j(C(X), \mathbb{C}), \quad j = 0, 1,$$

from geometric K-homology to analytic K-homology, defined in the following way. Let $D_M \otimes I_E$ denote the Dirac operator D_M for the Spin^c manifold M twisted by the vector bundle E. Then $\mu(M, E, \varphi)$ is the push-forward by φ of the analytic K-cycle $[D_M \otimes I_E] \in \mathrm{KK}^j(C(M), \mathbb{C})$,

$$\mu(M, E, \varphi) = \varphi_*(D_M \otimes I_E) \in \mathrm{KK}^j(C(X), \mathbb{C}).$$

For a finite CW complex X the map μ is an isomorphism [5].

In [3] Baum and Douglas asked the following question.

Problem. Let X be a finite CW complex. Given an element $\xi \in \mathrm{KK}^j(C(X), \mathbb{C})$ in analytic K-homology, explicitly compute the unique $\tilde{\xi} \in K_j^{\mathrm{top}}(X)$ in geometric K-homology corresponding to ξ.

As explained in [3], several well-known index theorems (including, of course, the Atiyah–Singer index formula for elliptic operators) can be understood in this framework. Moreover, if the construction of the (M, E, φ)-cycle for a given class of operators is sufficiently canonical, it will also solve the equivariant and families index problems for this class of operators.

In this paper we solve the index problem for the Heisenberg calculus in this context. In other words, we solve the following problem.

Problem. Consider a closed contact manifold X. A Heisenberg-elliptic (pseudo)differential operator P on X determines an element $[P] \in \mathrm{KK}^0(C(X), \mathbb{C})$ in analytic K-homology. Explicitly compute an (M, E, φ)-cycle in geometric K-homology $K_0^{\mathrm{top}}(X)$ with

$$\mu(M, E, \varphi) = [P].$$

Starting with $[P] \in \mathrm{KK}^0(C(X), \mathbb{C})$, the K-cycle (M, E, φ) which solves the index problem for P should be constructed so that, mutatis mutandis, the same K-cycle will solve the equivariant index problem (when a compact Lie group is acting) and the index problem for families. As an immediate corollary of the computation of the K-cycle, one obtains characteristic class formulas for the homology Chern character of P (including for equivariant operators and families of operators). Thus, the computation of the K-cycle gives a fully satisfactory answer to the index problem for the Heisenberg calculus on contact manifolds.

2.2. Contact manifolds

Definition 2.2.1. A *contact structure* on a manifold X of dimension $2n+1$ is a C^∞ subvector bundle $H \subset TX$ of fiber dimension $2n$ such that if θ is any 1-form on X with $H = \mathrm{Ker}\,\theta$ then the 2-form $d\theta$ restricted to H is symplectic in each fiber H_x.

In all that follows X will denote a closed smooth contact manifold of dimension $2n+1$. H is referred to as the contact hyperplane bundle, while θ (which has to be chosen) is a contact form. Note that θ is a contact form on X if and only if $\theta(d\theta)^n$ is a nowhere vanishing $2n+1$-form (i.e., a volume form) on X.

Once a contact form θ has been fixed, a number of geometric structures on X are then determined. The *Reeb vector field* T is the unique vector field on X with the properties

$$\theta(T) = 1 \quad \text{and} \quad d\theta(T, \cdot) = 0.$$

The Reeb field T is tranversal to the contact hyperplane bundle H.

We choose a complex structure J for H that is compatible with the symplectic form $d\theta$ in each fiber, i.e.,

$$J^2 = -1, \quad d\theta(Jv, Jw) = d\theta(v, w) \quad \text{and} \quad d\theta(Jv, v) \geqslant 0.$$

Such a choice of J is always possible, because the space of compatible complex structures on a symplectic vector space is contractible. The choice of J also defines a euclidean structure in the fibers of H, with inner product defined by $\langle v, w \rangle = d\theta(Jv, w)$. In addition, J and θ determine a hermitian inner product

$$\langle v, w \rangle = d\theta(Jv, w) + i d\theta(v, w).$$

2.3. Example: A hypoelliptic (but not elliptic) Fredholm operator

A *sublaplacian* for the given structure (X, θ, J) is a second-order differential operator on X constructed as follows. Locally, a sublaplacian is of the form

$$\Delta_H = \sum_{j=1}^{2n} -W_j^2.$$

Here $W_1, ..., W_{2n}$ is a collection of vector fields, defined in an open subset $U \subseteq X$, such that at each point $x \in U$ the vectors $W_j(x)$ form an orthonormal frame for the euclidean vector space H_x. A global sublaplacian on X is then constructed via a partition of unity.

To obtain an operator with interesting index theory we add a lower order term. Let r be a positive integer and let γ be a C^∞ map from X to the $\mathbb{C}$-vector space $M(r, \mathbb{C})$ of all $r \times r$ matrices,

$$\gamma \colon X \longrightarrow M(r, \mathbb{C}).$$

Examples of the kind of Fredholm operator to which our result applies are the second-order differential operators P_γ of the form

$$P_\gamma = I_r \otimes \Delta_H + i\gamma(I_r \otimes T) \colon C^\infty(X, \mathbb{C}^r) \longrightarrow C^\infty(X, \mathbb{C}^r),$$

where i is the imaginary unit and I_r is the $r \times r$ identity matrix. Operators of this type have been studied extensively. See for example [6, Chapter 1] or [13, Theorem 8.1] for the following result.

PROPOSITION 2.3.1. *The operator* P_γ *is invertible modulo smoothing operators if, for all* $x \in X$,

$$\gamma(x) - \lambda I_r \ \textit{is invertible for all} \ \lambda \in \{..., -n-4, -n-2, -n, n, n+2, n+4, ...\}.$$

The operators P_γ are not elliptic, but they are hypoelliptic and Fredholm. The parametrix (i.e., the inverse modulo smoothing operators) of these operators P_γ is not a classical pseudodifferential operator, but it is a pseudodifferential operator in the Heisenberg calculus. The invertibility requirement in Proposition 2.3.1 is equivalent to invertibility of the principal Heisenberg symbol of P_γ. We shall refer to hypoelliptic operators with a parametrix in the Heisenberg calculus as *Heisenberg-elliptic*. Like P_γ, any Heisenberg-elliptic differential operator consists of a highest-order part that only involves the H-directions, and is "elliptic" in the H-directions, plus terms of lower order. For further details about Heisenberg-elliptic operators, see §5 below.

2.4. The problem

In general, we consider a Heisenberg-elliptic operator

$$P \colon C^\infty(X, F^0) \longrightarrow C^\infty(X, F^1),$$

where F^0 and F^1 are two smooth $\mathbb{C}$-vector bundles on X. Although not elliptic in the usual sense, P has the same basic analytic properties as an elliptic operator. For example, if P is Heisenberg-elliptic, then the symmetric operator

$$D = \begin{pmatrix} 0 & P^* \\ P & 0 \end{pmatrix} \colon C^\infty(X, F^0 \oplus F^1) \longrightarrow C^\infty(X, F^0 \oplus F^1)$$

is essentially self-adjoint, has discrete spectrum and finite-dimensional eigenspaces consisting of smooth functions [15, §2.6]. The bounded operator $F = D(1+D^2)^{-1/2}$ is Fredholm, and commutators $[F, M_f]$ of F with functions $f \in C(X)$ are compact. Therefore F determines an element in the analytic K-homology of X,

$$[P] \in \mathrm{KK}^0(C(X), \mathbb{C}).$$

We briefly verify this. We use the standard convention that $\mathcal{K}$ denotes the separable C^*-algebra of compact operators.

LEMMA 2.4.1. *The operator $F = D(1+D^2)^{-1/2}$ is a bounded Fredholm operator on $L^2(X)^{\oplus 2r}$ that satisfies the pseudolocality condition $[F, M_f] \in \mathcal{K}(L^2(X)^{\oplus 2r})$ for multiplication operators $M_f \in \mathcal{L}(L^2(X)^{\oplus 2r})$, $f \in C(X)$.*

Proof. If P is Heisenberg-elliptic then F is Heisenberg-elliptic and has order zero in the Heisenberg pseudodifferential calculus. Order-zero operators in the Heisenberg calculus are bounded on $L^2(X)^{\oplus 2r}$ while operators of negative order are compact (by the Rellich lemma for the Heisenberg calculus). Therefore F is bounded and invertible modulo compact operators.

We now verify the pseudolocality condition. Let Ψ_H denote the norm completion of the algebra of order-zero Heisenberg operators. The principal Heisenberg symbol σ_H gives rise to a short exact sequence of C^*-algebras,

$$0 \longrightarrow \mathcal{K} \longrightarrow \Psi_H \xrightarrow{\ \sigma_H\ } S_H \longrightarrow 0.$$

Without going into the details of the Heisenberg calculus, it suffices to know that the symbol algebra S_H contains $C(X)$ as a central subalgebra. Moreover, for a continuous function $f \in C(X)$ the multiplication operator M_f on $L^2(X)$ is contained in Ψ_H, while its principal symbol is $\sigma_H(M_f) = f$, as usual. It follows that $[\sigma_H(F), \sigma_H(M_f)] = 0$ for any order-zero operator F, which is equivalent to $[F, M_f] \in \mathcal{K}$. □

Our aim is to solve the index problem for Heisenberg-elliptic operators within the K-homology framework formulated by Baum and Douglas [3], [4], i.e., given a Heisenberg-elliptic operator P to construct a K-cycle (M, E, φ) with

$$[P] = \mu(M, E, \varphi).$$

2.5. The solution

We now describe the solution of the index problem in K-homology for a Heisenberg-elliptic differential (or pseudodifferential) operator P.

The equatorial symbol.   Consider the *classical* principal symbol of P,

$$\sigma(P) \colon \pi^* F^0 \longrightarrow \pi^* F^1.$$

The symbol $\sigma(P)$ is a vector bundle homomorphism defined on the cosphere bundle $S(T^*X)$. For an elliptic operator the classical principal symbol is (by definition) a vector bundle isomorphism. For a Heisenberg-elliptic operator P, $\sigma(P)$ is not invertible on all of $S(T^*X)$.[1]

   The restriction $\sigma(P)|_{S(H^*)}$ of the principal symbol of P to the equator $S(H^*) \subset S(T^*X)$ is referred to as the *equatorial symbol* of P. Since a Heisenberg-elliptic differential operator P is "elliptic" in the H-directions, its equatorial symbol $\sigma(P)|_{S(H^*)}$ is invertible. The topological information contained in the equatorial symbol is essentially trivial, and is certainly insufficient for calculating the index of P. In fact, the equatorial symbol $\sigma(P)|_{S(H^*)}$ of a Heisenberg-elliptic operator is homotopic to the pull-back to $S(H^*)$ of an isomorphism of the vector bundles F^0 and F^1 on X,

$$\sigma_0 \colon F^0 \cong F^1.$$

For example, the classical principal symbol of the operators $P_\gamma = I_r \otimes \Delta_H + i\gamma(I_r \otimes T)$ only depends on the sublaplacian Δ_H, and the equatorial symbol of P_γ is the constant map from $S(H^*)$ to the identity matrix I_r.

   In order to state our result we fix an isomorphism $\sigma_0 \colon F^0 \cong F^1$ of vector bundles on X determined by the equatorial symbol. Having fixed σ_0, the construction of the geometric K-cycle $[(M, E, \varphi)] = \mu^{-1}(P)$ is as follows.

The Spin^c manifold M.   An interesting and unusual feature of the K-cycle determined by a Heisenberg-elliptic operator on a contact manifold is that it consists of *two* components.

   As a smooth manifold, M is a disjoint union of two copies of $X \times S^1$,

$$M = (X^+ \times S^1) \sqcup (X^- \times S^1).$$

Here X^+ and X^- denote two copies of the contact manifold X with opposite co-orientations. A *co-orientation* of the contact structure H is an orientation of the normal bundle TX/H. Each co-orientation of X gives rise to a symplectic structure for $X \times S^1$ (one the opposite of the other), and therefore a Spin^c structure for $X \times S^1$.

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[1] The only differential operators that are both elliptic and Heisenberg-elliptic are vector bundle isomorphisms.

The complex vector bundle E. The $\mathbb{C}$-vector bundle E on M is constructed from the principal Heisenberg symbol $\sigma_H(P)$ of P. E consists, of course, of two vector bundles: one on each copy of $X \times S^1$. Let θ be a contact form on X that agrees with the co-orientation on X^+, and let J be a complex structure of H compatible with $d\theta$. We denote by $H^{1,0}$ the $\mathbb{C}$-vector bundle on X^+ with underlying $\mathbb{R}$-vector bundle H on which the scalar $i \in \mathbb{C}$ acts as J, and $H^{0,1}$ is the conjugate bundle. If we choose a co-orientation for the contact structure of X, then the Bargmann–Fock representation of the Heisenberg group realizes the principal Heisenberg symbol $\sigma_H(P)$ as an invertible map between two (infinite-rank) $\mathbb{C}$-vector bundles on X,

$$\sigma_H^+(P) \colon F^0 \otimes V^+ \longrightarrow F^1 \otimes V^+.$$

Here V^+ denotes the bundle of Fock spaces

$$V^+ = \bigoplus_{j=0}^{\infty} \operatorname{Sym}^j H^{1,0}.$$

Reversing the co-orientation of X determines a second vector bundle isomorphism,

$$\sigma_H^-(P) \colon F^0 \otimes V^- \longrightarrow F^1 \otimes V^-,$$

where V^- is the bundle of conjugate Fock spaces

$$V^- = \bigoplus_{j=0}^{\infty} \operatorname{Sym}^j H^{0,1}.$$

Composing $\sigma_H^{\pm}(P)$ with the isomorphism $\sigma_0^{-1} \colon F^1 \to F^0$ extracted from the equatorial symbol $\sigma(P)|_{S(H^*)}$, we obtain vector bundle *automorphisms* of $F^0 \otimes V^{\pm}$, and therefore elements in K-theory $K^1(X)$,

$$[\sigma_0^{-1} \circ \sigma_H^+(P)] \in K^1(X) \quad \text{and} \quad [\sigma_0^{-1} \circ \sigma_H^-(P)] \in K^1(X).$$

Remark 2.5.1. The vector bundles $F^0 \otimes V^{\pm}$ are infinite direct sums of vector bundles of finite fiber dimension. The automorphisms $\sigma_0^{-1} \circ \sigma_H^{\pm}(P)$, when compressed to a *finite* sum $F^0 \otimes \bigoplus_{j=0}^{N} \operatorname{Sym}^j H$, become stably equivalent (i.e., K-theoretically equivalent) for sufficiently large values of N. Therefore, taking $N \gg 0$ sufficiently large, $[\sigma_0^{-1} \circ \sigma_H^{\pm}(P)]$ are well-defined elements in K-theory. In particular, the Chern characters of $[\sigma_0^{-1} \circ \sigma_H^{\pm}(P)]$ can be computed by the usual formalism taking $N \gg 0$ sufficiently large.

