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Geometric quantization for proper moment maps: the Vergne conjecture

by

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0. Introduction

The purpose of this paper is to establish a geometric quantization formula for a Hamiltonian action of a compact Lie group acting on a non-compact symplectic manifold with proper moment map. Our results provide a solution to a conjecture of Michèle Vergne in her ICM 2006 plenary lecture [26].

Let (M,ω) be a symplectic manifold with symplectic form ω and $\dim M = n$. We assume that (M,ω) is *prequantizable*, that is, there exists a complex line bundle L (called a *prequantum line bundle*) carrying a Hermitian metric h^L and a Hermitian connection ∇^L such that the associated curvature $R^L = (\nabla^L)^2$ satisfies

$$\frac{i}{2\pi} R^L = \omega. \tag{0.1}$$

Let TM be the tangent vector bundle of M. Let J^M be an almost-complex structure on TM such that

$$g^{TM}(u,v) = \omega(u, J^M v), \quad u,v \in TM, \tag{0.2}$$

defines a J^M-invariant Riemannian metric g^{TM} on TM.

Let G be a compact connected Lie group. Let $\mathfrak{g}$ denote the Lie algebra of G and $\mathfrak{g}^*$ denote the dual of $\mathfrak{g}$. Let G act on $\mathfrak{g}^*$ by the coadjoint action.

We assume that G acts on the left on M, that this action lifts to an action on L, and that G preserves g^{TM}, J^M, h^L and ∇^L.

For $K \in \mathfrak{g}$, let $K^M \in \mathscr{C}^\infty(M, TM)$ denote the vector field generated by K over M. The moment map $\mu: M \to \mathfrak{g}^*$ is defined [8] by the Kostant formula

$$2\pi i \mu(K) := \nabla^L_{K^M} - L_K, \quad K \in \mathfrak{g}. \tag{0.3}$$

Then, for any $K \in \mathfrak{g}$, we have

$$d\mu(K) = i_{K^M}\omega. \tag{0.4}$$

From now on, we make the following assumption.

Fundamental assumption. The moment map $\mu: M \to \mathfrak{g}^*$ is proper, i.e., for any compact subset $B \subset \mathfrak{g}^*$, the subset $\mu^{-1}(B) \subset M$ is compact.

Let T be a maximal torus of G, let $\mathfrak{t}$ be its Lie algebra and $\mathfrak{t}^*$ be the dual of $\mathfrak{t}$. The integral lattice $\Lambda \subset \mathfrak{t}$ is defined as the kernel of the exponential map $\exp: \mathfrak{t} \to T$, and the real weight lattice $\Lambda^* \subset \mathfrak{t}^*$ is defined by $\Lambda^* := \mathrm{Hom}(\Lambda, 2\pi\mathbb{Z})$. We fix a positive Weyl chamber $\mathfrak{t}^*_+ \subset \mathfrak{t}^*$. Then the set of finite-dimensional G-irreducible representations is parameterized by $\Lambda^*_+ := \Lambda^* \cap \mathfrak{t}^*_+$.

Recall that $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{r}$, with $\mathfrak{r} = [\mathfrak{t}, \mathfrak{g}]$, and so $\mathfrak{g}^* = \mathfrak{t}^* \oplus \mathfrak{r}^*$. Thus we identify Λ^*_+ with a subset of $\mathfrak{g}^*$. For $\gamma \in \Lambda^*_+$, we denote by V^G_γ the irreducible G-representation with highest weight γ. The V^G_γ, $\gamma \in \Lambda^*_+$, form a $\mathbb{Z}$-basis of the representation ring $R(G)$. Let $R[G]$ be the formal representation ring of G. For $W \in R[G]$, we denote by $W_\gamma \in \mathbb{Z}$ the multiplicity of V^G_γ in W.

Take $\gamma \in \Lambda^*_+$. If γ is a regular value of the moment map μ, then one can construct the Marsden–Weinstein symplectic reduction $(M_\gamma, \omega_\gamma)$, with $M_\gamma = G \backslash \mu^{-1}(G \cdot \gamma)$ being a *compact* orbifold (since μ is proper). Moreover, the line bundle L (resp. the almost-complex structure J^M) induces a prequantum line bundle L_γ (resp. an almost-complex structure J_γ) over $(M_\gamma, \omega_\gamma)$. One can then construct the associated Spin^c-Dirac operator (twisted by L_γ) $D_+^{L_\gamma}: \Omega^{0,\mathrm{even}}(M_\gamma, L_\gamma) \to \Omega^{0,\mathrm{odd}}(M_\gamma, L_\gamma)$ (cf. (1.5) and §2) on M_γ, the index of which is defined by

$$Q(L_\gamma) = \mathrm{Ind}(D_+^{L_\gamma}) := \dim \mathrm{Ker}(D_+^{L_\gamma}) - \dim \mathrm{Coker}(D_+^{L_\gamma}) \in \mathbb{Z}. \tag{0.5}$$

If $\gamma \in \Lambda_+^*$ is not a regular value of μ, then by a perturbation argument (cf. [17] and [18, §7.4]), one still gets a well-defined quantization number $Q(L_\gamma)$ extending the above definition.

We equip $\mathfrak{g}$ with an Ad_G-invariant scalar product. We will identify $\mathfrak{g}$ and $\mathfrak{g}^*$ by this scalar product. Let $\pi: TM \to M$ denote the projection from TM to M. We identify T^*M with TM by the scalar product g^{TM}.

Set $\mathcal{H} = |\mu|^2$. Let $X^{\mathcal{H}} = -J^M(d\mathcal{H})$ be the Hamiltonian vector field associated with $\mathcal{H}$. Then (see (2.5))

$$X^{\mathcal{H}} = 2\mu^M, \tag{0.6}$$

where $\mu^M \in \mathscr{C}^\infty(M, TM)$ is the vector field on M generated by $\mu: M \to \mathfrak{g}$, i.e., for any $x \in M$, $\mu^M(x) = (\mu(x))^M(x)$.

For $a \geqslant 0$, set

$$M_a := \mathcal{H}^{-1}([0,a]) = \{x \in M : \mathcal{H}(x) \leqslant a\}.$$

For any regular value $a > 0$ of $\mathcal{H}$, by (0.6), μ^M does not vanish on $\partial M_a = \mathcal{H}^{-1}(a)$, the boundary of the compact G-manifold M_a. According to Atiyah [1, §1 and §3] and Paradan [18, §3] (cf. also Vergne [24]), the triple (M_a, μ^M, L) defines a transversally elliptic symbol

$$\sigma_{L,\mu}^{M_a} := \pi^*(ic(\cdot + \mu^M) \otimes \mathrm{Id}_L): \pi^*(\Lambda(T^{*(0,1)}M_a) \otimes L) \longrightarrow \pi^*(\Lambda(T^{*(0,1)}M_a) \otimes L),$$

where $c(\cdot)$ is the Clifford action on $\Lambda(T^{*(0,1)}M)$ (cf. (2.3)).(1) Let $\mathrm{Ind}(\sigma_{L,\mu}^{M_a}) \in R[G]$ denote the corresponding transversal index in the sense of Atiyah [1, §1].

Theorem 0.1. (a) *For $\gamma \in \Lambda_+^*$, there exists $a_\gamma \geqslant 0$(2) such that $\mathrm{Ind}(\sigma_{L,\mu}^{M_a})_\gamma \in \mathbb{Z}$ does not depend on the regular value $a > a_\gamma$ of $\mathcal{H}$.*

(b) *$\mathrm{Ind}(\sigma_{L,\mu}^{M_a})_{\gamma=0} \in \mathbb{Z}$ does not depend on the regular value $a > 0$ of $\mathcal{H}$.*

(1) The symbol $\sigma_{L,\mu}^{M_a}$ is the (semi-classical) symbol of Tian–Zhang's [22], [23] deformed Dirac operator (2.11) in their approach to the Guillemin–Sternberg geometric quantization conjecture [7]. The associated symbol was used by Paradan [18], [19] in his approach to the same conjecture.

(2) In view of Theorem 2.1, we can take $a_\gamma = c_\gamma / 4\pi^2$ with c_γ being defined in (2.8).

By Theorem 0.1, for $\gamma\in\Lambda^*_+$, we can associate an integer $Q(L)_\gamma$ that is equal to $\mathrm{Ind}(\sigma^{M_a}_{L,\mu})_\gamma$ for large enough regular values $a>0$ of $\mathcal{H}$.

We can now state the main result of this paper.

THEOREM 0.2. *For $\gamma\in\Lambda^*_+$, the following identity holds*:

$$Q(L)_\gamma = Q(L_\gamma). \tag{0.7}$$

Remark 0.3. When M is compact, Theorem 0.2 is the Guillemin–Sternberg geometric quantization conjecture [7] which was first proved by Meinrenken [15] and Vergne [24] in the case where G is abelian, and by Meinrenken [16] and Meinrenken–Sjamaar [17] in the general case. We refer to [25] for a survey on the Guillemin–Sternberg geometric quantization conjecture.

If M is non-compact but the zero set of $X^{\mathcal{H}}$ is compact, then Theorem 0.1 is already contained in [19] and [26], while Theorem 0.2 was conjectured by Michèle Vergne in her ICM 2006 plenary lecture [26, §4.3]. Special cases of this conjecture, related to the discrete series of semi-simple Lie groups, have been proved by Paradan [19], [20].

Theorem 0.2 provides a solution to Michèle Vergne's conjecture even when the zero set of $X^{\mathcal{H}}$ is non-compact.

Theorem 0.2 is a consequence of a more general result that we will now describe.

Let (N,ω^N,J^N) be a compact symplectic manifold with compatible almost-complex structure J^N. Let (F,h^F,∇^F) be the prequantum line bundle over N carrying a Hermitian metric h^F and a Hermitian connection ∇^F satisfying

$$\frac{i}{2\pi}(\nabla^F)^2 = \omega^N.$$

We assume that G acts on N and F as above. Let $\eta\colon N\to\mathfrak{g}^*$ be the associated moment map.

Let $D^F_+\colon\Omega^{0,\mathrm{even}}(N,F)\to\Omega^{0,\mathrm{odd}}(N,F)$ be the associated Spin^c-Dirac operator on N. Then as a virtual representation of G, we have

$$\mathrm{Ind}(\sigma^N_{F,\eta}) = \mathrm{Ind}(D^F_+) := \mathrm{Ker}(D^F_+) - \mathrm{Coker}(D^F_+) \in R(G). \tag{0.8}$$

For $\gamma\in\Lambda^*_+$, let $Q(F)_{\gamma,*}$ be the multiplicity of the G-irreducible representation $(V^G_\gamma)^*$ in $\mathrm{Ind}(D^F_+)\in R(G)$.

Let $L\otimes F$ be the prequantum line bundle over $M\times N$ obtained by the tensor product of the natural lifts of L and F to $M\times N$.

THEOREM 0.4. *For the induced action of G on $(M\times N,\omega\oplus\omega^N)$ and $L\otimes F$, the following identity holds*:

$$Q((L\otimes F)_{\gamma=0}) = \sum_{\gamma\in\Lambda^*_+} Q(L)_\gamma\cdot Q(F)_{\gamma,*}. \tag{0.9}$$

For $\gamma\in\Lambda^*_+$, denote by $\mathcal{O}_\gamma=G\cdot\gamma$ the orbit of the coadjoint action of G on $\mathfrak{g}^*$. Let L^γ be the canonical prequantum holomorphic line bundle on $\mathcal{O}_\gamma$, such that the associated moment map is the inclusion $\mathcal{O}_\gamma\hookrightarrow\mathfrak{g}^*$. By the Borel–Weil–Bott theorem and the solution of the Guillemin–Sternberg geometric quantization conjecture for the compact manifold $\mathcal{O}_{\nu_1}\times\mathcal{O}_{\nu_2}$, one has that $\mathrm{Hom}_G(V^G_{\nu_3}, V^G_{\nu_1}\otimes V^G_{\nu_2})\neq 0$ if and only if $\nu_3\in G\cdot\nu_1+G\cdot\nu_2$. In particular, one has $|\nu_1|\leqslant|\nu_3|+|\nu_2|$. For $\nu_1,\nu_2\in\Lambda^*_+$, set

$$C^\gamma_{\nu_1,\nu_2}=\dim \mathrm{Hom}_G(V^G_\gamma, V^G_{\nu_1}\otimes V^G_{\nu_2}). \tag{0.10}$$

By taking $N=\mathcal{O}_\gamma$ and $F=(L^\gamma)^*$, we recover Theorem 0.2 from Theorem 0.4 by using the Borel–Weil–Bott theorem.

By applying Theorems 0.2 and 0.4 to $M\times N\times\mathcal{O}_\gamma$, we get the following result which is trivial in the compact case.

Corollary 0.5. *For any $\gamma\in\Lambda^*_+$ the following identity holds*:

$$Q(L\otimes F)_\gamma=\sum_{\nu_1,\nu_2\in\Lambda^*_+} C^\gamma_{\nu_1,\nu_2} Q(L)_{\nu_1}\cdot Q(F)_{\nu_2}, \tag{0.11}$$

where there are only finitely many non-vanishing terms in the right-hand side.

We now explain briefly the main ideas of the proofs of Theorems 0.1 and 0.4.

The first observation is that in the case when $\gamma=0$, both Theorems 0.1 and 0.2 are relatively easy to prove. On the other hand, in the case when $\gamma\neq 0$, one needs to establish the more general Theorem 0.4, in order to prove (0.7).

In fact, it is relatively easy to see that (cf. (4.1) and (4.2))

$$Q(L\otimes F)_{\gamma=0}=Q((L\otimes F)_{\gamma=0}). \tag{0.12}$$

Thus Theorem 0.4 is a consequence of (0.12) and the identity

$$Q(L\otimes F)_{\gamma=0}=\sum_{\gamma\in\Lambda^*_+} Q(L)_\gamma\cdot Q(F)_{\gamma,*}. \tag{0.13}$$

Assume that M is compact. Then (0.13) is trivial and this is why one only needs to prove (0.7) for $\gamma=0$, in order to establish (0.7).

However, if M is non-compact, although the geometric data on $M\times N$ have product structure and the associated moment map is $\theta(x,y)=\mu(x)+\eta(y)$, the vector field $\theta^{M\times N}$ on $M\times N$ induced by θ is not a sum of two vector fields lifted from M and N (cf. (3.7)). Thus one cannot compute directly $Q(L\otimes F)_{\gamma=0}$ as the right-hand side of (0.13).

To be more precise, let $a>0$ be a regular value of $\mathcal{H}$ so that μ^M does not vanish on ∂M_a. By the multiplicativity of the transversal index,

$$\sum_{\gamma\in\Lambda_+^*}\operatorname{Ind}(\sigma_{L,\mu}^{M_a})_\gamma\cdot Q(F)_{\gamma,*}=\operatorname{Ind}(\sigma_{L\otimes F,\mu}^{M_a\times N})_{\gamma=0}. \tag{0.14}$$

Let $b>0$ be a regular value of $\mathcal{H}'=|\theta|^2$. Then $\theta^{M\times N}\in T(M\times N)$ does not vanish on the boundary $\partial(M\times N)_b$ of $(M\times N)_b=\{(x,y)\in M\times N:|\theta(x,y)|^2\leqslant b\}$. By Theorem 0.1 (b), we have

$$Q(L\otimes F)_{\gamma=0}=\operatorname{Ind}(\sigma_{L\otimes F,\theta}^{(M\times N)_b})_{\gamma=0}. \tag{0.15}$$

We take $b>0$ large enough so that

$$M_a\times N\subset(M\times N)_b \quad\text{and}\quad \partial(M\times N)_b\cap\partial(M_a\times N)=\varnothing.$$

Denote by $\mathcal{M}_{a,b}$ the closure of $(M\times N)_b\setminus M_a\times N$. Then $\mathcal{M}_{a,b}$ is a manifold with boundary $\partial\mathcal{M}_{a,b}=\partial(M\times N)_b\cup\partial(M_a\times N)$.

Let $\Psi_{a,b}:\mathcal{M}_{a,b}\to\mathfrak{g}$ be a G-equivariant map such that

$$(\Psi_{a,b})|_{\partial(M_a\times N)}=\mu, \quad \text{while } (\Psi_{a,b})|_{\partial(M\times N)_b}=\theta.$$

From the additivity of the transversal index, we get

$$\operatorname{Ind}(\sigma_{L\otimes F,\Psi_{a,b}}^{\mathcal{M}_{a,b}})_{\gamma=0}=\operatorname{Ind}(\sigma_{L\otimes F,\theta}^{(M\times N)_b})_{\gamma=0}-\operatorname{Ind}(\sigma_{L\otimes F,\mu}^{M_a\times N})_{\gamma=0}. \tag{0.16}$$

We infer from (0.13)–(0.16) that Theorem 0.4 is equivalent to

$$\operatorname{Ind}(\sigma_{L\otimes F,\Psi_{a,b}}^{\mathcal{M}_{a,b}})_{\gamma=0}=0. \tag{0.17}$$

Let $a_1>0$ be another large enough regular value of $\mathcal{H}$. By the additivity and the homotopy invariance of the transversal index, we have

$$\operatorname{Ind}(\sigma_{L\otimes F,\Psi_{a,b}}^{\mathcal{M}_{a,b}})_{\gamma=0}-\operatorname{Ind}(\sigma_{L\otimes F,\Psi_{a_1,b}}^{\mathcal{M}_{a_1,b}})_{\gamma=0}=\operatorname{Ind}(\sigma_{L\otimes F,\mu}^{\mathcal{M}_{a_1}\times N})_{\gamma=0}-\operatorname{Ind}(\sigma_{L\otimes F,\mu}^{\mathcal{M}_a\times N})_{\gamma=0}. \tag{0.18}$$

By (0.14) and (0.18), and by taking $N=\mathcal{O}_\gamma$ and $F=(L^\gamma)^*$ for $\gamma\in\Lambda_+^*$, we find that Theorem 0.1 (a) is a consequence of the vanishing result (0.17).

Note that in the situations considered in [19] and [20], for $a,b>0$ large enough, one is able to find $\Psi_{a,b}:\mathcal{M}_{a,b}\to\mathfrak{g}$ such that $\Psi_{a,b}^{\mathcal{M}_{a,b}}\in T\mathcal{M}_{a,b}$ does not vanish on $\mathcal{M}_{a,b}$. From this, (0.17) follows tautologically. However, there is no canonical way to construct $\Psi_{a,b}$ such that $\Psi_{a,b}^{\mathcal{M}_{a,b}}\in T\mathcal{M}_{a,b}$ does not vanish on $\mathcal{M}_{a,b}$ in the general situation considered here.

Our proof of (0.17) consists of two steps. In the first step, we express the transversal index as an Atiyah–Patodi–Singer (APS) type index on corresponding manifolds with boundary. Then in a second step, we construct a specific deformation map $\Psi_{a,b}$, when a, $b>0$ are large enough, so that we can apply the analytic localization techniques developed in [3], [22] and [23] to the current problem. This allows us to show that the APS type index corresponding to the left-hand side of (0.17) vanishes.([3])

This paper is organized as follows. In §1, we express the transversal index as an APS type index. In §2, we establish Theorem 0.1, by applying the identification of the transversal index to an APS index that was established in §1, as well as the analytic localization techniques developed in [3], [22] and [23]. In §3, we present our proof of (0.17). Finally, in §4, we provide details of the proofs of (0.12) and (0.14), thus completing the proof of Theorem 0.4. We explain also the compatibility of quantization and its restriction to a subgroup.

The results of this paper, the first version of which was put on the arXiv as [12], have been announced in [13] (cf. also [10, §4]).

0.1. Notation

In the whole paper, G is a compact connected Lie group with Lie algebra $\mathfrak{g}$. Let Ad_g denote the adjoint action of $g\in G$ on $\mathfrak{g}$. We equip $\mathfrak{g}$ with an Ad_G-invariant scalar product, and we identify $\mathfrak{g}$ and $\mathfrak{g}^*$ by this scalar product. Let $V_1, ..., V_{\dim G}$ be an orthonormal basis of $\mathfrak{g}$.

If a Hilbert space H is a G-unitary representation space, by the Peter–Weyl theorem, one has the orthogonal decomposition of Hilbert spaces

$$H=\bigoplus_{\gamma\in\Lambda^*_+} H^\gamma, \quad \text{with } H^\gamma=\mathrm{Hom}_G(V^G_\gamma, H)\otimes V^G_\gamma. \tag{0.19}$$

We will call H^γ the *γ-component* of H. Moreover, if $W\subset H$ is a G-invariant linear subspace, for $\gamma\in\Lambda^*_+$, we set

$$W^\gamma=W\cap H^\gamma \tag{0.20}$$

and call it the *γ-component* of W. If $D\colon \mathrm{Dom}(D)\subset H\to H$ is a G-equivariant linear operator, where $\mathrm{Dom}(D)$ is a dense G-invariant subspace of H, for $\gamma\in\Lambda^*_+$ we denote by $D(\gamma)$ the restriction of D to $\mathrm{Dom}(D)^\gamma$ which is dense in H^γ.

