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The $\varkappa$ ring of the moduli of curves of compact type

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Contents

1. Introduction

1.1. Curves of compact type

Let C be a reduced and connected curve over $\mathbb{C}$ with at worst nodal singularities. The associated *dual graph* Γ_C has vertices corresponding to the irreducible components of C and edges corresponding to the nodes. The curve C is of *compact type* if Γ_C is a tree. Alternatively, C is of compact type if the Picard variety of line bundles of fixed multidegree on C is compact.

Standard marked points $p_1, ..., p_n$ on C must be distinct and lie in the non-singular locus. The pointed curve $(C, p_1, ..., p_n)$ is *stable* if the line bundle $\omega_C(p_1+...+p_n)$ is ample. Stability implies that the condition $2g-2+n>0$ holds. Let

$$M^c_{g,n} \subset \overline{M}_{g,n}$$

denote the open subset of genus-g, n-pointed stable curves of compact type. The complement

$$\overline{M}_{g,n} \setminus M^c_{g,n} = \delta_0$$

is the irreducible divisor of stable curves with a non-disconnecting node.

Since every non-singular curve is of compact type, the inclusion

$$M_{g,n} \subset M^c_{g,n}$$

is obtained. While the Torelli map

$$M_{g,n} \longrightarrow A_g$$

from the moduli of non-singular curves to the moduli of principally polarized Abelian varieties does not extend to $\overline{M}_{g,n}$, the extension

$$M_{g,n} \subset M^c_{g,n} \longrightarrow A_g$$

is easily defined.

1.2. $\varkappa$ classes

The $\varkappa$ classes in the Chow ring(1) $A^*(\overline{M}_{g,n})$ are defined by the following construction. Let

$$\varepsilon: \overline{M}_{g,n+1} \longrightarrow \overline{M}_{g,n}$$

be the universal curve viewed as the $(n+1)$-pointed space, let

$$\mathbb{L}_{n+1} \longrightarrow \overline{M}_{g,n+1}$$

be the line bundle obtained from the cotangent space of the last marking, and let

$$\psi_{n+1} = c_1(\mathbb{L}_{n+1}) \in A^1(\overline{M}_{g,n+1})$$

be the Chern class. The $\varkappa$ classes, first defined by Mumford, are

$$\varkappa_i = \varepsilon_*(\psi^{i+1}_{n+1}) \in A^i(\overline{M}_{g,n}), \quad i \geqslant 0.$$

(1) Since the moduli spaces here are Deligne–Mumford stacks, we will always take Chow rings with $\mathbb{Q}$ coefficients.

The simplest is $\varkappa_0$ which equals $2g-2+n$ times the unit in $A^0(\overline{M}_{g,n})$. The convention

$$\varkappa_{-1} = \varepsilon_*(\psi_{n+1}^0) = 0$$

is often convenient.

The $\varkappa$ classes on $M_{g,n}$ and $M_{g,n}^c$ are defined via restriction from $\overline{M}_{g,n}$. Define the $\varkappa$ rings

$$\varkappa^*(M_{g,n}) \subset A^*(M_{g,n}), \quad \varkappa^*(M_{g,n}^c) \subset A^*(M_{g,n}^c) \quad \text{and} \quad \varkappa^*(\overline{M}_{g,n}) \subset A^*(\overline{M}_{g,n})$$

to be the $\mathbb{Q}$-subalgebras generated by the $\varkappa$ classes. Of course, the $\varkappa$ rings are graded by degree.

Since $\varkappa_i$ is a tautological class(2), the $\varkappa$ rings are subalgebras of the corresponding tautological rings. For unpointed non-singular curves, the $\varkappa$ ring equals the tautological ring by definition [22],

$$\varkappa^*(M_g) = R^*(M_g).$$

The topic of the paper is the compact type case where the inclusion

$$\varkappa^*(M_{g,n}^c) \subset R^*(M_{g,n}^c)$$

is proper even for divisor classes.

1.3. Results

We present here several results about the rings $\varkappa^*(M_{g,n}^c)$. The first two yield a minimal set of generators in the $n>0$ case.

THEOREM 1.1. *$\varkappa^*(M_{g,n}^c)$ is generated over $\mathbb{Q}$ by the classes*

$$\varkappa_1, \varkappa_2, \ldots, \varkappa_{g-1+\lfloor n/2 \rfloor}.$$

THEOREM 1.2. *If $n>0$, there are no relations among*

$$\varkappa_1, \ldots, \varkappa_{g-1+\lfloor n/2 \rfloor} \in \varkappa^*(M_{g,n}^c)$$

in degrees $\leqslant g-1+\lfloor \frac{1}{2}n \rfloor$.

(2) A discussion of tautological classes is presented in §5.1.

Since $\varkappa^*(M^c_{g,n}) \subset R^*(M^c_{g,n})$, the socle and vanishing results for the tautological ring [7], [14] imply that

$$\varkappa^{2g-3+n}(M^c_{g,n}) = \mathbb{Q} \quad \text{and} \quad \varkappa^{>2g-3+n}(M^c_{g,n}) = 0. \tag{1}$$

By Theorem 1.2, all the interesting relations among the $\varkappa$ classes lie in degrees $g+\lfloor \frac{1}{2}n \rfloor$ to $2g-3+n$.

By Theorem 1.1, the classes $\varkappa_1, ..., \varkappa_{g-1}$ generate $\varkappa^*(M^c_g)$. Since M^c_g is excluded in Theorem 1.2, the possibility of a relation among the $\varkappa$ classes in degree $g-1$ is left open. However, no lower relations exist.

PROPOSITION 1.3. *There exist no relations among* $\varkappa_1, ..., \varkappa_{g-1} \in \varkappa^*(M^c_g)$ *in degrees* $\leqslant g-2$ *and at most a single relation in degree* $g-1$.

The structure of $\varkappa^*(M_g)$ has been studied for many years [22]. Faber [4] conjectured that the classes $\varkappa_1, ..., \varkappa_{\lfloor g/3 \rfloor}$ form a minimal set of generators for $\varkappa^*(M_g)$. The result was proven in cohomology by Morita [21], and a second proof, via admissible covers and valid in the Chow ring, was given by Ionel [16]. A uniform view of M_g, M^c_g and $\overline{M}_g$ was proposed in [6], but very few results in the latter two cases have been obtained.

1.4. Relations

Theorem 1.1 is proven by finding sufficiently many geometric relations among the $\varkappa$ classes. The method uses the virtual geometry of the moduli space of stable quotients introduced in [20] and reviewed in §2. Non-standard moduli spaces of pointed curves, arising naturally as subloci of the moduli space of stable quotients, are required for the construction.

Following the notation of [20], let $\overline{M}_{g,n|d}$ be the moduli space of genus-g stable curves with markings

$$\{p_1 ..., p_n\} \cup \{\hat{p}_1, ..., \hat{p}_d\} \in C$$

lying in the non-singular locus and satisfying the conditions

(i) the points p_i are distinct,

(ii) the points $\hat{p}_j$ are distinct from the points p_i,

with stability given by the ampleness of

$$\omega_C\left(\sum_{i=1}^n p_i + \varepsilon \sum_{j=1}^d \hat{p}_j\right)$$

for every strictly positive $\varepsilon\in\mathbb{Q}$. The conditions allow the points $\hat{p}_j$ and $\hat{p}_{j'}$ to coincide. The moduli space $\overline{M}_{g,n|d}$ is a non-singular, irreducible, Deligne–Mumford stack.(3)

Denote the open locus of curves of compact type by

$$M^c_{g,n|d}\subset\overline{M}_{g,n|d}.$$

Consider the universal curve

$$\pi\colon U\longrightarrow M^c_{g,n|d}.$$

The morphism π has sections $\sigma_1,...,\sigma_d$ corresponding to the markings $\hat{p}_1,...,\hat{p}_d$. Let $\sigma\subset U$ be the divisor obtained from the union of the d sections. The two rank-d bundles on $M^c_{g,n|d}$,

$$\mathbb{A}_d=\pi_*(\mathcal{O}_\sigma)\quad\text{and}\quad\mathbb{B}_d=\pi_*(\mathcal{O}_\sigma(\sigma)),$$

play important roles in the geometry.

The new relations studied here arise from the vanishing of the Chern classes of the virtual bundle $\mathbb{A}^*_d-\mathbb{B}_d$ on $M^c_{g,n|d}$ after push-forward via the proper forgetful map

$$\varepsilon^c\colon M^c_{g,n|d}\longrightarrow M^c_{g,n}.$$

THEOREM 1.4. *For all $k>n$,*

$$\varepsilon^c_*(c_{2g-2+k}(\mathbb{A}^*_d-\mathbb{B}_d))=0\in A^*(M^c_{g,n}).$$

The proofs of Theorem 1.4 and richer variants are given in §3. The ε^c push-forwards are calculated by simple rules explained in §3.5. In particular, we will see that Theorem 1.4 yields relations purely among the $\varkappa$ classes on the moduli space $M^c_{g,n}$.

Theorem 1.1 is proven for $M^c_{g,n}$ in §4 by examining the relations of Theorem 1.4. The coefficient of $\varkappa_i$ for $i>g-1+\lfloor\frac{1}{2}n\rfloor$ is shown to be non-zero. The method yields an effective evaluation of the relations. Theorem 1.2 and Proposition 1.3 are proven in §5 by intersection calculations in the tautological ring.

1.5. Genus zero

The strategy of Theorem 1.4 does *not* generate all the relations in $\varkappa^*(M^c_g)$. The first example of failure, occurring in genus 5, is discussed in §6.

Since all genus-zero curves are of compact type,

$$M^c_{0,n}=\overline{M}_{0,n}.$$

(3) In fact, $\overline{M}_{g,n|d}$ is a special case of the moduli of pointed curves with weights studied in [15] and [18]. Specifically, the points p_i are weighted with 1 and the points $\hat{p}_j$ are weighted with ε.

For emphasis here, we will use the notation $M^c_{0,n}$. The following universality property, motivated by the relations of Theorem 1.4, gives considerable weight to the genus-zero case.

Let $x_1, x_2, x_3, \ldots$ be variables with x_i of degree i. Let

$$f \in \mathbb{Q}[x_1, x_2, x_3, \ldots]$$

be *any* graded homogeneous polynomial.

THEOREM 1.5. *If* $f(\varkappa_i)=0\in\varkappa^*(M^c_{0,n})$, *then*

$$f(\varkappa_i) = 0 \in \varkappa^*(M^c_{g,n-2g})$$

for all genera g *for which* $n-2g\geqslant 0$.

Our proof of Theorem 1.5 uses relations constructed from the virtual geometry of the moduli spaces $\overline{M}_{g,n}(\mathbb{P}^1, d)$ of stable maps to $\mathbb{P}^1$. A crucial point is the calculation of the ranks of the vector spaces of relations. In fact, the stable maps relations will be shown to give *all* relations among $\varkappa$ classes in the ring $\varkappa^*(M^c_{0,n})$. The proof of Theorem 1.5 is given in §§7–9.

1.6. λ_g-formula

The rank-g Hodge bundle over the moduli space of curves

$$\mathbb{E} \longrightarrow \overline{M}_{g,n}$$

has fiber $H^0(C, \omega_C)$ over $[C, p_1, \ldots, p_n]$. Let $\lambda_k = c_k(\mathbb{E})$ be the Chern classes. Since λ_g vanishes when restricted to δ_0, we obtain a well-defined evaluation

$$\phi\colon A^*(M^c_{g,n}) \longrightarrow \mathbb{Q}$$

given by integration

$$\phi(\gamma) = \int_{\overline{M}_{g,n}} \bar{\gamma}\lambda_g,$$

where $\bar{\gamma}$ is any lift of $\gamma\in A^*(M^c_{g,n})$ to $A^*(\overline{M}_{g,n})$.