The $\mathbb{C}$-vector bundles E^+ and E^- on $X \times S^1$ are constructed by a familiar clutching construction. Quite generally, for a topological space W with vector bundle $V \to W$, an automorphism $\sigma \colon V \to V$ determines a $\mathbb{C}$-vector bundle $\nu(\sigma)$ on $W \times S^1$. Set $S^1 = S^1_+ \cup S^1_-$, where S^1_+ and S^1_- are the upper and lower hemispheres of $S^1 = \{z \,|\, |z| = 1\} \subset \mathbb{C}$. Then $\nu(\sigma)$ is the $\mathbb{C}$-vector bundle on $W \times S^1$ obtained as the quotient

$$\nu(\sigma) = (V \times S^1_+) \cup_\sigma (V \times S^1_-),$$

where the identification using σ is

$$(w, v, -1) \sim (w, v, -1) \quad \text{and} \quad (w, v, 1) \sim (w, \sigma(w)v, 1), \quad w \in W, \ v \in V_w, \ \pm 1 \in S^1.$$

Then on $X^+ \times S^1$ we set

$$E^+ = \nu(\sigma_0^{-1} \circ \sigma_H^+(P)),$$

while on $X^- \times S^1$ we have

$$E^- = \nu(\sigma_0^{-1} \circ \sigma_H^-(P)).$$

The continuous map φ.   The map $\varphi \colon M \to X$ is the projection on the first factor X for each of the two copies of $X \times S^1$.

THEOREM 2.5.2. *Let P be a Heisenberg-elliptic operator on a closed co-orientable contact manifold X,*

$$P \colon C^\infty(X, F^0) \longrightarrow C^\infty(X, F^1).$$

Then the element in K-homology determined by P is

$$[P] = [\sigma_0^{-1} \circ \sigma_H^+(P)] \cap [X^+] + [\sigma_0^{-1} \circ \sigma_H^-(P)] \cap [X^-].$$

Equivalently, a K-cycle that represents $[P]$ in geometric K-homology is

$$\mu^{-1}([P]) = ((X^+ \times S^1) \sqcup (X^- \times S^1), E^+ \sqcup E^-, \varphi).$$

2.6. Characteristic class formula

If X is a finite CW complex and $\xi \in \mathrm{KK}^0(C(X), \mathbb{C})$, then the homology Chern character $\mathrm{ch}(\xi) \in H_{\mathrm{ev}}(X; \mathbb{Q})$ of ξ is the unique element of $H_{\mathrm{ev}}(X; \mathbb{Q})$ such that

$$\mathrm{Index}(F \otimes \xi) = \varepsilon_*(\mathrm{ch}(F) \cap \mathrm{ch}(\xi))$$

whenever F is a $\mathbb{C}$-vector bundle on X. Here $H_{\mathrm{ev}}(X; \mathbb{Q})$ is the direct sum of the even rational homology groups of X,

$$H_{\mathrm{ev}}(X; \mathbb{Q}) = H_0(X; \mathbb{Q}) \oplus H_2(X; \mathbb{Q}) \oplus H_4(X; \mathbb{Q}) \oplus ...,$$

and $\varepsilon\colon X \to \bullet$ is the map of X to a point and $\varepsilon_*\colon H_*(X; \mathbb{Q}) \to H_*(\bullet; \mathbb{Q}) = \mathbb{Q}$ is the resulting map in rational homology. As usual in algebraic topology ch is the Chern character and $\cap$ is the cap product.

To explicitly construct a K-cycle (M, E, φ) on X with

$$\mu(M, E, \varphi) = \xi,$$

is to solve the index problem for ξ *integrally*. To solve the index problem for ξ *rationally* is to give an explicit formula for $\mathrm{ch}(\xi)$. If an (M, E, φ) has been constructed, then

$$\mathrm{ch}(\xi) = \varphi_*(\mathrm{ch}(E) \cup \mathrm{Td}(M) \cap [M]),$$

where $[M]$ is the fundamental cycle in $H_*(M; \mathbb{Z})$ of M, and $\varphi_*\colon H_*(M; \mathbb{Q}) \to H_*(X; \mathbb{Q})$ is the map of rational homology induced by $\varphi\colon M \to X$.

Thus, Theorem 2.5.2 immediately implies the following characteristic class formula.

COROLLARY 2.6.1. *The Fredholm index of a Heisenberg-elliptic operator P on a closed contact manifold X,*

$$P\colon C^\infty(X, F^0) \longrightarrow C^\infty(X, F^1),$$

is given by

$$\mathrm{Index}\, P = \int_X \mathrm{ch}(\sigma_0^{-1} \circ \sigma_H^+(P)) \wedge \mathrm{Td}(H^{1,0}) + (-1)^{n+1} \int_X \mathrm{ch}(\sigma_0^{-1} \circ \sigma_H^-(P)) \wedge \mathrm{Td}(H^{0,1}).$$

Here X is oriented (in both integrals) by the volume form $\theta(d\theta)^n$.

The factor $(-1)^{n+1}$ for the second term arises from the fact that $X^- \times S^1$ has the conjugate almost complex structure of $X^+ \times S^1$, and is of complex dimension $n+1$.

Remark 2.6.2. In several special cases, characteristic class formulas for the index of Heisenberg-elliptic operators have been derived before. A Toeplitz operator is (essentially) an order-zero pseudodifferential operator in the Heisenberg calculus, and the Toeplitz index formula of Boutet de Monvel [8] is a special case of our formula. We discuss this in detail in §6.4.

Epstein and Melrose prove a characteristic class formula for the index of the twisted sublaplacians P_γ, and a separate formula for a class of operators they refer to as Hermite

operators (a generalization of Toeplitz operators) [14]. van Erp's index formula in [16] only applies in the special case of *scalar* Heisenberg-elliptic operators. Before now, no explicit result existed that was generally applicable and thus unified these various results. In the absense of a general formula, the solution of the equivariant and families index problems was out of reach. All previously obtained formulas [8], [14], [16] are easily seen to be special cases of Corollary 2.6.1.

2.7. Examples

Below are examples which illustrate that the geometric K-cycle of Theorem 2.5.2 is explicitly computable and yields simple clear formulas.

Example 2.7.1. The K-cycle for a second-order Heisenberg-elliptic operator

$$P_\gamma = I_r \otimes \Delta_H + i\gamma(I_r \otimes T) \colon C^\infty(X, \mathbb{C}^r) \longrightarrow C^\infty(X, \mathbb{C}^r)$$

is made explicit if we compute the principal Heisenberg symbol of P_γ (see §6.5). The vector bundle $E^+ \to X^+ \times S^1$ in the K-cycle for P_γ is constructed from the winding of γ around the positive integers $n+2j$. For each $j=0,1,2,\dots$, the smooth map

$$\gamma - (n+2j) \colon X \longrightarrow \mathrm{GL}(r, \mathbb{C})$$

determines an automorphism of the trivial vector bundle $X \times \mathbb{C}^r$. If ν denotes the clutching construction described above, we obtain a $\mathbb{C}$-vector bundle $\nu(\gamma - (n+2j)I_r)$ on $X^+ \times S^1$. Then E^+ on $X^+ \times S^1$ is the $\mathbb{C}$-vector bundle

$$E^+ = \bigoplus_{j=0}^N \nu(\gamma - (n+2j)I_r) \otimes \varphi^* \operatorname{Sym}^j H^{1,0}.$$

Likewise, E^- on $X^- \times S^1$ is constructed from the winding of γ around the negative integers $-(n+2j)$, $j=0,1,2,\dots$,

$$E^- = \bigoplus_{j=0}^N \nu(\gamma + (n+2j)I_r) \otimes \varphi^* \operatorname{Sym}^j H^{0,1}.$$

Example 2.7.2. The odd-dimensional unit spheres S^{2n+1} of $\mathbb{R}^{2n+2}$ have standard contact form

$$\theta = \sum_{i=1}^{n+1} (x_{2i} dx_{2i-1} - x_{2i-1} dx_{2i}),$$

where $x_1, x_2, ..., x_{2n+2}$ are the standard coordinates on $\mathbb{R}^{2n+2}$. Theorem 2.5.2 applies to give a simple and clear index formula for the P_γ operators on S^{2n+1}. Any continuous map $f : S^{2n+1} \to \mathrm{GL}(r, \mathbb{C})$ determines an element in the homotopy group $\pi_{2n+1}(\mathrm{GL})$ of $\mathrm{GL} = \lim_{k \to \infty} \mathrm{GL}(k, \mathbb{C})$. Bott periodicity identifies $\pi_{2n+1}(\mathrm{GL})$ with the integers,

$$\beta : \pi_{2n+1}(\mathrm{GL}) \cong \mathbb{Z}.$$

Therefore an integer $\beta(f)$ has been assigned to f,

$$\beta(f) = \mathrm{ch}(f)[S^{2n+1}] = \int_{S^{2n+1}} \mathrm{Tr}\left(-\frac{f^{-1}df}{2\pi i} \right)^{2n+1}.$$

The formula for the index of P_γ is

$$\mathrm{Index}\, P_\gamma = \sum_{j=0}^{N} \binom{n+j-1}{j} [\beta(\gamma - (n+2j)I_r) + (-1)^{n+1}\beta(\gamma + (n+2j)I_r)],$$

where N is sufficiently large, as above. Observe that on an odd-dimensional sphere any complex vector bundle is stably trivial. Thus $H^{1,0}$ is stably trivial, and in the K-cycle for P_γ we may replace $\varphi^* \mathrm{Sym}^j H^{1,0}$ by a trivial vector bundle of the same rank, which is

$$\binom{n+j-1}{j}.$$

In particular, on S^3 we obtain the very simple formula

$$\mathrm{Index}\, P_\gamma = \sum_{k \text{ odd}} \beta(\gamma - kI_r).$$

3. Outline of the proof

In [15] we showed that the principal Heisenberg symbol $\sigma_H(P)$ of a Heisenberg-elliptic operator P determines an element in the K-theory group of a non-commutative C^*-algebra,

$$[\sigma_H(P)] \in K_0(C^*(T_H X)),$$

where $T_H X$ is the tangent bundle TX, and each fiber $T_x X = H_x \times \mathbb{R}$ has the structure of a nilpotent Lie group isomorphic to the Heisenberg group. $C^*(T_H X)$ is the convolution C^*-algebra of the groupoid $T_H X$. The Connes–Thom isomorphism in analytic K-theory gives a canonical isomorphism

$$\Psi : K^0(T^* X) \xrightarrow{\cong} K_0(C^*(T_H X)).$$

The proof of Theorem 2.5.2 will be accomplished by constructing from the K-theory element $[\sigma_H(P)]$ the relevant K-cycle. This will be done in three steps.

Step 1. In §4, by a direct construction, an isomorphism of abelian groups

$$b: K_0(C^*(T_H X)) \longrightarrow K_0^{\text{top}}(X)$$

will be defined such that there is commutativity in the triangle

$$
\begin{array}{ccc}
K^0(T^*X) & \xrightarrow{\;\;\Psi\;\;} & K_0(C^*(T_H X)) \\
 & {\scriptstyle c}\searrow \qquad \swarrow {\scriptstyle b} & \\
 & K_0^{\text{top}}(X), &
\end{array}
$$

where c is the standard Poincaré duality map.

Step 2. In §5 we prove that $b(\sigma_H(P))$ is the correct K-cycle, i.e.,

$$\mu^{-1}([P]) = b(\sigma_H(P)).$$

Step 3. Finally, in §6, $b(\sigma_H(P))$ is explicitly calculated, i.e., we prove that

$$b(\sigma_H(P)) = [\sigma^{-1} \circ \sigma_H^+(P)] \cap [X^+] + [\sigma^{-1} \circ \sigma_H^-(P)] \cap [X^-].$$

4. Non-commutative topology of contact structures

In this section, the noncommutative Poincaré duality map

$$b: K_0(C^*(T_H X)) \longrightarrow K_0^{\text{top}}(X)$$

will be defined. $T_H X$ is the tangent bundle TX with each fiber $T_x X$ viewed as a nilpotent Lie group. If θ is a contact form on X, with contact hyperplane bundle $H := \operatorname{Ker} \theta$, then the tangent space $T_x X = H_x \times \mathbb{R}$ at a point $x \in X$ has the structure of a nilpotent Lie group that is isomorphic to the Heisenberg group,

$$(v, t) \cdot (v', t') = \left(v + v', t + t' - \tfrac{1}{2} d\theta(v, v')\right), \quad v, v \in H_x, \ t, t' \in \mathbb{R}.$$

The type-I C^*-algebra $C^*(T_H X)$ decomposes into stably commutative factors, i.e., there is a short exact sequence of C^*-algebras

$$0 \longrightarrow I_H \longrightarrow C^*(T_H X) \longrightarrow C_0(H^*) \longrightarrow 0,$$

where $C_0(H^*)$ is commutative, while the ideal I_H is Morita equivalent to a commutative C^*-algebra. Moreover, this short exact sequence is the *quantization* of, and is K-theoretically equivalent to the short exact sequence of commutative C^*-algebras

$$0 \longrightarrow C_0(T^*X \setminus H^*) \longrightarrow C_0(T^*X) \longrightarrow C_0(H^*) \longrightarrow 0.$$

The definition and explicit calculation of b is made possible by this reduction to commutative C^*-algebras.

4.1. The C^*-algebra of the Heisenberg group

The C^*-algebra $C^*(T_H X)$ is a locally trivial bundle of C^*-algebras over X, whose fiber is the group C^*-algebra $C^*(G)$ of the Heisenberg group. We start with a careful description of the structure of this algebra.

Let $V = \mathbb{R}^{2n}$ be the standard symplectic vector space with symplectic form ω. Multiplication in the Heisenberg group $G = V \times \mathbb{R}$ is defined by

$$(v, t) \cdot (v', t') = \left(v + v', t + t' + \tfrac{1}{2}\omega(v, v')\right).$$

The factor $\frac{1}{2}$ is chosen so that for two vectors $v, w \in V$ the Lie bracket is given by the symplectic form $[v, w] = \omega(v, w)$.

Fourier transform in the $\mathbb{R}$-variable for functions $f \in C_c^\infty(G)$,

$$\tilde{f}(v, s) = \int_{\mathbb{R}} e^{-its} f(v, t)\, dt,$$

completes to a $*$-isomorphism,

$$C^*(G) \cong C^*(V \times \mathbb{R}^*, c_\omega).$$

Here $V \times \mathbb{R}^*$ denotes the vector bundle over $\mathbb{R}^*$ with fiber V. We may think of it as a smooth groupoid with object space $\mathbb{R}^*$. The expression c_ω refers to the groupoid cocycle

$$c_\omega : V \times V \times \mathbb{R}^* \longrightarrow U(1), \quad c_\omega((v, s), (w, s)) = \exp\left(is\omega(v, w)\right).$$

The C^*-algebra $C^*(V \times \mathbb{R}^*, c_\omega)$ is the completion of the groupoid convolution algebra $C_c^\infty(V \times \mathbb{R}^*)$ twisted by the cocycle c_ω,

$$(\tilde{f} *_c \tilde{g})(v, s) = \int_V e^{is\omega(v, w)} \tilde{f}(v - w, s) \tilde{g}(w, s)\, dw.$$

Elements in the C^*-algebra $C^*(G)$ can be identified with sections in a continuous field over $\mathbb{R}^*$. The fiber at $s \in \mathbb{R}^*$ is the twisted convolution C^*-algebra $C^*(V, s\omega)$ (now with fixed value of the parameter s appearing in the cocycle). At $s = 0$ the fiber is commutative, $C^*(V) \cong C_0(V^*)$. If $s \neq 0$ the field is trivial, and each fiber is isomorphic to the algebra of compact operators, $C^*(V, s\omega) \cong \mathcal{K}$. Restriction to $s = 0$ gives the decomposition

$$0 \longrightarrow C^*(V \times \mathbb{R}^\times, c_\omega) \longrightarrow C^*(G) \longrightarrow C_0(V^*) \longrightarrow 0.$$

A simple rescaling of the parameter $s \in \mathbb{R}^\times = \mathbb{R}^* \setminus \{0\}$ gives an isomorphism

$$C^*(V \times \mathbb{R}^\times, c_\omega) \cong C_0(-\infty, 0) \otimes C^*(V, -\omega) \oplus C_0(0, \infty) \otimes C^*(V, \omega).$$

We will identify

$$\mathbb{R}^\times = (-\infty, 0) \sqcup (0, \infty) \approx \mathbb{R} \sqcup \mathbb{R}$$

by the map $s \mapsto \log|s|$ for $s \in \mathbb{R}^\times$. This fixes an isomorphism

$$C^*(V \times \mathbb{R}^\times, s\omega) \cong C_0(\mathbb{R}) \otimes C^*(V, -\omega) \oplus C_0(\mathbb{R}) \otimes C^*(V, \omega).$$

While both C^*-algebras $C^*(V, \pm\omega)$ are isomorphic to $\mathcal{K}$, the algebra of compact operators on Hilbert space, we do not identify these two algebras, for reasons that will soon become apparent.