If G acts on the left on a manifold $\mathbf{M}$, for $K\in\mathfrak{g}$ we denote by

$$K^{\mathbf{M}}(x)=\frac{\partial}{\partial t}e^{tK}x\bigg|_{t=0}$$

([3]) In fact, the corresponding vanishing result for the APS index, in the case when N is a point and $\eta=0$, has already been proved in [23, Theorems 2.6 and 4.3].

the corresponding vector field on $\mathbf{M}$.

For any $\Phi\in\mathscr{C}^\infty(\mathbf{M},\mathfrak{g})$, we denote by Φ_j, $1\leqslant j\leqslant \dim G$, the smooth functions on $\mathbf{M}$ defined by

$$\Phi(x)=\sum_{j=1}^{\dim G}\Phi_j(x)V_j \quad \text{for } x\in\mathbf{M}. \tag{0.21}$$

Let $\Phi^{\mathbf{M}}$ denote the vector field over $\mathbf{M}$ such that, for any $x\in\mathbf{M}$,

$$\Phi^{\mathbf{M}}(x)=(\Phi(x))^{\mathbf{M}}(x)=\sum_{j=1}^{\dim G}\Phi_j(x)V_j^{\mathbf{M}}(x), \tag{0.22}$$

where $(\Phi(x))^{\mathbf{M}}$ is the vector field over $\mathbf{M}$ generated by $\Phi(x)\in\mathfrak{g}$.

Finally, when a subscript index appears two times in a formula, we sum up with this index unless other notification is given.

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1. Transversal index and APS index

In this section we express the transversal index as an Atiyah–Patodi–Singer (APS) type index which has been studied in [23] for the $\gamma=0$ component.

This section is organized as follows. In §1.1, we recall the definition of the transversal index in the sense of Atiyah [1] for manifolds with boundary. In §1.2, we consider instead an index problem on a manifold with boundary for a Dirac operator with APS boundary conditions. In §1.3, we prove that the corresponding Dirac operator on the boundary is invertible. This guarantees that the APS index of the Dirac operator is invariant under deformation. In §1.4, we show that the transversal index can be identified with the APS index by using a result by Braverman [4].

We use the same notation as in the introduction.

1.1. Transversal index

Let M be an even-dimensional compact oriented Spin^c-manifold with non-empty boundary ∂M and $\dim M = n$. In the following, the boundary ∂M carries the induced orientation. Let g^{TM} be a Riemannian metric on the tangent vector bundle $\pi: TM \to M$. Let E be a complex vector bundle over M.

We assume that the compact connected Lie group G acts isometrically on the left on M, and that this action lifts to an action of G on the Spin^c-principal bundle of TM and on E. Then the G-action also preserves ∂M.

We identify TM and T^*M by the G-invariant metric g^{TM}. Following [1, p. 7] (cf. [18, §3]), set

$$T_G M = \{(x, v) \in T_x M : x \in M \text{ and } \langle v, K^M(x) \rangle = 0 \text{ for all } K \in \mathfrak{g}\}. \tag{1.1}$$

Let $S(TM) = S_+(TM) \oplus S_-(TM)$ be the vector bundle of spinors associated with the spin^c-structure on TM and g^{TM}. For any $V \in TM$, the Clifford action $c(V)$ exchanges $S_+(TM)$ and $S_-(TM)$.

Let $\Psi: M \to \mathfrak{g}$ be a G-equivariant smooth map. Assume that Ψ^M does not vanish on ∂M, i.e., $\Psi^M(x) \neq 0$ for any $x \in \partial M$.

Let $\sigma^M_{E,\Psi} \in \mathrm{Hom}(\pi^*(S_+(TM) \otimes E), \pi^*(S_-(TM) \otimes E))$ be the symbol

$$\sigma^M_{E,\Psi}(x, v) = \pi^*(ic(v + \Psi^M) \otimes \mathrm{Id}_E)|_{(x,v)} \quad \text{for } x \in M \text{ and } v \in T_x M. \tag{1.2}$$

Since Ψ^M does not vanish on ∂M, the set $\{(x, v) \in T_G M : \sigma^M_{E,\Psi}(x, v) \text{ is non-invertible}\}$ is a compact subset of $T_G \widehat{M}$ (where $\widehat{M} = M \setminus \partial M$ is the interior of M), so that $\sigma^M_{E,\Psi}$ is a G-transversally elliptic symbol on $T_G \widehat{M}$ in the sense of Atiyah [1, §1 and §3] and Paradan [18, §3], [19, §3]. The associated transversal index can be written in the form

$$\mathrm{Ind}(\sigma^M_{E,\Psi}) = \bigoplus_{\gamma \in \Lambda^*_+} \mathrm{Ind}(\sigma^M_{E,\Psi})_\gamma \cdot V^G_\gamma \in R[G], \tag{1.3}$$

with each $\mathrm{Ind}(\sigma^M_{E,\Psi})_\gamma \in \mathbb{Z}$. Moreover, $\mathrm{Ind}(\sigma^M_{E,\Psi})$ only depends on the homotopy class of Ψ as long as Ψ^M does not vanish on ∂M, but does not depend on g^{TM}. Note that the number of $\gamma \in \Lambda^*_+$ such that $\mathrm{Ind}(\sigma^M_{E,\Psi})_\gamma \neq 0$ could be infinite.

1.2. The Atiyah–Patodi–Singer (APS) index

We make the same assumptions and use the same notation as in §1.1.

Let h^E be a G-invariant Hermitian metric on E and ∇^E be a G-invariant Hermitian connection on E with respect to h^E. Let $h^{S(TM)}$ be the G-invariant Hermitian metric on

$S(TM)$ induced by g^{TM} and a G-invariant metric on the line bundle defining the spin^c structure (cf. [9, Appendix D]). Let $h^{S(TM)\otimes E}$ be the metric on $S(TM)\otimes E$ induced by the metrics on $S(TM)$ and E.

Let $\nabla^{S(TM)}$ be the Clifford connection on $S(TM)$ induced by the Levi-Civita connection ∇^{TM} of g^{TM} and a G-invariant Hermitian connection on the line bundle defining the spin^c structure (cf. [9, Appendix D]). Let $\nabla^{S(TM)\otimes E}$ be the Hermitian connection on $S(TM)\otimes E$ induced by $\nabla^{S(TM)}$ and ∇^E.

Let dv_M denote the Riemannian volume form on (M, g^{TM}). The L^2-norm $\|s\|_0$ of $s\in\mathscr{C}^\infty(M, S(TM)\otimes E)$ is defined by

$$\|s\|_0^2 = \int_M |s(x)|^2 \, dv_M(x). \tag{1.4}$$

Let $\langle\cdot,\cdot\rangle$ denote the Hermitian product on $\mathscr{C}^\infty(M, S(TM)\otimes E)$ corresponding to $\|\cdot\|_0^2$, and let $L^2(M, S(TM)\otimes E)$ be the space of L^2-sections of $S(TM)\otimes E$ on M.

Let D_M^E be the Spin^c-Dirac operator defined by (cf. [9, Appendix D])

$$D_M^E = \sum_{j=1}^n c(e_j)\nabla_{e_j}^{S(TM)\otimes E} : \mathscr{C}^\infty(M, S(TM)\otimes E) \longrightarrow \mathscr{C}^\infty(M, S(TM)\otimes E), \tag{1.5}$$

where $\{e_j\}_{j=1}^n$ is an oriented orthonormal frame of TM.

Let $\varepsilon>0$ be less than the injectivity radius of g^{TM}. We use the inward geodesic flow to identify a neighborhood of the boundary ∂M with the collar $\partial M\times[0,\varepsilon]$, and we identify $\partial M\times\{0\}$ with the boundary ∂M.

Let e_n be the inward unit normal vector field perpendicular to ∂M. Let $e_1, ..., e_{n-1}$ be an oriented orthonormal frame of $T\partial M$ so that $\{e_j\}_{j=1}^n$ is an oriented orthonormal frame of $TM|_{\partial M}$. By using parallel transport with respect to ∇^{TM} along the unit speed geodesics perpendicular to ∂M, $e_1, ..., e_n$ give rise to an oriented orthonormal frame of TM over $\partial M\times[0,\varepsilon]$.

The operator D_M^E induces a Dirac operator on ∂M,

$$D_{\partial M}^E : \mathscr{C}^\infty(\partial M, (S(TM)\otimes E)|_{\partial M}) \longrightarrow \mathscr{C}^\infty(\partial M, (S(TM)\otimes E)|_{\partial M})$$

defined by (cf. [6, p. 142])

$$D_{\partial M}^E = -\sum_{j=1}^{n-1} c(e_n)c(e_j)\nabla_{e_j}^{S(TM)\otimes E} + \frac{1}{2}\sum_{j=1}^{n-1}\pi_{jj}, \tag{1.6}$$

where

$$\pi_{jk} = \langle \nabla_{e_j}^{TM} e_k, e_n\rangle|_{\partial M}, \quad 1\leqslant j,k\leqslant n-1, \tag{1.7}$$

is the second fundamental form of the isometric embedding $\imath_{\partial M}: \partial M \hookrightarrow M$. Let $D^E_{\partial M,\pm}$ be the restrictions of $D^E_{\partial M}$ to $\mathscr{C}^\infty(\partial M, (S_\pm(TM)\otimes E)|_{\partial M})$.

As in (1.4), we define the Riemannian volume form $dv_{\partial M}$ on ∂M, the Hermitian product $\langle\cdot,\cdot\rangle_{\partial M,0}$ and the L^2-norm $\|\cdot\|_{\partial M,0}$ on $\mathscr{C}^\infty(\partial M, (S(TM)\otimes E)|_{\partial M})$.

By [6, Lemma 2.2], $D^E_{\partial M}$ is a self-adjoint first-order elliptic differential operator defined on ∂M. Moreover, the following identity holds on ∂M:

$$D^E_{\partial M,\pm} = c(e_n)^{-1}(-D^E_{\partial M,\mp})c(e_n). \tag{1.8}$$

Since the G-action preserves ∂M, the restriction of Ψ^M to ∂M is a section of $T\partial M$, i.e.,

$$\Psi^M|_{\partial M} \in \mathscr{C}^\infty(\partial M, T\partial M). \tag{1.9}$$

For $T\in\mathbb{R}$, set

$$\begin{aligned} D^E_{M,T} &= D^E_M + iTc(\Psi^M),\\ D^E_{M,\pm,T} &= D^E_{M,T}|_{\mathscr{C}^\infty(M,S_\pm(TM)\otimes E)}, \end{aligned} \tag{1.10}$$

and

$$\begin{aligned} D^E_{\partial M,T} &= D^E_{\partial M} - iTc(e_n)c(\Psi^M),\\ D^E_{\partial M,\pm,T} &= D^E_{\partial M,T}|_{\mathscr{C}^\infty(\partial M,(S_\pm(TM)\otimes E)|_{\partial M})}. \end{aligned} \tag{1.11}$$

Then $D^E_{M,T}$ exchanges the spaces associated with $S_+(TM)\otimes E$ and $S_-(TM)\otimes E$, and by (1.9), $D^E_{\partial M,T}$ is self-adjoint and preserves $\mathscr{C}^\infty(\partial M, (S_\pm(TM)\otimes E)|_{\partial M})$.

Let $\operatorname{Spec}(D^E_{\partial M,\pm,T})$ be the spectrum of $D^E_{\partial M,\pm,T}$. For $\lambda\in\operatorname{Spec}(D^E_{\partial M,\pm,T})$, let $E_{\lambda,\pm,T}$ be the corresponding eigenspace. Let $P_{\geqslant 0,\pm,T}$ (resp. $P_{>0,\pm,T}$) be the orthogonal projection from $L^2(\partial M, (S_\pm(TM)\otimes E)|_{\partial M})$ onto $\bigoplus_{\lambda\geqslant 0} E_{\lambda,\pm,T}$ (resp. $\bigoplus_{\lambda>0} E_{\lambda,\pm,T}$). We will call $P_{\geqslant 0,+,T}$ (resp. $P_{>0,-,T}$) the APS projection associated with $D^E_{\partial M,+,T}$ (resp. $D^E_{\partial M,-,T}$).

For $T\in\mathbb{R}$, let $(D^E_{M,+,T}, P_{\geqslant 0,+,T})$ (resp. $(D^E_{M,-,T}, P_{>0,-,T})$) denote the corresponding operator with the APS boundary condition [2]. More precisely, the boundary condition of $D^E_{M,+,T}$ (resp. $D^E_{M,-,T}$) is $P_{\geqslant 0,+,T}(s|_{\partial M})=0$ for $s\in\mathscr{C}^\infty(M, S_+(TM)\otimes E)$ (resp. $P_{>0,-,T}(s|_{\partial M})=0$ for $s\in\mathscr{C}^\infty(M, S_-(TM)\otimes E)$).

Both $(D^E_{M,+,T}, P_{\geqslant 0,+,T})$ and $(D^E_{M,-,T}, P_{>0,-,T})$ are elliptic, and $(D^E_{M,-,T}, P_{>0,-,T})$ is the adjoint of $(D^E_{M,+,T}, P_{\geqslant 0,+,T})$ (cf. (1.8) and [6, Theorem 2.3]). In particular, they are Fredholm operators and they commute with the G-action.

Let $Q^M_{\mathrm{APS},T}(E,\Psi)_\gamma\in\mathbb{Z}$, $\gamma\in\Lambda^*_+$, be defined by

$$\begin{aligned} \bigoplus_{\gamma\in\Lambda^*_+} Q^M_{\mathrm{APS},T}(E,\Psi)_\gamma\cdot V^G_\gamma &= \operatorname{Ind}(D^E_{M,+,T}, P_{\geqslant 0,+,T})\\ &:= \operatorname{Ker}(D^E_{M,+,T}, P_{\geqslant 0,+,T}) - \operatorname{Ker}(D^E_{M,-,T}, P_{>0,-,T}) \in R(G). \end{aligned} \tag{1.12}$$

1.3. An invariance property of the APS index

PROPOSITION 1.1. *For $\gamma\in\Lambda_+^*$, there exist $C_\gamma>0$ and $T_\gamma\geqslant 0$ such that, for $T>T_\gamma$ and $s\in\mathscr{C}^\infty(\partial M,(S(TM)\otimes E)|_{\partial M})^\gamma$, we have*

$$\|D^E_{\partial M,T}s\|^2_{\partial M,0}\geqslant\|D^E_{\partial M}s\|^2_{\partial M,0}+C_\gamma T^2\|s\|^2_{\partial M,0}. \tag{1.13}$$

In particular, $D^E_{\partial M,T}(\gamma)$ is invertible.

Proof. From (1.6), (1.9) and (1.11), we get

$$\begin{aligned}(D^E_{\partial M,T})^2=(D^E_{\partial M})^2-iT\sum_{j=1}^{n-1}\pi_{jj}c(e_n)c(\Psi^M)\\ +iT\sum_{j=1}^{n-1}c(e_n)c(e_j)(\nabla^{S(TM)\otimes E}_{e_j}(c(e_n)c(\Psi^M)))\\ -2iT\nabla^{S(TM)\otimes E}_{\Psi^M}+T^2|\Psi^M|^2.\end{aligned} \tag{1.14}$$

For any $K\in\mathfrak{g}$, let L_K denote the Lie derivative of K acting on $\mathscr{C}^\infty(M,S(TM)\otimes E)$ and thus also on $\mathscr{C}^\infty(\partial M,(S(TM)\otimes E)|_{\partial M})$. Then

$$\mu^{S(TM)\otimes E}(K):=\nabla^{S(TM)\otimes E}_{K^M}-L_K\in\mathscr{C}^\infty(M,\mathrm{End}(S(TM)\otimes E)). \tag{1.15}$$

By (0.21) and (0.22), we have

$$\nabla^{S(TM)\otimes E}_{\Psi^M}=\sum_{j=1}^{\dim G}\Psi_jL_{V_j}+\sum_{j=1}^{\dim G}\Psi_j(\nabla^{S(TM)\otimes E}_{V_j^M}-L_{V_j}). \tag{1.16}$$

In view of (0.19), it is clear that each L_{V_j}, $1\leqslant j\leqslant\dim G$, acts as a bounded operator on $L^2(\partial M,(S(TM)\otimes E)|_{\partial M})^\gamma$.

On the other hand, since Ψ^M does not vanish on ∂M, there exists $C>0$ such that

$$|\Psi^M|^2\geqslant 4C\quad\text{on }\partial M. \tag{1.17}$$

We deduce from (1.14)–(1.17) that there exists $C'_\gamma>0$ such that, for any

$$s\in\mathscr{C}^\infty(\partial M,(S(TM)\otimes E)|_{\partial M})^\gamma,$$

we have

$$\|D^E_{\partial M,T}s\|^2_{\partial M,0}\geqslant\|D^E_{\partial M}s\|^2_{\partial M,0}-TC'_\gamma\|s\|^2_{\partial M,0}+4T^2C\|s\|^2_{\partial M,0}. \tag{1.18}$$

The inequality (1.18) implies that Proposition 1.1 holds with $T_\gamma=2C'_\gamma/C$. □

PROPOSITION 1.2. *For* $\gamma\in\Lambda_+^*$, *there exists* $T_\gamma\geqslant 0$ *such that* $Q^M_{\mathrm{APS},T}(E,\Psi)_\gamma$ *does not depend on* $T>T_\gamma$.

Proof. For $\gamma\in\Lambda_+^*$, let $(D^E_{M,+,T}(\gamma), P_{\geqslant 0,+,T}(\gamma))$ denote the corresponding operator with the APS boundary condition [2], which is just the restriction of $(D^E_{M,+,T}, P_{\geqslant 0,+,T})$ to the corresponding γ-component. Thus, $(D^E_{M,+,T}(\gamma), P_{\geqslant 0,+,T}(\gamma))$ is elliptic and defines a Fredholm operator, the index of which is given by (1.12),

$$\mathrm{Ind}(D^E_{M,+,T}(\gamma), P_{\geqslant 0,+,T}(\gamma)) = Q^M_{\mathrm{APS},T}(E,\Psi)_\gamma\cdot V^G_\gamma. \tag{1.19}$$

By Proposition 1.1, there exists $T_\gamma\geqslant 0$ such that $(D^E_{M,+,T}(\gamma), P_{\geqslant 0,+,T}(\gamma))$ forms a continuous family of Fredholm operators for $T>T_\gamma$. Therefore, $\mathrm{Ind}(D^E_{M,+,T}(\gamma), P_{\geqslant 0,+,T}(\gamma))$ does not depend on $T>T_\gamma$. By (1.19), this completes the proof of our proposition. □

Definition 1.3. By Proposition 1.2, with every $\gamma\in\Lambda_+^*$ we may associate an integer $Q^M_{\mathrm{APS}}(E,\Psi)_\gamma$ that is equal to

$$Q^M_{\mathrm{APS},T}(E,\Psi)_\gamma \quad \text{for } T>T_\gamma.$$

Remark 1.4. The same argument shows that the APS type index $Q^M_{\mathrm{APS}}(E,\Psi)_\gamma$ does not depend on the given metrics and connections. It only depends on the homotopy class of Ψ as long as $\Psi^M|_{\partial M}$ does not vanish over ∂M.

1.4. Transversal index and APS index

THEOREM 1.5. *For* $\gamma\in\Lambda_+^*$, *the following identity holds*:

$$\mathrm{Ind}(\sigma^M_{E,\Psi})_\gamma = Q^M_{\mathrm{APS}}(E,\Psi)_\gamma. \tag{1.20}$$

The proof of Theorem 1.5 consists of two steps. In the first step, by applying a result of Braverman [4, Theorem 5.5], we express $\mathrm{Ind}(\sigma^M_{E,\Psi})_\gamma$ as the L^2-index of a Dirac operator on $\widetilde{M}=M\cup(\partial M\times(-\infty,0])$, and show that the difference of the above L^2-index and $Q^M_{\mathrm{APS}}(E,\Psi)_\gamma$ is equal to an index on the cylindrical end. In the second step, we prove that the index on the cylindrical end is zero.

We start by deforming our geometric data to those on a manifold with cylindrical end.

Recall that $g^{T\partial M}$ is the Riemannian metric on ∂M induced by g^{TM}. We use the inward geodesic flow to identify a neighborhood of ∂M with the collar $\partial M\times[0,\varepsilon]$. As g^{TM} is G-invariant, the G-action on $\partial M\times[0,\varepsilon]$ is induced by the G-action on ∂M, and there exists a family of metrics $g^{T\partial M}(x_n)$ on $T\partial M$ satisfying

$$g^{TM}_{(y,x_n)} = g^{T\partial M}_y(x_n)+(dx_n)^2, \quad (y,x_n)\in\partial M\times[0,\varepsilon]. \tag{1.21}$$

For $(y,x_n)\in\partial M\times[0,\varepsilon]$, we identify $S(TM)_{(y,x_n)}$ and $E_{(y,x_n)}$ with $S(TM)_{(y,0)}$ and $E_{(y,0)}$ by using the parallel transport with respect to $\nabla^{S(TM)}$ and ∇^E along the geodesic $[0,1]\ni t\mapsto(y,tx_n)$. Thus, the restriction of $(S(TM),h^{S(TM)})$ (resp. (E,h^E)) to $\partial M\times[0,\varepsilon]$ is the pull-back of the restrictions $S(TM)|_{\partial M}$ and $h^{S(TM)}|_{\partial M}$ (resp. $E|_{\partial M}$ and $h^E|_{\partial M}$) to ∂M. Moreover, the G-actions on $S(TM)$ and E on $\partial M\times[0,\varepsilon]$ are induced by the G-actions on $S(TM)|_{\partial M}$ and $E|_{\partial M}$ under this identification.