A discussion of the evaluation ϕ and the associated Gorenstein conjecture for the tautological ring can be found in [7] and [24]. For background on integrating the classes λ_i on the moduli space of curves, see [5].

The evaluation ϕ is determined on $R^*(M^c_{g,n})$ by the λ_g-formula for descendent integrals,

$$\int_{\overline{M}_{g,n}} \psi_1^{a_1} \dots \psi_n^{a_n} \lambda_g = \binom{2g-3+n}{a_1, \dots, a_n} \int_{\overline{M}_{g,1}} \psi_1^{2g-2} \lambda_g,$$

discovered in [11] and proven in [7]. Theorem 1.5 is much stronger. The λ_g-formula is a direct consequence of Theorem 1.5 in the special case where f has degree equal to

$$\dim_{\mathbb{C}} M^c_{0,n} = n-3.$$

Theorem 1.5 may be viewed as an extension of the λ_g-formula from $\mathbb{Q}$ to cycle classes of all intermediate degrees.

1.7. Bases and Betti numbers

Let $P(d)$ be the set of partitions of d. Let $P(d,k) \subset P(d)$ be the set of partitions of d into at most k parts, and let $|P(d,k)|$ be its cardinality. With a partition([4])

$$\mathbf{p} = (p_1, \dots, p_l) \in P(d,k),$$

we associate a $\varkappa$ monomial by

$$\varkappa_{\mathbf{p}} = \varkappa_{p_1} \dots \varkappa_{p_l} \in \varkappa^d(M^c_{0,n}).$$

Theorem 1.6. *A $\mathbb{Q}$-basis of $\varkappa^d(M^c_{0,n})$ is given by*

$$\{\varkappa_{\mathbf{p}} : \mathbf{p} \in P(d, n-2-d)\}.$$

For example, if $d \leqslant \lfloor \frac{1}{2} n \rfloor - 1$, then $n-2-d \geqslant d$ and $P(d, n-2-d) = P(d)$. Hence, Theorem 1.6 agrees with Theorem 1.2. The Betti number calculation,

$$\dim_{\mathbb{Q}} \varkappa^d(M^c_{0,n}) = |P(d, n-2-d)|,$$

is implied by Theorem 1.6. The proof of Theorem 1.6 is given in §6.2.

The relations of Theorem 1.4 and variants provide an indirect approach for multiplication in the canonical basis of $\varkappa^*(M^c_{0,n})$ determined by Theorem 1.6.

Question 1.7. Does there exist a direct calculus for multiplication in the canonical basis of $\varkappa^*(M^c_{0,n})$?

([4]) The parts of $\mathbf{p}$ are posititve and satisfy $p_1 \geqslant \dots \geqslant p_l$.

1.8. Universality

The universality of Theorem 1.5 expresses the higher-genus structures as canonical *ring* quotients,

$$\varkappa^*(M^c_{0,2g+n}) \xrightarrow{\iota_{g,n}} \varkappa^*(M^c_{g,n}) \longrightarrow 0.$$

Theorem 1.8. *If $n>0$, then $\iota_{g,n}$ is an isomorphism.*

The rings $\varkappa^*(M^c_{g,n})$ for $n>0$ are determined by Theorem 1.8. For example,

$$\dim_{\mathbb{Q}} \varkappa^d(M^c_{g,n}) = |P(d, 2g-2+n-d)|$$

by Theorem 1.6 for $n>0$. The proof of Theorem 1.8 is presented in §10 via intersection calculations.

The quotient $\iota_{g,0}$ is not always an isomorphism. For example, a non-trivial kernel appears for $\iota_{5,0}$.

Question 1.9. What is the kernel of $\iota_{g,0}$?

Universality appears to be special to the moduli of compact-type curves. No similar phenomena have been found for M_g or $\overline{M}_g$.

1.9. Gorenstein conjecture

The tautological rings $R^*(M^c_{g,n}) \subset A^*(M^c_{g,n})$ have been conjectured in [6] and [24] to be Gorenstein algebras with socle in degree $2g-3+n$,

$$\phi\colon R^{2g-3+n}(M^c_{g,n}) \xrightarrow{\sim} \mathbb{Q}.$$

The following result, proven in §11, may be viewed as significant evidence for the Gorenstein conjecture for all $M^c_{g,n}$ with $n>0$.

Theorem 1.10. *If $n>0$ and $\xi \in \varkappa^d(M^c_{g,n}) \neq 0$, the linear function*

$$L_\xi\colon R^{2g-3+n-d}(M^c_{g,n}) \longrightarrow \mathbb{Q}$$

defined by the socle evaluation

$$L_\xi(\gamma) = \phi(\gamma\xi)$$

is non-trivial.

1.10. Acknowledgments

Theorem 1.4 was motivated by the study of stable quotients developed in [20]. Discussions with A. Marian and D. Oprea were very helpful. Easy exploration of the relations of Theorem 1.4 was made possible by code written by C. Faber. Conversations with C. Faber played an important role.

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2. Stable quotients

2.1. Stability

Our first set of relations in $\varkappa(M^c_{g,n})$ will be obtained from the virtual geometry of the moduli space of stable quotients $\overline{Q}_{g,n}(\mathbb{P}^1,d)$. We start by reviewing basic definitions and results of [20].

Let C be a curve([5]) with distinct markings $p_1,...,p_n$ in the non-singular locus C^{ns}. Let q be a quotient of the rank-N trivial bundle C,

$$\mathbb{C}^N\otimes\mathcal{O}_C\xrightarrow{q}Q\longrightarrow 0.$$

If the quotient subsheaf Q is locally free at the nodes and markings of C, then q is a *quasi-stable quotient*. Quasi-stability of q implies that the associated kernel

$$0\longrightarrow S\longrightarrow\mathbb{C}^N\otimes\mathcal{O}_C\xrightarrow{q}Q\longrightarrow 0$$

is a locally free sheaf on C. Let r denote the rank of S.

Let $(C,p_1,...,p_n)$ be a pointed curve equipped with a quasi-stable quotient q. The data $(C,p_1,...,p_n,q)$ determine a *stable quotient* if the $\mathbb{Q}$-line bundle

$$\omega_C(p_1+...+p_n)\otimes\left(\bigwedge^r S^*\right)^{\otimes\varepsilon}\tag{2}$$

is ample on C for every strictly positive $\varepsilon\in\mathbb{Q}$. Quotient stability implies that $2g-2+n\geqslant 0$.

Viewed in concrete terms, no amount of positivity of S^* can stabilize a genus-zero component $\mathbb{P}^1\cong P\subset C$, unless P contains at least two nodes or markings. If P contains exactly two nodes or markings, then S^* *must* have positive degree.

A stable quotient $(C,p_1,...,p_n,q)$ yields a rational map from the underlying curve C to the Grassmannian $\mathbb{G}(r,N)$. We will only require the $\mathbb{G}(1,2)=\mathbb{P}^1$ case for the proof of Theorem 1.4.

([5]) All curves here are reduced and connected with at worst nodal singularities.

2.2. Isomorphism

Let $(C, p_1, ..., p_n)$ be a pointed curve. Two quasi-stable quotients

$$\mathbb{C}^N \otimes \mathcal{O}_C \xrightarrow{q} Q \longrightarrow 0 \quad \text{and} \quad \mathbb{C}^N \otimes \mathcal{O}_C \xrightarrow{q'} Q' \longrightarrow 0 \tag{3}$$

on C are *strongly isomorphic* if the associated kernels $S, S' \subset \mathbb{C}^N \otimes \mathcal{O}_C$ are equal.

An *isomorphism* of quasi-stable quotients

$$\phi\colon (C, p_1, ..., p_n, q) \longrightarrow (C', p'_1, ..., p'_n, q')$$

is an isomorphism of curves

$$\phi\colon C \xrightarrow{\sim} C'$$

satisfying

(i) $\phi(p_i) = p'_i$ for $1 \leqslant i \leqslant n$;

(ii) the quotients q and $\phi^*(q')$ are strongly isomorphic.

Quasi-stable quotients (3) on the same curve C may be isomorphic without being strongly isomorphic.

The following result is proven in [20] by Quot scheme methods from the perspective of geometry relative to a divisor.

THEOREM 2.1. *The moduli space of stable quotients $\overline{Q}_{g,n}(\mathbb{G}(r, N), d)$ parameterizing the data*

$$(C, p_1, ..., p_n, 0 \longrightarrow S \longrightarrow \mathbb{C}^N \otimes \mathcal{O}_C \xrightarrow{q} Q \longrightarrow 0),$$

with $\operatorname{rank}(S) = r$ *and* $\deg(S) = -d$, *is a separated and proper Deligne–Mumford stack of finite type over* $\mathbb{C}$.

2.3. Structures

Over the moduli space of stable quotients, there is a universal curve

$$\pi\colon U \longrightarrow \overline{Q}_{g,n}(\mathbb{G}(r, N), d) \tag{4}$$

with n sections and a universal quotient

$$0 \longrightarrow S_U \longrightarrow \mathbb{C}^N \otimes \mathcal{O}_U \xrightarrow{q_U} Q_U \longrightarrow 0.$$

The subsheaf S_U is locally free on U because of the stability condition.

The moduli space $\overline{Q}_{g,n}(\mathbb{G}(r,N),d)$ is equipped with two basic types of maps. If $2g-2+n>0$, then the stabilization of $(C,p_1,...,p_m)$ determines a map

$$\nu: \overline{Q}_{g,n}(\mathbb{G}(r,N),d) \longrightarrow \overline{M}_{g,n}$$

by forgetting the quotient. For each marking p_i, the quotient is locally free over p_i, and hence determines an evaluation map

$$\mathrm{ev}_i: \overline{Q}_{g,n}(\mathbb{G}(r,N),d) \longrightarrow \mathbb{G}(r,N).$$

The general linear group $\mathrm{GL}_N(\mathbb{C})$ acts on $\overline{Q}_{g,n}(\mathbb{G}(r,N),d)$ via the standard action on $\mathbb{C}^N \otimes \mathcal{O}_C$. The structures π, q_U, ν and the evaluations maps are all $\mathrm{GL}_N(\mathbb{C})$-equivariant.

2.4. Obstruction theory

The moduli of stable quotients maps to the Artin stack of pointed domain curves

$$\nu^A: \overline{Q}_{g,n}(\mathbb{G}(r,N),d) \longrightarrow \mathcal{M}_{g,n}.$$

The moduli of stable quotients with fixed underlying curve

$$(C,p_1,...,p_n) \in \mathcal{M}_{g,n}$$

is simply an open set of the Quot scheme. The following result of [20, §3.2] is obtained from the standard deformation theory of the Quot scheme.

THEOREM 2.2. *The deformation theory of the Quot scheme determines a 2-term obstruction theory on* $\overline{Q}_{g,n}(\mathbb{G}(r,N),d)$ *relative to* ν^A *given by* $\mathrm{RHom}(S,Q)$.