4.2. $U(n)$ symmetry

A choice of contact 1-form θ and compatible complex structure J in the fibers of H amounts to a reduction of the structure group of TX to $U(n)$. For this reason it is useful to consider the $U(n)$ symmetry of the group C^*-algebra $C^*(G)$.

With the standard identification $V = \mathbb{R}^{2n} = \mathbb{C}^n$ the canonical action of the unitary group $U(n)$ on $V = \mathbb{C}^n$ induces automorphisms of the Heisenberg group $G = V \times \mathbb{R}$, and therefore of the C^*-algebra $C^*(G)$. Because the cocycle c_ω is invariant under the $U(n)$ action on V, $U(n)$ also acts in the obvious way on the C^*-algebra $C^*(V \times \mathbb{R}^*, c_\omega)$,

The $U(n)$ action on functions on $G = V \times \mathbb{R}$ commutes with the Fourier transform in the $\mathbb{R}$-variable. Therefore the isomorphism

$$C^*(G) \cong C^*(V \times \mathbb{R}^*, c_\omega)$$

is $U(n)$-equivariant. Likewise, the isomorphism

$$C^*(V \times \mathbb{R}^\times, c_\omega) \cong C(\mathbb{R}) \otimes C^*(V, \omega) \oplus C(\mathbb{R}) \otimes C^*(V, -\omega)$$

is $U(n)$-equivariant.

Now choose a representation

$$C^*(V, \omega) \cong \mathcal{K}(\mathcal{H}).$$

Up to unitary equivalence, this is a uniquely determined irreducible unitary representation of the Heisenberg group G. Below we will choose an explicit model for this representation. But first we consider the $U(n)$ action on $\mathcal{K}(\mathcal{H})$ induced by its action on $C^*(V, \omega)$. The automorphism group of $\mathcal{K}(\mathcal{H})$ is the projective unitary group $U(\mathcal{H})/U(1)$ of the separable Hilbert space $\mathcal{H}$. As we will see below, the action of $U(n)$ on $\mathcal{K}(\mathcal{H})$ lifts to a unitary representation of $U(n)$ on the Hilbert space $\mathcal{H}$. This specific fact about the

representation theory of the Heisenberg group is of crucial importance for our analysis, because it implies, later on, the vanishing of the Dixmier–Douady invariant of a specific continuous trace C^*-algebra over X, providing us with an important Morita equivalence.

For the opposite algebra $C^*(V, -\omega)$ we have the isomorphism

$$C^*(V, -\omega) \cong \mathcal{K}(\overline{\mathcal{H}}),$$

where $\overline{\mathcal{H}}$ denotes the Hilbert space that is conjugate (or dual) to $\mathcal{H}$, and that carries the dual representation of G as well as of $U(n)$. This isomorphism is equivariant with respect to the $U(n)$ actions, and the Hilbert space $\overline{\mathcal{H}}$ is *not* ismorphic to $\mathcal{H}$ *as a representation space* of $U(n)$.

In summary, we obtain a $U(n)$-equivariant isomorphism

$$C^*(V \times \mathbb{R}^\times, c_\omega) \cong C_0(\mathbb{R}) \otimes \mathcal{K}(\mathcal{H}) \oplus C_0(\mathbb{R}) \otimes \mathcal{K}(\overline{\mathcal{H}}).$$

It will be useful to have a concrete model for the Hilbert space $\mathcal{H}$ with its representations of the Heisenberg group G and the unitary group $U(n) \subset \mathrm{Aut}(G)$.

4.3. Bargmann–Fock space

The representation space of the Heisenberg group G that most clearly exhibits the $U(n)$ action is the Bargmann–Fock space. As a $U(n)$ representation, the Bargmann–Fock space is a completion of the space of symmetric tensors of $\mathbb{C}^n$,

$$\mathcal{H}^{\mathrm{BF}} \supset \mathrm{Sym}\,\mathbb{C}^n = \bigoplus_{j=0}^{\infty} \mathrm{Sym}^j \mathbb{C}^n$$

with its standard $U(n)$ action. If we identify symmetric tensors with complex polynomials on $\mathbb{C}^n$, then the Bargmann–Fock space is the Hilbert space of entire functions on $\mathbb{C}^n$ with inner product

$$\langle f, g \rangle = \pi^{-n} \int f(z) \overline{g(z)} e^{-|z|^2} \, dz.$$

If $Z_1, ..., Z_n$ is an orthonormal basis for $\mathbb{C}^n$, then the monomials $Z_1^{m_1} ... Z_n^{m_n} / \sqrt{m_1! ... m_n!}$ form an orthonormal basis for $\mathcal{H}^{\mathrm{BF}}$. In particular, the summands $\mathrm{Sym}^j \mathbb{C}^n$ are mutually orthogonal.

Let us briefly review how the Heisenberg group G acts on this space. To describe the representation it is convenient to pass to the complexified Lie group $G_{\mathbb{C}}$. In order not to get confused, let us denote by $J: V \to V$ the standard complex structure on $V = \mathbb{R}^{2n}$ and *not* identify V with $\mathbb{C}^n$ here. The complexified space $V \otimes \mathbb{C}$ splits into the $\pm i$ eigenspaces

for J as $V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}$. Correspondingly, the complexification of the Lie algebra $\mathfrak{g}$ of G splits as a direct sum

$$\mathfrak{g} \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1} \oplus \mathbb{C}.$$

Recall that $\omega(Ju, Jv) = \omega(u, v)$ and $\omega(Jv, v) > 0$ if $v \neq 0$. If we extend the bilinear form ω to a complex bilinear form on $V \otimes \mathbb{C}$ then the expression

$$\langle z, w \rangle := i\omega(z, \overline{w})$$

defines a hermitian inner product on $V^{1,0}$. Let $\{Z_1, ..., Z_n\}$ be a basis for $V^{1,0}$ that is orthonormal with respect to this hermitian product. For elements in the complexified Lie algebra $A, B \in \mathfrak{g} \otimes \mathbb{C}$ we have $[A, B] = \omega(A, B) = -i\langle A, \overline{B} \rangle$. Therefore

$$[Z_j, \overline{Z}_k] = \frac{1}{i}\delta_{jk}.$$

Since $V^{1,0}$ is orthogonal to $V^{0,1}$, we also obtain

$$[Z_j, Z_k] = [\overline{Z}_j, \overline{Z}_k] = 0.$$

In this picture, the Bargmann–Fock space $\mathcal{H}^{\mathrm{BF}}$ is the completion of $\mathrm{Sym}\, V^{1,0}$, and can be identified with a space of entire functions on $V^{0,1}$. The representation π of the complexified Lie algebra $\mathfrak{g} \otimes \mathbb{C}$ on $\mathcal{H}^{\mathrm{BF}}$ is given by

$$\pi(Z_j) = iz_j, \quad \pi(\overline{Z}_j) = i\frac{\partial}{\partial z_j} \quad \text{and} \quad \pi(1) = i.$$

The multiplication operator z_j and the complex derivative $\partial/\partial z_j$ are unbounded linear operators on $\mathcal{H}^{\mathrm{BF}}$. The subspace of polynomials $\mathrm{Sym}\, V^{1,0}$ is an invariant subspace for both operators, and hence also for the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the Lie algebra. The operators z_j and $\partial/\partial z_j$ are adjoints, which implies that $\pi(W)^* = -\pi(\overline{W})$ for all elements $W \in \mathfrak{g} \otimes \mathbb{C}$. Therefore $\pi(v)$ is skew adjoint for elements $v \in \mathfrak{g}$ in the real Lie algebra, and the corresponding representation for the group G is unitary.

In summary, we have $U(n)$-equivariant $*$-homomorphisms

$$
\begin{array}{ccc}
 & C^*(G) & \\
{\scriptstyle f \mapsto \tilde{f}(-,1)}\Big\downarrow & \searrow {\scriptstyle \pi} & \\
C^*(V, \omega) & \xrightarrow{\ \cong\ } & \mathcal{K}(\mathcal{H}^{\mathrm{BF}}).
\end{array}
$$

The horizontal arrow is the Bargmann integral transform, which assigns to a function on V (or its Fourier transform on V^*) an operator on $\mathcal{H}^{\mathrm{BF}}$ [2]. For the opposite algebra $C^*(V, -\omega)$ we obtain a representation on the conjugate space $\overline{\mathcal{H}}^{\mathrm{BF}}$, which is a completion of the space of symmetric tensors of $V^{0,1}$ carrying the dual (or conjugate) representation of $U(n)$. We will adopt the notation

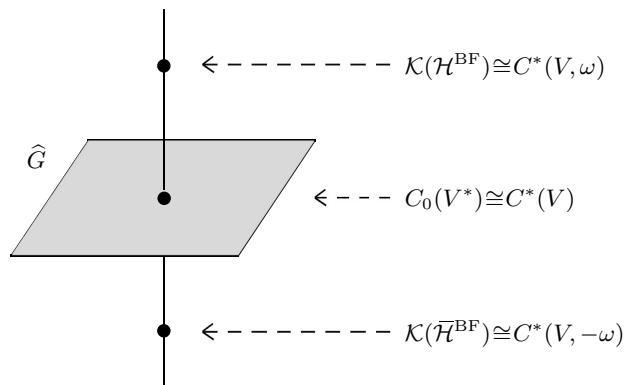$$\mathcal{H}^{\mathrm{BF}}_+ = \mathcal{H}^{\mathrm{BF}} \quad \text{and} \quad \mathcal{H}^{\mathrm{BF}}_- = \overline{\mathcal{H}}^{\mathrm{BF}}.$$

Figure 1. The short exact sequence $0 \to C_0(\mathbb{R}^\times, \mathcal{K}) \to C^*(G) \to C_0(V^*) \to 0$.

4.4. Quantization

Recall that the group C^*-algebra $C^*(G)$ is a quantization of the commutative algebra $C_0(\mathfrak{g}^*) = C_0(V^* \times \mathbb{R}^*)$. Consider the groupoid $V \times \mathbb{R}^* \times [0,1]$, which is, algebraically, a disjoint union of copies of the abelian group V parameterized by $(s, \varepsilon) \in \mathbb{R}^* \times [0,1]$. For this groupoid, define the groupoid 2-cocycle

$$c_{\varepsilon\omega} \colon V \times V \times \mathbb{R}^* \times [0,1] \longrightarrow U(1), \quad c_{\varepsilon\omega}((v,s,\varepsilon),(w,s,\varepsilon)) = \exp\left(is\varepsilon\omega(v,w)\right).$$

Then we have the twisted convolution C^*-algebra

$$A = C^*(V \times \mathbb{R}^* \times [0,1], c_{\varepsilon\omega}).$$

We can think of elements in this C^*-algebra A as sections in a continuous field $\{A_\varepsilon\}_\varepsilon$ with parameter $\varepsilon \in [0,1]$. For $\varepsilon \in (0,1]$ the field is trivial, with $A_\varepsilon \cong A_1 = C^*(G)$. At $\varepsilon = 0$ we have the commutative algebra

$$A_0 = C^*(V \times \mathbb{R}^*) \cong C_0(V^* \times \mathbb{R}^*).$$

We thus have a (strong deformation) quantization from A_0 to A_1. As is well known, this quantization induces an invertible E-theory element in $E(A_0, A_1)$ that, in turn, corresponds to an element in $\mathrm{KK}(C_0(\mathfrak{g}^*), C^*(G))$ that implements a KK-equivalence.

Now consider the decomposition discussed in §4.1,

$$0 \longrightarrow C^*(V \times \mathbb{R}^\times, c_\omega) \longrightarrow C^*(G) \longrightarrow C^*(V) \longrightarrow 0.$$

When restricted to the ideal

$$C^*(V \times \mathbb{R}^\times, c_\omega) \cong C_0(\mathbb{R}) \otimes \mathcal{K}(\mathcal{H}^{\mathrm{BF}}_-) \oplus C_0(\mathbb{R}) \otimes \mathcal{K}(\mathcal{H}^{\mathrm{BF}}_+),$$

the KK-equivalence $C^*(G) \sim C_0(\mathfrak{g}^*)$ amounts to the *Morita equivalence*

$$C_0(\mathbb{R}) \otimes \mathcal{K}(\mathcal{H}_-^{\mathrm{BF}}) \oplus C_0(\mathbb{R}) \otimes \mathcal{K}(\mathcal{H}_+^{\mathrm{BF}}) \sim C_0(\mathbb{R}) \otimes \mathbb{C} \oplus C_0(\mathbb{R}) \otimes \mathbb{C}$$

composed with the Bott isomorphisms $\beta_- \oplus \beta_+$,

$$\ldots \sim C_0(\mathbb{R}) \otimes C_0(V^*) \oplus C_0(\mathbb{R}) \otimes C_0(V^*) \cong C_0(V^* \times \mathbb{R}^\times).$$

Here β_+ denotes the canonical Bott generator for the complex vector space $V^{1,0} = \mathbb{C}^n$, while β_- denotes the Bott generator for the conjugate space $V^{0,1}$,

$$\beta_+ \in \mathrm{KK}(\mathbb{C}, C_0(V^{1,0})) \quad \text{and} \quad \beta_- \in \mathrm{KK}(\mathbb{C}, C_0(V^{0,1})).$$

We obtain the KK-equivalence of short exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C^*(V \times \mathbb{R}^\times, c_\omega) & \longrightarrow & C^*(G) & \longrightarrow & C^*(V) & \longrightarrow & 0 \\
 & & \mathrm{KK}^{U(n)} \downdownarrows \cong & & \mathrm{KK}^{U(n)} \downdownarrows \cong & & \downarrow \cong & & \\
0 & \longrightarrow & C_0(V^* \times \mathbb{R}^\times) & \longrightarrow & C_0(V^* \times \mathbb{R}^*) & \longrightarrow & C_0(V^*) & \longrightarrow & 0.
\end{array}
$$

Observe that all $*$-homomorphisms and KK-equivalences are $U(n)$-equivariant. The diagram can therefore be interpreted as a commutative diagram in the category $\mathrm{KK}^{U(n)}$.

4.5. Associated bundles

Our entire analysis carries over to associated bundles (of groups, Hilbert spaces or C^*-algebras) over a contact manifold X. As above, the contact 1-form θ and complex structure J in the fibers of H induce a reduction of the structure group of the tangent bundle TX to $U(n)$. Let P_U denote the principal $U(n)$ bundle of orthonormal frames in $H^{1,0}$ with respect to its hermitian structure associated with θ and J.

