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An inner amenable group whose von Neumann algebra does not have property Gamma

by

Stefaan Vaes

KU Leuven
Leuven, Belgium

1. Introduction

In order to exhibit two non-isomorphic II_1 factors, Murray and von Neumann defined in [MN] *property Gamma* as the existence of a non-trivial sequence of asymptotically central elements. They showed that the group von Neumann algebra $L\mathbb{F}_n$ of the free group $\mathbb{F}_n$, $n\geqslant 2$, does not have property Gamma, while the group von Neumann algebra LS_∞ of the group of finite permutations of $\mathbb{N}$ has property Gamma.

More precisely, a II_1 factor M with trace τ has property Gamma if there exists a sequence of unitary operators x_n in M satisfying $\tau(x_n)=0$ for all n and $\|x_n y - y x_n\|_2 \to 0$ for all $y\in M$. Here $\|\cdot\|_2$ denotes the L^2-norm on M given by $\|x\|_2=\sqrt{\tau(xx^*)}$.

In [E], Effros aims to express property Gamma for a group von Neumann algebra LG in terms of a group-theoretic property. In this respect, he introduced the notion of *inner amenability* for a countable group G, by requiring the existence of a mean on $G\setminus\{e\}$ which is invariant under all inner automorphisms. More precisely, G is inner amenable if there exists a finitely additive measure m on the subsets of $G\setminus\{e\}$, with total mass 1 and satisfying $m(gXg^{-1})=m(X)$ for all $X\subset G\setminus\{e\}$ and all $g\in G$. In [E], Effros proved that if LG is a II_1 factor with property Gamma, then G is inner amenable. He posed the question whether the converse holds: does LG have property Gamma whenever G is an inner amenable group with infinite conjugacy classes (icc)? This problem attracted a lot of attention over the years, see e.g. [H, Problem 2] and the survey [BH]. In attempts to answer Effros' question, several groups were first shown to be inner amenable (e.g. Thompson's group [J1] and Baumslag–Solitar groups [St]), but later shown to satisfy property Gamma as well (e.g. [J2]).

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We solve Effros' question in the negative by providing concrete examples of inner amenable icc groups G such that LG does not have property Gamma. Our construction is inspired by [Sc, Example 2.7], which provides examples of strongly ergodic group actions that do not have spectral gap.

2. Construction of the group G

Fix a sequence of distinct prime numbers p_n. We define as follows a countable group G. Define

$$H_n := \left(\frac{\mathbb{Z}}{p_n\mathbb{Z}}\right)^3 \quad \text{and} \quad K := \bigoplus_{n=0}^{\infty} H_n.$$

Put $\Lambda = \mathrm{SL}(3,\mathbb{Z})$, which acts on H_n by automorphisms in the natural way. We denote this action by $g\cdot x$ whenever $g\in\Lambda$ and $x\in H_n$. We let Λ act on K diagonally: $(g\cdot x)_n = g\cdot x_n$ for all $g\in\Lambda$ and $n\in\mathbb{N}$. For every $N\in\mathbb{N}$, define the subgroup $K_N < K$ as

$$K_N := \bigoplus_{n=N}^{\infty} H_n.$$

We put $G_0 = K\rtimes\Lambda$ and inductively define G_{N+1} as the following amalgamated free product:

$$G_N \hookrightarrow G_{N+1} := G_N *_{K_N} (K_N\times\mathbb{Z}).$$

Note here that we view K_N as a subgroup of G_N by considering $K_N < K < G_0 < G_N$. We finally define G as the inductive limit of the increasing sequence of groups $G_0\subset G_1\subset \dots$.

THEOREM 1. *The group G is inner amenable and has infinite conjugacy classes, while the* II_1 *factor LG does not have property Gamma.*

3. Proof of Theorem 1

We denote by LG the group von Neumann algebra of a countable group G, generated by the unitary operators $\{u_g\}_{g\in G}$. We denote by $\{\delta_g\}_{g\in G}$ the canonical orthonormal basis of $\ell^2(G)$. Then, $\ell^2(G)$ is an LG-LG-bimodule, given by $u_g\delta_k u_h = \delta_{gkh}$. On LG, we consider the usual trace given by $\tau(x) = \langle\delta_e, x\delta_e\rangle$.