More concretely, for the stable quotient,

$$0 \longrightarrow S \longrightarrow \mathbb{C}^N \otimes \mathcal{O}_C \xrightarrow{q} Q \longrightarrow 0,$$

the deformation and obstruction spaces relative to ν^A are $\mathrm{Hom}(S,Q)$ and $\mathrm{Ext}^1(S,Q)$, respectively. Since S is locally free, the higher obstructions

$$\mathrm{Ext}^k(S,Q) = H^k(C, S^* \otimes Q) = 0, \quad k > 1,$$

vanish since C is a curve. An absolute 2-term obstruction theory on $\overline{Q}_{g,n}(\mathbb{G}(r,N),d)$ is obtained from Theorem 2.2 and the smoothness of $\mathcal{M}_{g,n}$, see [2] and [12]. The analogue of Theorem 2.2 for the Quot scheme of a *fixed* non-singular curve was observed in [19].

The $\mathrm{GL}_N(\mathbb{C})$-action lifts to the obstruction theory, and the resulting virtual class is defined in $\mathrm{GL}_N(\mathbb{C})$-equivariant cycle theory,

$$[\overline{Q}_{g,n}(\mathbb{G}(r,N),d)]^{\mathrm{vir}} \in A_*^{\mathrm{GL}_N(\mathbb{C})}(\overline{Q}_{g,n}(\mathbb{G}(r,N),d)).$$

3. Relations via stable quotients

3.1. $\mathbb{C}^*$-equivariant geometry

Let $\mathbb{C}^*$ act on $\mathbb{C}^2$ with weights $[0,1]$ on the respective basis elements. Let $\mathbb{P}^1=\mathbb{P}(\mathbb{C}^2)$, and let $0,\infty\in\mathbb{P}^1$ be the $\mathbb{C}^*$-fixed points corresponding to the eigenspaces of weight 0 and 1, respectively.

There is an induced $\mathbb{C}^*$-action on $\overline{Q}_{g,n}(\mathbb{P}^1,d)$. As the virtual dimension of $\overline{Q}_{g,n}(\mathbb{P}^1,d)$ is $2g-2+2d+n$,

$$[\overline{Q}_{g,n}(\mathbb{P}^1,d)]^{\mathrm{vir}}\in A^{\mathbb{C}^*}_{2g-2+2d+n}(\overline{Q}_{g,n}(\mathbb{P}^1,d)),$$

see [20]. The $\mathbb{C}^*$-action lifts canonically([6]) to the universal curve

$$\pi\colon U\longrightarrow \overline{Q}_{g,n}(\mathbb{P}^1,d)$$

and to the universal subsheaf S_U. The higher direct image $R^1\pi_*(S_U)$ is a vector bundle of rank $g+d-1$ with top Chern class

$$\mathsf{e}(R^1\pi_*(S_U))\in A^{g+d-1}_{\mathbb{C}^*}(\overline{Q}_{g,n}(\mathbb{P}^1,d)).$$

3.2. Relations

The relations of Theorem 1.4 will be obtained by studying the class

$$\Phi_{g,n,d}=\left(\mathsf{e}(R^1\pi_*(S_U))\cup\prod_{i=1}^{n}\mathrm{ev}_i^*([\infty])\right)\cap[\overline{Q}_{g,n}(\mathbb{P}^1,d)]^{\mathrm{vir}}$$

on the moduli space of stable quotients. A dimension calculation shows that

$$\Phi_{g,n,d}\in A^{\mathbb{C}^*}_{g-1+d}(\overline{Q}_{g,n}(\mathbb{P}^1,d)).$$

Let $2g-2+n>0$ and consider the proper morphism

$$\nu\colon \overline{Q}_{g,n}(\mathbb{P}^1,d)\longrightarrow \overline{M}_{g,n}.$$

Let $[1]$ denote the trivial bundle with $\mathbb{C}^*$-weight 1, and let $\mathsf{e}([1])$ be the $\mathbb{C}^*$-equivariant first Chern class. Since the non-equivariant limit of $\mathsf{e}([1])$ is zero, the class

$$\nu_*(\Phi_{g,n,d}\mathsf{e}([1])^k)\in A_{g-1+d-k}(\overline{M}_{g,n}) \tag{5}$$

certainly vanishes in the non-equivariant limit for $k>0$.

We will calculate the push-forward (5) via $\mathbb{C}^*$-localization to find relations. Theorem 1.4 will be obtained after restriction to the moduli space

$$M^c_{g,n}\subset\overline{M}_{g,n}$$

of curves of compact type.

([6]) The particular $\mathbb{C}^*$-lift to S_U plays an important role in the calculation.

3.3. $\mathbb{C}^*$-fixed loci

Since $\Phi_{g,n,d}\mathsf{e}([1])^k$ is a $\mathbb{C}^*$-equivariant class, we may calculate the non-equivariant limit of the push-forward (5) by the virtual localization formula [12] as applied in [20]. We will be interested in the restriction of $\nu_*(\Phi_{g,n,d}\mathsf{e}([1])^k)$ to $M^c_{g,n}$.

The first step is to determine the $\mathbb{C}^*$-fixed loci of $\overline{Q}_{g,n}(\mathbb{P}^1,d)$. The full list of $\mathbb{C}^*$-fixed loci is indexed by decorated graphs described in [20]. However, we will see that most loci do not contribute to the localization calculation of

$$\nu_*(\Phi_{g,n,d}\mathsf{e}([1])^k)|_{M^c_{g,n}},$$

by our specific choices of $\mathbb{C}^*$-lifts.

The *principal* component of the $\mathbb{C}^*$-fixed point locus

$$\overline{Q}_{g,n}(\mathbb{P}^1,d)^{\mathbb{C}^*} \subset \overline{Q}_{g,n}(\mathbb{P}^1,d)$$

is defined as follows. Consider the quotient

$$\overline{M}_{g,n|d}/S_d, \tag{6}$$

where the symmetric group acts by permutation of the d non-standard markings. Given an element

$$[C,p_1,...,p_n,\hat{p}_1,...,\hat{p}_d] \in \overline{M}_{g,n|d},$$

there is a canonically associated sequence

$$0 \longrightarrow \mathcal{O}_C\left(-\sum_{j=1}^{d}\hat{p}_j\right) \longrightarrow \mathcal{O}_C \longrightarrow \widetilde{Q} \longrightarrow 0. \tag{7}$$

By including $\mathcal{O}_C$ above the *second* factor of $\mathbb{C}^2\otimes\mathcal{O}_C$, we obtain a stable quotient from (7),

$$0 \longrightarrow \mathcal{O}_C\left(-\sum_{j=1}^{d}\hat{p}_j\right) \longrightarrow \mathcal{O}_C\oplus\mathcal{O}_C \longrightarrow \mathcal{O}_C\oplus\widetilde{Q} \longrightarrow 0.$$

The corresponding S_d-invariant morphism

$$\iota\colon \overline{M}_{g,n|d} \longrightarrow \overline{Q}_{g,n}(\mathbb{P}^1,d)$$

surjects onto the principal component of $\overline{Q}_{g,n}(\mathbb{P}^1,d)^{\mathbb{C}^*}$.

Let $F\subset\overline{Q}_{g,n}(\mathbb{P}^1,d)^{\mathbb{C}^*}$ be a component of the $\mathbb{C}^*$-fixed locus, and $[C,p_1,...,p_n,q]\in F$ be a generic element of F.

(i) If an irreducible component of C lying over $0\in\mathbb{P}^1$ has genus $h>0$, then

$$\mathsf{e}(R^1\pi_*(S_U))$$

yields the class λ_h by the contribution formulas of [20]. Since $\lambda_h|_{M^c_{h,*}}=0$ by [9], such loci F have vanishing contribution to

$$\nu_*(\Phi_{g,n,d}\mathsf{e}([1])^k)|_{M^c_{g,n}}.$$

(ii) If an irreducible component of C lying over $0\in\mathbb{P}^1$ is incident to more than a single irreducible component dominating $\mathbb{P}^1$, then $\mathsf{e}(R^1\pi_*(S_U))$ vanishes on F by the zero-weight space in $\mathbb{C}^2$ associated with $0\in\mathbb{P}^1$.

(iii) If $p_i\in C$ lies over $0\in\mathbb{P}^1$, then $\mathrm{ev}_i^*([\infty])$ vanishes on F.

By the vanishing properties (i)–(iii) together with the stability conditions, we conclude that the principal locus (6) is the *only* $\mathbb{C}^*$-fixed component of $\overline{Q}_{g,n}(\mathbb{P}^1,d)$ which contributes to $\nu_*(\Phi_{g,n,d}\mathsf{e}([1])^k)|_{M^c_{g,n}}$.

3.4. Proof of Theorem 1.4

The contribution of the principal component of $\overline{Q}_{g,n}(\mathbb{P}^1,d)$ to the push-forward

$$\nu_*(\Phi_{g,n,d}\mathsf{e}([1])^k)|_{M^c_{g,n}}$$

is obtained from the localization formulas of [20] together with an analysis of

$$\mathsf{e}(R^1\pi_*(S_U)).$$

For $[C,p_1,...,p_n,\hat{p}_1,...,\hat{p}_d]\in\overline{M}_{g,n|d}$, the long exact sequence associated with (7) yields

$$0\longrightarrow H^0(C,\mathcal{O}_C)\longrightarrow H^0(C,\mathcal{O}_{\hat{p}_1+...+\hat{p}_d})\longrightarrow H^1(C,S)\longrightarrow H^1(C,\mathcal{O}_C)\longrightarrow 0.$$

We conclude that

$$\mathsf{e}(R^1\pi_*(S_U))=\frac{\mathsf{e}(\mathbb{E}^*\otimes[1])\mathsf{e}(\mathbb{A}_d\otimes[1])}{\mathsf{e}([1])}$$

on the principal component. The evaluation

$$\prod_{i=1}^n \mathrm{ev}_i^*([\infty])\mathsf{e}([1])^k=\mathsf{e}([-1])^n\mathsf{e}([1])^k$$

is immediate.

By [20], the full localization contribution of the principal component is therefore

$$\frac{\mathsf{e}(\mathbb{E}^*\otimes[1])\mathsf{e}(\mathbb{A}_d\otimes[1])}{\mathsf{e}([1])}\mathsf{e}([-1])^n\mathsf{e}([1])^k\frac{\mathsf{e}(\mathbb{E}^*\otimes[-1])}{\mathsf{e}([-1])}\frac{1}{\mathsf{e}(\mathbb{B}_d\otimes[-1])}.$$

Using the Mumford relation $c(\mathbb{E})c(\mathbb{E}^*)=1$, we conclude, in the non-equivariant limit, that

$$\nu_*(\Phi_{g,n,d}\mathsf{e}([1])^k)|_{M^c_{g,n}}=(-1)^{3g-3+d+n+k}\varepsilon^c_*(c_{2g-2+k}(\mathbb{A}^*_d-\mathbb{B}_d)).$$

As the non-equivariant limit of $\nu_*(\Phi_{g,n,d}\mathsf{e}([1])^k)|_{M^c_{g,n}}$ vanishes, the proof of Theorem 1.4 is complete.

3.5. Evaluation rules

3.5.1. Chern classes

Associated with each non-standard marking $\hat{p}_j$, there is a cotangent line bundle

$$\hat{\mathbb{L}}_j \longrightarrow M^c_{g,n|d}.$$

Let $\hat{\psi}_j = c_1(\hat{\mathbb{L}}_j)$ be the first Chern class.