If α denotes the standard action α of $U(n)$ on $\mathbb{C}^n$ then

$$H^{1,0} = P_U \times_\alpha \mathbb{C}^n.$$

Likewise, the tangent space TX can identified with

$$TX = P_U \times_{\alpha \oplus 1} (\mathbb{C}^n \oplus \mathbb{R}) = H^{1,0} \oplus \mathbb{R}.$$

This identification exhibits the stably almost complex (and hence Spin^c) structure of TX. The bundle of Heisenberg groups $T_H X$ identifies with

$$T_H X = P_U \times_{\alpha \times 1} G.$$

Here $\alpha \times 1$ denotes the $U(n)$ action on $G = \mathbb{C}^n \times \mathbb{R}$. The induced action on $C^*(G)$ gives the convolution C^*-algebra $C^*(T_H X)$ of the groupoid $T_H X$,

$$C^*(T_H X) = P_U \times_{\alpha \times 1} C^*(G).$$

Denote by ϱ the action of $U(n)$ on the Bargmann–Fock space $\mathcal{H}^{\mathrm{BF}} = \mathcal{H}^{\mathrm{BF}}_+$, the completion of the symmetric tensors $\mathrm{Sym}\,\mathbb{C}^n$ described above. We can form the associated bundle of Hilbert spaces

$$V^{\mathrm{BF}}_+ = P_U \times_\varrho \mathcal{H}^{\mathrm{BF}}_+.$$

Continuous sections in V^{BF}_+ form a Hilbert module over $C(X)$, and we have the dense subspace

$$\bigoplus_{j=0}^\infty \mathrm{Sym}^j H^{1,0} \subset V^{\mathrm{BF}}_+.$$

The conjugate representation to ϱ gives rise to the dual module V^{BF}_- with fiber $\mathcal{H}^{\mathrm{BF}}_-$ and

$$\bigoplus_{j=0}^\infty \mathrm{Sym}^j H^{0,1} \subset V^{\mathrm{BF}}_-.$$

If π denotes the Bargmann–Fock representation of G on $\mathcal{H}^{\mathrm{BF}}$, then for fixed $u \in U(n)$ the operator $\varrho(u) \in U(\mathcal{H}^{\mathrm{BF}})$ is an intertwiner of the representations π and $\pi \circ (\alpha \times 1)(u)$ of G for the automorphism $(\alpha \times 1)(u)$ of G,

$$\pi((\alpha \times 1)(u).g) = \varrho(u)\pi(g)\varrho(u)^{-1}.$$

This compatibility of $\alpha \times 1$, ϱ and π (respectively: $U(n)$ acting on G, $U(n)$ acting on $\mathcal{H}^{\mathrm{BF}}$, and G acting on $\mathcal{H}^{\mathrm{BF}}$) implies that π is well defined as a representation of a fiber $C^*(G_x)$ of the bundle $C^*(T_H X)$ on the fiber V^{BF}_x of the Hilbert module V^{BF}_+. The representation π therefore induces a $*$-homomorphism of $C(X)$-algebras,

$$\pi \colon C^*(T_H X) \longrightarrow \mathcal{K}(V^{\mathrm{BF}}_+).$$

Here $\mathcal{K}(V^{\mathrm{BF}}_+)$ denotes the 'compact operators' on the $C(X)$ Hilbert module V^{BF}_+. Our analysis of the structure of $C^*(G)$ carries over to $C^*(T_H X)$. We obtain a short exact sequence

$$0 \longrightarrow I_H \longrightarrow C^*(T_H X) \longrightarrow C_0(H^*) \longrightarrow 0,$$

where the ideal I_H can be identified with

$$I_H \cong P_U \times_{1 \otimes \mathrm{Ad}(\bar\varrho)} [C_0(\mathbb{R}) \otimes \mathcal{K}(\mathcal{H}^{\mathrm{BF}}_-)] \oplus P_U \times_{1 \otimes \mathrm{Ad}(\varrho)} [C_0(\mathbb{R}) \otimes \mathcal{K}(\mathcal{H}^{\mathrm{BF}}_+)]$$

$$\cong C_0(\mathbb{R}) \otimes \mathcal{K}(V^{\mathrm{BF}}_-) \oplus C_0(\mathbb{R}) \otimes \mathcal{K}(V^{\mathrm{BF}}_+).$$

In particular, the ideal I_H is Morita equivalent to the commutative algebra $C_0(X \times \mathbb{R}^\times)$, and we have an explicit imprimitivity bimodule, namely the disjoint union of the pull-back of V^{BF}_+ to $X \times (0, \infty)$ and the pull-back of V^{BF}_- to $X \times (-\infty, 0)$.

4.6. Inverting the Connes–Thom isomorphism

In [15] we discussed the crucial role of the isomorphism

$$\Psi \colon K^0(T^*X) \longrightarrow K_0(C^*(T_H X))$$

for our index problem. The isomorphism Ψ is essentially the Connes–Thom isomorphism in each fiber, which agrees with the KK-equivalence induced by the quantization from $C_0(\mathfrak{g}^*)$ to $C^*(G)$. We now obtain a better grip on this isomorphism by analyzing how the quantization behaves when we decompose $C^*(T_H X)$. Our analysis of $C^*(G)$ carries over to $C^*(T_H X)$ because of the $U(n)$ equivariance of all the relevant constructions.

Quite generally, a principal G bundle P for a compact group G and compact base $X = P/G$ induces a functor from the category KK^G (equivariant KK-theory) to the category KK^X (RKK-theory for $C(X)$-algebras) in the obvious way. The functor assigns to a G-C^*-algebra A the $C(X)$-C^*-algebra of G-equivariant continuous functions

$$P(A) = \{f \colon P \to A \mid f \text{ is continuous and } f(pg) = g^{-1} f(p)\}.$$

Similarly for the morphisms: to a G Hilbert module $\mathcal{E}$ over A the functor assigns the $C(X)$ Hilbert module of G-equivariant continuous functions $P \to \mathcal{E}$, etc.

In this way the $U(n)$-equivariant KK-equivalence of short exact sequences for $C^*(G)$ that we derived above implies KK-equivalence of the associated sequences for bundles over X. We obtain te commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & I_H & \longrightarrow & C^*(T_H X) & \longrightarrow & C^*(H) & \longrightarrow & 0 \\
& & \mathrm{KK}^X \Big\downarrow \cong & & \mathrm{KK}^X \Big\downarrow \cong & & \Big\downarrow \cong & & \\
0 & \longrightarrow & C_0(H^* \times \mathbb{R}^\times) & \longrightarrow & C_0(H^* \times \mathbb{R}^*) & \longrightarrow & C_0(H^*) & \longrightarrow & 0.
\end{array}
$$

Observe that the bottom sequence is just

$$0 \longrightarrow C_0(T^*X \setminus H^*) \longrightarrow C_0(T^*X) \longrightarrow C_0(H^*) \longrightarrow 0,$$

induced by the inclusion $H^* \subset T^*X$. Following our analysis of $C^*(G)$ we see that the KK-equivalence $I_H \sim C_0(H^* \times \mathbb{R}^\times)$ is a composition of the isomorphism of C^*-algebras

$$I_H \cong C^*(H \times \mathbb{R}^\times, c_\omega) \cong C_0(\mathbb{R}) \otimes \mathcal{K}(V_-^{\mathrm{BF}}) \oplus C_0(\mathbb{R}) \otimes \mathcal{K}(V_+^{\mathrm{BF}})$$

with the Morita equivalence

$$\ldots \sim C_0(X \times \mathbb{R}) \oplus C_0(X \times \mathbb{R})$$

and, finally, the two Thom isomorphisms $\tau^- \oplus \tau^+$,

$$... \sim C_0(H^{0,1} \times \mathbb{R}) \oplus C_0(H^{1,0} \times \mathbb{R}) \cong C_0(H^* \times \mathbb{R}^\times).$$

Here τ^+ and τ^- denote the Thom classes for the complex bundles $H^{1,0}$ and $H^{0,1}$, respectively.

Let us isolate the maps in K-theory that are relevant for our purposes. We summarize the conclusion of our analysis in the form of a proposition.

PROPOSITION 4.6.1. *The following diagram commutes*:

$$
\begin{array}{ccc}
K^0(T^*X \setminus H^*) & \longrightarrow & K^0(T^*X) \\
\big\uparrow{\scriptstyle \cup\tau^\pm \mid \text{Thom isom.}} & & \big\downarrow{\scriptstyle \Psi \mid \text{Connes–Thom}} \\
K^0(X \times \mathbb{R}^\times) & & \\
\big\uparrow{\scriptstyle \otimes V_\pm^{\mathrm{BF}} \mid \text{Morita equiv.}} & & \\
K_0(I_H) & \longrightarrow & K^0(C^*(T_H X)).
\end{array}
$$

All vertical maps in the diagram are isomorphisms, and the horizontal maps (induced by inclusions) are surjective. The vertical arrows on the left are explicitly given by

$$
\begin{array}{ccc}
K^0(H^{0,1} \times \mathbb{R}) \oplus K^0(H^{1,0} \times \mathbb{R}) & \xrightarrow{\;\cong\;} & K^0(T^*X \setminus H^*) \\
\big\uparrow{\scriptstyle \cup\tau^- \oplus \cup\tau^+ \mid \text{Thom isom.}} & & \big\uparrow \\
K^0(X \times \mathbb{R}) \oplus K^0(X \times \mathbb{R}) & \xrightarrow{\;\cong\;} & K^0(X \times \mathbb{R}^\times) \\
\big\uparrow{\scriptstyle \otimes V_-^{\mathrm{BF}} \oplus \otimes V_+^{\mathrm{BF}} \mid \text{Morita equiv.}} & & \big\uparrow \\
K_0(C_0(\mathbb{R}) \otimes \mathcal{K}(V_-^{\mathrm{BF}})) \oplus K_0(C_0(\mathbb{R}) \otimes \mathcal{K}(V_+^{\mathrm{BF}})) & \xrightarrow{\;\cong\;} & K_0(I_H).
\end{array}
$$

Proof. We only need to check the claim about surjectivity of the maps represented by the horizontal arrows. The inclusion of each connected component of $T^*X \setminus H^*$ into T^*X is a homotopy equivalence. Thus $K^0(T^*X \setminus H^*) \to K^0(T^*X)$ is surjective. As all vertical maps are isomorphisms, the same is true for the map $K_0(I_H) \to K^*(C^*(T_H X))$. □

4.7. Non-commutative Poincaré duality

Definition 4.7.1. The *non-commutative Poincaré duality* map

$$b\colon K_0(C^*(T_H X)) \longrightarrow K_0^{\mathrm{top}}(X)$$

is defined by choosing an arbitrary lift of an element in $K_0(C^*(T_H X))$ to an element in $K_0(I_H)$, followed by the composition of maps

$$K_0(I_H) \cong K^1(X) \oplus K^1(X) \xrightarrow{\cap [X^-] \oplus \cap [X^+]} K_0^{\text{top}}(X) \oplus K_0^{\text{top}}(X) \longrightarrow K_0^{\text{top}}(X).$$

From left to right, these maps are (1) Morita equivalence induced by the Bargmann–Fock Hilbert modules V_-^{BF} and V_+^{BF}, (2) Poincaré duality for the two natural Spin^c structures on X and (3) addition of K-homology classes.

Remark 4.7.2. Let us repeat here that the implicit isomorphism

$$K^0(X \times \mathbb{R}^\times) \cong K^0(X \times \mathbb{R}) \oplus K^0(X \times \mathbb{R}) \cong K^1(X) \oplus K^1(X)$$

is chosen such that the identifications $(-\infty, 0) \approx \mathbb{R}$ and $(0, \infty) \approx \mathbb{R}$ are given by the map $s \mapsto \log|s|$, i.e., each component of $\mathbb{R}^\times$ is oriented from 0 to $\pm\infty$.

Remark 4.7.3. The K-theory class of the symbol $[\sigma_H(P)] \in K_0(C^*(T_H X))$ only depends on the principal symbol of P in the Heisenberg calculus, and is uniquely defined by P, independent of arbitrary choices. In §6 we will see that lifting $[\sigma_H(P)]$ to $K_0(I_H)$ amounts to taking a distribution $\sigma_H(P) \in \mathcal{E}'(T_H X)$ that represents a *full* Heisenberg symbol for P, and possibly perturbing it by a compactly supported smooth function in $C_c^\infty(T_H X)$ in order to give it an extra property. Such a perturbation does not affect the *principal* Heisenberg symbol. The choice of such a perturbation is always possible, but not uniquely determined by P. However, the equivalence class of $b(\sigma_H(P))$ in the K-homology group $K_0^{\text{top}}(X)$ does not depend on the choice of this perturbation, and so the map b is well-defined. The proof of this fact is implicit in the proof of the following theorem.

LEMMA 4.7.4. *The following diagram commutes*:

$$
\begin{array}{ccc}
K^0(X \times \mathbb{R}) & \xrightarrow{\cup \tau_+} & K^0(T^* X) \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle c} \\
K^1(X) & \xrightarrow[\cap [X^+]]{} & K_0^{\text{top}}(X).
\end{array}
$$

Here the left vertical arrow is the suspension isomorphism in K-theory; the upper horizontal arrow is the Thom isomorphism resulting from the direct sum decomposition $T^* X = H^{1,0} \oplus \mathbb{R}$; the lower horizontal arrow is Poincaré duality on the Spin^c manifold X, i.e., is cap product with the K-homology fundamental cycle $[X^+]$; and the right vertical arrow is the "clutching construction", i.e., cap product with the K-homology fundamental cycle of the Spin^c manifold $T^* X$.

Proof. Commutativity of the diagram follows from commutativity of the two triangles that appear if we introduce the diagonal arrow which is the Thom isomorphism for the Spin^c vector bundle T^*X on X,

$$
\begin{array}{ccc}
K^0(X \times \mathbb{R}) & \xrightarrow{\ \cup \tau_+\ } & K^0(T^*X) \\
{\scriptstyle \cong} \uparrow & \nearrow & \downarrow {\scriptstyle c} \\
K^1(X) & \xrightarrow[\ \cap [X^+]\ ]{} & K_0^{\mathrm{top}}(X).
\end{array}
$$

Commutativity of the two triangles is standard algebraic topology, which we now briefly indicate. The upper triangle commutes because the Thom class for $T^*X = H^{1,0} \oplus \mathbb{R}$ is the product of the Thom classes used for the left vertical arrow and the upper horizontal arrow. For commutativity of the lower triangle, use the following notation: $\beta \in \mathrm{KK}^1(C(X), C_0(T^*X))$ is the Thom class for T^*X, and $\alpha \in \mathrm{KK}^1(C_0(T^*X), C(X))$ is the KK-element given by the family of Dirac operators for the fibers of T^*X. The Dirac-dual Dirac identity in this context is the assertion that the Kasparov product $\beta \# \alpha$ is the unit element of the ring $\mathrm{KK}^0(C(X), C(X))$,

$$
\beta \# \alpha = 1 \in \mathrm{KK}^0(C(X), C(X)).
$$

The total space of T^*X is itself a (non-compact) Spin^c manifold, and its K-homology fundamental class $[T^*X]$ is the Kasparov product of α and $[X^+]$,

$$
[T^*X] = \alpha \# [X^+] \in \mathrm{KK}^0(C_0(T^*X), \mathbb{C}).
$$

It now follows that

$$
\beta \# [T^*X] = \beta \# \alpha \# [X^+] = [X^+],
$$

which gives commutativity of the lower triangle. $\qquad\square$

THEOREM 4.7.5. *The following diagram commutes*:

$$
\begin{array}{ccc}
K^0(T^*X) & \xrightarrow{\ \Psi\ } & K_0(C^*(T_H X)) \\
\downarrow {\scriptstyle c} & \swarrow {\scriptstyle b} & \\
K_0^{\mathrm{top}}(X), & &
\end{array}
$$

where b is as in Definition 4.7.1, *and c is the "clutching construction" introduced in* [3, §22].