By the homotopy invariance of the transversal index $\operatorname{Ind}(\sigma^M_{E,\Psi})_\gamma$ (cf. (1.3)) and of the APS index $Q^M_{\mathrm{APS}}(E,\Psi)_\gamma$ (cf. Remark 1.4), to establish Theorem 1.5 we may and will assume that $\varepsilon=2$, that g^{TM}, $h^{S(TM)}$, $\nabla^{S(TM)}$, ∇^E and Ψ have product structures on $\partial M\times[0,2]$, and that the G-actions on objects such as E and $S(TM)$ on $\partial M\times[0,2]$ are the product of the G-actions on their restrictions to ∂M and the identity in the direction $[0,2]$.

We now attach an infinite cylinder $\partial M\times(-\infty,0]$ to M along the boundary ∂M and extend trivially all objects on M to $\widetilde M=M\cup(\partial M\times(-\infty,0])$. We decorate the extended objects on $\widetilde M$ by a tilde. Thus, for $(y,x_n)\in\partial M\times(-\infty,2]$, we have

$$\begin{aligned}\tilde\Psi(y,x_n)&=\Psi(y,0)\in\mathfrak{g},\\ g^{T\widetilde M}_{(y,x_n)}&=g^{T\partial M}_y+(dx_n)^2,\\ (S(T\widetilde M),h^{S(T\widetilde M)},\nabla^{S(T\widetilde M)})|_{\partial M\times(-\infty,2]}&=\pi_1^*(S(TM)|_{\partial M},h^{S(TM)}|_{\partial M},\nabla^{S(TM)}|_{\partial M}),\\ (\widetilde E,h^{\widetilde E},\nabla^{\widetilde E})|_{\partial M\times(-\infty,2]}&=\pi_1^*(E|_{\partial M},h^E|_{\partial M},\nabla^E|_{\partial M}),\end{aligned}\tag{1.22}$$

where $\pi_1\colon\partial M\times(-\infty,2]\to\partial M$ is the natural projection.

Let $D^{\widetilde E}_{\widetilde M}$ be the Spin^c-Dirac operator on $\mathscr{C}^\infty_0(\widetilde M,S(T\widetilde M)\otimes\widetilde E)$ defined as in (1.5). By (1.5), (1.6) and (1.22), on $\partial M\times(-\infty,2]$ we have

$$D^{\widetilde E}_{\widetilde M}=c(e_n)D^E_{\partial M}+c(e_n)\frac{\partial}{\partial x_n}.\tag{1.23}$$

For any $h\in\mathscr{C}^\infty(\widetilde M)$, let $D^{\widetilde E}_{\widetilde M,h}$ be the operator on $\mathscr{C}^\infty_0(\widetilde M,S(T\widetilde M)\otimes\widetilde E)$ defined by

$$D^{\widetilde E}_{\widetilde M,h}=D^{\widetilde E}_{\widetilde M}+ihc(\tilde\Psi^{\widetilde M}).\tag{1.24}$$

Let $\mathbf{H}^1_{\pm,h}(\widetilde M)$ be the Sobolev space obtained by completion of $\mathscr{C}^\infty_0(\widetilde M,S_\pm(T\widetilde M)\otimes\widetilde E)$ under the norm $\|\cdot\|_{h,1}$ defined by

$$\|s\|^2_{h,1}=\|s\|^2_0+\|D^{\widetilde E}_{\widetilde M,h}s\|^2_0.\tag{1.25}$$

Let f be a strictly positive G-invariant smooth function on $\widetilde M$ such that $f|_M\equiv1$, and such that, for $(y,x_n)\in\partial M\times(-\infty,0]$,

$$f(y,x_n)\text{ does not depend on }y\text{ for }x_n\leqslant0\quad\text{and}\quad f(y,x_n)=e^{-x_n}\text{ for }x_n\leqslant-1.\tag{1.26}$$

For $T>0$, Tf is an admissible function on $\widetilde{M}$ for the triple $(S(T\widetilde{M})\otimes\widetilde{E}, \nabla^{S(T\widetilde{M})\otimes\widetilde{E}}, \tilde{\Psi})$ in the sense of Braverman [4, Definition 2.6] as we are in the product case.

By a result of Braverman [4, Theorem 5.5] (cf. also [14]), for $T>0$ and $\gamma\in\Lambda^*_+$, $D^{\widetilde{E}}_{\widetilde{M},Tf}(\gamma)$ and $D^{\widetilde{E}}_{\widetilde{M},\pm,Tf}(\gamma)$ extend to bounded Fredholm operators, for which we keep the same notation,

$$D^{\widetilde{E}}_{\widetilde{M},\pm,Tf}(\gamma)\colon \mathbf{H}^1_{\pm,Tf}(\widetilde{M})^\gamma \longrightarrow L^2(\widetilde{M}, S_\mp(T\widetilde{M})\otimes\widetilde{E})^\gamma, \tag{1.27}$$

and the following identity holds:

$$\operatorname{Ind}(D^{\widetilde{E}}_{\widetilde{M},+,Tf}(\gamma)) = \operatorname{Ind}(\sigma^M_{E,\Psi})_\gamma \cdot V^G_\gamma. \tag{1.28}$$

Set

$$\begin{aligned} \widetilde{M}_1 &= \partial M\times(-\infty,1]\subset\widetilde{M},\\ \widetilde{M}_2 &= \partial M\times(-\infty,2]\subset\widetilde{M},\\ Z &= \partial M\times[0,2]\subset\widetilde{M}. \end{aligned} \tag{1.29}$$

Let $\xi\in\mathscr{C}^\infty([0,2])$ be such that

$$\xi|_{[0,1/2]}=1,\quad 0\leqslant \xi|_{[1/2,3/2]}\leqslant 1\quad\text{and}\quad \xi|_{[3/2,2]}=0, \tag{1.30}$$

and such that

$$\varphi = (1-\xi^2)^{1/2} \tag{1.31}$$

is smooth. Clearly, ξ extends to $\widetilde{M}_2$ by setting $\xi=1$ on $\widetilde{M}_0=\widetilde{M}_1\setminus(\partial M\times(0,1])$. It also extends to M by setting $\xi=0$ on $M\setminus(\partial M\times[0,2))$. Thus φ also extends to $\widetilde{M}_0$ and $M\setminus(\partial M\times[0,2))$. Set

$$\begin{aligned} H &= L^2(\widetilde{M}, S(T\widetilde{M})\otimes\widetilde{E})\oplus L^2(Z, (S(TM)\otimes E)|_Z),\\ H' &= L^2(\widetilde{M}_2, (S(T\widetilde{M})\otimes\widetilde{E})|_{\widetilde{M}_2})\oplus L^2(M, S(TM)\otimes E). \end{aligned} \tag{1.32}$$

Let $U\colon H\to H'$ be defined by

$$(s_1,s_2)\in H\longmapsto (\xi s_1-\varphi s_2, \varphi s_1+\xi s_2)\in H'. \tag{1.33}$$

Let $U^*\colon H'\to H$ be the adjoint of U. By (1.33), $U^*(s_1,s_2)=(\xi s_1+\varphi s_2, -\varphi s_1+\xi s_2)\in H$. On easily sees that U is unitary (cf. [5, §3.2]), that is

$$U^*U=\operatorname{Id}_H\quad\text{and}\quad UU^*=\operatorname{Id}_{H'}. \tag{1.34}$$

Fix $\gamma\in\Lambda_+^*$ and let $T>0$. If W is one of $\widetilde{M}_2$, M and Z, let $(D^{\widetilde{E}}_{W,+,Tf}(\gamma), P^W_{\geqslant 0,+,Tf}(\gamma))$ be the operator with the APS boundary condition

$$\begin{aligned}&\mathbf{H}^1_{+,Tf}(W, P^W_{\geqslant 0,+,Tf})^\gamma\\ &\quad=\{u\in\mathbf{H}^1_{+,Tf}(W)^\gamma : P^W_{\geqslant 0,+,Tf}(\gamma)(u|_{\partial W})=0\}\longrightarrow L^2(W,(S_-(T\widetilde{M})\otimes\widetilde{E})|_W)^\gamma.\end{aligned}\tag{1.35}$$

Since $f=1$ on M and Z, we know that, for $W=M$ and $W=Z$,

$$(D^{\widetilde{E}}_{W,+,Tf}(\gamma), P^W_{\geqslant 0,+,Tf}(\gamma))=(D^E_{W,+,T}(\gamma), P_{\geqslant 0,+,T}(\gamma)),\tag{1.36}$$

and they are Fredholm as explained in §1.2.

By (1.27), (1.34) and (1.36), we see that

$$\begin{aligned}&U[D^{\widetilde{E}}_{\widetilde{M},+,Tf}(\gamma)+(D^E_{Z,+,T}(\gamma), P_{\geqslant 0,+,T}(\gamma))]U^*:\\ &\quad\mathbf{H}^1_{+,Tf}(\widetilde{M}_2, P^{\widetilde{M}_2}_{\geqslant 0,+,Tf})^\gamma\oplus\mathbf{H}^1_{+,Tf}(M, P^M_{\geqslant 0,+,Tf})^\gamma\\ &\qquad\longrightarrow L^2(\widetilde{M}_2,(S_-(T\widetilde{M})\otimes\widetilde{E})|_{\widetilde{M}_2})^\gamma\oplus L^2(M, S_-(TM)\otimes E)^\gamma\end{aligned}\tag{1.37}$$

is Fredholm.

By the construction of U, it is clear that U preserves the APS boundary conditions on the corresponding boundary components. Moreover, the difference

$$\begin{aligned}&U[D^{\widetilde{E}}_{\widetilde{M},+,Tf}(\gamma)+(D^E_{Z,+,T}(\gamma), P_{\geqslant 0,+,T}(\gamma))]U^*\\ &\quad-(D^{\widetilde{E}}_{\widetilde{M}_2,+,Tf}(\gamma), P^{\widetilde{M}_2}_{\geqslant 0,+,Tf}(\gamma))-(D^E_{M,+,T}(\gamma), P_{\geqslant 0,+,T}(\gamma))\end{aligned}\tag{1.38}$$

is a zeroth-order differential operator with compact support,(4) which implies that it is a compact operator. Thus,

$$(D^{\widetilde{E}}_{\widetilde{M}_2,+,Tf}(\gamma), P^{\widetilde{M}_2}_{\geqslant 0,+,Tf}(\gamma))+(D^E_{M,+,T}(\gamma), P_{\geqslant 0,+,T}(\gamma))$$

is Fredholm. In particular, $(D^{\widetilde{E}}_{\widetilde{M}_2,+,Tf}, P^{\widetilde{M}_2}_{\geqslant 0,+,Tf}(\gamma))$ is Fredholm. Moreover, we have

$$\begin{aligned}&\operatorname{Ind}(D^{\widetilde{E}}_{\widetilde{M},+,Tf}(\gamma))+\operatorname{Ind}(D^E_{Z,+,T}(\gamma), P_{\geqslant 0,+,T}(\gamma))\\ &\quad=\operatorname{Ind}(D^{\widetilde{E}}_{\widetilde{M}_2,+,Tf}(\gamma), P^{\widetilde{M}_2}_{\geqslant 0,+,Tf}(\gamma))+\operatorname{Ind}(D^E_{M,+,T}(\gamma), P_{\geqslant 0,+,T}(\gamma)).\end{aligned}\tag{1.39}$$

Note that $\partial Z=(\partial M\times\{0\})\cup(-\partial M\times\{2\})$. By (1.22) and (1.23),

$$P_{\geqslant 0,+,T}|_{\partial M\times\{0\}}=P_{\geqslant 0,+,T},\quad P_{>0,-,T}|_{\partial M\times\{0\}}=P_{>0,-,T},$$

(4) Indeed, for any $s\in\mathscr{C}^\infty_0(\widetilde{M}_2,(S_+(T\widetilde{M})\otimes\widetilde{E})|_{\widetilde{M}_2})\oplus\mathscr{C}^\infty_0(M,(S_+(T\widetilde{M})\otimes\widetilde{E})|_M)$ which is supported in $\widetilde{M}_2\setminus(\partial M\times[0,2])$, the difference operator in (1.38) acts on s as a zero operator.

and $P_{\geqslant 0,+,T}|_{-\partial M\times\{2\}}$ (resp. $P_{>0,-,T}|_{-\partial M\times\{2\}}$) is the orthogonal projection from

$$L^2(\partial M,(S_+(TM)\otimes E)|_{\partial M})\quad(\text{resp. } L^2(\partial M,(S_-(TM)\otimes E)|_{\partial M}))$$

onto $\bigoplus_{\lambda\leqslant 0}E_{\lambda,+,T}$ (resp. $\bigoplus_{\lambda<0}E_{\lambda,-,T}$). Thus from the product structure on Z, we get (cf. [2, Proposition 3.11])

$$\begin{aligned}&\operatorname{Ker}(D^E_{Z,+,T}(\gamma),P_{\geqslant 0,+,T}(\gamma))=0,\\&\operatorname{Ker}(D^E_{Z,-,T}(\gamma),P_{>0,-,T}(\gamma))=\operatorname{Ker}(D^E_{\partial M,-,T}(\gamma)).\end{aligned}\tag{1.40}$$

Combining (1.40) with Proposition 1.1, for $T>T_\gamma$, we get

$$\operatorname{Ind}(D^E_{Z,+,T}(\gamma),P_{\geqslant 0,+,T}(\gamma))=0.\tag{1.41}$$

By Definition 1.3, (1.19), (1.28), (1.39) and (1.41), for any $T>T_\gamma$,

$$\operatorname{Ind}(D^{\widetilde{E}}_{\widetilde{M}_2,+,Tf}(\gamma),P^{\widetilde{M}_2}_{\geqslant 0,+,Tf}(\gamma))=(\operatorname{Ind}(\sigma^M_{E,\Psi})_\gamma-Q^M_{\mathrm{APS}}(E,\Psi)_\gamma)\cdot V^G_\gamma.\tag{1.42}$$

For the second step, we need to prove the following lemma.

LEMMA 1.6. *For $\gamma\in\Lambda^*_+$, there exists $T_2>T_\gamma$ such that for $T>T_2$ we have*

$$\operatorname{Ind}(D^{\widetilde{E}}_{\widetilde{M}_2,+,Tf}(\gamma),P^{\widetilde{M}_2}_{\geqslant 0,+,Tf}(\gamma))=0.\tag{1.43}$$

Proof. Following Bismut–Lebeau [3, pp. 115–116], let

$$U_1=\partial M\times(-\infty,1)\quad\text{and}\quad U_2=\partial M\times(0,2]$$

be an open covering of $\widetilde{M}_2$. Let h_1 and h_2 be two smooth G-invariant functions on $\widetilde{M}_2$ such that h_1^2 and h_2^2 form a partition of unity associated with the covering $\{U_1,U_2\}$.

By (0.22), (1.5), (1.15), (1.16) and (1.24), we deduce that

$$\begin{aligned}(D^{\widetilde{E}}_{\widetilde{M},Tf})^2=(D^{\widetilde{E}}_{\widetilde{M}})^2+iT\sum_{j=1}^n c(e_j)c(\nabla^{T\widetilde{M}}_{e_j}(f\tilde{\Psi}^{\widetilde{M}}))\\-2iTf\sum_{j=1}^{\dim G}\tilde{\Psi}_jL_{V_j}-2iTf\sum_{j=1}^{\dim G}\tilde{\Psi}_j\,\mu^{S(T\widetilde{M})\otimes\widetilde{E}}(V_j)+T^2|f\tilde{\Psi}^{\widetilde{M}}|^2.\end{aligned}\tag{1.44}$$

By (1.22), $\tilde{\Psi}_j$ and $\mu^{S(T\widetilde{M})\otimes\widetilde{E}}(V_j)$ are constant on x_n on $\widetilde{M}_2$, and thus, from (0.21), there exists $C_1>0$ such that

$$\bigg\|\sum_{j=1}^{\dim G}\tilde{\Psi}_jL_{V_j}\bigg\|+\bigg\|\sum_{j=1}^{\dim G}\tilde{\Psi}_j\mu^{S(T\widetilde{M})\otimes\widetilde{E}}(V_j)\bigg\|\leqslant C_1,\tag{1.45}$$

where the norm in (1.45) refers to operators acting on $L^2(\widetilde{M},S(T\widetilde{M})\otimes\widetilde{E})^\gamma$.

By (1.26), (1.44) and (1.45), there exists $C>0$ such that, for $T>0$ and

$$s\in\mathscr{C}_0^\infty(U_1,(S(T\widetilde{M})\otimes\widetilde{E})|_{\widetilde{M}_2})^\gamma,$$

we have

$$\|D^{\widetilde{E}}_{\widetilde{M}_2,Tf}s\|_0^2=\langle(D^{\widetilde{E}}_{\widetilde{M}_2,Tf})^2s,s\rangle\geqslant\|D^{\widetilde{E}}_{\widetilde{M}}s\|_0^2+T^2\big\|f|\tilde{\Psi}^{\widetilde{M}}|s\big\|_0^2-CT\|fs\|_0\|s\|_0. \tag{1.46}$$

Thus from (1.17), (1.22), (1.26) and (1.46), we see that there exist $T_1>T_\gamma$ and $C_2>0$ such that, for any $T>T_1$ and $s\in\mathscr{C}_0^\infty(U_1,(S(T\widetilde{M})\otimes\widetilde{E})|_{\widetilde{M}_2})^\gamma$, we have

$$\|D^{\widetilde{E}}_{\widetilde{M}_2,Tf}s\|_0^2\geqslant\|D^{\widetilde{E}}_{\widetilde{M}_2}s\|_0^2+C_2T^2\|s\|_0^2. \tag{1.47}$$

By Green's formula, (1.23) and (1.24) imply that, for $s\in\mathscr{C}_0^\infty(\widetilde{M}_2,(S(T\widetilde{M})\otimes\widetilde{E})|_{\widetilde{M}_2})$, we have

$$\begin{aligned}\|D^{\widetilde{E}}_{\widetilde{M}_2,Tf}s\|_0^2&=\int_{\widetilde{M}_2}\langle s,(D^{\widetilde{E}}_{\widetilde{M}_2,Tf})^2s\rangle\,dv_{\widetilde{M}_2}+\int_{\partial\widetilde{M}_2}\langle s,c(-e_n)D^{\widetilde{E}}_{\widetilde{M}_2,Tf}s\rangle\,dv_{\partial\widetilde{M}_2}\\&=\int_{\widetilde{M}_2}\langle s,(D^{\widetilde{E}}_{\widetilde{M}_2,Tf})^2s\rangle\,dv_{\widetilde{M}_2}-\int_{\partial\widetilde{M}_2}\langle s,\nabla^{S(T\widetilde{M})\otimes\widetilde{E}}_{-e_n}s\rangle\,dv_{\partial\widetilde{M}_2}\\&\quad-\int_{\partial\widetilde{M}_2}\langle s,D^{\widetilde{E}}_{\partial\widetilde{M}_2,Tf}s\rangle\,dv_{\partial\widetilde{M}_2}.\end{aligned} \tag{1.48}$$

By the Lichnerowicz formula (cf. [9, Appendix D]), we have

$$(D^{\widetilde{E}}_{\widetilde{M}})^2=-\Delta^{\widetilde{E}}+\mathcal{O}(1), \tag{1.49}$$

where $\Delta^{\widetilde{E}}$ is the Bochner Laplacian, and $\mathcal{O}(1)$ is an endomorphism of $S(T\widetilde{M})\otimes\widetilde{E}$. By (1.22), the fiberwise norm of this endomorphism has a uniform upper bound over $\widetilde{M}$.