The non-standard markings are allowed by the stability conditions to be coincident. The *diagonal* $D_{ij} \subset M^c_{g,n|d}$ is defined to be the locus where $\hat{p}_i = \hat{p}_j$. Let

$$S_{ij} = \{l : l \neq i, j\} \cup \{\star\}.$$

The basic isomorphism $D_{ij} \cong M^c_{g,n|S_{ij}}$ gives the diagonal geometry a recursive structure compatible with the cotangent line classes,

$$\hat{\psi}_l|_{D_{ij}} = \hat{\psi}_l \quad \text{and} \quad \hat{\psi}_i|_{D_{ij}} = \hat{\psi}_j|_{D_{ij}} = \hat{\psi}_\star.$$

The intersection of distinct diagonals leads to smaller diagonals

$$D_{ij} \cap D_{jk} = D_{ijk}$$

in the obvious sense. The self-intersection is determined by

$$[D_{ij}]^2 = -\hat{\psi}_\star|_{D_{ij}}. \tag{8}$$

For convenience, let

$$\Delta_i = D_{1i} + D_{2i} + \ldots + D_{i-1,i}$$

with the convention that $\Delta_1 = 0$.

The Chern classes of $\mathbb{A}_d$ and $\mathbb{B}_d$ are easily obtained inductively from the sequences

$$0 \longrightarrow \mathcal{O}_{\sigma_1+\ldots+\sigma_{d-1}}(-\sigma_d) \longrightarrow \mathcal{O}_\sigma \longrightarrow \mathcal{O}_{\sigma_d} \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}_{\sigma_1+\ldots+\sigma_{d-1}}(\sigma_1+\ldots+\sigma_{d-1}) \longrightarrow \mathcal{O}_\sigma(\sigma) \longrightarrow \mathcal{O}_{\sigma_d}(\sigma) \longrightarrow 0$$

on the universal curve U over $M^c_{g,n}$. We find that

$$c(\mathbb{A}_d) = \prod_{j=1}^{d} (1-\Delta_j) \quad \text{and} \quad c(\mathbb{B}_d) = \prod_{j=1}^{d} (1-\hat{\psi}_j+\Delta_j), \tag{9}$$

see [20] for similar calculations.

3.5.2. Push-forward

From the Chern class formulas (9) and the diagonal intersection rules of §3.5.1,

$$\varepsilon^c_*(c_{2g-2+k}(\mathbb{A}^*_d-\mathbb{B}_d))\in A^*(M^c_{g,n})$$

is canonically a sum of push-forwards of the type

$$\varepsilon^c_*(\hat{\psi}_1^{j_1+1}\dots\hat{\psi}_s^{j_s+1})\in A^*(M^c_{g,n})$$

along the forgetful maps $\varepsilon^c_*\colon M^c_{g,n|s}\to M^c_{g,n}$ associated with the various diagonals.

Lemma 3.1. *$\varepsilon_*(\hat{\psi}_1^{j_1+1}\dots\hat{\psi}_s^{j_s+1})=\varkappa_{j_1}\dots\varkappa_{j_d}$ in $A^*(M^c_{g,n})$.*

Proof. There are forgetful maps

$$\gamma_j\colon M^c_{g,n|s}\longrightarrow M^c_{g,n|1}=M^c_{g,n+1}$$

associated with each non-standard marking where the isomorphism on the right follows from the definition of stability. Taking the fiber product over $M^c_{g,n}$ of all the γ_j yields a birational morphism

$$\gamma\colon M^c_{g,n|s}\longrightarrow M^c_{g,n+1}\times_{M^c_{g,n}}M^c_{g,n+1}\times_{M^c_{g,n}}\dots\times_{M^c_{g,n}}M^c_{g,n+1}.$$

The morphism γ is a small resolution. The exceptional loci are at most codimension-2 in $M^c_{g,n|s}$. Hence, $\mu^*(\psi_j)=\hat{\psi}_j$ for each non-standard marking. We see that

$$\mu_*(\hat{\psi}_1^{j_1+1}\dots\hat{\psi}_s^{j_s+1})=\psi_1^{j_1+1}\dots\psi_s^{j_s+1}.$$

The result then follows after push-forward to $M^c_{g,n}$ by the definition of the $\varkappa$ classes. □

By Lemma 1, the relations of Theorem 1.4 are purely among the $\varkappa$ classes in $A^*(M^c_{g,n})$.

3.5.3. Example

The $d=1$ case of Theorem 1.4 immediately yields the relations

$$\varkappa_{2g-2+k}=0\in A^*(M^c_{g,n})\quad\text{for all } k>n$$

implied also by the vanishing results (1).

More interesting relations occur for $d=2$. By the Chern class calculation (9),

$$c(\mathbb{A}^*_2-\mathbb{B}_2)=\frac{1+\Delta_1}{1-\hat{\psi}_1+\Delta_1}\,\frac{1+\Delta_2}{1-\hat{\psi}_2+\Delta_2}.$$

Using the series expansion

$$\frac{1+x}{1-y+x} = 1+\sum_{r=0}^{\infty} y(y-x)^r \tag{10}$$

and the diagonal intersection rules, we obtain

$$c(\mathbb{A}_2^*-\mathbb{B}_2) = \left(1+\sum_{r=0}^{\infty}\hat{\psi}_1^{r+1}\right)\left(1+\sum_{r=0}^{\infty}\hat{\psi}_2(\hat{\psi}_2^r-(2^r-1)\hat{\psi}_2^{r-1}\Delta_2)\right).$$

In genus 3 with $n=0$, the $k=1$ case of Theorem 1.4 concerns

$$c_5(\mathbb{A}_2^*-\mathbb{B}_2) = \sum_{r_1+r_2=5}\hat{\psi}_1^{r_1}\hat{\psi}_2^{r_2} - \sum_{r=1}^{4}(2^r-1)\hat{\psi}_\star^4\Delta_2.$$

The push-forward is easily evaluated

$$\varepsilon_*^c(c_5(\mathbb{A}_2^*-\mathbb{B}_2)) = 4\varkappa_3+\varkappa_1\varkappa_2+\varkappa_2\varkappa_1+4\varkappa_3-(1+3+7+15)\varkappa_3 = -18\varkappa_3+2\varkappa_1\varkappa_2.$$

We obtain the non-trivial relation

$$-18\varkappa_3+2\varkappa_1\varkappa_2 = 0\in A^*(M_3^c).$$

3.6. Richer relations

The proof of Theorem 1.4 naturally yields a richer set of relations among the $\varkappa$ classes. The universal curve $\pi: U\to M_{g,n|d}^c$ carries the basic divisor classes

$$s = c_1(S_U^*) \quad \text{and} \quad \omega = c_1(\omega_\pi)$$

obtained from the universal subsheaf S_U and the π-relative dualizing sheaf.

PROPOSITION 3.2. *For all* $a_i, b_i \geqslant 0$ *and* $k>n$,

$$\varepsilon_*\left(\prod_{i=1}^{m}\pi_*(s^{a_i}\omega^{b_i})c_{2g-2+k}(\mathbb{A}_d^*-\mathbb{B}_d)\right) = 0\in A^*(M_{g,n}^c).$$

The proof of Proposition 3.2 exactly follows the proof of Theorem 1.4. We leave the details to the reader. By the rules of §3.5, the relations of Proposition 3.2 are also purely among the $\varkappa$ classes.

4. Evaluation of the stable quotient relations

4.1. Overview

Our goal here is to explicitly evaluate the relations of Theorem 1.4 as polynomials in the $\varkappa$ classes. By examining the coefficients, we will obtain a proof of Theorem 1.1.

4.2. Term counts

Consider the total Chern class

$$c(\mathbb{A}_d^* - \mathbb{B}_d) = \prod_{i=1}^{d} \frac{1+\Delta_i}{1-\hat{\psi}_i+\Delta_i}. \tag{11}$$

After substituting $\Delta_i = D_{1i} + ... + D_{i-1,i}$, we may expand the right-hand side of (11) fully. The resulting expression is a formal series in the $d + \binom{d}{2}$ variables([7])

$$\hat{\psi}_1, ..., \hat{\psi}_d \quad \text{and} \quad -D_{12}, -D_{13}, ..., -D_{d-1,d}.$$

Let M_r^d denote the coefficient in degree r, so that

$$c(\mathbb{A}_d^* - \mathbb{B}_d) = \sum_{r=0}^{\infty} M_r^d(\hat{\psi}_i, -D_{ij}).$$

Lemma 4.1. *After setting all the variables to* 1, *one has*

$$\sum_{r=0}^{\infty} M_r^d(\hat{\psi}_i = 1, -D_{ij} = 1) t^r = \frac{1}{1-dt}.$$

Proof. After setting the variables to 1 in (11), we find that

$$c_t(\mathbb{A}_d^* - \mathbb{B}_d) = \prod_{i=1}^{d} \frac{1-(i-1)t}{1-it},$$

which is a telescoping product. □

Lemma 4.1 may be viewed counting the number of terms in the expansion of (11):

$$M_r^d(\hat{\psi}_i = 1, -D_{ij} = 1) = d^r.$$

This simple answer will play a crucial role in the analysis.

([7]) The sign on the diagonal variables is chosen because of the self-intersection formula (8).

4.3. Connected counts

A monomial in the diagonal variables

$$D_{12}, D_{13}, ..., D_{d-1,d} \tag{12}$$

determines a set partition of $\{1, ..., d\}$ by the diagonal associations. For example, the monomial $3D_{12}^2 D_{13} D_{56}^3$ determines the set partition

$$\{1,2,3\}\cup\{4\}\cup\{5,6\}$$

in the case $d=6$. A monomial in the variables (12) is *connected* if the corresponding set partition consists of a single part with d elements.

A monomial in the variables

$$\hat{\psi}_1, ..., \hat{\psi}_d \quad \text{and} \quad -D_{12}, -D_{13}, ..., -D_{d-1,d}$$

is connected if the corresponding monomial in the diagonal variables obtained by setting all $\hat{\psi}_i=1$ is connected. Let C_r^d be the summand of $M_r^d(\hat{\psi}_i=1, -D_{ij}=1)$ consisting of the contributions of only the connected monomials.

LEMMA 4.2. *We have*

$$\sum_{d=1}^{\infty}\sum_{r=0}^{\infty} C_r^d t^r \frac{z^d}{d!} = \log\left(1+\sum_{d=1}^{\infty}\sum_{r=0}^{\infty} d^r t^r \frac{z^d}{d!}\right).$$

Proof. By a standard application of Wick's theorem, the connected and disconnected counts are related by exponentiation:

$$\exp\left(\sum_{d=1}^{\infty}\sum_{r=0}^{\infty} C_r^d t^r \frac{z^d}{d!}\right) = 1+\sum_{d=1}^{\infty}\sum_{r=0}^{\infty} M_r^d(\hat{\psi}_i=1, -D_{ij}=1) t^r \frac{z^d}{d!}.$$

The right-hand side is then evaluated by Lemma 4.1. □

4.4. C_r^d for $r \leqslant d$

We may write the series inside the logarithm in Lemma 4.2 in the form

$$F(t,z) = 1+\sum_{d=1}^{\infty}\sum_{r=0}^{\infty} d^r t^r \frac{z^d}{d!} = \left(1-tz\frac{d}{dz}\right)^{-1} e^z.$$

Expanding the exponential of the differential operator by order in t yields

$$F(t,z) = e^z + tze^z + t^2(z^2+z)e^z + t^3(z^3+3z^2+z)e^z + t^4(z^4+6z^3+7z^2+z)e^z + \ldots .$$

We have proven the following result.

Lemma 4.3. *We have*

$$F(t,z)=e^z\sum_{r=0}^{\infty}t^r p_r(z),$$

where

$$p_r(z)=\sum_{s=0}^{r}c_{r,s}z^{r-s}$$

is a degree-r polynomial.