Proof. If we identify $X \times (0, \infty) \cong X \times \mathbb{R}$, then Lemma 4.7.4 gives commutativity of

$$
\begin{array}{ccc}
K^0(X \times (0, \infty)) & \xrightarrow{\cup \tau_+} & K^0(T^*X) \\
\Big\downarrow{\cong} & & \Big\downarrow{c} \\
K^1(X) & \xrightarrow[\cap[X^+]]{} & K_0^{\mathrm{top}}(X).
\end{array}
$$

Similarly, for the conjugate Spin^c structure on X we have

$$
\begin{array}{ccc}
K^0(X \times (-\infty, 0)) & \xrightarrow{\cup \tau_-} & K^0(T^*X) \\
\Big\downarrow{\cong} & & \Big\downarrow{c} \\
K^1(X) & \xrightarrow[\cap[X^-]]{} & K_0^{\mathrm{top}}(X).
\end{array}
$$

The *sign* of the isomorphism $K^0(X \times (0, \infty)) \cong K^1(X)$ depends on the orientation of $(0, \infty)$, i.e., on the orientation of the normal bundle $N = X \times \mathbb{R}$, which in turn depended on the choice of contact form. In order to make the second diagram commute we must orient $(-\infty, 0)$ from 0 to $-\infty$. This is because the reversed orientation of the normal bundle N is built into the definition of the fundamental cycle $[X^-]$, and also affects the choice of the isomorphism

$$
K^0(X \times (-\infty, 0)) \cong K^0(X \times \mathbb{R}) \cong K^1(X).
$$

Combining the two components of $X \times \mathbb{R}^\times$, we obtain a single diagram

$$
\begin{array}{ccccc}
K^0(X \times (-\infty, 0)) \oplus K^0(X \times (0, \infty)) & \xrightarrow{\cup \tau^- \oplus \cup \tau^+} & K^0(T^*X \setminus H^*) & \longrightarrow & K^0(T^*X) \\
\Big\downarrow{\cong} & & & & \Big\downarrow{c} \\
K^1(X) \oplus K^1(X) & \xrightarrow[\cap[X^-] \oplus \cap[X^+]]{} & K_0^{\mathrm{top}}(X) \oplus K_0^{\mathrm{top}}(X) & \longrightarrow & K_0^{\mathrm{top}}(X).
\end{array}
$$

Comparing this diagram with Corollary 4.6.1 proves the proposition. $\qquad \square$

5. The index theorem as a commutative triangle

The symbol $\sigma_H(P)$ of a hypoelliptic operator in the Heisenberg calculus naturally determines an element in $K_0(C^*(T_H X))$. In this section we prove that the Poincaré dual $b(\sigma_H(P)) \in K_0^{\mathrm{top}}(X)$ of the Heisenberg symbol is the desired K-cycle

$$
\mu^{-1}(P) = b(\sigma_H(P)).
$$

For convenience of the reader, we start with a brief sketch of the main features of the Heisenberg calculus. References are [21], [6] and [13].

5.1. The Heisenberg filtration

Consider a differential operator P on a smooth manifold X, given in local coordinates by an expression

$$P = \sum_{|\alpha| \leqslant d} a_\alpha \partial^\alpha, \quad \alpha = (\alpha_1, ..., \alpha_n), \ |\alpha| = \sum_{j=1}^n \alpha_j, \ \partial^\alpha = \prod_{j=1}^n \left(\frac{\partial}{\partial x_j} \right)^{\alpha_j},$$

where the coefficients a_α are smooth functions. The highest-order part of P is not well defined as a differential operator on X. The algebra of differential operators is only filtered, not graded. But the highest-order part at a point $x \in X$,

$$P_x = \sum_{|\alpha| = d} a_\alpha(x) \partial^\alpha,$$

can be interpreted as a constant coefficient operator on the tangent fiber $T_x X$. As such it is well defined and independent of coordinate choices.

The root of the *Heisenberg calculus* is a simple but important idea, proposed by Folland and Stein in [19], to equip the algebra of differential operators with an alternative filtration. The filtration on the algebra of differential operators proposed by Folland and Stein is generated by a filtration on the Lie algebra of vector fields, defined as follows: All vector fields in the direction of the contact hyperplane bundle H have order 1, as usual, but *any* vector field that is not everywhere tangent to H is given order 2. For example, a sublaplacian Δ_H has order 2, as in the classical calculus. But in the Heisenberg calculus the Reeb vector field T is also a second-order operator.

What sort of object is the *highest-order part* of a differential operator if we adopt this Heisenberg filtration? Abstractly, for any filtered algebra, the notion of highest-order part refers to an element in the associated graded algebra. Observe that in the associated graded algebra smooth functions f commute with all vector fields W, because the commutator $[W, f] = W.f$ is of order zero. This implies that elements in the graded algebra can be *localized* at points $x \in X$. Therefore, just as in the classical case, the highest-order part of a differential operator P will consist of a family of operators P_x parameterized by $x \in X$. But they are not exactly constant-coefficient operators here. Instead, they are translation-invariant operators for a certain nilpotent group structure on $T_x X$.

In all that follows, X denotes a closed contact manifold with a contact hyperplane bundle $H \subset TX$. We denote by $N = TX/H$ the quotient line bundle, and assume that there exists a global contact form θ, so that we can identify $N = X \times \mathbb{R}$. Also, the Reeb vector field provides us with a section $N \subset TX$, so that we can identify $T_x X = H_x \times \mathbb{R}$.

Moreover, $d\theta$ restricted to H_x is a symplectic form and the tangent space $G_x = H_x \times \mathbb{R}$ is then a Heisenberg group with group operation

$$(v,t) \cdot (v',t') = \left(v + v', t + t' - \tfrac{1}{2} d\theta(v,v')\right), \quad v, v' \in H_x, \ t, t' \in \mathbb{R}.$$

Algebraically, the smooth groupoid $T_H X$ is the disjoint union of Heisenberg groups

$$T_H X = \bigsqcup_{x \in X} G_x.$$

The highest-order part $\{P_x \,|\, x \in X\}$ of a differential operator P in the Heisenberg calculus can be interpreted as a smooth family of translation-invariant operators P_x on the Heisenberg groups $G_x = H_x \times \mathbb{R} \cong T_x X$, or, equivalently, a right-invariant differential operator on the Lie groupoid $T_H X$.

For example, for the second-order operators $P_\gamma = \Delta_H + i\gamma T$ that are locally represented as

$$P = -\sum_{j=1}^{2n} W_j^2 + i\gamma T,$$

freezing coefficients at a point $x \in X$ results in

$$P_x = -\sum_{j=1}^{2n} W_j(x)^2 + i\gamma(x)T(x).$$

This formal polynomial in the tangent vectors $W_j(x), T(x) \in T_x X$ should be interpreted as an element in the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_x)$ of the Heisenberg Lie algebra $\mathfrak{g}_x = H_x \oplus \mathbb{R}$, or, equivalently, as an invariant differential operator on the group G_x. In other words, the vector $W_j(x)$ should *not* be identified with a vector field on $T_x X$ that is translation-invariant for the usual vector space structure on $T_x X$, but rather with a vector field on $T_x X$ that is translation-invariant for the Heisenberg group structure on $T_x X$.

5.2. Heisenberg pseudodifferential calculus

When we restrict attention to differential operators the Heisenberg calculus is fairly straightforward. Constructing the corresponding $\mathbb{Z}$-filtered algebra of Heisenberg *pseudodifferential* operators requires more work (see [21], [6] and [13]). In this section we sketch one possible approach.

Just like the classical pseudodifferential calculus, the Heisenberg algebra consists of pseudolocal continuous linear operators

$$P \colon C^\infty(X) \longrightarrow C^\infty(X).$$

In other words, they are operators with a Schwartz kernel $k(x, y)$ that is a smooth function off the diagonal in $X \times X$. The Heisenberg calculus (as well as its filtration) is defined by asymptotic expansions of the singularity of k in the direction transversal to the diagonal.

Choose a connection ∇ on TX that preserves the distribution H (this is an important technical condition). Consider the exponential map $\exp^\nabla$ associated with this connection ∇,

$$h: T_H X \longrightarrow X \times X,$$

$$(x, v) \longmapsto (\exp_x^\nabla(v), x).$$

The map h is a local diffeomorphism of a neighborhood of the zero section of $T_H X$ with a neighborhood of the diagonal in $X \times X$. We pull back the distribution k to the groupoid $T_H X$ by the map h, and then chop it by a cut-off function. Let ϕ be an arbitrary smooth function on $T_H X$ that is compactly supported and equals 1 in a neighborhood of the zero section. At a point $x \in X$ let k_x be the distribution on the Heisenberg group $G_x = T_x X$ defined by

$$k_x(v) = \phi(x, v)k(h(x, -v)), \quad v \in T_x X.$$

The smooth family of distributions $\{k_x | x \in X\}$ defines a compactly supported distribution on $T_H X$. The action of the operator P with kernel k is approximated (in a neighborhood of the point $x \in X$) by convolution with the compactly supported distribution k_x on the Heisenberg group G_x, in a sense that can be made precise. This was the basic idea introduced by Folland and Stein in [19].

Because the kernel k is smooth off the diagonal, each distribution k_x is regular, i.e., it is a smooth function when restricted to $G_x \setminus \{0\}$. We say that P is a pseudodifferential operator of order d if each distribution k_x has an asymptotic expansion near $0 \in G_x$,

$$k_x \sim k_x^0 + k_x^1 + k_x^2 + \dots.$$

The asymptotic expansion should be interpreted in the usual way: the remainder

$$k - \sum_{j=0}^{N} k^j$$

becomes more regular as N increases, and the entire expansion determines k_x modulo compactly supported smooth functions $C_c^\infty(T_H X)$. The defining feature of the Heisenberg calculus is that the terms k_x^j in this expansion must be homogeneous with respect to the 'parabolic' dilation structure of the Heisenberg group $G_x = T_x X = H_x \times \mathbb{R}$,

$$\delta_s: H_x \times \mathbb{R} \longrightarrow H_x \times \mathbb{R},$$

$$(v, t) \longmapsto (sv, s^2 t), \quad s > 0.$$

Formally, the term k_x^j in the expansion must satisfy

$$k_x^j(\delta_s v) = s^{j-(2n+2)}\, k_x^j(v).$$

This notion of homogeneity based on the dilations δ_s corresponds to the Heisenberg filtration for differential operators that assigns order 1 to vector fields tangent to H and order 2 to vector fields transveral to H. But we now obtain a filtered algebra of pseudodifferential operators that differs from the classical pseudodifferential algebra.

The existence of an asymptotic expansion is independent of the choice of exponential map $\exp^\nabla$ and cut-off function ϕ, as long as the exponential map satisfies the technical condition that it preserves sections in H.[2] We denote the operator of convolution with the distribution k_x by P_x, and we write

$$\sigma_H(P) = \{P_x \mid x \in X\}.$$

We regard $\sigma_H(P)$ as an element in the convolution algebra $\mathcal{E}'(T_H X)$ of compactly supported distributions on the groupoid $T_H X$, and we can think of it as the 'full symbol' of P. While the value of $\sigma_H(P)$ depends on the choice of exponential $\exp^\nabla$ and cut-off ϕ, the highest-order part $\sigma_H^d(P) = \{k_x^0 \mid x \in X\}$ in the asymptotic expansion of $\sigma_H(P)$ is invariantly defined as a smooth family of convolution operators on $T_H X$, independent of $\exp^\nabla$ and ϕ.

The collection of all Heisenberg pseudodifferential operators on X is a $\mathbb{Z}$-filtered algebra: if P and Q are Heisenberg pseudodifferential operators of order a and b, respectively, then PQ is a Heisenberg pseudodifferential operator of order $a+b$. Moreover, the leading term in the asymptotic expansion of $\sigma_H(PQ)$ is obtained by convolution of the leading terms in the expansions of $\sigma_H(P)$ and $\sigma_H(Q)$, i.e.,

$$\sigma_H^{a+b}(PQ) = \sigma_H^a(P) * \sigma_H^b(Q).$$

As usual, if the leading term $\sigma_H^d(P)$ of an order-d symbol $\sigma_H(P)$ is invertible (in the convolution algebra), then we can derive an asymptotic expansion for an inverse of the full symbol $\sigma_H(P)$ modulo $C_c^\infty(T_H X)$. This implies that P itself is invertible in the Heisenberg algebra modulo smoothing operators, and hence a hypoelliptic Fredholm operator on X. It is precisely for such operators that we prove our index formula.

---

[2] In the literature, the class of exponential maps that is implicit in the definition of the Heisenberg calculus differs from author to author. The underlying geometric properties of these exponential maps are clarified in [18].

5.3. The Heisenberg symbol in *K*-theory

In [15, Definition 15] we constructed a *K*-theoretic symbol in $K_0(C^*(T_HX))$ for a hypoelliptic differential operator P in the Heisenberg calculus. As we have seen, the principal symbol $\sigma^d_H(P)$ is then a family of differential operators P_x on G_x, where each P_x is obtained from P by a simple procedure of 'freezing coefficients' at $x \in X$. The result is independent of a choice of exponential map $T_HX \to X \times X$ or cut-off function ϕ. In fact, the construction given in [15] makes no reference to the Heisenberg pseudodifferential calculus at all. In [16] we gave an alternative definition that works for order-zero operators in the Heisenberg calculus.

Perhaps the easiest way to define the *K*-theory element

$$[\sigma_H(P)] \in K_0(C^*(T_HX))$$

is to follow the general ideas set out by Connes in [10, §II.9.α]. The resulting construction works for arbitrary hypoelliptic operators in the Heisenberg algebra.

Compactly supported smooth functions $C^\infty_c(T_HX)$ form a two-sided ideal in the convolution algebra $\mathcal{E}'(T_HX)$ of compactly supported distributions on the groupoid T_HX. The asymptotic expansion for products of full symbols in the Heisenberg calculus implies that if the principal symbol of an operator P is invertible, then the *full* symbol $\sigma_H(P)$ has an inverse modulo the ideal $C^\infty_c(T_HX)$. By a general argument explained in [10, §II.9.α], this implies that $\sigma_H(P)$ has an 'index' in the *K*-theory group $K_0(C^*(T_HX))$. First, the full symbol $\sigma_H(P)$ determines an element in *algebraic* relative *K*-theory,

$$[\sigma_H(P)] \in K_0(\mathcal{E}'(T_HX), C^\infty_c(T_HX)).$$

By excision, we have

$$K_0(\mathcal{E}'(T_HX), C^\infty_c(T_HX)) \cong K_0(C^\infty_c(T_HX)),$$

and, finally, the inclusion of rings $C^\infty_c(T_HX) \subset C^*(T_HX)$ maps the element $[\sigma_H(P)]$ to the *K*-theory group $K_0(C^*(T_HX))$ (which is the same in algebraic and C^*-algebraic *K*-theory). To make this more explicit, if Q is a parametrix for P (an inverse modulo smoothing operators), then the *K*-theory element

$$[\sigma_H(P)] = [e] \ominus [e_0] \in K_0(C^*(T_HX))$$

is represented by the formal difference of the idempotents

$$e = \begin{pmatrix} S_0^2 & S_0\sigma_H(Q) \\ S_1(1+S_1)\sigma_H(P) & 1-S_1^2 \end{pmatrix} \quad \text{and} \quad e_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

where $S_0 = 1 - \sigma_H(Q)\sigma_H(P)$ and $S_1 = 1 - \sigma_H(P)\sigma_H(Q)$. (See [10, §II.9.$\alpha$] for the details of the general construction.)