LEMMA 2. *For every $g\in G\setminus K$, the set $\{hgh^{-1} : h\in\Lambda\}$ is infinite. Also, G has infinite conjugacy classes.*

Proof. If $g \in G \setminus G_0$, take $N \geqslant 0$ such that $g \in G_{N+1} \setminus G_N$. From the description of G_{N+1} as the amalgamated free product $G_{N+1} = G_N *_{K_N} (K_N \times \mathbb{Z})$, it follows that the elements hgh^{-1}, $h \in G_N$, are all distinct. In particular, $\{hgh^{-1} : h \in \Lambda\}$ is infinite. If $g \in G_0 \setminus K$, the set $\{hgh^{-1} : h \in \Lambda\}$ is infinite because Λ has infinite conjugacy classes.

Finally, assume that $g \neq e$ has a finite conjugacy class. By the first part of the proof, $g \in K$. Taking N large enough, $g \in K \setminus K_N$. So, $g \in G_N \setminus K_N$ and we arrive at the contradiction that g has a finite conjugacy class in $G_N *_{K_N} (K_N \times \mathbb{Z})$. □

Denote by (A_n, τ) the tracial von Neumann algebra with $A_n \cong \mathbb{C}^2$ and with minimal projections e_n and $1 - e_n$ such that $\tau(e_n) = p_n^{-3}$.

LEMMA 3. *Define* $(A, \tau) := \overline{\bigotimes}_{n=0}^{\infty} (A_n, \tau)$. *There is a unique trace-preserving bijective isomorphism*

$$\alpha : A \longrightarrow LG \cap (L\Lambda)'$$

satisfying

$$\alpha(e_n) = p_n^{-3} \sum_{h \in H_n} u_h \quad \text{for all } n.$$

Moreover, $\alpha(e_n)$ *lies in the center of* LG_{n+1}.

Proof. By Lemma 2, $LG \cap (L\Lambda)' = LK \cap (L\Lambda)'$. Put $B_n = \ell^\infty(H_n)$ and define the trace τ on B_n given by the normalized counting measure. View $A_n \subset B_n$ in a trace-preserving way and such that e_n corresponds to the function $\chi_{\{0\}}$.

Define $\Lambda \overset{\theta}{\curvearrowright} B_n$ by $(\theta_g(F))(x) = F(g^{-1} \cdot x)$ for all $g \in \Lambda$, $x \in H_n$ and $F \in B_n$. Define $\Lambda \overset{\sigma}{\curvearrowright} LK$ by $\sigma_g(u_x) = u_{g \cdot x}$ for all $g \in \Lambda$ and $x \in K$. We have $H_n \cong \widehat{H}_n$, and the Fourier transform yields a trace-preserving isomorphism $\alpha_n : B_n \to LH_n$ satisfying $\alpha_n \circ \theta_g = \sigma_{(g^{-1})^T} \circ \alpha_n$. Here, g^T denotes the transpose of $g \in \Lambda = \mathrm{SL}(3, \mathbb{Z})$.

Put $(B, \tau) = \overline{\bigotimes}_{n=1}^{\infty} (B_n, \tau)$ and define $\Lambda \overset{\theta}{\curvearrowright} B$ diagonally. The isomorphisms α_n combine into a trace-preserving isomorphism $\alpha : B \to LK$ satisfying $\alpha \circ \theta_g = \sigma_{(g^{-1})^T} \circ \alpha$ for all $g \in \Lambda$. We view A as a von Neumann subalgebra of B. In order to prove that α is an isomorphism of A onto $LK \cap (L\Lambda)'$, we have to show that $B^\Lambda = A$, where, by definition, $B^\Lambda = \{b \in B : \theta_g(b) = b \text{ for all } g \in \Lambda\}$.

The orbits of the diagonal action $\Lambda \curvearrowright H_0 \times ... \times H_N$ are precisely the sets $\mathcal{U}_0 \times ... \times \mathcal{U}_N$, where every $\mathcal{U}_i$ is either $\{0\}$ or $H_i \setminus \{0\}$. Hence,

$$\left(\bigotimes_{n=0}^{N} B_n \right)^{\Lambda} = \bigotimes_{n=0}^{N} A_n.$$

Letting $N \to \infty$, we get that $B^\Lambda = A$.