By Lemma 4.3 and the coefficient evaluation $c_{r,0}=1$, we see that

$$\log F(t,z)=z+\log\frac{1}{1-tz}+\ldots,$$

where the dots stand for terms of the form $t^r z^d$ with $r>d$. We obtain the following result.

Proposition 4.4. *The only non-vanishing C_r^d for $r\leqslant d$ are $C_0^1=1$ and*

$$\sum_{r=1}^{\infty}C_r^r t^r\frac{z^r}{r!}=-\log(1-tz).$$

4.5. Evaluation

Let g and n be fixed. We are interested in calculating

$$R_{g,n}(t,z)=\sum_{d=1}^{\infty}\varepsilon_*^c(c_t(\mathbb{A}_d^*-\mathbb{B}_d))\frac{z^d}{d!}.$$

By a straightforward application of the evaluation rules of §3.5, we find that

$$R_{g,n}(t,z)=\exp\biggl(\sum_{d=1}^{\infty}\sum_{r=d}^{\infty}(-1)^{d-1}C_r^d\varkappa_{r-d}t^r z^d\biggr). \tag{13}$$

We rewrite (13) after separating out the $r=d$ terms using Proposition 4.4 and the evaluation $\varkappa_0=2g-2+n$:

$$\begin{aligned}R_{g,n}(t,-z)&=\exp\biggl(-\sum_{d=1}^{\infty}\sum_{r=d}^{\infty}C_r^d\varkappa_{r-d}t^r z^d\biggr)\\&=(1-tz)^{2g-2+n}\exp\biggl(-\sum_{d=1}^{\infty}\sum_{r=d+1}^{\infty}C_r^d\varkappa_{r-d}t^r z^d\biggr).\end{aligned}$$

The $t^r z^d$ coefficient of $R_{g,n}$ is a valid relation in $A^*(M_{g,n}^c)$ if $r>2g-2+n$. The above formula, taken together with Lemma 4.2, provides a very effective approach to the relations of Theorem 1.4.

4.6. Proof of Theorem 1.1

The generating series for the coefficients of the singleton $\varkappa_{l>0}$ in the $t^{d+l}z^d$ terms of $R_{g,n}(t,-z)$ is

$$R^l_{g,n}(t,-z) = -(1-tz)^{2g-2+n}\sum_{d=1}^{\infty} C^d_{d+l}(tz)^d t^l. \tag{14}$$

In order to analyze the right-hand side of (14), we will use Lemma 4.3. For $l\geqslant 0$, let

$$G_l(t,z) = \sum_{d=1}^{\infty} c_{d+l,l}(tz)^d. \tag{15}$$

By Lemma 4.3 and Proposition 4.4, we have

$$\begin{aligned}\sum_{l=0}^{\infty}\sum_{d=1}^{\infty} C^d_{d+l}(tz)^d t^l &= \log\sum_{l=0}^{\infty} G_l(t,z)t^l = \log\left(\frac{1}{1-tz}+\sum_{l=1}^{\infty} G_l(t,z)t^l\right)\\ &= \log\frac{1}{1-tz}+\log\left(1+(1-zt)\sum_{l=1}^{\infty} G_l(t,z)t^l\right).\end{aligned}$$

So, for $l>0$,

$$\sum_{d=1}^{\infty} C^d_{d+l}(tz)^d = \operatorname{Coeff}_l\left(\log\left(1+(1-zt)\sum_{l=1}^{\infty} G_l(t,z)t^l\right)\right). \tag{16}$$

Here, Coeff_l extracts all the terms of the form $t^{*+l}z^*$ and divides by t^l.

The behavior of the coefficients $c_{r,s}$ is easily determined by induction on s.

LEMMA 4.5. *For $r\geqslant s$, $c_{r,s}=f_s(r)$, where $f_s(r)$ is a polynomial of degree $2s$ with leading term*

$$f_s(r) = \frac{1}{2^s s!} r^{2s}+\dots .$$

For example, $f_0(r)=1$ and

$$f_1(r) = \tfrac{1}{2}r^2+\tfrac{1}{2}r.$$

We leave the elementary proof of Lemma 4.5 to the reader.

By (15) and Lemma 4.5, we conclude that, for $l>0$,

$$G_l(t,z) = \frac{1}{2^l l!}\frac{(2l)!}{(1-tz)^{2l+1}}+\sum_{i=0}^{2l}\frac{\tilde{c}_{i,l}}{(1-tz)^i}$$

with $\tilde{c}_{i,l}\in\mathbb{Q}$. Then, by (16),

$$\sum_{d=1}^{\infty} C^d_{d+l}(tz)^d = \operatorname{Coeff}_l\left(\log\left(1+\sum_{l=1}^{\infty}\frac{(2l-1)!!}{(1-tz)^{2l}}t^l\right)\right)\dots, \tag{17}$$

where the dots stand for finitely many terms of the form $(1-tz)^{-j}$ with $j<2l$. By Proposition 4.6 proven in §4.7 below,

$$\sum_{d=1}^{\infty} C_{d+l}^{d}(tz)^d = \frac{\alpha_l}{(1-zt)^{2l}} + \ldots, \tag{18}$$

with $\alpha_l \neq 0$.

We now return to the coefficients of the singleton $\varkappa_{l>0}$ in the $t^{d+l}z^d$ terms of $R_{g,n}(t,-z)$. By (14),

$$R_{g,n}^{l}(t,-z) = -\alpha_l(1-tz)^{2g-2+n-2l}t^l + \ldots, \tag{19}$$

where the dots stand for finitely many terms of the form $(1-tz)^j t^l$ with $j>2g-2+n-2l$. If

$$2g-2+n-2l < 0, \tag{20}$$

then the coefficient of $(tz)^d t^l$ in $R_{g,n}^{l}$ will be non-zero for all large d. Once

$$d+l > 2g-2+n,$$

the corresponding $\varkappa$ relation is valid by Theorem 1.4. If (20) is satisfied, $\varkappa_l$ lies in the subring of $\varkappa^*(M_{g,n}^c)$ generated by $\varkappa_1, \ldots, \varkappa_{l-1}$.

4.7. Series analysis

Define the double factorial by

$$(2l-1)!! = \frac{(2l)!}{2^l l!} = (2l-1)(2l-3)\cdot \ldots \cdot 3\cdot 1$$

and let

$$\phi(x) = 1 + \sum_{l=1}^{\infty} (2l-1)!!x^l = 1 + x + 3x^2 + 15x^3 + \ldots$$

be the generating series. Define $\alpha_l \in \mathbb{Q}$ for $l>0$ by

$$\log \phi(x) = \sum_{l=1}^{\infty} \alpha_l x^l.$$

Series expansion yields the first terms

$$\log \phi(x) = x + \tfrac{5}{2}x^2 + \tfrac{37}{3}x^3 + \tfrac{353}{4}x^4 + \ldots.$$

To complete the proof of Theorem 1.4, we must prove the following result.

PROPOSITION 4.6. *We have $\alpha_l \neq 0$ for all $l>0$.*

Let $x=y^2$. Then $\phi(x(y))$ satisfies the differential equation

$$y^2 \frac{d}{dy}(y\phi) = \phi - 1.$$

Equivalently,

$$y^3 \frac{d}{dy} \log \phi + y^2 - 1 = -\frac{1}{\phi}$$

Changing variables back to x, we find that

$$2x^2 \frac{d}{dx} \log \phi + x - 1 = -\frac{1}{\phi}. \tag{21}$$

Let β_l denote the coefficients of the inverse series,

$$\phi(x)^{-1} = 1 + \sum_{l=0}^{\infty} \beta_l x^l = 1 - x - 2\alpha_1 x^2 - 4\alpha_2 x^3 - 6\alpha_3 x^4 - ...,$$

where the second equality is obtained from (21).

LEMMA 4.7. *We have $\beta_l \neq 0$ for all $l>0$.*

Proof. Series expansion yields

$$\phi(x)^{-1} = 1 - x - 2x^2 - 10x^3 - 74x^4 - \dots.$$

We will establish the following two properties for $l>0$ by joint induction:

(i) $\beta_l < 0$;

(ii) $|\beta_l| \leqslant (2l-1)!!$.

By inspection, the conditions hold in the base case $l=1$.

Let $l>1$ and assume that conditions (i) and (ii) hold for all $l'<l$. Since $\phi\phi^{-1}=1$,

$$(2l-1)!! + \beta_l = -\sum_{k=1}^{l-1} (2k-1)!!\beta_{l-k} \leqslant \sum_{k=1}^{l-1} (2k-1)!!(2l-2k-1)!!, \tag{22}$$

where the second line uses (ii). For $\frac{1}{2}l \leqslant k \leqslant l-1$,

$$(2k-1)!!(2l-2k-1)!! = (2l-1)!! \frac{1}{2l-1} \frac{3}{2l-3} \dots \frac{2l-2k-1}{2k+1} \leqslant (2l-1)!! \frac{1}{2l-1}.$$

By putting the two inequalities above together, we obtain that

$$(2l-1)!! + \beta_l \leqslant (l-1)(2l-1)!! \frac{1}{2l-1} < (2l-1)!!.$$

Hence, $\beta_l < 0$. Since also $(2l-1)!! + \beta_l > 0$ by the first equality of (22) and (i), we see that $|\beta_l| < (2l-1)!!$. □

Lemma 4.7 and the relation $-2l\alpha_l = \beta_{l+1}$ complete the proof of Proposition 4.6.

5. Independence

5.1. Tautological classes

The moduli space $M^c_{g,n}$ has an algebraic stratification by topological type. The push-forward of the $\varkappa$ and ψ classes from the strata generate the *tautological ring*

$$R^*(M^c_{g,n}) \subset A^*(M^c_{g,n}),$$

see [13]. Following the Gorenstein philosophy explained in [6], we will study the independence of

$$\varkappa_1, ..., \varkappa_{g-1+\lfloor n/2 \rfloor} \in R^*(M^c_{g,n})$$

through degree $g-1+\left\lfloor \frac{1}{2}n \right\rfloor$ by pairing with strata classes.

5.2. Case $n=1$

We first prove Theorem 1.2 for $M^c_{g,1}$. By stability, $g \geqslant 1$. With each partition $\mathbf{p} \in P(d)$, we associate a $\varkappa$ monomial

$$\varkappa_{\mathbf{p}} = \varkappa_{p_1} \varkappa_{p_2} ... \varkappa_{p_l} \in R^*(M^c_{g,1}).$$

Theorem 1.2 is equivalent to the independence of the $|P(g-1)|$ monomials

$$\{\varkappa_{\mathbf{p}} : \mathbf{p} \in P(g-1)\}$$

in $R^*(M^c_{g,1})$.

With each partition $\mathbf{p} \in P(g-1)$ of length l, we associate a stratum $S_{\mathbf{p}} \subset M^c_{g,1}$ of codimension $g-1$ by the following construction. Start with a chain of elliptic curves E_i of length $l+1$ with the marking on the first

$$E_1^* \text{——} E_2 \text{——} E_3 \text{——} \;...\; \text{——} E_l \text{——} E_{l+1} \tag{23}$$

(the $*$ indicates the marking).

Since $l \leqslant g-1$, such a chain does not exceed genus g. Next, we add elliptic tails([8]) to the first l elliptic components. To the curve E_i, we add p_i-1 elliptic tails. Let C be the resulting curve. The total genus of C is

$$l+1+(g-1)-l = g.$$

([8]) An elliptic tail is an unmarked elliptic curve meeting the rest of the curve in exactly one point.