5.4. The "choose an operator" maps

In this section we prove the analog of the Poincaré duality (Theorem 4.7.5) for analytic K-homology $\mathrm{KK}(C(X),\mathbb{C})$. The proof of this theorem closely follows the steps of [15, §3.7], generalizing everything from a statement in K-theory to a statement about the functor $\mathrm{KK}(C(X),-)$. We refer the reader to [15] for details of the proof in K-theory, and indicate here how the argument can be stengthened to prove Theorem 5.4.1.

The Poincaré duality map

$$\mathrm{Op}_e\colon K^0(T^*X) \longrightarrow \mathrm{KK}^0(C(X),\mathbb{C})$$

is defined as follows. Given $\sigma\in K^0(T^*X)$, choose an elliptic order-zero pseudodifferential operator P on X such that in $K^0(T^*X)$,

$$[\sigma(P)]=\sigma.$$

Then $\mathrm{Op}_e(\sigma)$ is the element in $\mathrm{KK}^0(C(X),\mathbb{C})$ determined by P. The non-trivial point here is that if two elliptic pseudodifferential operators P and Q satisfy $[\sigma(P)]=[\sigma(Q)]$ in $K^0(T^*X)$, then P and Q determine the same element in $\mathrm{KK}^0(C(X),\mathbb{C})$—i.e., a homotopy of symbols can be lifted to a homotopy of operators. (See [1] and [3, §23].) Instead of attempting a direct construction for the Heisenberg pseudodifferential calculus, we will, in the proof below, construct a map

$$\mathrm{Op}_H\colon K_0(C^*(T_HX)) \longrightarrow \mathrm{KK}(C(X),\mathbb{C}),$$

and then prove a posteriori that if P is an order-zero operator in the Heisenberg calculus with principal symbol $\sigma_H(P)$, then

$$\mathrm{Op}_H([\sigma_H(P)])=[P].$$

This justifies referring to the map Op_H as the "choose a hypoelliptic operator" map for the Heisenberg calculus.

THEOREM 5.4.1. *There exists a map*

$$\mathrm{Op}_H\colon K_0(C^*(T_HX)) \longrightarrow \mathrm{KK}^0(C(X),\mathbb{C}),$$

such that (i) *if P is an order-zero operator in the Heisenberg algebra with invertible principal symbol $\sigma_H(P)$ then $\mathrm{Op}_H(\sigma_H(P))=[P]$ in $\mathrm{KK}^0(C(X),\mathbb{C})$, and* (ii) *the diagram*

$$
\begin{array}{ccc}
K^0(T^*X) & \xrightarrow{\ \ \Psi\ \ } & K_0(C^*(T_HX)) \\
\Big\downarrow{\scriptstyle\mathrm{Op}_e} & & \diagup{\scriptstyle\mathrm{Op_H}} \\
\mathrm{KK}^0(C(X),\mathbb{C}) & &
\end{array}
$$

commutes.

Proof. The proof is a strengthening of the groupoid arguments developed in [15]. The main difference is that we need to replace the K-theory $\mathrm{KK}(\mathbb{C}, -)$ by the functor $\mathrm{KK}(C(X), -)$. What makes this possible is that all C^*-algebras that play a role in our argument are of type I, and hence nuclear. Therefore all ideals are semi-split (by [9]), and since $C(X)$ is separable (X is a compact manifold) the $\mathrm{KK}(C(X), -)$ functor has 6-term exact sequences [7, Theorem 19.5.7]. As a result, the quotients by various contractible ideals that play a role in the tangent groupoid argument induce isomorphisms not only in $K_0(-) \cong \mathrm{KK}(\mathbb{C}, -)$, but also in $\mathrm{KK}(C(X), -)$.

We briefly review the argument and indicate how it can be strengthened. For a differential Heisenberg operator P with invertible principal symbol we constructed in [15, §3.6] an element in the K-theory of the parabolic tangent groupoid,

$$[\mathbb{P}] \in K_0(C^*(\mathbb{T}_H X)).$$

The 'parabolic tangent groupoid' $\mathbb{T}_H X$ is algebraically the disjoint union of $T_H X$ and the family of pair groupoids $X \times X \times (0, 1]$, similar to Connes' tangent groupoid. The 'blow-up' of the diagonal in $X \times X$ is defined using the parabolic dilations δ_s in the Heisenberg groups G_x [15, §3]. The class $[\mathbb{P}]$ combines the principal Heisenberg symbol of P with P itself into a single K-theory element for the tangent groupoid.

We must modify this construction and apply it to order-zero operators. This is actually a considerable simplification, especially if we construct $[\mathbb{P}]$ in KK-theory. Let $\mathcal{E} = C^*(\mathbb{T}_H X) \oplus C^*(\mathbb{T}_H X)$ be the obvious $\mathbb{Z}_2$-graded Hilbert module over $C^*(\mathbb{T}_H X)$. The adjointable operator

$$\mathbb{P} \in \mathcal{L}(\mathcal{E}) \cong M(C^*(\mathbb{T}_H X)) \otimes M_2(\mathbb{C})$$

restricts to the principal symbol

$$\begin{pmatrix} 0 & P_x^* \\ P_x & 0 \end{pmatrix} \in M(C^*(G_x)) \otimes M_2(\mathbb{C})$$

for the fiber at $s = 0$ and $x \in X$, and to

$$\begin{pmatrix} 0 & P^* \\ P & 0 \end{pmatrix} \in \mathcal{L}(L^2(M) \oplus L^2(M)) \cong M(C^*(X \times X)) \otimes M_2(\mathbb{C})$$

for all other values $s > 0$. The principal symbol is, by definition, the highest-order part in an asymptotic expansion of the operator kernel of P, and these asymptotics agree precisely with the parabolic 'blow-up' of the diagonal in $X \times X$ that defines $\mathbb{T}_H X$.

What we did not realize in [15] is that $(\mathcal{E}, \mathbb{P})$ actually defines an element in the KK-group,

$$[\mathbb{P}] \in \mathrm{KK}(C(X), C^*(\mathbb{T}_H X)).$$

There is an obvious diagonal representation

$$\phi\colon C(X) \longrightarrow \mathcal{L}(\mathcal{E}) = M(C^*(\mathbb{T}_H X)) \otimes M_2(\mathbb{C}).$$

At $s=0$ it restricts to the identification of $C(X)$ with the center of $M(C^*(T_H X))$, while at $s>0$ we have the representation of continuous functions on X as multiplication operators in $M(C^*(X \times X)) \cong \mathcal{L}(L^2(X))$. One easily verifies compactness of commutators $[\mathbb{P}, \phi(f)] \in \mathcal{K}(\mathcal{E})$ for continuous functions $f \in C(X)$.

Just as in [15], the restriction map at $s=0$ has a contractible ideal and therefore induces an isomorphism

$$e_0\colon \mathrm{KK}(C(X), C^*(\mathbb{T}_H X)) \cong \mathrm{KK}(C(X), C^*(T_H X)).$$

Observe that, by construction of the Fredholm module $[\mathbb{P}]$, we have $e_0([\mathbb{P}]) = [\sigma_H(P)]$.

Restriction to the fiber at $s=1$ induces a map

$$e_1\colon \mathrm{KK}(C(X), C^*(\mathbb{T}_H X)) \longrightarrow \mathrm{KK}(C(X), C^*(X \times X)) \cong \mathrm{KK}(C(X), \mathbb{C}).$$

Again, by construction, $e_1([\mathbb{P}]) = [P]$. It follows that the combined map

$$e_1 \circ e_0^{-1}\colon \mathrm{KK}(C(X), C^*(T_H X)) \longrightarrow \mathrm{KK}(C(X), \mathbb{C}),$$

is the "choose a hypoelliptic operator" map,

$$e_1 \circ e_0^{-1}([\sigma_H(P)]) = [P].$$

The exact same argument, using the tangent groupoid $\mathbb{T}X = TX \sqcup (X \times X \times (0,1])$ of Connes, works for elliptic order-zero operators.

Having proven that each "choose an operator" map is induced by the appropriate tangent groupoid, we can now show that these two maps commute with the isomorphism Ψ. The argument from [15, §3.8] involving the *adiabatic groupoid* of the parabolic tangent groupoid $\mathbb{T}_H X$ (a deformation of a deformation) applies, without essential change, to show that the diagram

$$\mathrm{KK}(C(X), C_0(T^*X)) \xrightarrow{\ \Psi\ } \mathrm{KK}(C(X), C^*(T_H X))$$
$$\mathrm{KK}((C(X), \mathbb{C})$$

commutes.

If $\mathcal{H}_A$ denotes the standard countably generated free Hilbert A-module, then any $*$-homomorphism $\phi\colon C(X)\to Z(M(A))$ induces, in the obvious way, a map $\tilde{\phi}\colon C(X)\to Z(\mathcal{L}(\mathcal{H}_A))$. Thus, for a $C(X)$-C^*-algebra A there is a natural homomorphism

$$\alpha_X\colon \mathrm{KK}(\mathbb{C}, A) \longrightarrow \mathrm{KK}(C(X), A).$$

The map α_X is a natural transformation from the $\mathrm{KK}(\mathbb{C}, -)$ functor to the $\mathrm{KK}(C(X), -)$ functor on the category of $C(X)$-C^*-algebras. In fact, it is a natural transformation for the KK^X-category.

Therefore, since the isomorphism Ψ is implemented by a KK^X-element

$$\Psi \in \mathrm{KK}^X(C_0(T^*X), C^*(T_H X)),$$

we have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{KK}(\mathbb{C}, C_0(T^*X)) & \xrightarrow{\ \Psi\ } & \mathrm{KK}(\mathbb{C}, C^*(T_H X)) \\
\alpha_X \big\downarrow & & \big\downarrow \alpha_X \\
\mathrm{KK}(C(X), C_0(T^*X)) & \xrightarrow{\ \Psi\ } & \mathrm{KK}(C(X), C^*(T_H X)).
\end{array}
$$

Composing this with our result above yields the proposition. $\qquad\square$

5.5. The hypoelliptic index theorem in K-homology

The Atiyah–Singer theorem for elliptic operators amounts to commutativity of the triangle [3, Theorem 23.1],

$$
\begin{array}{ccc}
& K^0(T^*X) & \\
{}^{c}\swarrow & & \searrow^{\mathrm{Op_e}} \\
K_0(X)\mu \xrightarrow{\qquad\ \ \tilde{}\ \qquad} & & \mathrm{KK}_0(C(X), \mathbb{C}).
\end{array}
$$

We obtain an analogous theorem for the Heisenberg calculus.

THEOREM 5.5.1. *Let (X, H) be a closed contact manifold. Then there is commutativity in the diagram*

$$
\begin{array}{ccc}
& K_0(C^*(T_H X)) & \\
{}^{b}\swarrow & & \searrow^{\mathrm{Op_H}} \\
K_0(X)\mu \xrightarrow{\qquad\ \ \tilde{}\ \qquad} & & \mathrm{KK}_0(C(X), \mathbb{C}).
\end{array}
$$

Equivalently, the geometric K-cycle that corresponds to the analytic K-cycle $[P]$ determined by a Heisenberg-elliptic operator is

$$\mu^{-1}([P]) = b(\sigma_H(P)).$$

Proof. The theorem is implied by commutativity of the three smaller triangles in the diagram

$$
\begin{array}{c}
K_0(C^*(T_H X)) \\
\end{array}
$$

b Ψ $\mathrm{Op_H}$

$$K^0(T^*X)$$

c $\mathrm{Op_e}$

$$K_0^{\mathrm{top}}(X) \xrightarrow{\hspace{2cm} \mu \hspace{2cm}} \mathrm{KK}^0(C(X), \mathbb{C}). \qquad \square$$

Modulo an explicit computation of the Poincaré dual $b(\sigma_H(P))$ of the Heisenberg symbol of a hypoelliptic operator, Theorem 5.5.1 solves the general index problem for hypoelliptic operators in the Heisenberg calculus. In §6 we discuss the explicit computation of the (M, E, φ)-cycle that corresponds to a Heisenberg-elliptic operator, and apply it in concrete examples.

6. Computation of the K-cycle

In this section we explicitly calculate $\mu^{-1}([P]) = b(\sigma_H(P))$. A number of examples are then considered.

6.1. Perturbing the symbol

Consider a Heisenberg-elliptic operator

$$P \colon C^\infty(X, F^0) \longrightarrow C^\infty(X, F^1)$$

acting on sections in a $\mathbb{C}$-vector bundle F^0, whose range consists of sections in a vector bundle F^1. We wish to compute the image $b(\sigma_H(P))$ of the K-theory element $[\sigma_H(P)]$ associated with the Heisenberg symbol of a Heisenberg-elliptic operator P under the non-commutative Poincaré duality map

$$b \colon K_0(C^*(T_H X)) \longrightarrow K_0(X).$$

Following Definition 4.7.1, the first step is to lift the element $[\sigma_H(P)]$ in the K-theory group $K_0(C^*(T_H X))$ to an element in $K_0(I_H)$.

The full Heisenberg symbol of P is a smooth family $\sigma_H(P) = \{P_x \,|\, x \in X\}$ of operators

$$P_x \colon C_c^\infty(G_x, F_x^0) \longrightarrow C_c^\infty(G_x, F_x^1).$$

Each P_x is an operator of convolution with a compactly supported distribution on G_x, where $G_x = H_x \times \mathbb{R}$ is the Heisenberg group that is the fiber at $x \in X$ of $T_H X$. Taken together, the operators $\{P_x | x \in X\}$ correspond to a compactly supported distributional section $\sigma_H(P)$ of the bundle $\mathrm{Hom}(\pi^* F^0, \pi^* F^1)$ on $T_H X$. Recall that the operator P determines the distribution $\sigma_H(P)$ on $T_H X$ (its 'full symbol') up to a perturbation by a compactly supported smooth section defined on $T_H X$.

Let π_0 denote the representation of the Heisenberg group that assembles all scalar unitary representations. At the level of the group algebra, π_0 is the composition of the Fourier transform $C_c^\infty(G) \to C^\infty(V^* \times \mathbb{R}^*)$ with the restriction to $s = 0$ in $\mathbb{R}^*$,

$$C_c^\infty(G) \longrightarrow C^\infty(V^*).$$

For scalar distributions on $T_H X$ this representation induces the algebra homomorphism

$$\pi_0 \colon \mathcal{E}'(T_H X) \longrightarrow C^\infty(H^*),$$

where $\mathcal{E}'(T_H X)$ is a convolution algebra, while $C^\infty(H^*)$ is the algebra with pointwise multiplication of functions.

If we take the vector bundles F^0 and F^1 into account, then the scalar representations of P_x assemble to a function

$$\pi_0(P_x) \colon H_x^* \longrightarrow \mathrm{Hom}(F_x^0, F_x^1),$$

and what we get is the classical symbol of P restricted to H^*,

$$\pi_0(\sigma_H(P)) \in C^\infty(H^*, \mathrm{Hom}(\pi^* F^0, \pi^* F^1)),$$

where $\pi^* F^j$ is the pull-back of F^j from X to H^*. This classical part of the Heisenberg symbol is 'elliptic' in the sense that it is invertible outside a compact set in H^*.