By Green's formula, we have, for any $s\in\mathscr{C}_0^\infty(\widetilde{M}_2,(S(T\widetilde{M})\otimes\widetilde{E})|_{\widetilde{M}_2})$,

$$\int_{\widetilde{M}_2}\langle-\Delta^{\widetilde{E}}s,s\rangle\,dv_{\widetilde{M}_2}-\int_{\partial\widetilde{M}_2}\langle s,\nabla^{S(T\widetilde{M})\otimes\widetilde{E}}_{-e_n}s\rangle\,dv_{\partial\widetilde{M}_2}=\|\nabla^{S(T\widetilde{M})\otimes\widetilde{E}}s\|_0^2. \tag{1.50}$$

Note that $f=1$ on $\partial\widetilde{M}_2$. By (1.13), for any $T>T_\gamma$ and $s\in\mathscr{C}_0^\infty(\widetilde{M}_2,(S(T\widetilde{M})\otimes\widetilde{E})|_{\widetilde{M}_2})^\gamma$ with $P^{\widetilde{M}_2}_{\geqslant0,\pm,Tf}(s|_{\partial\widetilde{M}_2})=0$, we have

$$\int_{\partial\widetilde{M}_2}\langle s,D^{\widetilde{E}}_{\partial\widetilde{M}_2,Tf}s\rangle\,dv_{\partial\widetilde{M}_2}\leqslant-\sqrt{C_\gamma}\,T\big\|s|_{\partial\widetilde{M}_2}\big\|_{\partial\widetilde{M}_2,0}^2\leqslant0. \tag{1.51}$$

As h_2 has compact support in $\partial M\times(0,2]\subset\widetilde{M}_2$, on which $f\equiv 1$, by (1.17), (1.22), (1.44), (1.45) and (1.48)–(1.51), there exist constants $C_3, C_4, C_5>0$ such that, for $T>1$ and $s\in\mathscr{C}_0^\infty(\widetilde{M}_2,(S(T\widetilde{M})\otimes\widetilde{E})|_{\widetilde{M}_2})^\gamma$ with $P^{\widetilde{M}_2}_{\geqslant 0,\pm,Tf}(s|_{\partial\widetilde{M}_2})=0$, we have

$$\|D^{\widetilde{E}}_{\widetilde{M}_2,Tf}(h_2 s)\|_0^2\geqslant C_3\|D^{\widetilde{E}}_{\widetilde{M}_2}(h_2 s)\|_0^2-C_4T\|h_2 s\|_0^2+C_5T^2\|h_2 s\|_0^2. \tag{1.52}$$

Since $h_1^2+h_2^2\equiv 1$, for any $s\in\mathscr{C}_0^\infty(\widetilde{M}_2,(S(T\widetilde{M})\otimes\widetilde{E})|_{\widetilde{M}_2})^\gamma$ with $P^{\widetilde{M}_2}_{\geqslant 0,\pm,Tf}(s|_{\partial\widetilde{M}_2})=0$, we obtain

$$\begin{aligned}\|D^{\widetilde{E}}_{\widetilde{M}_2,Tf}s\|_0^2&=\|h_1D^{\widetilde{E}}_{\widetilde{M}_2,Tf}s\|_0^2+\|h_2D^{\widetilde{E}}_{\widetilde{M}_2,Tf}s\|_0^2\\&\geqslant\tfrac12\|D^{\widetilde{E}}_{\widetilde{M}_2,Tf}(h_1s)\|_0^2+\tfrac12\|D^{\widetilde{E}}_{\widetilde{M}_2,Tf}(h_2s)\|_0^2\\&\quad-\|c((dh_1)^*)s\|_0^2-\|c((dh_2)^*)s\|_0^2,\end{aligned} \tag{1.53}$$

where $(dh_j)^*\in T\widetilde{M}_2$ is the dual of dh_j with respect to $g^{T\widetilde{M}}$.

By (1.47), (1.52) and (1.53), there exist $C_6, C_7>0$ such that for any $T>T_1>T_\gamma$ and $s\in\mathscr{C}_0^\infty(\widetilde{M}_2,(S(T\widetilde{M})\otimes\widetilde{E})|_{\widetilde{M}_2})^\gamma$ with $P^{\widetilde{M}_2}_{\geqslant 0,\pm,Tf}(s|_{\partial\widetilde{M}_2})=0$, we have

$$\|D^{\widetilde{E}}_{\widetilde{M}_2,Tf}s\|_0^2\geqslant\tfrac12\|D^{\widetilde{E}}_{\widetilde{M}}(h_1s)\|_0^2+\tfrac12C_3\|D^{\widetilde{E}}_{\widetilde{M}_2}(h_2s)\|_0^2-C_6T\|s\|_0^2+C_7T^2\|s\|_0^2. \tag{1.54}$$

By

$$D^{\widetilde{E}}_{\widetilde{M}_2}(h_js)=h_jD^{\widetilde{E}}_{\widetilde{M}_2}s+c((dh_j)^*)s,\quad h_1^2+h_2^2\equiv 1$$

and (1.54), there exist $T_2>T_\gamma$ and $C_8, C_9>0$ such that, for $T>T_2$ and

$$s\in\mathscr{C}_0^\infty(\widetilde{M}_2,(S(T\widetilde{M})\otimes\widetilde{E})|_{\widetilde{M}_2})^\gamma,\quad\text{with } P^{\widetilde{M}_2}_{\geqslant 0,\pm,Tf}(s|_{\partial\widetilde{M}_2})=0,$$

we have that

$$\|D^{\widetilde{E}}_{\widetilde{M}_2,Tf}s\|_0^2\geqslant C_8\|D^{\widetilde{E}}_{\widetilde{M}_2}s\|_0^2+C_9T^2\|s\|_0^2. \tag{1.55}$$

By Proposition 1.1, (1.19) and (1.55), we get Lemma 1.6. □

By (1.42) and Lemma 1.6, the proof of Theorem 1.5 is complete.

2. Quantization for proper moment maps: proof of Theorem 0.1

The purpose of this section is to give a proof of Theorem 0.1. This proof consists of two steps. In the first step, we reduce Theorem 0.1 to a vanishing result for the transversal index and then use Theorem 1.5 to interpret the latter as a vanishing result for the APS type index. In the second step, we apply the analytic localization method developed in [3], [22] and [23] to prove the vanishing of this APS type index.

We use the assumptions and the notation in the introduction. Also, for any real 1-form υ on a Riemannian manifold, we denote by υ^* the corresponding vector field on this manifold.

Recall that (M,ω,J^M) is a non-compact symplectic manifold of dimension n with a compatible almost-complex structure J^M, and $g^{TM}=\omega(\cdot\,,J^M\cdot)$ is the associated Riemannian metric on M. We have the canonical splitting $TM\otimes_{\mathbb{R}}\mathbb{C}=T^{(1,0)}M\oplus T^{(0,1)}M$, for the complexification of TM, with

$$\begin{aligned} T^{(1,0)}M &= \{u\in TM\otimes_{\mathbb{R}}\mathbb{C} : J^M u = iu\},\\ T^{(0,1)}M &= \{u\in TM\otimes_{\mathbb{R}}\mathbb{C} : J^M u = -iu\}. \end{aligned} \tag{2.1}$$

Let $T^{*(0,1)}M$ be the dual of $T^{(0,1)}M$.

The almost-complex structure J^M on TM determines a canonical spin^c-structure on TM with the associated Hermitian line bundle $\det(T^{(1,0)}M):=\Lambda^{n/2}(T^{(1,0)}M)$. Moreover, we have

$$\begin{aligned} S(TM) &= \Lambda(T^{*(0,1)}M),\\ S_+(TM) &= \Lambda^{\mathrm{even}}(T^{*(0,1)}M),\\ S_-(TM) &= \Lambda^{\mathrm{odd}}(T^{*(0,1)}M). \end{aligned} \tag{2.2}$$

For any $W\in TM$, we write $W=w+\bar{w}\in T^{(1,0)}M\oplus T^{(0,1)}M$. Let $w^*\in T^{*(0,1)}M$ correspond to w so that $(w^*,\bar{u})=g^{TM}(w,\bar{u})$ for any $\bar{u}\in T^{(0,1)}M$. Then

$$c(W) = \sqrt{2}(w^*\wedge - i_{\bar{w}}) \tag{2.3}$$

defines the Clifford action of W on $\Lambda(T^{*(0,1)}M)$. It interchanges

$$\Lambda^{\mathrm{even}}(T^{*(0,1)}M) \quad \text{and} \quad \Lambda^{\mathrm{odd}}(T^{*(0,1)}M).$$

The Levi-Civita connection ∇^{TM} together with the almost-complex structure J^M induces by projection a canonical Hermitian connection $\nabla^{T^{(1,0)}M}$ on $T^{(1,0)}M$. This induces a Hermitian connection $\nabla^{\det}$ on $\det(T^{(1,0)}M)$. The Clifford connection $\nabla^{\Lambda(T^{*(0,1)}M)}$ on $\Lambda(T^{*(0,1)}M)$ is induced by the Levi-Civita connection ∇^{TM} and the connection $\nabla^{\det}$ (cf. [9, Appendix D], [11, §1.3] and [22, §1a)]).

We take $E=L$, where L is the prequantum line bundle on M in the introduction, and set $\Omega^{0,\bullet}(M,L)=\mathscr{C}^\infty(M,\Lambda(T^{*(0,1)}M)\otimes L)$. Let D^L_M be the corresponding Dirac operator defined as in (1.5).

Recall that the moment map $\mu: M\to\mathfrak{g}^*$ is assumed to be proper. Let $X^{\mathcal{H}}$ be the Hamiltonian vector field of $\mathcal{H}=|\mu|^2$, i.e.,

$$i_{X^{\mathcal{H}}}\omega = d\mathcal{H}. \tag{2.4}$$

By (0.2), (0.4), (0.21) and (2.4), we find (cf. [22, (1.19)]),

$$X^{\mathcal{H}} = -J^M (d\mathcal{H})^* = -2J^M \sum_{j=1}^{\dim G} \mu_j (d\mu_j)^* = 2 \sum_{j=1}^{\dim G} \mu_j V_j^M = 2\mu^M. \tag{2.5}$$

For any regular value $a>0$ of $\mathcal{H}=|\mu|^2$, denote by M_a the compact G-manifold with boundary defined by

$$M_a = \{x \in M : \mathcal{H}(x) \leqslant a\}. \tag{2.6}$$

By (2.5), μ^M does not vanish on the boundary $\partial M_a = \mathcal{H}^{-1}(a)$ of M_a.

Let $a'>a>0$ be two regular values of $\mathcal{H}$. Let $M_{a,a'}$ denote the compact G-manifold with boundary $M_{a,a'} = \overline{M_{a'} \setminus M_a}$. By the additivity of the transversal index (cf. [1, Theorem 3.7, §6] and [18, Proposition 4.1]), we have, for $\gamma \in \Lambda_+^*$,

$$\operatorname{Ind}(\sigma_{L,\mu}^{M_{a'}})_\gamma - \operatorname{Ind}(\sigma_{L,\mu}^{M_a})_\gamma = \operatorname{Ind}(\sigma_{L,\mu}^{M_{a,a'}})_\gamma. \tag{2.7}$$

Let $\operatorname{Cas}_G = -\sum_{j=1}^{\dim G} V_j V_j$ be the Casimir operator associated with G. Let $c_\gamma \geqslant 0$ be defined by

$$\operatorname{Cas}_G|_{V_\gamma^G} = c_\gamma \operatorname{Id}_{V_\gamma^G}. \tag{2.8}$$

Clearly, $c_{\gamma=0}=0$. As $\operatorname{Cas}_G|_{V_\gamma^G} = -\sum_{j=1}^{\dim G} L_{V_j}(\gamma)^2$, from (1.45) and (2.8) we get

$$c_\gamma = \left\| \sum_{j=1}^{\dim G} L_{V_j}(\gamma)^2 \right\|. \tag{2.9}$$

By Theorem 1.5, (2.7) and (2.9), the following result is a reformulation of Theorem 0.1, with a more precise form of the bound a_γ.

THEOREM 2.1. *Fix $\gamma \in \Lambda_+^*$. Then for any regular values a' and a of $\mathcal{H}$ with*

$$a' > a > \frac{c_\gamma}{4\pi^2},$$

one has

$$Q_{\mathrm{APS}}^{M_{a,a'}}(L,\mu)_\gamma = 0. \tag{2.10}$$

Proof. If $\gamma=0$, (2.10) has been proved in [23, Theorems 2.6 and 4.3]. The proof for general $\gamma \in \Lambda_+^*$ is a modification of the proof of [23, Theorem 2.6], where it is assumed that $\gamma=0$. Let $\gamma \in \Lambda_+^*$ and $a'>a>c_\gamma/4\pi^2$ be fixed.

By (2.5), (1.10) becomes in the current situation (cf. [22, (1.20)] and [23, (1.19)]),

$$D_{M,T}^L = D_M^L + \frac{iT}{2} c(X^{\mathcal{H}}) \colon \Omega^{0,\bullet}(M_{a,a'}, L) \longrightarrow \Omega^{0,\bullet}(M_{a,a'}, L). \tag{2.11}$$

Let $e_1,...,e_n$ be an oriented orthonormal frame of $TM_{a,a'}$. By [22, Theorem 1.6], the following formula holds:

$$\begin{aligned}(D^L_{M,T})^2 = (D^L_M)^2 + \frac{iT}{4}\sum_{k=1}^{n} c(e_k)c(\nabla^{TM}_{e_k}X^{\mathcal{H}}) - \frac{iT}{2}\,\mathrm{Tr}[\nabla^{TM}X^{\mathcal{H}}|_{T^{(1,0)}M}] \\ + \frac{T}{2}\sum_{j=1}^{\dim G}(ic(J^M V_j^M)c(V_j^M) + |V_j^M|^2) \\ + 4\pi T\mathcal{H} - 2iT\sum_{j=1}^{\dim G}\mu_j L_{V_j} + \frac{T^2}{4}|X^{\mathcal{H}}|^2.\end{aligned} \tag{2.12}$$

Let $\mathcal{U}$ be a G-invariant open neighborhood of $\partial M_{a,a'}$ in $M_{a,a'}$ such that $X^{\mathcal{H}}$ does not vanish on $\bar{\mathcal{U}}$. Since $X^{\mathcal{H}}$ does not vanish on $\partial M_{a,a'}$, the existence of $\mathcal{U}$ is clear. Let $\mathcal{U}'$ be a G-invariant open subset of $M_{a,a'}$ such that $\bar{\mathcal{U}}'\cap\partial M_{a,a'}=\varnothing$ and $\mathcal{U}\cup\mathcal{U}'=M_{a,a'}$.

By the fact that L_{V_j} acts as a bounded operator on $L^2(M_{a,a'},\Lambda(T^{*(0,1)}M)\otimes L)^\gamma$, by using (1.13) instead of [23, Theorem 2.1], and by proceeding in exactly the same way as in [23, Proof of Proposition 2.4] (cf. also the proof of (1.52)), we know that there exist $T_1>0$ and $C_1>0$ (depending on $\mathcal{U}$ and γ) such that, for any $T>T_1$ and $s\in\Omega^{0,\bullet}(M_{a,a'},L)^\gamma$ with $\mathrm{supp}(s)\subset\mathcal{U}$ and $P_{\geqslant 0,\pm,T}(s|_{\partial M_{a,a'}})=0$, the following inequality holds:

$$\|D^L_{M,T}s\|_0^2 \geqslant C_1(\|D^L_M s\|_0^2 + T^2\|s\|_0^2). \tag{2.13}$$

For any $\varepsilon>0$, set

$$G^L_{T,\varepsilon} = (D^L_{M,T})^2 - (4\pi-\varepsilon)T\mathcal{H} + 2iT\sum_{j=1}^{\dim G}\mu_j L_{V_j}. \tag{2.14}$$

Clearly, $G^L_{T,\varepsilon}$ is of the same form as F^L_T in [22, (2.6)], with $4\pi T\mathcal{H}$ in [22, (2.6)] being replaced by $\varepsilon T\mathcal{H}$.

By replacing $4\pi\mathcal{H}$ in [22, (2.26)] by $\varepsilon\mathcal{H}$ in the proof of [22, Proposition 2.2, case 2], from (2.12) and (2.14), we know that the analogue of [22, Proposition 2.2] holds for the operator $G^L_{T,\varepsilon}$: for any $x\in M_{a,a'}\setminus\partial M_{a,a'}$, there exist $C_x>0$, $b_x>0$ and an open neighborhood $U_x\subset M_{a,a'}\setminus\partial M_{a,a'}$ of x such that, for any $T>1$ and $s\in\Omega^{0,\bullet}(M_{a,a'},L)$ with $\mathrm{supp}(s)\subset U_x$, we have

$$\langle G^L_{T,\varepsilon}s, s\rangle \geqslant C_x(\|D^L_M s\|_0^2 + (T-b_x)\|s\|_0^2). \tag{2.15}$$

From (2.15), as explained in [22, §2 (c)], there exist $C_2>0$ and $b_1>0$ such that, for any $T>1$ and $s\in\Omega^{0,\bullet}(M_{a,a'},L)$ with $\mathrm{supp}(s)\subset\mathcal{U}'$, we have

$$\langle G^L_{T,\varepsilon}s, s\rangle \geqslant C_2(\|D^L_M s\|_0^2 + (T-b_1)\|s\|_0^2). \tag{2.16}$$

LEMMA 2.2. *There exists* $0<\varepsilon<4\pi$ *such that, for any* $s\in\Omega^{0,\bullet}(M_{a,a'},L)^\gamma$, *one has*

$$\left\langle \left((4\pi-\varepsilon)\mathcal{H}-2i\sum_{j=1}^{\dim G}\mu_j L_{V_j}\right)s,s\right\rangle \geqslant 0. \tag{2.17}$$

Proof. Since $a'>a>c_\gamma/4\pi^2$, there exists $\varepsilon\in(0,4\pi)$ such that the following inequality holds on $M_{a,a'}$:

$$\mathcal{H}\geqslant \frac{4c_\gamma}{(4\pi-\varepsilon)^2}. \tag{2.18}$$

By the Cauchy inequality and (2.9), we have that, for any $s\in\Omega^{0,\bullet}(M_{a,a'},L)^\gamma$,

$$\begin{aligned}\left|\left\langle \sum_{j=1}^{\dim G}\mu_j L_{V_j}s,s\right\rangle\right| &\leqslant \frac{1}{2}\sum_{j=1}^{\dim G}\left(\frac{4\pi-\varepsilon}{2}|\mu_j s|_0^2+\frac{2}{4\pi-\varepsilon}\|L_{V_j}s\|_0^2\right)\\ &=\frac{4\pi-\varepsilon}{4}\langle\mathcal{H}s,s\rangle+\frac{c_\gamma}{4\pi-\varepsilon}\|s\|_0^2.\end{aligned} \tag{2.19}$$

From (2.18) and (2.19), we obtain, for any $s\in\Omega^{0,\bullet}(M_{a,a'},L)^\gamma$,

$$\left\langle \left((4\pi-\varepsilon)\mathcal{H}-2i\sum_{j=1}^{\dim G}\mu_j L_{V_j}\right)s,s\right\rangle \geqslant \left\langle\left(\frac{4\pi-\varepsilon}{2}\mathcal{H}-\frac{2c_\gamma}{4\pi-\varepsilon}\right)s,s\right\rangle\geqslant 0. \tag{2.20}$$

The proof of Lemma 2.2 is complete. □

Let $\varepsilon>0$ be fixed as in Lemma 2.2. By Lemma 2.2, (2.14) and (2.16), we have that, for any $T>1$ and $s\in\Omega^{0,\bullet}(M_{a,a'},L)^\gamma$ with $\operatorname{supp}(s)\subset\mathcal{U}'$,

$$\|D^L_{M,T}s\|_0^2=\langle (D^L_{M,T})^2s,s\rangle\geqslant\langle G^L_{T,\varepsilon}s,s\rangle\geqslant C_2(\|D^L_M s\|_0^2+(T-b_1)\|s\|_0^2). \tag{2.21}$$

Let h_1 and h_2 be two smooth G-invariant functions on $M_{a,a'}$ such that $\{h_1^2,h_2^2\}$ is a partition of unity associated with the G-invariant open covering $\{\mathcal{U}',\mathcal{U}\}$ of $M_{a,a'}$.(5)

Let $s\in\Omega^{0,\bullet}(M_{a,a'},L)^\gamma$ with $P_{\geqslant 0,\pm,T}(s|_{\partial M_{a,a'}})=0$. Clearly, h_1s and h_2s still belong to $\Omega^{0,\bullet}(M_{a,a'},L)^\gamma$ with $\operatorname{supp}(h_2s)\subset\mathcal{U}$ and $P_{\geqslant 0,\pm,T}((h_2s)|_{\partial M_{a,a'}})=0$, while $\operatorname{supp}(h_1s)\subset\mathcal{U}'$. By applying (2.13) to h_2s, (2.21) to h_1s, and by proceeding as in (1.53)–(1.55) (cf. [3, pp. 115–116]), we obtain constants $C_3>0$ and $b_2>0$ such that, for any $T>T_1$ and $s\in\Omega^{0,\bullet}(M_{a,a'},L)^\gamma$ with $P_{\geqslant 0,\pm,T}(s|_{\partial M_{a,a'}})=0$, one has

$$\langle (D^L_{M,T})^2s,s\rangle\geqslant C_3(\|D^L_M s\|_0^2+(T-b_2)\|s\|_0^2). \tag{2.22}$$

By Proposition 1.1, (2.5), (2.11) and (2.22), we have $Q^{M_{a,a'}}_{\mathrm{APS},T}(L,\mu)_\gamma=0$ for $T>0$ large enough. Combining this with Definition 1.3, we get Theorem 2.1.

By Theorems 1.5 and 2.1 and identity (2.7), we get Theorem 0.1. □

(5) We can take as h_2 a radical function with respect to $|\mu|^2$ near $\partial M_{a,a'}$ as in (1.30), then h_1 and h_2 are automatically G-invariant.

3. A vanishing result for the APS index

In this section, we prove the vanishing result (0.17).

This section is organized as follows. In §3.1, we state (0.17) as a vanishing theorem on the APS index, Theorem 3.2. In §3.2, we construct a suitable function $\psi: M\times N\to\mathfrak{g}$ which is homotopy equivalent to the function Y in Theorem 3.2 such that the APS index associated with ψ vanishes. In §3.3, we prove the invertibility of the operator associated with ψ, Theorem 3.7, up to a pointwise estimate, Lemma 3.9. In §§3.4–3.6, we prove Lemma 3.9.