The number of nodes of C is

$$l+(g-1)-l=g-1.$$

Hence, C determines a stratum $S_{\mathbf{p}}\subset M^c_{g,1}$ of codimension $g-1$.

The moduli in $S_{\mathbf{p}}$ is found mainly on the first l components of the original chain (23). Each such E_i has p_i+1 moduli parameters. All other components (including E_{l+1}) are elliptic tails with one moduli parameter each.

The λ_g-evaluation on $R^*(M^c_{g,1})$ discussed in §1.6 yields the following pairing on partitions $\mathbf{p},\mathbf{q}\in P(g-1)$:

$$\mu_g(\mathbf{p},\mathbf{q})=\int_{\overline{M}_{g,1}}\varkappa_{\mathbf{p}}[S_{\mathbf{q}}]\lambda_g\in\mathbb{Q}.$$

LEMMA 5.1. *For all $g\geqslant 1$, the matrix μ_g is non-singular.*

Proof. To evaluate the pairing, we first restrict λ_g to $S_{\mathbf{q}}$ by distributing a λ_1 to each elliptic component. To pair $\varkappa_{\mathbf{p}}$ with the class $[S_{\mathbf{q}}]\lambda_g$, we must distribute the factors $\varkappa_{p_i}$ to the components E_j of $S_{\mathbf{q}}$ in all possible ways. By the dimension constraints imposed by the moduli parameters of the components of $S_{\mathbf{q}}$, we immediately conclude that

$$\mu_g(\mathbf{p},\mathbf{q})=0$$

unless $l(\mathbf{p})\geqslant l(\mathbf{q})$. Moreover, if $l(\mathbf{p})=l(\mathbf{q})$, the pairing vanishes unless $\mathbf{p}=\mathbf{q}$.

We have already shown μ_g to be upper-triangular with respect to the length partial ordering on $P(g-1)$. To establish the non-singularity of μ_g, we must show that the diagonal entries $\mu_g(\mathbf{p},\mathbf{p})$ do not vanish. Since $\mu_g(\mathbf{p},\mathbf{p})$ is a product of factors of the form

$$\int_{\overline{M}_{1,p+1}}\varkappa_p\lambda_1=\frac{1}{24},$$

the required non-vanishing holds. □

By Lemma 5.1, the $\varkappa$ monomials of degree $g-1$ are independent. The proof of Theorem 1.2 for $M^c_{g,1}$ is complete.

5.3. Case $n=2$

We now consider Theorem 1.2 for $M^c_{g,2}$. By stability, $g\geqslant 1$. We must prove the independence of the $|P(g)|$ monomials

$$\{\varkappa_{\mathbf{p}}:\mathbf{p}\in P(g)\}$$

in $R^*(M^c_{g,2})$.

With each partition $\mathbf{p}\in P(g)$ of length l, we associate a stratum $T_{\mathbf{p}}\subset M_{g,1}^c$ of codimension $g-1$ by the following construction. Start with a chain of elliptic curves E_i of length l with the markings on the first and last curves

$$E_1^* \text{——} E_2 \text{——} E_3 \text{——} \;...\; \text{——} E_l^*. \tag{24}$$

Since $l\leqslant g$, such a chain does not exceed genus g. Next, we add elliptic tails to the l elliptic components of (24). To the curve E_i, we add p_i-1 elliptic tails. Let C be the resulting curve. The total genus of C is

$$l+g-l=g.$$

The number of nodes of C is

$$l-1+g-l=g-1.$$

Hence, C determines a stratum $T_{\mathbf{p}}\subset M_{g,1}^c$ of codimension $g-1$.

As before, the λ_g-evaluation on $R^*(M_{g,2}^c)$ yields a pairing on partitions $\mathbf{p},\mathbf{q}\in P(g)$,

$$\nu_g(\mathbf{p},\mathbf{q})=\int_{\overline{M}_{g,2}} \varkappa_{\mathbf{p}}[T_{\mathbf{q}}]\lambda_g\in\mathbb{Q}.$$

Lemma 5.2. *For all $g\geqslant 1$, the matrix ν_g is non-singular.*

The proof is identical to the proof of Lemma 5.1. We leave the details to the reader. The proof of Theorem 1.2 for $M_{g,2}^c$ is complete.

5.4. Proof of Theorem 1.2

To complete the proof of Theorem 1.2, we must consider the case $n\geqslant 3$ and prove the independence of the monomials

$$\left\{\varkappa_{\mathbf{p}}:\mathbf{p}\in P\left(g-1+\left\lfloor\tfrac{1}{2}n\right\rfloor\right)\right\}$$

in $R^*(M_{g,n}^c)$.

We will relate the question to the established cases with one and two markings. Let

$$\hat{g}=g+\left\lfloor\tfrac{1}{2}(n-1)\right\rfloor \quad\text{and}\quad \hat{n}=n-2\left\lfloor\tfrac{1}{2}(n-1)\right\rfloor.$$

If n is odd, then $\hat{n}=1$. If n is even, then $\hat{n}=2$. Note that

$$\hat{g}-1+\left\lfloor\tfrac{1}{2}\hat{n}\right\rfloor=g-1+\left\lfloor\tfrac{1}{2}n\right\rfloor.$$

To start, assume that $\hat{n}=1$. We have constructed strata classes in $M^c_{\hat{g},1}$ which show the independence of the monomials

$$\{\varkappa_{\mathbf{p}} : \mathbf{p} \in P(\hat{g}-1)\}$$

in $R^*(M^c_{\hat{g},1})$. For each $\mathbf{q}\in P(\hat{g}-1)$, the stratum

$$S_{\mathbf{q}} \subset M^c_{\hat{g},1}$$

consists of a configuration of $\hat{g}$ elliptic curves. We construct a corresponding stratum

$$S'_{\mathbf{q}} \subset M^c_{g,n}$$

by the following method. Choose any subset(9) of $\lfloor \frac{1}{2}(n-1) \rfloor$ elliptic components of $S_{\mathbf{q}}$. For each component E selected, replace E by a rational component carrying two additional markings.(10) The construction trades $\lfloor \frac{1}{2}(n-1) \rfloor$ genus for $2\lfloor \frac{1}{2}(n-1) \rfloor$ markings.

Theorem 1.2 is implied by the non-singularity of the λ_g-pairing between the $\varkappa$ monomials of degree $\hat{g}-1$ and the strata classes $[S'_{\mathbf{q}}]$. The proof of the non-singularity is identical to the proof of Lemma 5.1.

The $\hat{n}=2$ case proceeds by exactly the same method. Again, elliptic components of the strata $T_{\mathbf{q}}\subset M^c_{\hat{g},2}$ are traded for rational components with two additional markings. Theorem 1.2 is deduced by non-singularity of the λ_g-pairing.

5.5. Proof of Proposition 1.3

Consider M^c_g for $g\geqslant 2$. Let

$$P^*(g-1) = P(g-1) \setminus \{(1, ..., 1)\}$$

be the subset excluding the longest partition. We will first prove the independence of the monomials

$$\{\varkappa_{\mathbf{p}} : \mathbf{p} \in P^*(g-1)\}$$

in $R^*(M^c_g)$. The result shows that there can be at most one $\varkappa$ relation in degree $g-1$.

With each partition $\mathbf{p}\in P^*(g-1)$ of length $l\leqslant g-2$, we associate a stratum $U_{\mathbf{p}}\subset M^c_g$ of codimension $g-2$ by the following construction. Start with a chain of curves of length $l+1$,

$$X \text{ --- } E_2 \text{ --- } E_3 \text{ --- } \ldots \text{ --- } E_l \text{ --- } E_{l+1},$$

(9) The particular choice of subset is not important.
(10) The particular markings chosen are not important.

where X has genus 2 and all the E_i are elliptic curves. Since $l \leqslant g-2$, such a chain does not exceed genus g. Next, we add elliptic tails to the first l components. Since p_1 is the greatest part of $\mathbf{p}$, we have $p_1 \geqslant 2$. To the curve X, we add p_1-2 elliptic tails. To the curve E_i, we add p_i-1 elliptic tails for $2 \leqslant i \leqslant l$. Let C be the resulting curve. The total genus of C is

$$2+l+(g-1)-l-1=g.$$

The number of nodes of C is

$$l+(g-1)-l-1=g-2.$$

Hence, C determines a stratum $U_{\mathbf{p}} \subset M_g^c$ of codimension $g-2$.

The λ_g-evaluation on $R^*(M_g^c)$ yields a pairing on $\mathbf{p}, \mathbf{q} \in P^*(g-1)$,

$$\omega_g(\mathbf{p}, \mathbf{q}) = \int_{\overline{M}_g} \varkappa_{\mathbf{p}} [U_{\mathbf{q}}] \lambda_g \in \mathbb{Q}.$$

The argument of Lemma 5.1 yields the following result.

LEMMA 5.3. *For all $g \geqslant 2$, the matrix ω_g is non-singular.*

The independence of the $\varkappa$ monomials in degrees at most $g-2$ is easier and proven in a similar way. With each partition $\mathbf{p} \in P(g-2)$ of length l, we associate a stratum $U'_{\mathbf{p}} \subset M_g^c$ of codimension $g-1$ by the following construction. Start with a chain of elliptic curves of length $l+2$,

$$E_0 \text{ —— } E_1 \text{ —— } E_2 \text{ —— } E_3 \text{ —— } \ldots \text{ —— } E_l \text{ —— } E_{l+1}.$$

Since $l \leqslant g-2$, such a chain does not exceed genus g. Next, we add elliptic tails to the components. To E_i, for $1 \leqslant i \leqslant l$, we add p_i-1 elliptic tails. To E_0 and E_{l+1}, we add nothing. Let C be the resulting curve. The total genus of C is

$$l+2+(g-2)-l=g$$

The number of nodes of C is

$$l+1+(g-2)-l=g-1.$$

Hence, C determines a stratum $U'_{\mathbf{p}} \subset M_g^c$ of codimension $g-1$.

The λ_g-evaluation on $R^*(M_g^c)$ yields a pairing on $\mathbf{p}, \mathbf{q} \in P(g-2)$,

$$\omega'_g(\mathbf{p}, \mathbf{q}) = \int_{\overline{M}_g} \varkappa_{\mathbf{p}} [U'_{\mathbf{q}}] \lambda_g \in \mathbb{Q}.$$

Again, the argument of Lemma 5.1 yields the required result.

LEMMA 5.4. *For all $g \geqslant 2$, the matrix ω'_g is non-singular.*

Together, Lemmas 5.3 and 5.4 complete the proof of Proposition 1.3.

6. Universality of genus zero

6.1. Genus 5

Do the relations of Theorem 1.4 generate the entire ideal of relations in $\varkappa^*(M_g^c)$? Since Proposition 3.2 contains the relations of Theorem 1.4, we may ask the same question of the richer system. The answer to these questions is no. The first example occurs in $\varkappa^6(M_5^c)$.