As a class in $K^0(H^*)$ the partial symbol $\pi_0(\sigma_H(P))$ is trivial. To see this, consider the exact sequence in K-theory

$$K^0(T^* X \setminus H^*) \longrightarrow K^0(T^* X) \longrightarrow K^0(H^*).$$

The first map is surjective, and so the second map is trivial. Therefore the same is true for the isomorphic sequence

$$K_0(I_H) \longrightarrow K_0(C^*(T_H X)) \xrightarrow{\pi_0} K_0(C^*(H)).$$

Since $\pi_0(\sigma_H(P))$ is K-theoretically trivial, we may assume that, after a compactly supported perturbation of $\pi_0(\sigma_H(P))$, we have

$$\pi_0(\sigma_H(P)) \in C^\infty(H^*, \mathrm{Iso}(\pi^* F^0, \pi^* F^1)).$$

(We may have to stabilize F^0 and F^1.) Note that we can extend such a perturbation to the full Heisenberg symbol $\sigma_H(P)$. We shall assume, from now on, that $\pi_0(\sigma_H(P))$ is invertible on all of H^*.

We make one further modification to the symbol. If we restrict the (perturbed) $\pi_0(\sigma_H(P))$ to the zero section in H^*, the resulting section in $\mathrm{Hom}(F^0, F^1)$ is an isomorphism of vector bundles

$$\sigma_0 \colon F^0 \longrightarrow F^1.$$

We compose the operator P with the vector bundle isomorphism σ_0^{-1}, and obtain a new operator

$$\sigma_0^{-1} \circ P \colon C^\infty(X, F^0) \longrightarrow C^\infty(X, F^0)$$

that represents, of course, the same K-homology class as P. The point of this modification is that the equatorial symbol of $\sigma_0^{-1} \circ P$ (i.e., its value at the spherical boundary $S(H^*)$ of H^*) is homotopic to the constant map from $S(H^*)$ to the identity operator in the fibers of F^0. In fact, we can choose the isomorphism $\sigma_0 \colon F^0 \to F^1$ to be *any* isomorphism with this property.

With these modifications, the full Heisenberg symbol $\sigma_0^{-1}\sigma_H(P)$ defines an element in the convolution algebra $\mathcal{E}'(T_H X, \mathrm{End}(\pi^* F^0))$, and it is invertible modulo the ideal

$$I_H^\infty := \{ x \in C_c^\infty(T_H X, \mathrm{End}(\pi^* F^0)) \mid \pi_0(x) = 0 \}.$$

Since the closure of I_H^∞ is I_H, the formal difference of idempotents associated with $\sigma_0^{-1}\sigma_H(P)$, as defined in §5.3, is the desired element in $K_0(I_H)$.

6.2. Suspension in K-theory

The next step in Definition 4.7.1 is the isomorphism

$$K_0(I_H) \cong K^1(X) \oplus K^1(X),$$

which is a composition of Morita equivalence and suspension. We may reverse the order of the Morita equivalence and the suspension isomorphisms. Let B be the C^*-algebra

$$B = \mathcal{K}(V_-^{\mathrm{BF}}) \oplus \mathcal{K}(V_+^{\mathrm{BF}}).$$

In §4 we defined an explicit isomorphism

$$I_H \cong C_0(\mathbb{R}) \otimes B.$$

We will first prove a general lemma in K-theory that allows us to compute the suspension isomorphism

$$K_0(C_0(\mathbb{R}) \otimes B) \cong K_1(B).$$

We subsequently consider the effect of the Morita equivalence $B \sim C(X) \oplus C(X)$.

LEMMA 6.2.1. *Let B be a separable C^*-algebra. Let $u \in M_n(B)^+$ be a unitary matrix that represents a class $[u] \in K_1(B)$, and let θ denote the suspension isomorphism,*

$$\theta \colon K_1(B) \longrightarrow K_0(C_0(0,1) \otimes B).$$

If $F(t)$, $t \in [0,1]$, is the norm-continuous family that linearly interpolates between $F(0)=1$ and $F(1)=u$,

$$F(t) = 1 - t + tu \in M_n(B)^+,$$

then $F \in C_0([0,1], M_n(B))^+$ is invertible modulo $C_0((0,1), M_n(B))$, and defines a relative class

$$[F] \in K_0(C_0(0,1) \otimes B),$$

with $[F] = \theta([u])$.

Proof. We will prove this by making use of the flexibility of KK-theory. For ease of notation, let us write $A = C_0((0,1), B)$. Since $M_n(B)^+ \subset M_n(M(A)) = \mathcal{L}(A^n)$, we can think of F as an adjointable operator on the Hilbert A-module A^n. The fact that $F_0 = 1$, $F_1 = u$ and $F_t = 1$ modulo $M_n(B)$ for all $t \in [0,1]$ implies that F is invertible modulo compact operators $\mathcal{K}(A^n) = C_0((0,1), M_n(B))$. In other words, the relative K-theory class defined by F can also be represented as a Kasparov module

$$[F \colon A^n \to A^n] \in \mathrm{KK}(\mathbb{C}, A).$$

We will prove that this Fredholm module corresponds to $\theta([u])$ under the standard identification $\mathrm{KK}(\mathbb{C}, A) \cong K_0(A)$.

Choose a continous family of unitary operators $z(t) \in M_{2n}(B^+)$, $t \in [0,1]$, such that

$$z_0 = \begin{pmatrix} 1_n & 0 \\ 0 & 1_n \end{pmatrix} \quad \text{and} \quad z_1 = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}.$$

Let

$$p_n = \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix},$$

and let e denote the continuous family of projections

$$e_t = z_t p_n z_t^{-1}.$$

Observe that $e_0 = e_1 = p_n$. Then the proof of [7, Theorem 8.2.2] implies that

$$\theta([u]) = [e] - [p_n] \in K_0(C_0((0,1), B)).$$

Consider the family of partial isometries

$$G(t) = z(t)p_n.$$

Observe that $G(0){=}p_n$, $G(1){=}u$ and that $G(1){=}p_n$ modulo $\mathcal{K}(A^{2n})$, while

$$1 - G^*G = 1 - p_n \quad \text{and} \quad 1 - GG^* = 1 - e.$$

Let $\phi \colon \mathbb{C} \to \mathcal{L}(B^{2n})$ denote the homomorphism with $\phi(1){=}p_n$. With this choice of a non-unital map ϕ the operator G satisfies the Fredholm property, and we obtain a Kasparov module

$$z[G \colon A^{2n} \to A^{2n}, \phi] \in \mathrm{KK}(\mathbb{C}, A).$$

Moreover, the module $[G \colon A^{2n} \to A^{2n}, \phi]$ is homotopic to $[F \colon A^n \to A^n, p_n\phi p_n]$, where

$$p_n \phi p_n \colon \mathbb{C} \longrightarrow \mathcal{L}(A^n)$$

is just the standard unital map. But unlike F the operator G has closed range, and we can take its index

$$\mathrm{Index}\, G = [1 - G{*}G] - [1 - GG^*] = [e] - [p_n] = \theta([u]).$$

It follows that $[G]$ corresponds to $\theta([u])$, and therefore so does $[F]$. $\qquad\square$

6.3. The general formula

We now derive an explicit formula for the K-homology element $\mu^{-1}([P]){=}b(\sigma_H(P))$ for an arbitrary Heisenberg-elliptic operator P.

THEOREM 6.3.1. *Let P be a Heisenberg-elliptic operator on a closed oriented contact manifold that acts on sections in a complex vector bundle,*

$$P \colon C^\infty(X, F^0) \longrightarrow C^\infty(X, F^1).$$

Let $\sigma_0 \colon F_0 \to F_1$ be a vector bundle isomorphism such that the pull-back of σ_0 to the sphere bundle $S(H^*)$ is homotopic to the equatorial symbol $\sigma(P)|_{S(H^*)}$.

Then the element in K-homology determined by the Fredholm operator P is

$$[P] = \pi_+(\sigma_0^{-1}\sigma_H^+(P)) \cap [X^+] + \pi_-(\sigma_0^{-1}\sigma_H^-(P)) \cap [X^-].$$

Here $\pi_\pm$ denote the Bargmann–Fock representations on the Hilbert modules $V_\pm^{\mathrm{BF}} \otimes F^0$ of the two components of the principal Heisenberg symbol of the operator $\sigma_0^{-1} \circ P$.

Proof. Choose hermitian structures for the vector bundles F^0 and F^1 such that $\sigma_0 \colon F^0 \to F^1$ is a unitary isomorphism. For technical reasons, replace P by

$$\widetilde{P} = \sigma_0^{-1} \circ P(1+P^*P)^{-1/2}.$$

Then $\widetilde{P}$ is an order-zero Heisenberg pseudodifferential operator that represents the same K-homology class as P. But the *principal* Heisenberg symbol of $\widetilde{P}$ is unitary.

Let $B = \mathcal{K}(V_-^{\mathrm{BF}}) \oplus \mathcal{K}(V_+^{\mathrm{BF}})$ and

$$I_H \cong C_0(\mathbb{R}, B) \cong C_0((0,1), B),$$

where we choose an arbitrary orientation-preserving homeomorphism $\mathbb{R} \approx (0,1)$.

Recall that the equatorial symbol of $\widetilde{P}$ is homotopic to the identity. We first assume, for simplicity, that the equatorial symbol of $\widetilde{P}$ is *equal* to the identity. Then we can define the unitary element of the C^*-algebra B^+,

$$u = \pi_-(\sigma_H^-(\widetilde{P})) \oplus \pi_+(\sigma_H^+(\widetilde{P})) \in B^+.$$

The difference between the full symbol $\sigma_H(\widetilde{P})$ and $1-t+tu$ is an element in $C_0((0,1), B)$, so that $\sigma_H(\widetilde{P})$ and $1-t+tu$ represent the same element in $K_0(I_H)$. Therefore, by Lemma 6.2.1, the K-theory element

$$[u] \in K_1(B) \cong K_1(\mathcal{K}(V_-^{\mathrm{BF}})) \oplus K_1(\mathcal{K}(V_+^{\mathrm{BF}}))$$

corresponds, under the suspension isomorphism, to $[\sigma_H(\widetilde{P})] \in K_0(I_H)$. Completing the steps in Definition 4.7.1, we obtain the formula stated in the theorem.

In general, the equatorial symbol of $\widetilde{P}$ is *homotopic* to the constant map from $S(H^*)$ to the identity operator on F^0, and so essentially the same argument applies. $\square$

Remark 6.3.2. In Theorem 6.3.1 we did not explicitly address the Morita equivalence that is part of the definition of the map b. Strictly speaking, we only defined elements

$$[\pi_\pm(\sigma_0^{-1} \sigma_H^\pm(P))] \in K_1(\mathcal{K}(\mathcal{H}_\pm^{\mathrm{BF}})),$$

but we have not yet indicated what to do about the Morita equivalences

$$K_1(\mathcal{K}(\mathcal{H}_\pm^{\mathrm{BF}})) \cong K^1(X).$$

But this is a standard procedure in K-theory. Let $V_\pm^N \to X$ denote the complex vector bundles on X defined by

$$V_+^N = \bigoplus_{j=0}^N \operatorname{Sym}^j H^{1,0} \quad \text{and} \quad V_-^N = \bigoplus_{j=0}^N \operatorname{Sym}^j H^{0,1},$$

and let

$$S_{\pm}^{N} \colon V_{\pm}^{\mathrm{BF}} \longrightarrow V_{\pm}^{N}$$

be the family of orthogonal projections in each fiber. Let us denote by $\pi_{\pm}^{N}$ the Bargmann–Fock representations compressed by the projections S^{N},

$$\pi_{\pm}^{N}(a) := S_{\pm}^{N} \pi_{\pm}(a) S_{\pm}^{N}.$$

For sufficiently large values of the integer N, we have

$$[\pi_{\pm}(\sigma_{0}^{-1}\sigma_{H}^{\pm}(P))] = [S^{N}\pi_{\pm}(\sigma_{0}^{-1}\sigma_{H}^{\pm}(P))S^{N} + (1 - S^{N})] \in K_{1}(\mathcal{K}(\mathcal{H}_{\pm}^{\mathrm{BF}})),$$

where the correct value of N depends on $\pi_{\pm}(\sigma_{0}^{-1}\sigma_{H}^{\pm}(P))$. Observe that with this choice of N, the elements $\pi_{\pm}^{N}(\sigma_{0}^{-1}\sigma_{H}^{\pm}(P))$ define automorphisms of the vector bundles $V_{\pm}^{N}$, so that

$$[V^{N}, \pi_{\pm}^{N}(\sigma_{0}^{-1}\sigma_{H}^{\pm}(P))] \in K^{1}(X).$$

The compression

$$K_{1}(\mathcal{K}(\mathcal{H}_{\pm}^{\mathrm{BF}})) \longrightarrow K^{1}(X),$$
$$[\pi_{\pm}(\sigma_{0}^{-1}\sigma_{H}^{\pm}(P))] \longmapsto [V_{\pm}^{N}, \pi_{\pm}^{N}(\sigma_{0}^{-1}\sigma_{H}^{\pm}(P))],$$

implements the Morita equivalence (where the size of N depends on P, as mentioned). So we have, more precisely,

$$[P] = [\pi_{-}^{N}(\sigma_{0}^{-1}\sigma_{H}^{-}(P))] \cap [X^{-}] + [\pi_{+}^{N}(\sigma_{0}^{-1}\sigma_{H}^{+}(P))] \cap [X^{+}].$$

6.4. Toeplitz operators

The index formula for Toeplitz operators of Boutet de Monvel [8] is a special case of Theorem 6.3.1. If the contact manifold X is the boundary of a strictly pseudoconvex complex domain, then the *Szegő projector* S is defined as the projection of $L^{2}(X)$ onto the Hardy space $H^{2}(X)$. Let f be a smooth map

$$f \colon X \longrightarrow \mathrm{GL}(r, \mathbb{C}).$$

In an evident fashion, the function f defines a multiplication operator M_{f} on $L^{2}(X)^{\oplus r}$. The Toeplitz operator T_{f} is the compression of M_{f} to $H^{2}(X)^{\oplus r}$, i.e.,

$$T_{f} = S_{r} M_{f} S_{r},$$

where $S_r = S^{\oplus r}$. The Toeplitz operator T_f is a Fredholm operator, and Boutet de Monvel's formula is

$$\text{Index}\, T_f = \langle \text{ch}(f) \cup \text{Td}(X), [X] \rangle,$$

where $\text{Td}(X)$ is the Todd class of the Spin^c manifold X^+, and $[f]$ is viewed as an element in $K^1(X)$.