We make the same assumptions and we use the same notation as in the introduction and in §2.

3.1. A vanishing theorem for the APS index

For convenience, we recall the basic setting. Let (M,ω) and (N,ω^N) be two symplectic manifolds with symplectic forms ω and ω^N, and $\dim M=n$. We assume that M is non-compact and that N is compact.

Let J^M and J^N be almost-complex structures on TM and TN such that $\omega(\cdot\,,J^M\cdot)$ defines a metric g^{TM} on TM, and $\omega^N(\cdot\,,J^N\cdot)$ defines a metric g^{TN} on TN. Let (L,h^L,∇^L) be a prequantum line bundle on (M,ω), and (F,h^F,∇^F) be a prequantum line bundle on (N,ω^N) (cf. (0.1)).

Suppose that G acts (on the left) on M and N, and its actions on M and N lift to L and F. Moreover, we assume that these G-actions preserve the above metrics and the connections on TM, TN, L, F, J^M and J^N.

Let the moment map $\mu: M\to\mathfrak{g}$ be defined as in (0.3). Let $\eta: N\to\mathfrak{g}$ be the moment map defined in the same way for (N,ω^N) and (F,h^F,∇^F).

We will keep the same notation for the natural lifts of the objects on M and N to $M\times N$. In particular, $L\otimes F$ is the Hermitian line bundle on $M\times N$ induced by L and F with the Hermitian connection $\nabla^{L\otimes F}$ induced by ∇^L and ∇^F.

The G-action on $M\times N$ is defined by $g\cdot(x,y)=(gx,gy)$ for $(x,y)\in M\times N$. We define the symplectic form Ω and the almost-complex structure J on $M\times N$ by

$$\Omega(x,y)=\omega(x)+\omega^N(y) \quad\text{and}\quad J=(J^M,J^N). \tag{3.1}$$

The induced moment map $\theta: M\times N\to\mathfrak{g}$ is given by

$$\theta(x,y)=\mu(x)+\eta(y). \tag{3.2}$$

Since $\mu: M\to\mathfrak{g}$ is proper, $\theta: M\times N\to\mathfrak{g}$ is also proper.

For $A>0$, set

$$\begin{aligned}\mathcal{M}_1 &= \{(x,y)\in M\times N : |\mu(x)|^2 = A\} = \partial M_A\times N,\\ \mathcal{M}_2 &= \{(x,y)\in M\times N : |\theta(x,y)|^2 = 2A\},\\ \mathcal{M} &= \{(x,y)\in M\times N : |\mu(x)|^2 \geqslant A \text{ and } |\theta(x,y)|^2 \leqslant 2A\}\subset M\times N,\end{aligned} \tag{3.3}$$

where ∂M_A is the boundary of M_A defined in (2.6). As μ and θ are proper and M is non-compact, $|\mu(M)|^2$ and $|\theta(M\times N)|^2$ contain a half line of $\mathbb{R}$, and thus for A large enough, $\mathcal{M}_1$ and $\mathcal{M}_2$ are non-empty.

Remark 3.1. Since N is a compact manifold, there exists $C_0>0$ such that

$$|\eta|\leqslant C_0 \quad \text{on } N. \tag{3.4}$$

By (3.2) and (3.4), we have $|\theta|\leqslant|\mu|+C_0$. Set

$$A_0 = \left(\frac{C_0}{\sqrt{2}-\sqrt{5/3}}\right)^2.$$

By (3.3), for $A>A_0$, we have

$$|\mu|\geqslant\sqrt{2A}-C_0\geqslant\sqrt{\tfrac{5}{3}A} \quad \text{on } \mathcal{M}_2. \tag{3.5}$$

Thus, for any $A>A_0$, we have $\mathcal{M}_1\cap\mathcal{M}_2=\varnothing$.

By Sard's theorem, given $C>0$, there exists $C'>C$ which is a regular value of the functions $|\mu|^2$ and $\frac{1}{2}|\theta|^2$ on $M\times N$.

From now on, let $A>A_0$ be a regular value of $|\mu|^2$ and $\frac{1}{2}|\theta|^2$. By Remark 3.1 and (3.3), $\mathcal{M}$ is a smooth G-manifold with boundary $\partial\mathcal{M}=\mathcal{M}_1\cup\mathcal{M}_2$.

From (0.4), (0.21) and (3.2), for any $1\leqslant j\leqslant\dim G$, we have

$$V_j^{M\times N} = V_j^M + V_j^N, \quad \theta_j = \mu_j+\eta_j, \quad (d\mu_j)^* = J^M V_j^M \quad \text{and} \quad (d\eta_j)^* = J^N V_j^N. \tag{3.6}$$

By (0.22) and the first equation of (3.6), we get

$$\mu^{M\times N} = \mu^M + \mu^N \quad \text{and} \quad \theta^{M\times N} = \theta^M + \theta^N. \tag{3.7}$$

By (2.5), μ^M does not vanish on $\mathcal{M}_1$, so that $\mu^{M\times N}$ also does not vanish on $\mathcal{M}_1$. Similarly, $\theta^{M\times N}$ does not vanish on $\mathcal{M}_2$.

Let $Y:\mathcal{M}\to\mathfrak{g}$ be a G-equivariant smooth map such that

$$Y|_{\mathcal{M}_1} = \mu|_{\mathcal{M}_1} \quad \text{and} \quad Y|_{\mathcal{M}_2} = \theta|_{\mathcal{M}_2}. \tag{3.8}$$

Then $Y^{\mathcal{M}}\in\mathscr{C}^\infty(\mathcal{M},T\mathcal{M})$ does not vanish on $\partial\mathcal{M}$.

The main result of this section can be stated as follows.

THEOREM 3.2. *There exists $A_1 \geqslant A_0$ such that for any regular value $A > A_1$ of $|\mu|^2$ and $\frac{1}{2}|\theta|^2$, the following identity holds*:

$$Q^{\mathcal{M}}_{\mathrm{APS}}(L\otimes F, Y)_{\gamma=0} = 0. \tag{3.9}$$

Remark 3.3. By Theorem 1.5, (3.9) is equivalent to (0.17) with $a = \frac{1}{2}b = A$.

3.2. Proof of Theorem 3.2

LEMMA 3.4. *There exist two real smooth functions $\widetilde{\alpha}, \tilde{\phi} \in \mathscr{C}^\infty(\mathbb{R})$ satisfying the following properties*:

$$\begin{gathered}\widetilde{\alpha}(t) = \begin{cases} t^2, & \text{for } t \leqslant \frac{1}{3}, \\ 1, & \text{for } t \geqslant \frac{2}{3}, \end{cases} \qquad \tilde{\phi}(t) = \begin{cases} 1 - t^3, & \text{for } t \leqslant \frac{4}{11}, \\ 2(1-t), & \text{for } t \geqslant \frac{2}{3}, \end{cases} \\ \widetilde{\alpha}(t) + \tilde{\phi}(t) \geqslant \tfrac{29}{27} \quad \text{for } \tfrac{1}{3} \leqslant t \leqslant \tfrac{2}{3}, \qquad \tilde{\phi}'(t) < 0 \quad \text{for } 0 < t \leqslant 1.\end{gathered} \tag{3.10}$$

Proof. We may set $\widetilde{\alpha}_0(t) = t^2$ and $\tilde{\phi}_0(t) = 1 - t^3$ on $t \leqslant \frac{3}{8}$; $\widetilde{\alpha}_0(t) = 1$ and $\tilde{\phi}_0(t) = 2(1-t)$ on $t \geqslant \frac{5}{8}$; and assume that $\widetilde{\alpha}_0$ and $\tilde{\phi}_0$ are linear on $\frac{3}{8} \leqslant t \leqslant \frac{5}{8}$. By smoothing out the linear interpolation, starting from $\widetilde{\alpha}_0$ and $\tilde{\phi}_0$, we get $\widetilde{\alpha}$ and $\tilde{\phi}$ satisfying (3.10). □

Let $A > A_0$ be a regular value of $|\mu|^2$ and $\frac{1}{2}|\theta|^2$. Set

$$\alpha_A(t) = \widetilde{\alpha}\left(\frac{t}{A} - 1\right) \quad \text{and} \quad \phi_A(t) = \tilde{\phi}\left(\frac{t}{A} - 1\right). \tag{3.11}$$

The following identities hold:

$$\alpha'_A(t) = \frac{1}{A}\widetilde{\alpha}'\left(\frac{t}{A} - 1\right) \quad \text{and} \quad \phi'_A(t) = \frac{1}{A}\tilde{\phi}'\left(\frac{t}{A} - 1\right). \tag{3.12}$$

Let $\beta_A \in \mathscr{C}^\infty(M \times N)$ be defined by

$$\beta_A = |\mu|^2 + \alpha_A(|\mu|^2)(|\theta|^2 - |\mu|^2). \tag{3.13}$$

Let $\varrho_A, \gamma_A, \psi_A : M \times N \to \mathfrak{g}$ be the G-equivariant smooth maps defined by

$$\varrho_A = \theta - \phi_A(\beta_A)\eta, \tag{3.14a}$$

$$\gamma_A = 2[1 + \alpha'_A(|\mu|^2)(|\theta|^2 - |\mu|^2)]\mu + 2\alpha_A(|\mu|^2)\eta, \tag{3.14b}$$

$$\psi_A = \varrho_A - \phi'_A(\beta_A)\langle \varrho_A, \eta\rangle \gamma_A. \tag{3.14c}$$

For any function f on $M \times N$, we denote by $d^M f$ and $d^N f$ its differentials along M and N, respectively.

The following lemma partly motivates our choice of ψ_A (compare with (2.5)).

LEMMA 3.5. *The following identity holds*:

$$2\psi_A^M = -J^M (d^M|\varrho_A|^2)^*. \tag{3.15}$$

Proof. By (0.21), (3.2) and (3.13)–(3.14b), we have

$$\begin{aligned} d\beta_A &= 2[1+\alpha'_A(|\mu|^2)(|\theta|^2-|\mu|^2)-\alpha_A(|\mu|^2)]\mu_j\, d^M\mu_j + 2\alpha_A(|\mu|^2)\theta_j\, d\theta_j \\ &= \gamma_{Aj}\, d^M\mu_j + 2\alpha_A(|\mu|^2)\theta_j\, d^N\eta_j, \\ d\varrho_{Aj} &= d\theta_j - \phi'_A(\beta_A)\eta_j d\beta_A - \phi_A(\beta_A)\, d^N\eta_j. \end{aligned} \tag{3.16}$$

From (3.6) and (3.16), we get

$$\begin{aligned} (d\theta_j)^* &= J^M V_j^M + J^N V_j^N, \\ (d\beta_A)^* &= J^M \gamma_A^M + 2\alpha_A(|\mu|^2) J^N \theta^N, \\ (d\varrho_{Aj})^* &= J^M V_j^M - \phi'_A(\beta_A)\eta_j J^M \gamma_A^M + (1-\phi_A(\beta_A)) J^N V_j^N - 2\phi'_A(\beta_A)\alpha_A(|\mu|^2)\eta_j J^N \theta^N. \end{aligned} \tag{3.17}$$

From (0.22), (3.14c) and the third equality in (3.17), we get

$$2\psi_A^M = 2\varrho_{Aj}[V_j^M - \phi'_A(\beta_A)\eta_j\gamma_A^M] = -2J^M \varrho_{Aj}(d^M\varrho_{Aj})^* = -J^M(d^M|\varrho_A|^2)^*. \tag{3.18}$$

The proof of Lemma 3.5 is complete. □

LEMMA 3.6. *There exists* $A_2 \geqslant A_0$ *such that, for any regular value* $A > A_2$ *of* $|\mu|^2$ *and* $\frac{1}{2}|\theta|^2$, *the following identities hold*:

$$\begin{aligned} \psi_A|_{\mathcal{M}_1} &= \mu, & \beta_A|_{\mathcal{M}_1} &= A, \\ \psi_A|_{\mathcal{M}_2} &= \left(1+\frac{4}{A}\langle\theta,\eta\rangle\right)\theta, & \beta_A|_{\mathcal{M}_2} &= 2A. \end{aligned} \tag{3.19}$$

Moreover, *the following inequality holds*:

$$1+\frac{4}{A}\langle\theta,\eta\rangle \geqslant \frac{1}{2} \quad \text{on } \mathcal{M}_2. \tag{3.20}$$

In particular, $\psi_A^{\mathcal{M}}$ *does not vanish on* $\partial\mathcal{M}$.

Proof. On $\mathcal{M}_1$, we have $|\mu|^2 = A$. By (3.10)–(3.14a), we deduce that, on $\mathcal{M}_1$,

$$\beta_A = A, \quad \phi_A(\beta_A) = 1, \quad \phi'_A(\beta_A) = \alpha_A(|\mu|^2) = 0 \quad \text{and} \quad \varrho_A = \mu. \tag{3.21}$$

The first two equalities in (3.19) follow from (3.14c) and (3.21).

From (3.5) and (3.10)–(3.14b), for $A>A_0$, we have, on $\mathcal{M}_2=(|\theta|^2)^{-1}(2A)$,

$$\begin{aligned}\alpha_A(|\mu|^2)=1,\quad \alpha'_A(|\mu|^2)=0,\quad \gamma_A=2\theta,\\ \beta_A=2A,\quad \phi_A(\beta_A)=0,\quad \varrho_A=\theta,\quad \phi'_A(\beta_A)=-\frac{2}{A}.\end{aligned}\tag{3.22}$$

By (3.14c) and (3.22), the last two identities in (3.19) hold. Since $|\theta|=\sqrt{2A}$ on $\mathcal{M}_2$, (3.4) implies that there exists $A_2\geqslant A_0$ such that (3.20) holds on $\mathcal{M}_2$ for $A>A_2$.

We have seen just after (3.7) that $\mu^{\mathcal{M}}$ and $\theta^{\mathcal{M}}$ do not vanish on $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. Hence, by (0.22), (3.19) and (3.20), $\psi_A^{\mathcal{M}}$ does not vanish on $\partial\mathcal{M}$ when $A>A_2$. □

Let $D^{L\otimes F}:\Omega^{0,\bullet}(M\times N,L\otimes F)\to\Omega^{0,\bullet}(M\times N,L\otimes F)$ be the Spin^c-Dirac operator on $M\times N$ (cf. (1.5) and §2). Following (1.10), let $D_{\mathcal{M},T}$ be the operator defined for $T\in\mathbb{R}$, by

$$D_{\mathcal{M},T}=D^{L\otimes F}+iTc(\psi_A^{\mathcal{M}}):\Omega^{0,\bullet}(\mathcal{M},L\otimes F)\longrightarrow\Omega^{0,\bullet}(\mathcal{M},L\otimes F).\tag{3.23}$$

Let $P_{\geqslant 0,\pm,T}$ be the APS projections associated to $D_{\partial\mathcal{M},\pm,T}$ induced by $D_{\mathcal{M},T}$ (cf. (1.11)).

THEOREM 3.7. *There exists $A_1\geqslant A_2$ such that if $A>A_1$ is a regular value of $|\mu|^2$ and $\frac{1}{2}|\theta|^2$, then there exist $C>0$ and $T_0>0$ such that, for any $T>T_0$ and G-invariant element s of $\Omega^{0,\bullet}(\mathcal{M},L\otimes F)$ with $P_{\geqslant 0,\pm,T}(s|_{\partial\mathcal{M}})=0$, the following inequality holds:*

$$\|D_{\mathcal{M},T}s\|_0^2\geqslant C(\|D^{L\otimes F}s\|_0^2+T\|s\|_0^2).\tag{3.24}$$

Proof of Theorem 3.2. Let $A>A_1$ be a regular value of $|\mu|^2$ and $\frac{1}{2}|\theta|^2$. Then, by Theorem 3.7, $(D_{\mathcal{M},\pm,T}(\gamma=0),P_{\geqslant 0,\pm,T}(\gamma=0))$ is invertible for $T>T_0$. By Propositions 1.1 and 1.2, and Definition 1.3, this implies that

$$Q_{\mathrm{APS}}^{\mathcal{M}}(L\otimes F,\psi_A)_{\gamma=0}=0.\tag{3.25}$$

We connect the map Y defined in (3.8) and ψ_A via

$$\psi_{At}=(1-t)Y+t\psi_A,\quad 0\leqslant t\leqslant 1.$$

Lemma 3.6 shows that $\psi_{At}^{\mathcal{M}}\in\mathscr{C}^\infty(\mathcal{M},T\mathcal{M})$ generated by ψ_{At} via (0.22) does not vanish on $\partial\mathcal{M}$ for every $0\leqslant t\leqslant 1$. By the homotopy invariance of the APS index (cf. Remark 1.4) and (3.25), we get (3.9). □

The rest of the section is devoted to the proof of Theorem 3.7.

3.3. Proof of Theorem 3.7

Let $\{e_k\}_{k=1}^n$ (resp. $\{f_l\}_{l=1}^{\dim N}$) be an oriented orthonormal frame of TM (resp. TN). Then $\{e_a\}_{a=1}^{\dim \mathcal{M}}=\{e_k\}\cup\{f_l\}$ is an oriented orthonormal frame of $T\mathcal{M}$. Set

$$\begin{aligned}
&I_{A1}=\tfrac{1}{2}c((d^M\psi_{Aj})^*)c(V_j^M+2V_j^N)+c((d^N\psi_{Aj})^*)c(V_j^M),\\
&I_{A2}=\tfrac{1}{2}\langle(1-iJ^M)V_j^M,(d^M\psi_{Aj})^*\rangle=\operatorname{Tr}[(d^M\psi_{Aj})|_{T^{(1,0)}M}\otimes V_j^M],\\
&I_{A3}=c((d^N\psi_{Aj})^*)c(V_j^N).
\end{aligned}\tag{3.26}$$

THEOREM 3.8. *The following formula holds*:

$$\begin{aligned}
D^2_{\mathcal{M},T}=D^{L\otimes F,2}&+iT\bigg(\frac{1}{2}\sum_{k=1}^n c(e_k)c(\nabla^{TM}_{e_k}\psi_A^M)-\operatorname{Tr}[(\nabla^{TM}\psi_A^M)|_{T^{(1,0)}M}]\bigg)\\
&+iT\bigg(\frac{1}{2}\sum_{l=1}^{\dim N} c(f_l)c(\nabla^{TN}_{f_l}V_j^N)-\operatorname{Tr}[(\nabla^{TN}V_j^N)|_{T^{(1,0)}N}]\bigg)\psi_{Aj}\\
&+4\pi T\langle\psi_A,\theta\rangle+iT(I_{A1}+I_{A2}+I_{A3})-2iT\psi_{Aj}L_{V_j}+T^2|\psi_A^{\mathcal{M}}|^2.
\end{aligned}\tag{3.27}$$

Proof. Let $\nabla^{\Lambda^{0,\bullet}}$ be a brief notation for $\nabla^{\Lambda(T^{*(0,1)}\mathcal{M})\otimes L\otimes F}$. By (3.23), we deduce as in (1.14) and (1.44) that

$$D^2_{\mathcal{M},T}=D^{L\otimes F,2}+iT\sum_{a=1}^{\dim\mathcal{M}}c(e_a)c(\nabla^{T\mathcal{M}}_{e_a}\psi_A^{\mathcal{M}})-2iT\nabla^{\Lambda^{0,\bullet}}_{\psi_A^{\mathcal{M}}}+T^2|\psi_A^{\mathcal{M}}|^2.\tag{3.28}$$

From (3.6), the definition of the moment map, and $L_KX=\nabla^{T\mathcal{M}}_{K^{\mathcal{M}}}X-\nabla^{T\mathcal{M}}_XK^{\mathcal{M}}$ for $K\in\mathfrak{g}$ and $X\in\mathscr{C}^\infty(\mathcal{M},T\mathcal{M})$, we get (cf. [22, Lemma 1.5] and (2.12))

$$\begin{aligned}
\nabla^{\Lambda^{0,\bullet}}_{\psi_A^{\mathcal{M}}}&=\psi_{Aj}\nabla^{\Lambda^{0,\bullet}}_{V_j^{\mathcal{M}}}\\
&=\psi_{Aj}L_{V_j}+2\pi i\langle\psi_A,\theta\rangle+\frac{1}{4}\sum_{k=1}^n c(e_k)c(\nabla^{TM}_{e_k}V_j^M)\psi_{Aj}+\frac{1}{4}\sum_{l=1}^{\dim N}c(f_l)c(\nabla^{TN}_{f_l}V_j^N)\psi_{Aj}\\
&\quad+\frac{1}{2}\psi_{Aj}\operatorname{Tr}[(\nabla^{TM}V_j^M)|_{T^{(1,0)}M}]+\frac{1}{2}\psi_{Aj}\operatorname{Tr}[(\nabla^{TN}V_j^N)|_{T^{(1,0)}N}].
\end{aligned}\tag{3.29}$$

By (3.26), we get

$$\begin{aligned}
&\frac{1}{2}\sum_{k=1}^n c(e_k)c(\nabla^{TM}_{e_k}V_j^M)\psi_{Aj}=\frac{1}{2}\sum_{k=1}^n c(e_k)c(\nabla^{TM}_{e_k}\psi_A^M)-\frac{1}{2}c((d^M\psi_{Aj})^*)c(V_j^M),\\
&\psi_{Aj}\operatorname{Tr}[(\nabla^{TM}V_j^M)|_{T^{(1,0)}M}]=\operatorname{Tr}[(\nabla^{TM}\psi_A^M)|_{T^{(1,0)}M}]-I_{A2}.
\end{aligned}\tag{3.30}$$

Also by (0.21) and (3.6), we have

$$\sum_{a=1}^{\dim\mathcal{M}} c(e_a)c(\nabla^{T\mathcal{M}}_{e_a}\psi^{\mathcal{M}}_A)=\sum_{k=1}^{n} c(e_k)c(\nabla^{TM}_{e_k}\psi^M_A)+c((d^M\psi_{Aj})^*)c(V^N_j) + \sum_{l=1}^{\dim N} c(f_l)c(\nabla^{TN}_{f_l}V^N_j)\psi_{Aj}+c((d^N\psi_{Aj})^*)c(V^M_j+V^N_j). \tag{3.31}$$

By (3.26) and (3.28)–(3.31), we get (3.27). □

Lemma 3.9. *There exists $A_1\geqslant A_2$ such that if $A>A_1$ is a regular value for $|\mu|^2$ and $\frac{1}{2}|\theta|^2$, then, for any $z\in\mathcal{M}$ with $\psi^{\mathcal{M}}_A(z)=0$ and any $f\in(\Lambda(T^{*(0,1)}\mathcal{M})\otimes(L\otimes F)|_{\mathcal{M}})|_z$, the following inequality holds at z:*

$$\begin{aligned}\operatorname{Re}\Big\langle i\Big(\frac{1}{2}\sum_{l=1}^{\dim N} c(f_l)c(\nabla^{TN}_{f_l}V^N_j)-\operatorname{Tr}[(\nabla^{TN}V^N_j)|_{T^{(1,0)}N}]\Big)\psi_{Aj}f,f\Big\rangle\\ +\operatorname{Re}\langle(4\pi\langle\psi_A,\theta\rangle+i(I_{A1}+I_{A2}+I_{A3}))f,f\rangle\geqslant\pi A|f|^2.\end{aligned} \tag{3.32}$$

Lemma 3.9 will be proved in §§3.4–3.6.