Denote by $\mathcal{Z}(M)$ the center of a von Neumann algebra M. Observe that

$$\mathcal{Z}(LG_m) \cap LK_m \subset \mathcal{Z}(LG_{m+1}).$$

Since $\alpha(e_n) \in \mathcal{Z}(LG_0)$ and $\alpha(e_n) \in LK_m$ for all $m \leqslant n$, we have $\alpha(e_n) \in \mathcal{Z}(LG_{n+1})$. □

Proof of Theorem 1. We saw in Lemma 2 that G has infinite conjugacy classes.

Embed $LG \hookrightarrow \ell^2(G)$ by $x \mapsto x\delta_e$. Define $\xi_n = p_n^{3/2} \alpha(e_n)\delta_e$. Then, ξ_n is a sequence of unit vectors in $\ell^2(G)$ satisfying $\langle \delta_e, \xi_n \rangle = p_n^{-3/2} \to 0$. Moreover, by Lemma 3, we have $\alpha(e_n) \in \mathcal{Z}(LG_{n+1})$, so that $u_g \xi_n u_g^* = \xi_n$ whenever $g \in G_N$ and $N \leqslant n+1$. Hence, for every $g \in G$, the sequence $\|u_g \xi_n u_g^* - \xi_n\|_2$ is eventually zero. It follows that the adjoint representation of G on $\ell^2(G) \ominus \mathbb{C}\delta_e$ weakly contains the trivial representation. Hence, G is inner amenable (see e.g. [BH, Théorème 1]).

Suppose that x_n is a sequence of unitary operators in LG, such that $\|x_n y - y x_n\|_2 \to 0$ for all $y \in LG$. We have to prove that $\|x_n - \tau(x_n)1\|_2 \to 0$. Denote by $\pi: G \to \mathcal{U}(\ell^2(G))$ the adjoint representation, defined by $\pi(g)\xi = u_g \xi u_g^*$. Then $\xi_n := x_n \delta_e$ is a sequence of almost π-invariant unit vectors. Denote by P the orthogonal projection of $\ell^2(G)$ onto the closed subspace of $\pi(\Lambda)$-invariant vectors. Since Λ has property (T), it follows that

$$\|\xi_n - P(\xi_n)\|_2 \to 0.$$

This means that $\|x_n - y_n\|_2 \to 0$, where $y_n := E_{LG \cap (L\Lambda)'}(x_n)$ and $E_{LG \cap (L\Lambda)'}$ denotes the unique trace-preserving conditional expectation of LG onto $LG \cap (L\Lambda)'$.

As we have seen in the proof of Lemma 3, we have $LG \cap (L\Lambda)' = LK \cap (L\Lambda)'$. In particular, y_n belongs to the unit ball of LK. Recall that we inductively defined

$$G_{N+1} = G_N *_{K_N} (K_N \times \mathbb{Z}).$$

Denote by g_{N+1} the canonical generator of the copy of $\mathbb{Z}$ appearing in this definition of G_{N+1}. Using the fact that $u_{g_{N+1}}$ commutes with LK_N, we get that

$$\begin{aligned} u_{g_{N+1}} y_n u_{g_{N+1}}^* - y_n &= u_{g_{N+1}} (y_n - E_{LK_N}(y_n)) u_{g_{N+1}}^* + u_{g_{N+1}} E_{LK_N}(y_n) u_{g_{N+1}}^* - y_n \\ &= u_{g_{N+1}} (y_n - E_{LK_N}(y_n)) u_{g_{N+1}}^* + E_{LK_N}(y_n) - y_n. \end{aligned}$$

Because the sets $g_{N+1}(K \setminus K_N) g_{N+1}^{-1}$ and K are disjoint, we have that the elements $u_{g_{N+1}} \big(y_n - E_{LK_N}(y_n)\big) u_{g_{N+1}}^*$ and $E_{LK_N}(y_n) - y_n$ are orthogonal. Hence,

$$\|u_{g_{N+1}} y_n u_{g_{N+1}}^* - y_n\|_2 \geqslant \|u_{g_{N+1}} (y_n - E_{LK_N}(y_n)) u_{g_{N+1}}^*\|_2 = \|y_n - E_{LK_N}(y_n)\|_2.$$

So, for every N, we get that $\|y_n - E_{LK_N}(y_n)\|_2 \to 0$ as $n \to \infty$.