There are 11 $\varkappa$ monomials of degree 6. By the evaluation rules of §3.5, the $\varkappa$ relations in codimension 6 generated by Proposition 3.2 are the *same* for all the rings

$$\varkappa^*(M_5^c), \quad \varkappa^*(M_{4,2}^c), \quad \varkappa^*(M_{3,4}^c), \quad \varkappa^*(M_{2,6}^c), \quad \varkappa^*(M_{1,8}^c) \quad \text{and} \quad \varkappa^*(M_{0,10}^c).$$

On $M_{0,10}^c$, there are 4 types([11]) of boundary divisors determined by the point splittings:

$$8+2, \quad 7+3, \quad 6+4 \quad \text{and} \quad 5+5.$$

The pairings of these divisors with the $\varkappa$ monomials

$$\varkappa_6, \quad \varkappa_5\varkappa_1, \quad \varkappa_4\varkappa_2 \quad \text{and} \quad \varkappa_3^2$$

on $M_{0,10}^c$ are easily seen to determine a non-singular 4×4 matrix. Hence, the number of independent $\varkappa$ relations in $\varkappa^6(M_{0,10}^c)$ is at most 7. In fact, Proposition 3.2 generates 7 independent relations.

The number of divisor classes in $R^*(M_5^c)$ is 3 given by $\varkappa_1$ and the two boundary divisors with genus splittings $4+1$ and $3+2$. The Gorenstein conjecture for M_5^c predicts $R^6(M_5^c)$ to have rank 3. The rank of $R^6(M_5^c)$ can be proven to be 3 via an application([12]) of Getzler's relation [10], [23]. Therefore, there *must* be at least 8 relations among the $\varkappa$ monomials of degree 6 in M_5^c. We have proven that the method of Proposition 3.2 does not yield all the $\varkappa$ relations in $R^6(M_5^c)$.

6.2. Genus zero

In §§7–9 below, a set of relations obtained from the virtual geometry of the moduli space of stable maps will be proven to generate all the $\varkappa$ relations in the rings $\varkappa^*(M_{0,n}^c)$.

([11]) There are several actual divisors of each type depending on the marking distribution. We select one of each type.

([12]) The crucial geometry here is the gluing map

$$\overline{M}_{1,4}\times\prod_{i=1}^{4}\overline{M}_{1,1}\longrightarrow\overline{M}_5.$$

Getzler's codimension-2 relation on the factor $\overline{M}_{1,4}$ pushes-forward to a codimension-6 relation on $\overline{M}_5$ which may be restricted to M_5^c. The result is a non-trivial relation in $R^6(M_5^c)$ not implied by the 7 relations of Proposition 3.2. We thank C. Faber for pointing out the argument.

Question 6.1. Does Proposition 3.2 generate all $\varkappa$ relations in the rings $\varkappa^*(M^c_{0,n})$?

The answer to Question 6.1 is affirmative at least for $n\leqslant 12$. We list below the Betti polynomials $B_n(t)$ of $\varkappa^*(M^c_{0,n})$ for $n\leqslant 12$:

$$\begin{aligned}
B_3 &= 1,\\
B_4 &= 1+t,\\
B_5 &= 1+t+t^2,\\
B_6 &= 1+t+2t^2+t^3,\\
B_7 &= 1+t+2t^2+2t^3+t^4,\\
B_8 &= 1+t+2t^2+3t^3+3t^4+t^5,\\
B_9 &= 1+t+2t^2+3t^3+4t^4+3t^5+t^6,\\
B_{10} &= 1+t+2t^2+3t^3+5t^4+5t^5+4t^6+t^7,\\
B_{11} &= 1+t+2t^2+3t^3+5t^4+6t^5+7t^6+4t^7+t^8,\\
B_{12} &= 1+t+2t^2+3t^3+5t^4+7t^5+9t^6+8t^7+5t^8+t^9.
\end{aligned}$$

From the table of Betti numbers, a formula is easily guessed. Let $P(d,k)\subset P(d)$ be the subset of partitions of d of length at most k, and let $|P(d,k)|$ be its cardinality. We see that

$$\dim_{\mathbb{Q}} \varkappa^d(M^c_{0,n}) = |P(d, n-d-2)|$$

holds in all the above cases.

THEOREM 6.2. *A $\mathbb{Q}$-basis of $\varkappa^d(M^c_{0,n})$ is given by*

$$\{\varkappa_{\mathbf{p}} : \mathbf{p}\in P(d, n-2-d)\}.$$

Proof. In order for $P(d,n-d-2)$ to be non-empty, we must have

$$d\leqslant n-3.$$

We first prove the independence of the $\varkappa$ monomials associated with $P(d,n-d-2)$ by intersection with strata classes in $R^{n-3-d}(M^c_{0,n})$. With each partition

$$\mathbf{p}\in P(d, n-d-2),$$

we associate a stratum $V_{\mathbf{p}}\subset M^c_{0,n}$ of codimension $n-3-d$ by the following construction. We write the parts of $\mathbf{p}$ as

$$(p_1, ..., p_l, p_{l+1}, ..., p_{n-d-2}),$$

where $p_{l+\delta}=0$ for $\delta>0$. Start with a chain of rational curves of length $n-d-2$,

$$R_1 —— R_2 —— R_3 —— \ldots —— R_{n-d-2}.$$

Next, we add markings([13]) to the components:

- p_1+2 markings to R_1;
- p_i+1 markings to R_i for $2\leqslant i\leqslant n-d-3$;
- $p_{n-d-2}+2$ markings to R_{n-d-2}.

Let C be the resulting curve. The total number of markings of C is

$$2+d+n-d-2=n.$$

The number of nodes of C is $n-3-d$. Hence, C determines a stratum $V_{\mathbf{p}}\subset M^c_{0,n}$ of codimension $n-3-d$.

A simple analysis following the strategy of the proof of Lemma 5.1 shows that the paring on $P(d,n-d-2)$ given by

$$(\mathbf{p},\mathbf{q})\longmapsto\int_{M^c_{0,n}}\varkappa_{\mathbf{p}}[V]_{\mathbf{q}}$$

is upper-triangular and non-singular. We conclude that the $\varkappa$ monomials associated with $P(d,n-d-2)$ are linearly independent.

The strata of $M^c_{0,n}$ are indexed by marked trees. Given a marked tree Γ with $n-d-2$ vertices, the associated stratum $S_\Gamma\subset M^c_{0,n}$ parameterizes curves C with marked dual graph Γ. In other words, C is a tree of marked rational components $R_1,\ldots,R_{n-2-d}$. With S_Γ, we associate a partition $\mathbf{q}(\Gamma)\in P(d,n-d-2)$ by the following construction. Let $\mathsf{m}(R_i)$ and $\mathsf{n}(R_i)$ denote the numbers of markings and nodes incident to R_i, respectively. Let

$$q_i=\mathsf{m}(R_i)+\mathsf{n}(R_i)-3.$$

By stability, $q_i\geqslant 0$. After reordering by size,

$$\mathbf{q}(\Gamma)=(q_1,\ldots,q_{n-d-2})\in P(d,n-d-2).$$

Let $\mathbf{p}\in P(d)$. The intersection of $\varkappa_{\mathbf{p}}$ with a stratum class S is obtained by distributing the factors $\varkappa_{p_i}$ to the components of S. We conclude that, for all $\mathbf{p}\in P(d)$,

$$\int_{M^c_{0,n}}\varkappa_{\mathbf{p}}S_\Gamma=\int_{M^c_{0,n}}\varkappa_{\mathbf{p}}V_{\mathbf{q}(\mathbf{\Gamma})}. \tag{25}$$

([13]) The particular markings chosen are not important.

By Poincaré duality[14], the dimension of $\varkappa^d(M^c_{0,n})$ is the rank of the intersection pairing

$$\varkappa^d(M^c_{0,n})\times A^{n-3-d}(M^c_{0,n})\longrightarrow \mathbb{Q}.$$

The classes of strata generate $A^{n-3-d}(M^c_{0,n})$. Moreover, only the special strata $V_{\mathbf{q}}$ need to be considered by (25). So,

$$\dim_{\mathbb{Q}} \varkappa^d(M^c_{0,n})\leqslant |P(d,n-d-2)|.$$

The independence property together with the above dimension estimate yields the basis result. □

7. Strategy for Theorem 1.5

7.1. Overview

By Theorem 1.6, we know that

$$\dim_{\mathbb{Q}} \varkappa^d(M^c_{0,n}) = |P(d,n-2-d)|.$$

Hence, Theorem 1.5 is a consequence of the following result.

PROPOSITION 7.1. *Let $\zeta>0$ be fixed. The space of relations among $\varkappa$ monomials of degree d valid simultaneously in* all *the rings*

$$\{\varkappa^*(M^c_{g,n}) : 2g-2+n=\zeta\}$$

has rank at least $|P(d)|-|P(d,\zeta-d)|$.

Proposition 7.1 is proven in §8 and §9 by constructing universal relations in $\varkappa^*(M^c_{g,n})$ via the virtual geometry of the moduli space of stable maps. The interplay between stable quotients and stable maps is an interesting aspect of the study of $\varkappa^*(M^c_{g,n})$.

7.2. ψ classes

Consider the cotangent line classes $\psi_{n+1},...,\psi_{n+l}\in A^1(M^c_{g,n+l})$ at the last l marked points. Let $\varepsilon^c\colon M^c_{g,n+l}\to M^c_{g,n}$ be the proper forgetful map. With each partition $\mathbf{p}\in P(d)$ of length l, we associate the class

$$\varepsilon^c_*(\psi^{1+p_1}_{n+1}\,...\,\psi^{1+p_l}_{n+l})\in A^d(M^c_{g,n}).$$

[14] For $M^c_{0,n}$, singular cohomology and the Chow ring agree.

The relation between the above push-forwards of ψ monomials and the $\varkappa$ classes is easily obtained. For $\mathbf{p}=(d)$, we have

$$\varepsilon_*^c(\psi_{n+1}^{1+d}) = \varkappa_d$$

by definition. The standard cotangent line comparison formulas yield the length-2 case,

$$\varepsilon_*^c(\psi_{n+1}^{1+p_1}\psi_{n+2}^{1+p_2}) = \varkappa_{p_1}\varkappa_{p_2} + \varkappa_{p_1+p_2}.$$

The formula for arbitrary $\mathbf{p}$, due to Faber, is

$$\varepsilon_*^c(\psi_{n+1}^{1+p_1} \dots \psi_{n+l}^{1+p_l}) = \sum_{\sigma\in S_l} \varkappa_{\sigma(\mathbf{p})}, \tag{26}$$

where the sum is over the symmetric group S_l. For $\sigma\in S_l$, let $\sigma=\gamma_1 \dots \gamma_r$ be the canonical cycle decomposition (including the 1-cycles), and let $\sigma(\mathbf{p})_i$ be the sum of the parts of $\mathbf{p}$ with indices in the cycle γ_i. Then,

$$\varkappa_{\sigma(\mathbf{p})} = \varkappa_{\sigma(\mathbf{p})_1} \dots \varkappa_{\sigma(\mathbf{p})_r}.$$

A discussion of (26) can be found in [1], see equation (1.13) there.

LEMMA 7.2. *The sets of classes in $A^d(M_{g,n}^c)$ defined by*

$$\{\varepsilon_*^c(\psi_{n+1}^{1+p_1} \dots \psi_{n+l}^{1+p_l}) : \mathbf{p}\in P(d)\} \quad \textit{and} \quad \{\varkappa_{\mathbf{p}} : \mathbf{p}\in P(d)\}$$

are related by an invertible linear transformation independent of g and n.