Let $F_{\mathcal{M},T}:\Omega^{0,\bullet}(\mathcal{M},L\otimes F)\to\Omega^{0,\bullet}(\mathcal{M},L\otimes F)$ be defined by

$$F_{\mathcal{M},T}=D^2_{\mathcal{M},T}+2iT\psi_{Aj}L_{V_j}. \tag{3.33}$$

Proposition 3.10. *Let $A_1>0$ be as in Lemma 3.9. If $A>A_1$ is a regular value for $|\mu|^2$ and $\frac{1}{2}|\theta|^2$, then, for any $z\in\mathcal{M}\setminus\partial\mathcal{M}$, there exist an open neighborhood U_z of z in $\mathcal{M}$, with $\overline{U}_z\cap\partial\mathcal{M}=\varnothing$, and $C_z>0$ and $b_z>0$ such that, for any $T\geqslant1$ and $s\in\Omega^{0,\bullet}(\mathcal{M},L\otimes F)$ with $\operatorname{supp}(s)\subset U_z$, we have*

$$\operatorname{Re}\langle F_{\mathcal{M},T}s,s\rangle\geqslant C_z(\|D^{L\otimes F}s\|_0^2+(T-b_z)\|s\|_0^2). \tag{3.34}$$

Proof. Let $A>A_1$ be a fixed regular value for $|\mu|^2$ and $\frac{1}{2}|\theta|^2$, and fix $z\in\mathcal{M}\setminus\partial\mathcal{M}$.

If $\psi^{\mathcal{M}}_A(z)\neq0$, then, by (3.27) and (3.33), we see that Proposition 3.10 holds.

From now on we assume that $\psi^{\mathcal{M}}_A(z)=0$. We write $z=(x_0,y_0)$ with $x_0\in M$ and $y_0\in N$. From (0.22), $\psi^{\mathcal{M}}_A(z)=\psi^M_A(z)+\psi^N_A(z)$ with $\psi^M_A(z)\in TM$ and $\psi^N_A(z)\in TN$. Thus

$$\psi^{\mathcal{M}}_A(z)=0 \quad\text{if and only if}\quad \psi^M_A(z)=0 \text{ and } \psi^N_A(z)=0. \tag{3.35}$$

Let $x'=(x_1,...,x_n)$ be the normal coordinate system with respect to $\{e_j|_{x_0}\}_{j=1}^n$ near $x_0\in M$. Let $y'=(y_1,...,y_{\dim N})$ be the normal coordinate system near $y_0\in N$ associated with $\{f_l|_{y_0}\}_{l=1}^{\dim N}$.

By (3.15), $\psi_A^M(z)=0$ implies that $(d^M|\varrho_A|^2)(z)=0$. Thus we can choose the orthonormal frame $\{e_l\}_{l=1}^n$ so that the function $|\varrho_A(\cdot,y_0)|^2$ has the following expression near x_0:

$$|\varrho_A(x',y_0)|^2=|\varrho_A(x_0,y_0)|^2+\sum_{j=1}^n a_j x_j^2+\mathcal{O}(|x'|^3). \tag{3.36}$$

The following lemma is an analogue of [22, Lemma 2.3].

LEMMA 3.11. *The following inequality holds at the point* (x_0,y_0):

$$\frac{i}{2}\sum_{k=1}^n c(e_k)c(\nabla_{e_k}^{TM}\psi_A^M)-i\,\mathrm{Tr}[(\nabla^{TM}\psi_A^M)|_{T^{(1,0)}M}]\geqslant-\sum_{j=1}^n|a_j|, \tag{3.37}$$

and the inequality is strict if at least one of the a_j*'s is negative.*

Proof. Set

$$\psi_A^M(x',y')=-\sum_{k=1}^n t_k(x',y')J^M e_k. \tag{3.38}$$

Then Lemma 3.5 and (3.36) imply that

$$t_k(x',y_0)=a_k x_k+\mathcal{O}(|x'|^2). \tag{3.39}$$

Let $e_j=e_j^{1,0}+e_j^{0,1}\in T^{(1,0)}M\oplus T^{(0,1)}M$. By (2.3), (3.38) and (3.39), we deduce that, at the point (x_0,y_0), one has

$$\begin{aligned}\frac{i}{2}\sum_{k=1}^n & c(e_k)c(\nabla_{e_k}^{TM}\psi_A^M)-i\,\mathrm{Tr}[(\nabla^{TM}\psi_A^M)|_{T^{(1,0)}M}]\\ &=-\frac{i}{2}\sum_{j=1}^n a_j c(e_j)c(J^M e_j)-\frac{i}{2}\sum_{j=1}^n\langle(1-iJ^M)(-a_jJ^Me_j),e_j\rangle\\ &=-2\sum_{j=1}^n a_j i_{e_j^{0,1}}e_j^{1,0*}\wedge\\ &\geqslant-\sum_{j=1}^n|a_j|,\end{aligned} \tag{3.40}$$

where the last inequality is strict if at least one of the a_j's is negative. □

Let Δ^M and Δ^N be the Bochner Laplacians on M and N acting on $\Omega^{0,\bullet}(M,L)$ and $\Omega^{0,\bullet}(N,F)$, respectively. We still denote by Δ^M and Δ^N the induced operators acting on $\Omega^{0,\bullet}(M\times N,L\otimes F)$. Then $\Delta^{M\times N}=\Delta^M+\Delta^N$ is the Bochner Laplacian on $M\times N$. Clearly, they are non-positive operators acting on $\Omega^{0,\bullet}(M\times N,L\otimes F)$. From the

Lichnerowicz formula for $D^{L\otimes F,2}$ (cf. [9, Appendix D] and [11, Theorem 1.3.5]), we get, on $\mathcal{M}$,

$$D^{L\otimes F,2} = -\Delta^{M\times N} + \mathcal{O}(1), \tag{3.41}$$

where $\mathcal{O}(1)$ is an endomorphism of $\Lambda(T^{*(0,1)}\mathcal{M})\otimes L\otimes F$.

Let $F^*_{\mathcal{M},T}$ be the formal adjoint of $F_{\mathcal{M},T}$. Note that $|\psi_A^{\mathcal{M}}|^2 = |\psi_A^M|^2 + |\psi_A^N|^2$. From (3.27), (3.32), (3.33), (3.37), (3.38) and (3.41), we find that $\frac{1}{2}(F_{\mathcal{M},T}+F^*_{\mathcal{M},T})+\Delta^{M\times N}$ is an operator of order 0, and near $z=(x_0,y_0)$,

$$\begin{aligned}\frac{1}{2}(F_{\mathcal{M},T}+F^*_{\mathcal{M},T})+\Delta^{M\times N} \geqslant -T\sum_{j=1}^n |a_j| + T^2\sum_{j=1}^n t_j(x',y')^2 \\ + T^2|\psi_A^N(x',y')|^2 + \pi TA + \mathcal{O}(1+T|x'|+T|y'|).\end{aligned} \tag{3.42}$$

Let $\varepsilon_0>0$ be sufficiently small so that the orthonormal frame $\{e_j\}_{j=1}^n$ is well defined over the ball $B^M_{\varepsilon_0}(x_0)=\{x'\in M : d(x',x_0)<\varepsilon_0\}$, and $\big(\overline{B^M_{\varepsilon_0}(x_0)}\times\overline{B^N_{\varepsilon_0}(y_0)}\big)\cap\partial\mathcal{M}=\varnothing$. For any $1\leqslant j\leqslant n$, let $(\nabla_{e_j})^*$ be the formal adjoint of $\nabla^{\Lambda^{0,\bullet}}_{e_j}$. We have (cf. [11, (1.2.9)])

$$(\nabla_{e_j})^* = -\nabla^{\Lambda^{0,\bullet}}_{e_j} + \langle e_j, \nabla^{TM}_{e_l} e_l\rangle. \tag{3.43}$$

Set

$$-\Delta^M_T = \sum_{j=1}^n ((\nabla_{e_j})^* + T(\operatorname{sgn} a_j)t_j(x',y'))(\nabla^{\Lambda^{0,\bullet}}_{e_j} + T(\operatorname{sgn} a_j)t_j(x',y')). \tag{3.44}$$

Clearly, $-\Delta^M_T$ is non-negative near $z=(x_0,y_0)$. We verify using (3.39) that

$$-\Delta^M_T = -\Delta^M - T\sum_{j=1}^n |a_j| + T^2\sum_{j=1}^n t_j(x',y')^2 + \mathcal{O}(1+T|x'|+T|y'|). \tag{3.45}$$

By (3.42), (3.44) and (3.45), the following identity holds for any $k>1$, when both sides act on sections with compact support in $B^M_{\varepsilon_0}(x_0)\times B^N_{\varepsilon_0}(y_0)$:

$$\begin{aligned}\frac{1}{2}(F_{\mathcal{M},T}+F^*_{\mathcal{M},T}) &\geqslant -\Delta^N - \Delta^M_T + \pi TA + \mathcal{O}(1+T|x'|+T|y'|) \\ &\geqslant -\frac{1}{k}\Delta^N - \frac{1}{k}\Delta^M - \frac{T}{k}\sum_{j=1}^n |a_j| + \pi TA + \mathcal{O}(1+T|x'|+T|y'|).\end{aligned} \tag{3.46}$$

By (3.41) and (3.46), there exist $C_2>0$ and $C_3>0$ such that, for any $0<\varepsilon<\varepsilon_0$ and any $s\in\Omega^{0,\bullet}(\mathcal{M},L\otimes F)$ with $\operatorname{supp}(s)\subset B^M_\varepsilon(x_0)\times B^N_\varepsilon(y_0)$, we have

$$\operatorname{Re}\langle F_{\mathcal{M},T}s, s\rangle \geqslant \frac{1}{k}\|D^{L\otimes F}s\|_0^2 + \left[T\left(\pi A - \frac{1}{k}\sum_{j=1}^n |a_j| - C_3\varepsilon\right) - \left(\frac{C_2}{k}+C_3\right)\right]\|s\|_0^2. \tag{3.47}$$

We take k large enough and choose ε small enough so that

$$\frac{A}{2}-\frac{1}{k}\sum_{j=1}^{n}|a_j|>0 \quad \text{and} \quad \frac{A}{2}-C_3\varepsilon>0. \tag{3.48}$$

With ε chosen as in (3.48), the conclusion of Proposition 3.10 follows from (3.47) in the case where $\psi_A^{\mathcal{M}}(z)=0$. The proof of Proposition 3.10 is complete. ☐

By Proposition 3.10 and the gluing trick due to Bismut–Lebeau [3, pp. 115–117] (which has been used in the proof of (2.16)), we obtain the following fact: for any open subset $\mathcal{U}'\subset\mathcal{M}$ with $\bar{\mathcal{U}}'\cap\partial\mathcal{M}=\varnothing$, there exist $C_6>0$ and $b_1>0$ such that, for any $s\in\Omega^{0,\bullet}(\mathcal{M},L\otimes F)$ with $\operatorname{supp}(s)\subset\mathcal{U}'$, we have

$$\operatorname{Re}\langle F_{\mathcal{M},T}s,s\rangle \geqslant C_6(\|D^{L\otimes F}s\|_0^2+(T-b_1)\|s\|_0^2). \tag{3.49}$$

Let $\mathcal{U}$ be a G-invariant open neighborhood of $\partial\mathcal{M}$ in $\mathcal{M}$ such that $\psi_A^{\mathcal{M}}$ does not vanish on $\bar{\mathcal{U}}$. As $\psi_A^{\mathcal{M}}$ does not vanish on $\partial\mathcal{M}$, the existence of $\mathcal{U}$ is clear. Then one can proceed in exactly the same way as in the proof of (1.52) (or [23, Proposition 2.4]), to see that there exist $T_2>0$ and $C_7>0$ such that, for any $T>T_2$ and $s\in\Omega^{0,\bullet}(\mathcal{M},L\otimes F)^{\gamma=0}$ with $\operatorname{supp}(s)\subset\mathcal{U}$ and $P_{\geqslant 0,\pm,T}(s|_{\partial\mathcal{M}})=0$, we have

$$\|D_{\mathcal{M},T}s\|_0^2 \geqslant C_7(\|D^{L\otimes F}s\|_0^2+T^2\|s\|_0^2). \tag{3.50}$$

In view of (3.33), (3.49) and (3.50), one can then proceed as in the proof of (2.22), which goes back to [3, pp. 115–117], to see that Theorem 3.7 holds.

3.4. Proof of Lemma 3.9 (I): uniform estimates on functions

We first give uniform estimates for some functions appearing in the definition of γ_A and ψ_A when $A\to\infty$.

Recall that $A_2>0$ was determined in Lemma 3.6. Let $A>A_2$ be a regular value for $|\mu|^2$ and $\frac{1}{2}|\theta|^2$. Set

$$\begin{aligned}
\tau_{A1} &= 1+\alpha_A'(|\mu|^2)(|\theta|^2-|\mu|^2),\\
\tau_{A2} &= 1-2\phi_A'(\beta_A)\langle\varrho_A,\eta\rangle\tau_{A1},\\
\tau_{A4} &= 1-\phi_A(\beta_A)-2\phi_A'(\beta_A)\alpha_A(|\mu|^2)\langle\varrho_A,\eta\rangle,\\
\tau_{A5} &= [1-\phi_A(\beta_A)]\tau_{A1}-\alpha_A(|\mu|^2)\\
&= 1-\phi_A(\beta_A)-\alpha_A(|\mu|^2)+[1-\phi_A(\beta_A)]\alpha_A'(|\mu|^2)(|\theta|^2-|\mu|^2).
\end{aligned} \tag{3.51}$$

Then

$$\tau_{A5} = \tau_{A1}\tau_{A4} - \alpha_A(|\mu|^2)\tau_{A2}. \tag{3.52}$$

From (3.14b) and (3.51), we obtain

$$\gamma_A = 2\tau_{A1}\mu + 2\alpha_A(|\mu|^2)\eta. \tag{3.53}$$

From (3.2), (3.14a), (3.14c), (3.51) and (3.53), we get

$$\psi_A = \mu + [1-\phi_A(\beta_A)]\eta - \phi_A'(\beta_A)\langle \varrho_A, \eta\rangle [2\tau_{A1}\mu + 2\alpha_A(|\mu|^2)\eta] = \tau_{A2}\mu + \tau_{A4}\eta. \tag{3.54}$$

In the following, for $s\in\mathbb{R}$ and a function f_A on $\mathcal{M}$, we write $f_A = \mathcal{O}_0(A^s)$ if there exists $C>0$ (independent of A) such that its $\mathscr{C}^0$-norm on $\mathcal{M}$ can be controlled by CA^s.

The following lemma contains basic asymptotic estimates for these τ functions.

LEMMA 3.12. *There exists $A_6 \geqslant A_2$ such that, for $A > A_6$, we have*

$$A < \beta_A < 2A \quad \text{on } \mathcal{M}\setminus\partial\mathcal{M}. \tag{3.55}$$

Thus

$$0 < \phi_A(\beta_A) < 1 \quad \text{on } \mathcal{M}\setminus\partial\mathcal{M}. \tag{3.56}$$

Moreover,

$$\tau_{A1} = 1 + \mathcal{O}_0(A^{-1/2}), \tag{3.57a}$$

$$\tau_{A2} = 1 + \mathcal{O}_0(A^{-1/2}), \tag{3.57b}$$

$$\tau_{A4} = [1-\phi_A(\beta_A)](1 + \mathcal{O}_0(A^{-1/2})), \tag{3.57c}$$

$$\tau_{A5} = [1-\phi_A(\beta_A) - \alpha_A(|\mu|^2)](1 + \mathcal{O}_0(A^{-1/2})). \tag{3.57d}$$

Finally, for any $A > A_6$, we have

$$1-\phi_A(\beta_A) - \alpha_A(|\mu|^2) \begin{cases} <0, & \text{if } (x,y)\in\mathcal{M}\setminus\partial\mathcal{M}, \\ =0, & \text{if } (x,y)\in\partial\mathcal{M}. \end{cases} \tag{3.58}$$

Proof. From (3.2)–(3.4), for $A > A_2 \geqslant A_0$ we have, on $\mathcal{M}$,

$$\begin{gathered} A^{1/2} \leqslant |\mu| \leqslant |\theta| + |\eta| \leqslant \sqrt{2}A^{1/2} + |C_0| \leqslant (2\sqrt{2} - \sqrt{5/3})A^{1/2}, \\ |\theta|^2 - |\mu|^2 = 2\,\langle \mu, \eta\rangle + |\eta|^2 = \mathcal{O}_0(A^{1/2}). \end{gathered} \tag{3.59}$$

From (3.10)–(3.13) and (3.59), for $A > A_2$ we have, on $\mathcal{M}$,

$$\alpha_A(|\mu|^2) = \mathcal{O}_0(1), \quad \beta_A = |\mu|^2 + \mathcal{O}_0(A^{1/2}) \quad \text{and} \quad \alpha_A'(|\mu|^2) = \phi_A'(\beta_A) = \mathcal{O}_0(A^{-1}). \tag{3.60}$$

If $|\mu|^2 \leqslant \frac{4}{3}A$, then (3.10), (3.11), (3.13) and (3.59) yield

$$\begin{aligned}\frac{\beta_A}{A}-1&=\left(\frac{|\mu|^2}{A}-1\right)\left[1+\frac{1}{A}\left(\frac{|\mu|^2}{A}-1\right)(|\theta|^2-|\mu|^2)\right]\\&=\left(\frac{|\mu|^2}{A}-1\right)\left(1+\mathcal{O}_0(A^{-1/2})\right).\end{aligned}\tag{3.61}$$

If $|\mu|^2 \geqslant \frac{4}{3}A$, then, by (3.60), for $A>A_2$ large enough we have

$$\beta_A \geqslant \tfrac{4}{3}A+\mathcal{O}_0(A^{1/2}) > \tfrac{6}{5}A.\tag{3.62}$$

By (3.3), (3.61) and (3.62), we have $\beta_A>A$ on $\mathcal{M}\setminus\partial\mathcal{M}$ for $A>A_2$ large enough.

On the other hand, if $|\mu|^2\leqslant\frac{5}{3}A$, then by (3.60), for $A>A_2$ large enough, $\beta_A<2A$.

By (3.10), (3.11) and (3.13), if $|\mu|^2\geqslant\frac{5}{3}A$, then

$$\alpha_A(|\mu|^2)=1,\quad \alpha'_A(|\mu|^2)=0\quad\text{and}\quad \beta_A=|\theta|^2.\tag{3.63}$$

Combining with (3.3) we have $\beta_A<2A$ on $\mathcal{M}\setminus\partial\mathcal{M}$ for $A>A_2$ large enough. Thus there exists $A_7\geqslant A_2$ such that (3.55) holds for $A>A_7$. Note that $\tilde\phi(0)=1$, $\tilde\phi(1)=0$ and $\tilde\phi'<0$ on $(0,1]$. Thus (3.11) and (3.55) imply (3.56).