Fix N. As y_n commutes with $L\Lambda$, also $E_{LK_N}(y_n)$ commutes with $L\Lambda$. By Lemma 3, take a sequence a_n in the unit ball of $\overline{\bigotimes}_{k=N}^{\infty}(A_k,\tau)$ such that $E_{LK_N}(y_n)=\alpha(a_n)$. Since the sequence p_n^{-3} is summable, the product of the projections $1-e_n$, $n\geqslant N$, converges to a minimal projection f_N in $\overline{\bigotimes}_{k=N}^{\infty}(A_k,\tau)$, with

$$\tau(f_N)=\prod_{n=N}^{\infty}(1-p_n^{-3}).$$

Put $\varepsilon_N=1-\tau(f_N)$. An arbitrary a in the unit ball of $\overline{\bigotimes}_{k=N}^{\infty}(A_k,\tau)$ then satisfies

$$\|a-\tau(a)1\|_2\leqslant 4\sqrt{\varepsilon_N}.$$

Since $\tau(a_n)=\tau(E_{LK_N}(y_n))=\tau(y_n)$, it follows that for all N and n we have

$$\|E_{LK_N}(y_n)-\tau(y_n)1\|_2\leqslant 4\sqrt{\varepsilon_N}.$$

As $\varepsilon_N\to 0$ when $N\to\infty$, and since for every fixed N we have $\|y_n-E_{LK_N}(y_n)\|_2\to\infty$ when $n\to\infty$, we conclude that $\|y_n-\tau(y_n)1\|_2\to 0$. So, since $\|x_n-y_n\|_2\to 0$, also

$$\|x_n-\tau(x_n)1\|_2\to 0. \qquad \square$$

4. Concluding remarks

The group G constructed above is not finitely generated. It seems impossible to modify our construction to provide finitely generated counterexamples G, although we strongly believe that such examples exist.

The construction in this paper is inspired by the following similar phenomenology in ergodic theory of group actions, exhibited by Schmidt [Sc, Example 2.7]. Let $G\curvearrowright(X,\mu)$ be a measure-preserving action of a countable group G on a standard non-atomic probability space (X,μ). The action $G\curvearrowright(X,\mu)$ is said to be *strongly ergodic* if the following implication holds: whenever $\mathcal{U}_n\subset X$ is a sequence of almost invariant measurable subsets (i.e. $\mu(g\cdot\mathcal{U}_n\triangle\mathcal{U}_n)\to 0$ for all $g\in G$), then $\mu(\mathcal{U}_n)(1-\mu(\mathcal{U}_n))\to 0$. The action $G\curvearrowright(X,\mu)$ is said to have *spectral gap* if the Koopman representation $G\to\mathcal{U}(L^2(X)\ominus\mathbb{C}1)$ does not weakly contain the trivial representation. It is easy to see (e.g. [BHV, Proposition 6.3.2]) that spectral gap implies strong ergodicity. In [Sc, Example 2.7], Schmidt shows that the converse can fail.

Finally, we illustrate the subtlety of the difference between inner amenability and property Gamma. Let G be an icc group and consider the Hilbert space $\ell^2(G)$ as an LG-LG-bimodule. We denote by C_r^*G the reduced group C^*-algebra of G, viewed as a weakly dense C^*-subalgebra of LG.

The following are true:

• G is inner amenable if and only if there exists a sequence ξ_n of unit vectors in $\ell^2(G)$ such that $\xi_n(e)=0$ for all n and $\|a\xi_n-\xi_n a\|_2\to 0$ for all $a\in C_r^*G$.

This follows from the fact that G is inner amenable if and only if the adjoint representation of G on $\ell^2(G\setminus\{e\})$ weakly contains the trivial representation (see e.g. [BH, Théorème 1]).

• LG has property Gamma if and only if there exists a sequence ξ_n of unit vectors in $\ell^2(G)$ such that $\xi_n(e)=0$ for all n and $\|a\xi_n-\xi_n a\|_2\to 0$ for all $a\in LG$.

This follows from the characterization of property Gamma in [C, Theorem 2.1 (c)].

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Stefaan Vaes
Department of Mathematics
KU Leuven
Celestijnenlaan 200B
BE-3001 Leuven
Belgium
stefaan.vaes@wis.kuleuven.be

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