As shown in [13], the operator

$$\widetilde{T}_f = T_f + 1 - S_r$$

is an order-zero operator in the Heisenberg calculus. It is immediate that $\widetilde{T}_f$ is a Fredholm operator on $L^2(X)^{\oplus r}$ with $\text{Index}\, \widetilde{T}_f = \text{Index}\, T_f$. To derive Boutet de Monvel's result from our general formula, we use the calculation in [13] of the principal Heisenberg symbol of $\widetilde{T}_f$. Within the Bargmann–Fock space $\mathcal{H}_+^{\text{BF}}$ there is the vacuum summand $\mathbb{C} = \text{Sym}^0 \mathbb{C}^n$. Hence, the Hilbert module V_+^{BF} contains the trivial line bundle $\underline{\mathbb{C}} = \text{Sym}^0 H^{1,0}$. The Bargmann–Fock representations $\pi_\pm$ of the Heisenberg symbol of $\widetilde{T}_f$ are

$$\pi_+(\sigma_H(\widetilde{T}_f)) = f \quad \text{acting on } \underline{\mathbb{C}}^r,$$
$$\pi_+(\sigma_H(\widetilde{T}_f)) = 1 \quad \text{on the orthogonal complement of } \underline{\mathbb{C}}^r,$$
$$\pi_-(\sigma_H(\widetilde{T}_f)) = 1.$$

The equatorial symbol of $\widetilde{T}_f$ is 1, and the formula of Theorem 6.3.1,

$$[T_f] = [\pi_+(\sigma_H^+(\widetilde{T}_f))] \cap [X^+] + [\pi_-(\sigma_H^-(\widetilde{T}_f))] \cap [X^-],$$

reduces to

$$[T_f] = [f] \cap [X^+].$$

Equivalently, the K-cycle which solves the index problem for $\widetilde{T}_f$ is $(X^+ \times S^1, E_f, \varphi)$, where E_f is the vector bundle on $X \times S^1$ obtained from a clutching construction—using trivial vector bundles of fiber dimension r—with f, and $\varphi \colon X \times S^1 \to X$ is the projection. In particular, the formula of Boutet de Monvel is now a corollary.

6.5. Second-order scalar operators

For Heisenberg-elliptic *differential* operators P, the principal Heisenberg symbol and its action on the Bargmann–Fock spaces can be explicitly computed by an algorithmic procedure. Theorem 6.3.1 then gives an explicit (M, E, φ)-cycle that corresponds to $[P]$. We will illustrate this by computing the geometric K-cycle for the Heisenberg-elliptic operators $P_\gamma = \Delta_H + i\gamma T$. The procedure is essentially the same for all *differential* Heisenberg-elliptic operators, but it is most easily illustrated by an explicit example.

Because of Darboux's theorem the contact manifold X can locally be identified with an open subset of the Heisenberg group $\mathbb{R}^{2n+1}$. Let X_j, Y_j and T be the standard right-invariant vector fields on the Heisenberg group with $[X_j, Y_j]=T$. In such local coordinates, the operator P_γ can be written as

$$P_\gamma = \sum_{j=1}^{n} -(X_j^2+Y_j^2)+i\gamma T+...,$$

where we ignore first order terms in X_j and Y_j. These lower-order terms may appear because the sublaplacian $\Delta_H = \sum -(X_j^2+Y_j^2)+...$ is unique only up to lower-order terms.

We have $JX_j=Y_j$, and so $Z_j=(X_j-iY_j)/\sqrt{2}$. A simple computation shows that $2Z_j\bar{Z}_j=X_j^2+Y_j^2+i[X_j,Y_j]$, and so

$$\sum_{j=1}^{n} -2Z_j\bar{Z}_j = \sum_{j=1}^{n} -(X_j^2+Y_j^2)+inT.$$

We can therefore write

$$P_\gamma = \sum_{j=1}^{n} -2Z_j\bar{Z}_j -i(n-\gamma)T+....$$

The Bargmann–Fock representation on $(\mathcal{H}_+^{\mathrm{BF}})_x$ of the symbol $\sigma_H(P)=\{P_x\,|\,x\in X\}$ is

$$\pi_+(P_x) = \sum_{j=1}^{n} 2z_j\frac{\partial}{\partial z_j}+n-\gamma(x).$$

The symmetric powers of $H^{1,0}$ are eigenspaces of $\pi_+(P_x)$. To see this, observe that

$$\pi_+(P_x)z^\alpha = (2|\alpha|+n-\gamma(x))z^\alpha,$$

where $z^\alpha=z_1^{\alpha_1}...z_n^{\alpha_n}$ and $|\alpha|=\alpha_1+...+\alpha_n$. Therefore, on $\mathrm{Sym}^j H^{1,0}$, the operator $\pi_+(P_x)$ acts as the scalar

$$a_j(x) = 2j+n-\gamma(x).$$

In the dual Bargmann–Fock representation the roles of Z and $\bar{Z}$ are reversed. So, from

$$\sum_{j=1}^{n} -2\bar{Z}_j Z_j = \sum_{j=1}^{n} -(X_j^2+Y_j^2)-inT,$$

we get

$$P_\gamma = \sum_{j=1}^{n} -2\bar{Z}_j Z_j +i(n+\gamma)T.$$

The representation of $\sigma_H(P)$ on the conjugate Bargmann–Fock spaces $(\mathcal{H}_-^{\mathrm{BF}})_x$ is then given by

$$\pi_-(P_x) = \sum_{j=1}^{n} 2\bar{z}_j \frac{\partial}{\partial \bar{z}_j} + n + \gamma(x),$$

and we find that $\pi_-(P_x)$ acts on $\mathrm{Sym}^j H^{0,1}$ as the scalar

$$b_j = 2j + n + \gamma(x).$$

Replacing P by

$$\widetilde{P} = P(1 + P^*P)^{-1/2},$$

we obtain an order-zero operator whose principal symbols act on the symmetric powers of $H^{1,0}$ by the scalar $a_j/\sqrt{1+|a_j|^2}$, and similarly for $H^{0,1}$. For large values of j these renormalized scalars are close to 1, and so they contribute only trivial summands to the K-cycle and can be ignored. We can use the non-normalized scalars, because as vector bundle automorphisms they are homotopic to their normalized versions. The result is the following formula in K-homology.

PROPOSITION 6.5.1. *Let $P_\gamma = \Delta_\gamma + i\gamma T$ be a Heisenberg-elliptic operator. Then the element in K-homology represented by P_γ is*

$$\sum_{j=0}^{\infty} ([\mathrm{Sym}^j H^{1,0}] \cup [2j+n-\gamma]) \cap [X^+] + \sum_{j=0}^{\infty} ([\mathrm{Sym}^j H^{0,1}] \cup [2j+n+\gamma]) \cap [X^-].$$

Here the vector bundles $\mathrm{Sym}^j H^{1,0}$ and $\mathrm{Sym}^j H^{0,1}$ represent elements in the K-theory group $K^0(X)$, while the functions $2j+n-\gamma$ and $2j+n+\gamma$ represent elements in odd K-theory $K^1(X)$.

COROLLARY 6.5.2. *The Chern character of the K-cycle $[P_\gamma]$ is the Poincaré dual of the cohomology class*

$$\mathrm{P.D.\ ch}([P_\gamma]) = \sum_{j=0}^{N} \mathrm{ch}(2j+n-\gamma) \cup \mathrm{ch}(\mathrm{Sym}^j H^{1,0}) \cup e^{c_1(H^{1,0})/2} \cup \hat{A}(X)$$
$$+ (-1)^{n+1} \sum_{j=0}^{N} \mathrm{ch}(2j+n+\gamma) \cup \mathrm{ch}(\mathrm{Sym}^j H^{0,1}) \cup e^{c_1(H^{0,1})/2} \cup \hat{A}(X).$$

Here the odd Chern character $\mathrm{ch}(f) \in H^1(X, \mathbb{Z}) = [X, S^1]$ refers to the 1-cocycle associated with the continuous map $f: X \to \mathbb{C}^\times \sim S^1$.

Example 6.5.3. In [17] we derived an explicit topological formula for the Fredholm index of P_γ in the simplest possible case where X is a 3-manifold. We now see that this index formula, while correct, was incomplete. Let us calculate the K-cycle, and compare it with the index formula in [17].

On a 3-manifold X we have $\hat{A}(X){=}1$, while the bundle $H^{1,0}$ is a line bundle. Then $\mathrm{Sym}^j H^{1,0}{=}(H^{1,0})^{\otimes j}$. Writing $c_1{=}c_1(H^{1,0})$, we have

$$\mathrm{ch}(\mathrm{Sym}^j H^{1,0})e^{c_1/2} = (1{+}jc_1)\big(1{+}\tfrac{1}{2}c_1\big) = 1{+}\tfrac{1}{2}(2j{+}1)c_1.$$

Denote by

$$W_k = \mathrm{ch}(\gamma{-}k) = \left[-\frac{1}{2\pi i}\frac{d\gamma}{\gamma{-}k}\right] \in H^1(X,\mathbb{Z})$$

the 1-cocycle that encodes the winding of the coefficient γ around the odd integer k,

$$\gamma\colon X \longrightarrow \mathbb{C}\backslash\{\text{odd integers}\}.$$

We find that

$$\text{P.D. } \mathrm{ch}(P_\gamma) = \sum_{k\text{ odd}} W_k + \sum_{k\text{ odd}} kW_k\cup\frac{c_1}{2} \in H^1(X,\mathbb{Z})\oplus H^3(X,\mathbb{Z}).$$

Here all cocycles are integer cocycles. The Poincaré dual of the 3-cocycle

$$\sum_{k\text{ odd}} kW_k\cup\frac{c_1}{2}$$

is a 0-cycle whose image under the map $H_0(X,\mathbb{Z}){\to}H_0(\mathrm{pt},\mathbb{Z}){=}\mathbb{Z}$ is just the Fredholm index of P_γ. This term in our K-cycle contains exactly the same information as the index formula of [17]. The Poincaré dual of the term $\sum_{k\text{ odd}} W_k$ is a 2-cycle. If we twist P_γ by a complex vector bundle $F{\to}X$, then this 2-cycle will pair with $c_1(F)$, and the curvature of F will contribute to the index of $F{\otimes}P_\gamma$. The information contained in the term $\sum_{k\text{ odd}} W_k$ cannot be derived or guessed from the formulas of [16] and [17]. The following formula highlights the gap between the result of the present paper and the earlier formula,

$$\mathrm{Index}(F{\otimes}P_\gamma) = \mathrm{rank}(F)\,\mathrm{Index}\,P_\gamma + \sum_{k\text{ odd}} \int_X \mathrm{ch}(\gamma{-}k)\wedge c_1(F).$$

Example 6.5.4. The formula for the K-cycle of P_γ easily extends to the case where P_γ acts on sections in a trivial vector bundle. With X as above, let r be a positive integer and let γ be a C^∞ map from X to the $\mathbb{C}$-vector space $M(r,\mathbb{C})$ of all $r{\times}r$ matrices,

$$\gamma\colon X \longrightarrow M(r,\mathbb{C}).$$

Then—entirely analogous to the case $r=1$—a differential operator P_γ is given by

$$P_\gamma = \Delta_H \otimes I_r + iT \otimes \gamma \colon C^\infty(X, \mathbb{C}^r) \longrightarrow C^\infty(X, \mathbb{C}^r),$$

where (as usual) I_r is the $r \times r$ identity matrix. P_γ is elliptic in the Heisenberg calculus if and only if, for all $x \in X$,

$$\gamma(x) - \lambda I_r \text{ is invertible for all } \lambda \in \{..., -n-4, -n-2, -n, n, n+2, n+4, ...\}.$$

As above, P_γ determines an element $[P_\gamma]$ in $\mathrm{KK}^0(C(X), \mathbb{C})$. The K-cycle which solves the index problem for $[P_\gamma]$ is $(X^+ \times S^1, E^+, \varphi) \sqcup (X^- \times S^1, E^-, \varphi)$, where $(X^+ \times S^1) \sqcup (X^- \times S^1)$ and φ are as above, and

$$E^+ = \bigoplus_{j=0}^{N} (\nu_{2j+n}^\gamma \otimes \varphi^* \operatorname{Sym}^j H^{1,0}),$$

$$E^- = \bigoplus_{j=0}^{N} (\nu_{-(2j+n)}^\gamma \otimes \varphi^* \operatorname{Sym}^j H^{0,1}).$$

Here ν_{2j+n}^γ (resp. $\nu_{-(2j+n)}^\gamma$) is the $\mathbb{C}$-vector bundle of fiber dimension r on $X \times S^1$ obtained by doing a clutching construction using $\gamma - (2j+n)I_r$ (resp. $\gamma + (2j+n)I_r$).

References

[1] ATIYAH, M. F. & SINGER, I. M., The index of elliptic operators. I. *Ann. of Math.*, 87 (1968), 484–530.

[2] BARGMANN, V., On a Hilbert space of analytic functions and an associated integral transform. *Comm. Pure Appl. Math.*, 14 (1961), 187–214.

[3] BAUM, P. & DOUGLAS, R. G., *K* homology and index theory, in *Operator Algebras and Applications* (Kingston, Ont., 1980), Proc. Sympos. Pure Math., 38, Part I, pp. 117–173. Amer. Math. Soc., Providence, RI, 1982.

[4] — index theory, bordism, and *K*-homology, in *Operator Algebras and K-theory* (San Francisco, CA, 1981), Contemp. Math., 10, pp. 1–31. Amer. Math. Soc., Providence, RI, 1982.

[5] BAUM, P., HIGSON, N. & SCHICK, T., On the equivalence of geometric and analytic *K*-homology. *Pure Appl. Math. Q.*, 3 (2007), 1–24.

[6] BEALS, R. & GREINER, P., *Calculus on Heisenberg Manifolds.* Annals of Mathematics Studies, 119. Princeton University Press, Princeton, NJ, 1988.

[7] BLACKADAR, B., *K-Theory for Operator Algebras.* Mathematical Sciences Research Institute Publications, 5. Cambridge University Press, Cambridge, 1998.

[8] BOUTET DE MONVEL, L., On the index of Toeplitz operators of several complex variables. *Invent. Math.*, 50 (1978/79), 249–272.

[9] CHOI, M. D. & EFFROS, E. G., The completely positive lifting problem for C^*-algebras. *Ann. of Math.*, 104 (1976), 585–609.

[10] CONNES, A., *Noncommutative Geometry.* Academic Press, San Diego, CA, 1994.

[11] CONNES, A. & MOSCOVICI, H., The local index formula in noncommutative geometry. *Geom. Funct. Anal.*, 5 (1995), 174–243.

[12] — Hopf algebras, cyclic cohomology and the transverse index theorem. *Comm. Math. Phys.*, 198 (1998), 199–246.

[13] EPSTEIN, C. L., Lectures on indices and relative indices on contact and CR-manifolds, in *Woods Hole Mathematics*, Ser. Knots Everything, 34, pp. 27–93. World Scientific, Hackensack, NJ, 2004.

[14] EPSTEIN, C. L. & MELROSE, R., The Heisenberg algebra, index theory and homology. Preprint, 2004.

[15] VAN ERP, E., The Atiyah–Singer index formula for subelliptic operators on contact manifolds. Part I. *Ann. of Math.*, 171 (2010), 1647–1681.

[16] — The Atiyah–Singer index formula for subelliptic operators on contact manifolds. Part II. *Ann. of Math.*, 171 (2010), 1683–1706.

[17] — Noncommutative topology and the world's simplest index theorem. *Proc. Natl. Acad. Sci. USA*, 107 (2010), 8549–8556.

[18] — The intrinsic geometry of the osculating structures that underlie the Heisenberg calculus. Preprint, 2010. arXiv:1007.4759 [math.AP].

[19] FOLLAND, G. B. & STEIN, E. M., Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group. *Comm. Pure Appl. Math.*, 27 (1974), 429–522.

[20] MELROSE, R., Homology and the Heisenberg algebra, in *Séminaire sur les Équations aux Dérivées Partielles*, 1996–1997, Exp. No. XII. École Polytechnique, Palaiseau, 1997.

[21] TAYLOR, M. E., Noncommutative microlocal analysis. I. *Mem. Amer. Math. Soc.*, 52 (1984).

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