The identity (3.57a) follows immediately from (3.51), (3.59) and (3.60).

From (3.14a), (3.56) and (3.59), we obtain, for $A>A_7$,

$$|\varrho_A|\leqslant|\theta|+|\eta|<2A^{1/2}\quad\text{on } \mathcal{M}.\tag{3.64}$$

From (3.4), (3.51), (3.57a), (3.60) and (3.64), we get (3.57b).

We now prove (3.57c). If $|\mu|^2\leqslant\frac{4}{3}A$, then by (3.60), we have $\beta_A<\frac{15}{11}A$ for $A>A_7$ large enough. Then (3.10), (3.11) and (3.61) imply

$$\begin{aligned}\alpha_A(|\mu|^2)&=\left(\frac{|\mu|^2}{A}-1\right)^2,\\1-\phi_A(\beta_A)&=\left(\frac{\beta_A}{A}-1\right)^3=\left(\frac{|\mu|^2}{A}-1\right)^3(1+\mathcal{O}_0(A^{-1/2})),\\\phi'_A(\beta_A)&=-\frac{3}{A}\left(\frac{\beta_A}{A}-1\right)^2.\end{aligned}\tag{3.65}$$

From (3.4), (3.51), (3.61), (3.64) and (3.65), we deduce that

$$\begin{aligned}\tau_{A4}&=(1-\phi_A(\beta_A))\left[1+\frac{6}{A}\left(\frac{|\mu|^2}{A}-1\right)(1+\mathcal{O}_0(A^{-1/2}))\langle\varrho_A,\eta\rangle\right]\\&=(1-\phi_A(\beta_A))(1+\mathcal{O}_0(A^{-1/2})).\end{aligned}\tag{3.66}$$

If $|\mu|^2 \geqslant \frac{4}{3}A$, then, by (3.10), (3.11) and (3.62), we have

$$1-\phi_A(\beta_A) \geqslant 1-\phi_A\left(\tfrac{6}{5}A\right) = 5^{-3} > 0,$$

from which (3.57c) holds, since, in view of (3.4), (3.60) and (3.64),

$$\phi'_A(\beta_A)\alpha_A(|\mu|^2)\langle \varrho_A, \eta\rangle = \mathcal{O}_0(A^{-1/2}).$$

Together with (3.66), this implies (3.57c).

For the proof of (3.57d) and (3.58), we first consider the region $|\mu|^2 \geqslant \frac{5}{3}A$ in $\mathcal{M}$. By (3.51) and (3.63), we get

$$\tau_{A5} = 1-\phi_A(\beta_A)-\alpha_A(|\mu|^2) = -\phi_A(\beta_A). \tag{3.67}$$

Thus (3.57d) holds. From (3.22), (3.56) and (3.67), we get (3.58).

By (3.10), (3.12) and (3.60), we find that, for $A > A_7$,

$$\phi_A(\beta_A) = \phi_A(|\mu|^2)+\mathcal{O}_0(A^{-1/2}) \quad \text{on } \mathcal{M}. \tag{3.68}$$

If $\frac{4}{3}A \leqslant |\mu|^2 \leqslant \frac{5}{3}A$, then, from (3.10) and (3.68), for A large enough we have

$$1-\phi_A(\beta_A)-\alpha_A(|\mu|^2) \leqslant -\tfrac{1}{27}. \tag{3.69}$$

By (3.51), (3.59), (3.60) and (3.69), we get (3.57d) and (3.58).

Finally, if $|\mu|^2 \leqslant \frac{4}{3}A$, by (3.51), (3.57a) and (3.65), the following identities hold for $A > A_7$ large enough:

$$\begin{aligned} 1-\phi_A(\beta_A)-\alpha_A(|\mu|^2) &= -\left(\frac{|\mu|^2}{A}-1\right)^2\left[1-\left(\frac{|\mu|^2}{A}-1\right)(1+\mathcal{O}_0(A^{-1/2}))\right], \\ \tau_{A5} &= [1-\phi_A(\beta_A)](1+\mathcal{O}_0(A^{-1/2}))-\alpha_A(|\mu|^2) \\ &= [1-\phi_A(\beta_A)-\alpha_A(|\mu|^2)](1+\mathcal{O}_0(A^{-1/2})). \end{aligned} \tag{3.70}$$

From (3.21) and the first identity in (3.70), we get (3.58) in this case.

Combining the three cases discussed above, we conclude that there exists $A_6 \geqslant A_7$ such that (3.57d) and (3.58) hold for $A > A_6$. The proof of Lemma 3.12 is complete. □

The following lemma will also be used in the proof of Lemma 3.9.

Lemma 3.13. *There exists $A_8 \geqslant A_6$ such that, for any $A > A_8$,*

$$1 < \frac{(1-\phi_A(\beta_A))^2-\alpha_A(|\mu|^2)}{1-\phi_A(\beta_A)-\alpha_A(|\mu|^2)} < 12 \quad \text{on } \mathcal{M}\setminus\partial\mathcal{M}. \tag{3.71}$$

Proof. By (3.56) and (3.58), we have

$$(1-\phi_A(\beta_A))^2-\alpha_A(|\mu|^2)<1-\phi_A(\beta_A)-\alpha_A(|\mu|^2)<0 \quad \text{on } \mathcal{M}\backslash\partial\mathcal{M}. \tag{3.72}$$

To complete the proof of (3.71), we have to show that

$$11-10\phi_A(\beta_A)-11\alpha_A(|\mu|^2)-\phi_A(\beta_A)^2<0 \quad \text{on } \mathcal{M}\backslash\partial\mathcal{M}. \tag{3.73}$$

We examine three cases. First, if $|\mu|^2\geqslant\frac{5}{3}A$, then (3.73) follows from (3.56) and (3.63). Secondly, if $|\mu|^2\leqslant\frac{4}{3}A$, then by (3.65) we get

$$\begin{aligned}11-10\phi_A(\beta_A)-11\alpha_A(|\mu|^2)-\phi_A(\beta_A)^2 &\leqslant -11\alpha_A(|\mu|^2)+12(1-\phi_A(\beta_A))\\ &\leqslant \left(\frac{|\mu|^2}{A}-1\right)^2(-7+\mathcal{O}_0(A^{-1/2})).\end{aligned} \tag{3.74}$$

By (3.74), we see that (3.73) holds for A large enough.

Thirdly, if $\frac{4}{3}A\leqslant|\mu|^2\leqslant\frac{5}{3}A$, then from (3.69), for $A>0$ large enough, we have

$$11-10\phi_A(\beta_A)-11\alpha_A(|\mu|^2)-\phi_A(\beta_A)^2\leqslant-\tfrac{11}{27}+\phi_A(\beta_A)-\phi_A(\beta_A)^2\leqslant-\tfrac{17}{108}. \tag{3.75}$$

This completes the proof of Lemma 3.13. □

By (3.57b), we may and will assume that A is large enough so that $\tau_{A2}>\frac{1}{2}$. Set

$$\begin{aligned}\tau_{A6} &= -2\phi_A'(\beta_A)\alpha_A''(|\mu|^2)\langle\varrho_A,\eta\rangle(|\theta|^2-|\mu|^2)\left(\frac{\tau_{A4}}{\tau_{A2}}\right)^2+4\phi_A'(\beta_A)\alpha_A'(|\mu|^2)\langle\varrho_A,\eta\rangle\frac{\tau_{A4}}{\tau_{A2}}\\ &\quad +2[-\phi_A''(\beta_A)\langle\varrho_A,\eta\rangle+\phi_A'(\beta_A)^2|\eta|^2]\left(\frac{\tau_{A5}}{\tau_{A2}}\right)^2+2\phi_A'(\beta_A)\frac{\tau_{A5}}{\tau_{A2}},\\ \tau_{A7} &= 2\phi_A'(\beta_A)\alpha_A'(|\mu|^2)\langle\varrho_A,\eta\rangle\frac{(\tau_{A2}-\tau_{A4})\tau_{A4}}{\tau_{A2}^2}\\ &\quad -2[-\phi_A''(\beta_A)\langle\varrho_A,\eta\rangle+\phi_A'(\beta_A)^2|\eta|^2]\alpha_A(|\mu|^2)\frac{(\tau_{A2}-\tau_{A4})\tau_{A5}}{\tau_{A2}^2}\\ &\quad +\phi_A'(\beta_A)\left[\left(\frac{\tau_{A2}-\tau_{A4}}{\tau_{A2}}+1-2\phi_A(\beta_A)\right)\frac{\tau_{A5}}{\tau_{A2}}-\alpha_A(|\mu|^2)\frac{\tau_{A2}-\tau_{A4}}{\tau_{A2}}\right].\end{aligned} \tag{3.76}$$

LEMMA 3.14. *For $A>0$ large enough, the following identities hold on $\mathcal{M}$:*

$$\begin{aligned}\tau_{A6} &= 2\phi_A'(\beta_A)[1-\phi_A(\beta_A)-\alpha_A(|\mu|^2)](1+\mathcal{O}_0(A^{-1/2})),\\ \tau_{A7} &= \phi_A'(\beta_A)[(1-\phi_A(\beta_A))^2-\alpha_A(|\mu|^2)](1+\mathcal{O}_0(A^{-1/2})).\end{aligned} \tag{3.77}$$

In particular,

$$\begin{cases}\tau_{A6}>0 \text{ and } \tau_{A7}>0, & \text{if } (x,y)\in\mathcal{M}\backslash\partial\mathcal{M},\\ \tau_{A6}=0 \text{ and } \tau_{A7}=0, & \text{if } (x,y)\in\partial\mathcal{M}=\mathcal{M}_1\cup\mathcal{M}_2,\end{cases} \tag{3.78}$$

and

$$\tau_{A7}\leqslant 6\tau_{A6}(1+\mathcal{O}_0(A^{-1/2})). \tag{3.79}$$

Proof. Note that, from (3.4), (3.10), (3.11), (3.60) and (3.64), on $\mathcal{M}$ we have

$$\begin{aligned} \alpha_A'(|\mu|^2)\langle\varrho_A,\eta\rangle=\mathcal{O}_0(A^{-1/2}), \quad \alpha_A''(|\mu|^2)\langle\varrho_A,\eta\rangle=\mathcal{O}_0(A^{-3/2}), \\ -\phi_A''(\beta_A)\langle\varrho_A,\eta\rangle+\phi_A'(\beta_A)^2|\eta|^2=\mathcal{O}_0(A^{-3/2}). \end{aligned} \tag{3.80}$$

Recall that $\tilde{\phi}'<0$ on $(0,1]$. By (3.12), (3.55) and the second equation of (3.60), there exist $C>0$ and $A_{10}>0$ such that, for $A>A_{10}$,

$$\begin{cases} \phi_A'(\beta_A)<0, & \text{on } \mathcal{M}\setminus\partial\mathcal{M}, \\ |\phi_A'(\beta_A)|\geqslant \dfrac{C}{A}, & \text{if } |\mu|^2\geqslant\dfrac{4A}{3}. \end{cases} \tag{3.81}$$

By Lemma 3.12, (3.59), (3.76) and (3.80), we get

$$\begin{aligned} \tau_{A6}=\phi_A'(\beta_A)\mathcal{O}_0(A^{-1})[1-\phi_A(\beta_A)]^2 \\ +\phi_A'(\beta_A)\mathcal{O}_0(A^{-1/2})[1-\phi_A(\beta_A)]+\mathcal{O}_0(A^{-3/2})[1-\phi_A(\beta_A)-\alpha_A(|\mu|^2)]^2 \\ +2\phi_A'(\beta_A)[1-\phi_A(\beta_A)-\alpha_A(|\mu|^2)](1+\mathcal{O}_0(A^{-1/2})). \end{aligned} \tag{3.82}$$

By (3.61), (3.65) and the first equation of (3.70), there exists $C>0$ such that, for $A>0$ large enough, if $|\mu|^2\leqslant\frac{4}{3}A$ then

$$\begin{aligned} 0\leqslant 1-\phi_A(\beta_A)\leqslant C|1-\phi_A(\beta_A)-\alpha_A(|\mu|^2)|, \\ |1-\phi_A(\beta_A)-\alpha_A(|\mu|^2)|\leqslant C|A\phi_A'(\beta_A)|. \end{aligned} \tag{3.83}$$

Due to (3.56), (3.60), (3.69) and (3.81), if $\frac{4}{3}A\leqslant|\mu|^2\leqslant\frac{5}{3}A$, then (3.83) still holds for some constant $C>0$. By (3.83), the first three terms in (3.82) can be controlled by

$$|\phi_A'(\beta_A)(1-\phi_A(\beta_A)-\alpha_A(|\mu|^2))|\mathcal{O}_0(A^{-1/2})$$

if $|\mu|^2\leqslant\frac{5}{3}A$. Thus, from (3.82), the first identity in (3.77) holds when $|\mu|^2\leqslant\frac{5}{3}A$.

For $|\mu|^2\geqslant\frac{5}{3}A$, by (3.63), $\alpha_A''(|\mu|^2)=\alpha_A'(|\mu|^2)=0$, and thus the first two terms of τ_{A6} are zero. By (3.57a)–(3.57d), (3.67), (3.76) and the third equation in (3.80), we have

$$\tau_{A6}=\mathcal{O}_0(A^{-3/2})\phi_A(\beta_A)^2-2\phi_A'(\beta_A)\phi_A(\beta_A)(1+\mathcal{O}_0(A^{-1/2})). \tag{3.84}$$

From (3.56), (3.67), (3.81) and (3.84), the first identity in (3.77) holds when $|\mu|^2\geqslant\frac{5}{3}A$.

From (3.58), the first identity in (3.77) and (3.81), we get (3.78) for τ_{A6}.

For the second identity in (3.77), by Lemma 3.12 and (3.80), we obtain the asymptotics of the terms of τ_{A7} in (3.76) as follows:

$$\begin{aligned} \tau_{A7}=\phi_A'(\beta_A)(1-\phi_A(\beta_A))\mathcal{O}_0(A^{-1/2})+\alpha_A(|\mu|^2)(1-\phi_A(\beta_A)-\alpha_A(|\mu|^2))\mathcal{O}_0(A^{-3/2}) \\ +\phi_A'(\beta_A)[(1-\phi_A(\beta_A)+\mathcal{O}_0(A^{-1/2}))(1-\phi_A(\beta_A)-\alpha_A(|\mu|^2)) \\ -\alpha_A(|\mu|^2)(\phi_A(\beta_A)+\mathcal{O}_0(A^{-1/2}))]. \end{aligned} \tag{3.85}$$

The factor $1-\phi_A(\beta_A)$ in the first term of the right-hand side of (3.85) is from τ_{A4}, while the factor $1-\phi_A(\beta_A)-\alpha_A(|\mu|^2)$ is from τ_{A5}.

If $|\mu|^2\leqslant\frac{5}{3}A$, by (3.72), the first equation of (3.83) (which holds for $|\mu|^2\leqslant\frac{5}{3}A$ as explained after (3.83)), for $A>0$ large enough we get

$$\big|\alpha_A(|\mu|^2)\big|\leqslant(C+1)\big|(1-\phi_A(\beta_A))^2-\alpha_A(|\mu|^2)\big|. \tag{3.86}$$

Thus, by (3.72), (3.83) for $|\mu|^2\leqslant\frac{5}{3}A$ and (3.86), the first two terms of (3.85) are bounded by $|\phi'_A(\beta_A)((1-\phi_A(\beta_A))^2-\alpha_A(|\mu|^2))|\mathcal{O}_0(A^{-1/2})$. From (3.72), (3.85) and (3.86), the second identity in (3.77) holds for $|\mu|^2\leqslant\frac{5}{3}A$.

If $|\mu|^2\geqslant\frac{5}{3}A$, then, by (3.51), (3.63) and (3.67), we have

$$\tau_{A1}=1,\quad \tau_{A2}-\tau_{A4}=\phi_A(\beta_A)\quad\text{and}\quad \tau_{A5}=-\phi_A(\beta_A). \tag{3.87}$$

By (3.57b), (3.63), (3.76), (3.80) and (3.87), we get that the first term of τ_{A7} is zero and

$$\begin{aligned}\tau_{A7}=\phi_A(\beta_A)^2\mathcal{O}_0(A^{-3/2})+\phi'_A(\beta_A)[-\phi_A(\beta_A)^2(1+\mathcal{O}_0(A^{-1/2}))\\ -(1-2\phi_A(\beta_A))\phi_A(\beta_A)(1+\mathcal{O}_0(A^{-1/2}))-\phi_A(\beta_A)(1+\mathcal{O}_0(A^{-1/2}))].\end{aligned} \tag{3.88}$$

From (3.56), (3.81) and (3.88), we get that, if $|\mu|^2\geqslant\frac{5}{3}A$, then

$$\tau_{A7}=-\phi'_A(\beta_A)\phi_A(\beta_A)(2-\phi_A(\beta_A))(1+\mathcal{O}_0(A^{-1/2})). \tag{3.89}$$

Now (3.63) and (3.89) imply the second identity in (3.77) for $|\mu|^2\geqslant\frac{5}{3}A$. By (3.58), (3.71) and (3.81), we get (3.78) for τ_{A7}. From Lemma 3.13, (3.77) and (3.78), we get (3.79). This concludes the proof of Lemma 3.14. □

3.5. Proof of Lemma 3.9 (II): evaluation of $I_{A\cdot}$ over zero$(\psi_A^{\mathcal{M}})$

In this subsection, we evaluate the terms $I_{A\cdot}$ in (3.26) on zero$(\psi_A^{\mathcal{M}})$, the zero set of $\psi_A^{\mathcal{M}}$. The main point is that we use η^N (resp. η^M) to replace μ^N, θ^N and γ_A^N (resp. μ^M and γ_A^M) which are difficult to control over $\mathcal{M}$.

LEMMA 3.15. *On* $\{z\in\mathcal{M}:\psi_A^{\mathcal{M}}(z)=0\}$, *the following identities hold*:

$$\tau_{A2}\mu^M=-\tau_{A4}\eta^M,\qquad \tau_{A2}\gamma_A^M=-2\tau_{A5}\eta^M, \tag{3.90}$$

$$\tau_{A2}\mu^N=-\tau_{A4}\eta^N,\qquad \tau_{A2}\gamma_A^N=-2\tau_{A5}\eta^N\quad\textit{and}\quad \tau_{A2}\theta^N=(\tau_{A2}-\tau_{A4})\eta^N. \tag{3.91}$$

Proof. Let $z\in\mathcal{M}$ be such that $\psi_A^{\mathcal{M}}(z)=0$. In view of (3.54), the equation $\psi_A^M(z)=0$ in (3.35) is equivalent to the first equation of (3.90). Similarly, the equation $\psi_A^N(z)=0$ in (3.35) is equivalent to the first equation of (3.91).

By (3.51)–(3.53) and the first equation of (3.90), we get, at z,

$$\begin{aligned}\tau_{A2}\gamma_A^M &= 2\tau_{A1}\tau_{A2}\mu^M + 2\alpha_A(|\mu|^2)\tau_{A2}\,\eta^M \\ &= -2\tau_{A1}\tau_{A4}\eta^M + 2\alpha_A(|\mu|^2)\tau_{A2}\eta^M = -2\tau_{A5}\eta^M.\end{aligned} \tag{3.92}$$

The second equation in (3.91) follows similarly. By (3.6) and the first equation in (3.91), we get the third equation in (3.91). □

For any $x\in M$, $y\in N$, $W\in T_xM$ and $V\in T_yN$, let $B(W)\in \mathrm{End}(\Lambda(T^{*(0,1)}(M\times N))_{(x,y)}$ be defined by

$$B(W) = ic(J^M W)c(W) + |W|^2. \tag{3.93}$$

Clearly, the endomorphisms $B(W)$ and $ic(W)c(V)$ of $\Lambda(T^{*(0,1)}(M\times N))_{(x,y)}$ are self-adjoint, and $B(J^M W) = B(W) = B(-W)$.

Lemma 3.16. *On $\{z\in\mathcal{M}:\psi_A^{\mathcal{M}}(z)=0\}$, the following identities hold for $I_{A\cdot}$ in (3.26):*

$$\begin{aligned} i(I_{A1}+I_{A2}) = \frac{\tau_{A2}}{2}\sum_{j=1}^{\dim G} & B(V_j^M) + \tau_{A6}B(\eta^M) \\ & + i\tau_{A2}c(J^M V_j^M)c(V_j^N) + 2i\tau_{A6}c(J^M\eta^M)c(\eta^N) \\ & + i\tau_{A4}c(J^N V_j^N)c(V_j^M) + 2i\tau_{A7}c(J^N\eta^N)c(\eta^M), \\ I_{A3} = \tau_{A4}c(J^N V_j^N)&c(V_j^N) + 2\tau_{A7}c(J^N\eta^N)c(\eta^N). \end{aligned} \tag{3.94}$$

Proof. Let $z\in\mathcal{M}$ be such that $\psi_A^{\mathcal{M}}(z)=0$. By (3.6) and (3.51), we get

$$\begin{aligned} (d^M\tau_{A1})^* &= 2\alpha_A''(|\mu|^2)(|\theta|^2-|\mu|^2)J^M\mu^M + 2\alpha_A'(|\mu|^2)J^M\eta^M, \\ (d^N\tau_{A1})^* &= 2\alpha_A'(|\mu|^2)J^N\theta^N. \end{aligned} \tag{3.95}$$

Using (3.6), (3.17) and (3.90), we infer, at z,

$$\begin{aligned} (d^M\beta_A)^* &= J^M\gamma_A^M = -\frac{2\tau_{A5}}{\tau_{A2}}J^M\eta^M, \\ (d^M\langle\varrho_A,\eta\rangle)^* &= J^M\eta^M + 2\phi_A'(\beta_A)|\eta|^2\frac{\tau_{A5}}{\tau_{A2}}J^M\eta^M. \end{aligned} \tag{3.96}$$

By (3.6), (3.51), (3.90), (3.95) and (3.96), at z, we get

$$\begin{aligned} (d^M\tau_{A2})^* &= -2\phi_A'(\beta_A)\langle\varrho_A,\eta\rangle(d^M\tau_{A1})^* \\ &\qquad -2\phi_A''(\beta_A)\langle\varrho_A,\eta\rangle\tau_{A1}(d^M\beta_A)^* - 2\phi_A'(\beta_A)\tau_{A1}(d^M\langle\varrho_A,\eta\rangle)^* \\ &= \Big[4\phi_A'(\beta_A)\langle\varrho_A,\eta\rangle\Big(\alpha_A''(|\mu|^2)(|\theta|^2-|\mu|^2)\frac{\tau_{A4}}{\tau_{A2}} - \alpha_A'(|\mu|^2)\Big) \\ &\qquad -4(-\phi_A''(\beta_A)\langle\varrho_A,\eta\rangle + \phi_A'(\beta_A)^2|\eta|^2)\frac{\tau_{A5}}{\tau_{A2}}\tau_{A1} - 2\phi_A'(\beta_A)\tau_{A1}\Big]J^M\eta^M, \end{aligned} \tag{3.97}$$

and

$$\begin{aligned}(d^M\tau_{A4})^* &= -2\phi_A'(\beta_A)\langle\varrho_A,\eta\rangle\alpha_A'(|\mu|^2)2J^M\mu^M\\ &\quad+(-\phi_A'(\beta_A)-2\phi_A''(\beta_A)\langle\varrho_A,\eta\rangle\alpha_A(|\mu|^2))(d^M\beta_A)^*\\ &\quad-2\phi_A'(\beta_A)\alpha_A(|\mu|^2)(d^M\langle\varrho_A,\eta\rangle)^*\\ &=\bigg[4\phi_A'(\beta_A)\alpha_A'(|\mu|^2)\langle\varrho_A,\eta\rangle\frac{\tau_{A4}}{\tau_{A2}}+2\phi_A'(\beta_A)\frac{\tau_{A5}}{\tau_{A2}}\\ &\quad-4(-\phi_A''(\beta_A)\langle\varrho_A,\eta\rangle+\phi_A'(\beta_A)^2|\eta|^2)\alpha_A(|\mu|^2)\frac{\tau_{A5}}{\tau_{A2}}\\ &\quad-2\phi_A'(\beta_A)\alpha_A(|\mu|^2)\bigg]J^M\eta^M.\end{aligned}\tag{3.98}$$

From (3.6) and (3.54), we get

$$\begin{aligned}(d^M\psi_{Aj})^* &= (d^M\tau_{A2})^*\,\mu_j+(d^M\tau_{A4})^*\,\eta_j+\tau_{A2}J^MV_j^M,\\ (d^N\psi_{Aj})^* &= (d^N\tau_{A2})^*\,\mu_j+(d^N\tau_{A4})^*\,\eta_j+\tau_{A4}J^NV_j^N.\end{aligned}\tag{3.99}$$

From (3.52), (3.76), the first equation of (3.90) and (3.97)–(3.99), we get, at z,

$$\begin{aligned}c((d^M\psi_{Aj})^*)c(V_j^M) &= \tau_{A2}c(J^MV_j^M)c(V_j^M)+2\tau_{A6}c(J^M\eta^M)c(\eta^M),\\ \langle(1-iJ^M)V_j^M,(d^M\psi_{Aj})^*\rangle &= -i\bigg(\tau_{A2}\sum_{j=1}^{\dim G}|V_j^M|^2+2\tau_{A6}|\eta^M|^2\bigg).\end{aligned}\tag{3.100}$$

Using (3.76), the first equation of (3.91) and (3.97)–(3.99), we get, at z,

$$c((d^M\psi_{Aj})^*)c(V_j^N)=\tau_{A2}c(J^MV_j^M)c(V_j^N)+2\tau_{A6}c(J^M\eta^M)c(\eta^N).\tag{3.101}$$

By (3.17), (3.91) and (3.95), it follows that, at z,

$$(d^N\beta_A)^*=2\alpha_A(|\mu|^2)\frac{\tau_{A2}-\tau_{A4}}{\tau_{A2}}J^N\eta^N,\tag{3.102a}$$

$$(d^N\tau_{A1})^*=2\alpha_A'(|\mu|^2)\frac{\tau_{A2}-\tau_{A4}}{\tau_{A2}}J^N\eta^N,\tag{3.102b}$$

$$(d^N\varrho_{Aj})^*=(1-\phi_A(\beta_A))J^NV_j^N-2\phi_A'(\beta_A)\alpha_A(|\mu|^2)\eta_j\frac{\tau_{A2}-\tau_{A4}}{\tau_{A2}}J^N\eta^N.\tag{3.102c}$$

From (3.6), (3.14a), (3.91) and (3.102c), we have

$$\begin{aligned}(d^N\langle\varrho_A,\eta\rangle)^* &= (\theta_j-\phi_A(\beta_A)\eta_j)(d^N\eta_j)^*+\eta_j(d^N\varrho_{Aj})^*\\ &=\bigg[1-2\phi_A(\beta_A)+(1-2\phi_A'(\beta_A)\alpha_A(|\mu|^2)|\eta|^2)\frac{\tau_{A2}-\tau_{A4}}{\tau_{A2}}\bigg]J^N\eta^N.\end{aligned}\tag{3.103}$$

By (3.51), (3.90), (3.95) and (3.102a)–(3.103), we get, at z,

$$\begin{aligned}(d^N\tau_{A2})^* &= -2\phi_A'(\beta_A)\langle\varrho_A,\eta\rangle(d^N\tau_{A1})^*\\ &\quad -2\phi_A''(\beta_A)\langle\varrho_A,\eta\rangle\tau_{A1}(d^N\beta_A)^* - 2\phi_A'(\beta_A)\tau_{A1}(d^N\langle\varrho_A,\eta\rangle)^*\\ &= \bigg[-4\phi_A'(\beta_A)\langle\varrho_A,\eta\rangle\alpha_A'(|\mu|^2)\frac{\tau_{A2}-\tau_{A4}}{\tau_{A2}}\\ &\quad +4(-\phi_A''(\beta_A)\langle\varrho_A,\eta\rangle+\phi_A'(\beta_A)^2|\eta|^2)\alpha_A(|\mu|^2)\frac{\tau_{A2}-\tau_{A4}}{\tau_{A2}}\tau_{A1}\\ &\quad -2\phi_A'(\beta_A)\tau_{A1}\left(1-2\phi_A(\beta_A)+\frac{\tau_{A2}-\tau_{A4}}{\tau_{A2}}\right)\bigg]J^N\eta^N\end{aligned} \tag{3.104}$$

and

$$\begin{aligned}(d^N\tau_{A4})^* &= (-\phi_A'(\beta_A)-2\phi_A''(\beta_A)\langle\varrho_A,\eta\rangle\alpha_A(|\mu|^2))(d^N\beta_A)^*\\ &\quad -2\phi_A'(\beta_A)\alpha_A(|\mu|^2)(d^N\langle\varrho_A,\eta\rangle)^*\\ &= \bigg[4(-\phi_A''(\beta_A)\langle\varrho_A,\eta\rangle+\phi_A'(\beta_A)^2|\eta|^2)\alpha_A(|\mu|^2)^2\frac{\tau_{A2}-\tau_{A4}}{\tau_{A2}}\\ &\quad -2\phi_A'(\beta_A)\alpha_A(|\mu|^2)\left(1-2\phi_A(\beta_A)+2\frac{\tau_{A2}-\tau_{A4}}{\tau_{A2}}\right)\bigg]J^N\eta^N.\end{aligned} \tag{3.105}$$

From (3.52), (3.76), (3.90) (3.91), (3.99), (3.104) and (3.105), we get, at z,

$$\begin{aligned}c((d^N\psi_{Aj})^*)c(V_j^M) &= \tau_{A4}c(J^NV_j^N)c(V_j^M)+2\tau_{A7}c(J^N\eta^N)c(\eta^M),\\ c((d^N\psi_{Aj})^*)c(V_j^N) &= \tau_{A4}c(J^NV_j^N)c(V_j^N)+2\tau_{A7}c(J^N\eta^N)c(\eta^N).\end{aligned} \tag{3.106}$$

By (3.26), (3.93), (3.100), (3.101) and (3.106), we get (3.94). □

Lemma 3.17. *For any $k>0$, the following inequalities hold for $W\in TM$ and $V\in TN$:*

$$B(W)\geqslant 0 \quad \textit{and} \quad ic(W)c(V)\geqslant -\frac{1}{2k}B(W)-k|V|^2. \tag{3.107}$$

Proof. It is enough to prove this for $V=v+\bar v$ and $W=w+\bar w$, with $\{v,w\}$ being an orthonormal basis of $\mathbb{C}^2$ with the standard Hermitian product. Using (2.3) and (3.93), we find that

$$B(W) = -2(w^*\wedge+i_{\bar w})(w^*\wedge-i_{\bar w})+2 = 4w^*\wedge i_{\bar w}. \tag{3.108}$$

Thus the first inequality in (3.107) holds (cf. [22, (2.9) and (2.13)]).

For any $\sigma\in\Lambda\overline{\mathbb{C}^2}^*$, we write $\sigma=\sigma_1w^*\wedge v^*+\sigma_2w^*+\sigma_3v^*+\sigma_4$, where $\sigma_1,\sigma_2,\sigma_3,\sigma_4\in\mathbb{C}$. By (2.3), we get

$$ic(W)c(V)\sigma = 2i(-\sigma_1+\sigma_2v^*-\sigma_3w^*+\sigma_4w^*\wedge v^*). \tag{3.109}$$

From (3.108) and (3.109), we find that, for any $k>0$,

$$\begin{aligned}\langle ic(W)c(V)\sigma,\sigma\rangle &= 4\,\mathrm{Im}(\sigma_1\bar\sigma_4-\sigma_2\bar\sigma_3)\\ &\geqslant -\frac{2}{k}(|\sigma_1|^2+|\sigma_2|^2)-2k|\sigma|^2 = -\frac{1}{2k}\langle B(W)\sigma,\sigma\rangle-2k|\sigma|^2.\end{aligned} \tag{3.110}$$

From (3.110), we get the second inequality of (3.107). □

3.6. Proof of Lemma 3.9 (III): final step

Recall that $z\in\mathcal{M}$ satisfies $\psi_A^{\mathcal{M}}(z)=0$. By Lemma 3.6, $z\in\mathcal{M}\setminus\partial\mathcal{M}$.

By Lemma 3.12, $\tau_{A2},\tau_{A4}>0$ on $\mathcal{M}$ for A large enough. Thus by (3.78), (3.94) and the second equation in (3.107) with $k=8$, we get

$$\begin{aligned} i(I_{A1}+I_{A2}) \geqslant \frac{1}{2}\left(\frac{7}{8}\tau_{A2}-\frac{1}{8}\tau_{A4}\right)\sum_{j=1}^{\dim G} B(V_j^M)-(8\tau_{A2}+8\tau_{A4})\sum_{j=1}^{\dim G}|V_j^N|^2 \\ +\left(\frac{7}{8}\tau_{A6}-\frac{1}{8}\tau_{A7}\right)B(\eta^M)-(16\tau_{A6}+16\tau_{A7})|\eta^N|^2. \end{aligned} \tag{3.111}$$

By Lemma 3.12, for $A>0$ large enough we obtain

$$\tfrac{7}{8}\tau_{A2}-\tfrac{1}{8}\tau_{A4}=\tfrac{3}{4}+\tfrac{1}{8}\phi_A(\beta_A)+\mathcal{O}_0(A^{-1/2})\geqslant\tfrac{1}{2}. \tag{3.112}$$

By Lemma 3.14, for $A>0$ large enough, as $z\in\mathcal{M}\setminus\partial\mathcal{M}$,

$$\tfrac{7}{8}\tau_{A6}-\tfrac{1}{8}\tau_{A7}\geqslant\tfrac{1}{8}\tau_{A6}(1+\mathcal{O}_0(A^{-1/2}))>0. \tag{3.113}$$

Recall that V_j^N and η are defined on the compact manifold N. By Lemma 3.12, (3.77), (3.107) and (3.111)–(3.113), there exists $C'>0$ such that, for $A>0$ large enough,

$$i(I_{A1}+I_{A2})\geqslant -C'\,\mathrm{Id} \quad \text{on } \{z\in\mathcal{M}:\psi_A^{\mathcal{M}}(z)=0\}. \tag{3.114}$$

By (3.57c), (3.77) and (3.94), there exists $C''>0$ such that, for $A>0$ large enough, we have

$$|I_{A3}|\leqslant C'' \quad \text{on } \{z\in\mathcal{M}:\psi_A^{\mathcal{M}}(z)=0\}. \tag{3.115}$$

By Lemma 3.12, (3.4), (3.54) and (3.59), for $A>0$ large enough we get, over $\mathcal{M}$,

$$\begin{aligned} &2\langle\psi_A,\theta\rangle=2\tau_{A2}|\mu|^2+2\tau_{A4}|\eta|^2+2(\tau_{A2}+\tau_{A4})\langle\mu,\eta\rangle\geqslant 2A+\mathcal{O}_0(A^{1/2})\geqslant A,\\ &|\psi_A|=\mathcal{O}_0(A^{1/2}). \end{aligned} \tag{3.116}$$

By (3.114)–(3.116), we get (3.32). This completes the proof of Lemma 3.9.

4. Functoriality of quantization

This section is organized as follows. In §4.1, we establish the product formula for quantization, Theorem 0.4. In §4.2, we explain the compatibility of quantization and its restriction to a subgroup.

We will use the assumptions and notation in the introduction and in §3.1.

4.1. Proof of Theorem 0.4

Let $c>0$ be a regular value of $|\theta|^2$. By [23, Theorem 4.3] and [18, Proposition 7.10] (cf. also Theorems 1.5 and 2.1), the following identity holds:

$$\mathrm{Ind}(\sigma_{L\otimes F,\theta}^{(M\times N)_c})_{\gamma=0}=Q((L\otimes F)_{\gamma=0}). \tag{4.1}$$

Here 0 need not to be a regular value of θ.

On the other hand, by Theorem 0.1 (b), we have

$$\mathrm{Ind}(\sigma_{L\otimes F,\theta}^{(M\times N)_c})_{\gamma=0}=Q(L\otimes F)_{\gamma=0}. \tag{4.2}$$

Therefore, by (4.1) and (4.2), we get (0.12). Thus, to prove Theorem 0.4, we only need to prove the following identity, which was stated in (0.13):

$$Q(L\otimes F)_{\gamma=0}=\sum_{\gamma\in\Lambda_+^*}Q(L)_\gamma\cdot Q(F)_{\gamma,*}. \tag{4.3}$$

We first establish the following lemma, which was stated in (0.14).

Lemma 4.1. *There exists $a'\geqslant 0$ such that for any regular value $a>a'$ of $|\mu|^2: M\to\mathbb{R}$,*

$$\sum_{\gamma\in\Lambda_+^*}Q(L)_\gamma\cdot Q(F)_{\gamma,*}=\mathrm{Ind}(\sigma_{L\otimes F,\mu}^{M_a\times N})_{\gamma=0}. \tag{4.4}$$

Proof. We denote the finite set $\{\gamma\in\Lambda_+^*: Q(F)_{\gamma,*}\neq 0\}$ by $\Lambda_+^*(F)$. By Theorem 0.1, there exists $a_1\geqslant 0$ such that, for any regular value $a>a_1$ of $|\mu|^2$, we have

$$Q(L)_\gamma=\mathrm{Ind}(\sigma_{L,\mu}^{M_a})_\gamma \quad \text{for any } \gamma\in\Lambda_+^*(F). \tag{4.5}$$

Let $a>a_1$ be a regular value of $|\mu|^2$. For $0\leqslant t\leqslant 1$, let σ_t be the symbol on $M_a\times N$ defined to be a deformation of $\sigma_{L\otimes F,\mu}^{M_a\times N}$ as follows:

$$\sigma_t=\sigma_{L\otimes F,\mu}^{M_a\times N}-(1-t)i\pi^*c(\mu^N), \tag{4.6}$$

where $\pi: T(M_a\times N)\to M_a\times N$ is the canonical projection (cf. (1.2)).

By (1.2) and (3.7), when $t=0$, σ_0 is the external product of $\sigma_{L,\mu}^{M_a}$ and $\sigma_{F,0}^N$ in the sense of [1] (cf. [18, (3.11)]). Then, by the multiplicativity of the transversal index ([1, Theorem 3.5], [18, (3.12)]) and by (4.5), we get

$$\sum_{\gamma\in\Lambda_+^*}Q(L)_\gamma\cdot Q(F)_{\gamma,*}=\mathrm{Ind}(\sigma_0)_{\gamma=0}. \tag{4.7}$$

For $0 \leqslant t \leqslant 1$, set

$$V_t = \mu^{M_a \times N} - (1-t)\mu^N. \tag{4.8}$$

Then, by (3.7) and (4.8), we have

$$V_t = \mu^M + t\mu^N. \tag{4.9}$$

As $a > a_1$ is a regular value of $|\mu|^2$, μ^M does not vanish on ∂M_a. From (4.9), $\mu^{M_a \times N}$ and V_t do not vanish on $\partial(M_a \times N) = \partial M_a \times N$ for every $0 \leqslant t \leqslant 1$.

By (1.2), (4.6), (4.8) and (4.9), the set

$$\{(z,v) \in T_G(M_a \times N) : \text{there exists } 0 \leqslant t \leqslant 1 \text{ such that } \sigma_t(z,v) \text{ is non-invertible}\},$$

which is a subset of

$$\{(x,y,0) \in T_G(M_a \times N) : \mu^M(x) = 0, x \in M_a \text{ and } y \in N\},$$

is a compact subset of $T_G(\widehat{M_a \times N})$. Thus σ_t forms a continuous family of transversally elliptic symbols in the sense of [1] and [18, §3]. Hence, by (4.6), (4.7) and the homotopy invariance of the transversal index (cf. [1, Theorems 2.6 and 3.7], [18, §3]), we get (4.4). The proof of Lemma 4.1 is complete. □

Let $A > 0$ be a regular value of both $|\mu|^2$ and $\frac{1}{2}|\theta|^2$. We may and will assume that $A > 0$ is large enough so that both Theorem 3.2 and Lemma 4.1 hold.

Let $Y : \mathcal{M} \to \mathfrak{g}$ be a G-equivariant map such that (3.8) holds. By the additivity of the transversal index (cf. [1, Theorem 3.7, §6] and [18, Proposition 4.1]), we have

$$\operatorname{Ind}(\sigma_{L\otimes F,\theta}^{(M\times N)_{2A}})_{\gamma=0} = \operatorname{Ind}(\sigma_{L\otimes F,Y}^{\mathcal{M}})_{\gamma=0} + \operatorname{Ind}(\sigma_{L\otimes F,\mu}^{M_A\times N})_{\gamma=0}. \tag{4.10}$$

By Theorems 1.5 and 3.2, we find that

$$\operatorname{Ind}(\sigma_{L\otimes F,Y}^{\mathcal{M}})_{\gamma=0} = 0. \tag{4.11}$$

By Theorem 0.1 (b), (4.4), (4.10) and (4.11), we get (4.3). The proof of Theorem 0.4 is complete.

4.2. Restriction commutes with quantization

Set

$$Q_G(L)^{-\infty} = \bigoplus_{\gamma \in \Lambda_+^*} Q(L)_\gamma \cdot V_\gamma^G \in R[G]. \tag{4.12}$$

By Theorem 0.2, $Q_G(L)^{-\infty}$ is equal to the formal geometric quantization in the sense of Weitsman [27, Definition 4.1] (where the fundamental properness assumption of the moment map was introduced into the framework of geometric quantization) and Paradan [21, Definition 1.2].

On the other hand, let H be a compact connected subgroup of G such that the moment map of the induced action of H on M is also proper. By combining Theorem 0.2 and (4.12) with [21, Theorem 1.3], one gets the following relation between $Q_G(L)^{-\infty}$ and $Q_H(L)^{-\infty}$.

Theorem 4.2. *Any irreducible representation of H has a finite multiplicity in $Q_G(L)^{-\infty}$. Moreover, when both sides are viewed as virtual representation spaces of H, one has*

$$Q_G(L)^{-\infty}|_H = Q_H(L)^{-\infty}. \tag{4.13}$$

It would be interesting to give a direct proof of Theorem 4.2.

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