Proof. Formula (26) defines a universal transformation independent of g and n. Since the transformation is triangular in the partial ordering of $P(d)$ by length (with 1's on the diagonal), the invertibility is clear. □

7.3. Bracket classes

Let $\mathbf{p}\in P(d)$ be a partition of length l. Let

$$\langle \mathbf{p}\rangle = \varepsilon_*^c\left[\prod_{i=1}^{l} \frac{1}{1-p_i\psi_{n+i}}\right]^{l+d} \in A^d(M_{g,n}^c). \tag{27}$$

The superscript in the inhomogeneous expression

$$\left[\prod_{i=1}^{l} \frac{1}{1-p_i\psi_{n+i}}\right]^{l+d}$$

indicates the summand in $A^{l+d}(M^c_{g,n+l})$.

We can easily expand definition (27) to express the class $\langle \mathbf{p}\rangle$ linearly in terms of the classes

$$\{\varepsilon^c_*(\psi^{1+p_1}_{n+1} \dots \psi^{1+p_l}_{n+l}) : \mathbf{p} \in P(d)\}.$$

Since the string and dilation equation must be used to remove the ψ^0_{n+i} and ψ^1_{n+i} factors, the transformation depends upon g and n only through $2g-2+n$.

LEMMA 7.3. *The sets of classes in $A^d(M^c_{g,n})$ defined by*

$$\{\langle \mathbf{p}\rangle : \mathbf{p} \in P(d)\} \quad \textit{and} \quad \{\varepsilon^c_*(\psi^{1+p_1}_{n+1} \dots \psi^{1+p_l}_{n+l}) : \mathbf{p} \in P(d)\}$$

are related by an invertible linear transformation depending only upon $2g-2+n$.

Proof. Only the invertibility remains to be established. The result exactly follows from the proof of Proposition 3 in [7]. □

By Lemmas 7.2 and 7.3, the bracket classes lie in the $\varkappa$ ring:

$$\langle \mathbf{p}\rangle \in \varkappa^d(M^c_{g,n}).$$

We will prove Proposition 7.1 in the following equivalent form.

PROPOSITION 7.4. *Let $\zeta > 0$ be fixed. The space of relations among the classes*

$$\{\langle \mathbf{p}\rangle : \mathbf{p} \in P(d)\}$$

valid in all *the rings*

$$\{\varkappa^*(M^c_{g,n}) : 2g-2+n = \zeta\}$$

has rank at least $|P(d)| - |P(d, \zeta - d)|$.

8. Relations via stable maps

8.1. Moduli of stable maps

Let $\overline{M}_{g,n+m}(\mathbb{P}^1, d)$ denote the moduli of stable maps(15) to $\mathbb{P}^1$ of degree d, and let

$$\nu: \overline{M}_{g,n+m}(\mathbb{P}^1, d) \longrightarrow \overline{M}_{g,n}$$

be the morphism forgetting the map and the last m markings. The moduli space

$$M^c_{g,n+m}(\mathbb{P}^1, d) \subset \overline{M}_{g,n+m}(\mathbb{P}^1, d)$$

(15) Stable maps were defined in [17], see [8] for an introduction.

is defined by requiring the domain curve to be of compact type. The restriction

$$\nu^c: M^c_{g,n+m}(\mathbb{P}^1, d) \longrightarrow M^c_{g,n}$$

is proper and equivariant with respect to the symmetries of $\mathbb{P}^1$.

We will find relations in $A^*(M^c_{g,n})$ by localizing ν^c push-forwards which vanish geometrically. A complete analysis in the socle $A^{2g-3}(M^c_g)$ was carried out in [7], but much more will be required for Theorem 1.5. While the relations in $A^*(M^c_{g,n})$ of Theorem 1.4 via stable quotients are more elegantly expressed, the ranks of the relations via stable maps appear easier to compute.

8.2. Construction of relations

8.2.1. Indexing

Let $d \leqslant 2g-3+n$ and let $\delta = 2g-3+n-d$. We will construct a series of relations $I(g,d,\alpha)$ in $A^d(M^c_{g,n})$, where $\alpha = (\alpha_1, ..., \alpha_m)$ is a (non-empty) vector of non-negative integers satisfying the following two conditions:

(i) $|\alpha| = \sum_{i=1}^m \alpha_i \leqslant d-2-\delta$;

(ii) $\alpha_i > 0$ for $i > 1$.

By condition (i), $d-2-\delta \geqslant 0$ so $d > g-1+\lfloor \frac{1}{2} n \rfloor$. Condition (ii) implies that α_1 is the only integer permitted to vanish. The relation $I(g,d,\alpha)$ will be a variant of the equations considered in [7].

8.2.2. Formulas

Let Γ denote the data type

$$(p_1, ..., p_m) \cup \{p_{m+1}, ..., p_l\}, \tag{28}$$

satisfying

$$p_i > 0 \quad \text{and} \quad \sum_{i=1}^{l} p_i = d.$$

The first part of Γ is an ordered m-tuple $(p_1, ..., p_m)$. The second part $\{p_{m+1}, ..., p_l\}$ is an unordered set. Let $\mathrm{Aut}(\{p_{m+1}, ..., p_l\})$ be the group which permutes equal parts. The group of automorphisms $\mathrm{Aut}(\Gamma)$ equals $\mathrm{Aut}(\{p_{m+1}, ..., p_l\})$.

THEOREM 8.1. *For all vectors α satisfying conditions* (i) *and* (ii), *one has*

$$\sum_{\Gamma} \frac{1}{|\mathrm{Aut}(\Gamma)|} \left(\prod_{i=1}^{m} p_i^{-\alpha_i} \right) \left(\prod_{i=m+1}^{l} \frac{1}{-p_i} \right) \left(\prod_{i=1}^{l} \frac{p_i^{p_i}}{p_i!} \right) \langle p_1, ..., p_l \rangle = 0 \in A^d(M^c_{g,n}),$$

where the sum is over all Γ of type (28).

The bracket $\langle p_1,...,p_l\rangle \in A^d(M^c_{g,n})$ denotes the class associated with the partition defined by the union of all the parts p_i of Γ.

8.3. Proof of Theorem 8.1

8.3.1. Torus actions

The first step is to define the appropriate torus actions. Let $\mathbb{P}^1 = \mathbb{P}(V)$, where $V = \mathbb{C}\oplus\mathbb{C}$. Let $\mathbb{C}^*$ act on V by

$$\xi\cdot(v_1,v_2) = (v_1, \xi\cdot v_2). \tag{29}$$

Let p_1 and p_2 be the fixed points $[1,0]$ and $[0,1]$ of the corresponding action on $\mathbb{P}(V)$. An equivariant lifting of $\mathbb{C}^*$ to a line bundle L over $\mathbb{P}(V)$ is uniquely determined by the weights $[l_1,l_2]$ of the fiber representations at the fixed points

$$L_1 = L|_{\mathsf{p}_1} \quad \text{and} \quad L_2 = L|_{\mathsf{p}_2}.$$

The canonical lifting of $\mathbb{C}^*$ to the tangent bundle $T_{\mathbb{P}^1}$ has weights $[1,-1]$. We will utilize the equivariant liftings of $\mathbb{C}^*$ to $\mathcal{O}_{\mathbb{P}(V)}(1)$ and $\mathcal{O}_{\mathbb{P}(V)}(-1)$ with weights $[1,0]$ and $[0,1]$, respectively.

Over the moduli space of stable maps $\overline{M}_{g,n+m}(\mathbb{P}(V),d)$, we have

$$\pi: U \longrightarrow \overline{M}_{g,n+m}(\mathbb{P}(V),d) \quad \text{and} \quad \mu: U \longrightarrow \mathbb{P}(V),$$

where U is the universal curve and μ is the universal map. The representation (29) canonically induces $\mathbb{C}^*$-actions on U and $\overline{M}_{g,n+m}(\mathbb{P}(V),d)$ compatible with the maps π and μ. The $\mathbb{C}^*$-equivariant virtual class

$$[\overline{M}_{g,n+m}(\mathbb{P}(V),d)]^{\mathrm{vir}} \in A^{\mathbb{C}^*}_{2g+2d-2+n+m}(\overline{M}_{g,n+m}(\mathbb{P}(V),d))$$

will play an important role.

8.3.2. Equivariant classes

Three types of equivariant Chow classes on $\overline{M}_{g,n+m}(\mathbb{P}(V),d)$ will be considered here:

- The linearization $[0,1]$ on $\mathcal{O}_{\mathbb{P}(V)}(-1)$ defines a $\mathbb{C}^*$-action on the rank-$(d+g-1)$ bundle

$$\mathbf{R} = R^1\pi_*(\mu^*\mathcal{O}_{\mathbb{P}(V)}(-1))$$

on $\overline{M}_{g,n+m}(\mathbb{P}(V),d)$. Let

$$c_{\mathrm{top}}(\mathbf{R}) \in A^{g+d-1}_{\mathbb{C}^*}(\overline{M}_{g,n+m}(\mathbb{P}(V),d))$$

be its top Chern class.

- For each marking i, let $\psi_i \in A^1_{\mathbb{C}^*}(\overline{M}_{g,n+m}(\mathbb{P}(V),d))$ denote the first Chern class of the canonically linearized cotangent line corresponding to i.
- Denote the ith evaluation morphism by

$$\mathrm{ev}_i\colon \overline{M}_{g,n+m}(\mathbb{P}(V),d) \longrightarrow \mathbb{P}(V).$$

With $\mathbb{C}^*$-linearization $[1,0]$ on $\mathcal{O}_{\mathbb{P}(V)}(1)$, let

$$\varrho_i = c_1(\mathrm{ev}_i^*\,\mathcal{O}_{\mathbb{P}(V)}(1)) \in A^1_{\mathbb{C}^*}(\overline{M}_{g,n+m}(\mathbb{P}(V),d)).$$

With $\mathbb{C}^*$-linearization $[0,-1]$ on $\mathcal{O}_{\mathbb{P}(V)}(1)$, let

$$\tilde{\varrho}_i = c_1(\mathrm{ev}_i^*\,\mathcal{O}_{\mathbb{P}(V)}(1)) \in A^1_{\mathbb{C}^*}(\overline{M}_{g,n+m}(\mathbb{P}(V),d)).$$

In the non-equivariant limit, $\varrho_i^2=0$. Our notation here closely follows [7].

8.3.3. Vanishing integrals

The forgetful morphism

$$\nu\colon \overline{M}_{g,n+m}(\mathbb{P}(V),d) \longrightarrow \overline{M}_{g,n}$$

is $\mathbb{C}^*$-equivariant with respect to the trivial action on $\overline{M}_{g,n}$. As in §8.2.1, let

$$d \leqslant 2g-3+n \quad \text{and} \quad \delta = 2g-3+n-d,$$

and let $\alpha=(\alpha_1,...,\alpha_m)$ satisfy

(i) $|\alpha|=\sum_{i=1}^m \alpha_i \leqslant d-2-\delta$;

(ii) $\alpha_i>0$ for $i>1$.

Let $I(g,d,\alpha)$ be the $\mathbb{C}^*$-equivariant push-forward

$$\nu_*\left(\varrho_{n+1}^{d-1-\delta-|\alpha|}\left(\prod_{i=1}^m \varrho_{n+i}\psi_{n+i}^{\alpha_i}\right)\left(\prod_{j=1}^n \tilde{\varrho}_j\right) c_{\mathrm{top}}(\mathbf{R}) \cap [\overline{M}_{g,n+m}(\mathbb{P}(V),d)]^{\mathrm{vir}}\right).$$

The degree of the class

$$\varrho_{n+1}^{d-1-\delta-|\alpha|}\left(\prod_{i=1}^m \varrho_{n+i}\psi_{n+i}^{\alpha_i}\right)\left(\prod_{j=1}^n \tilde{\varrho}_j\right) c_{\mathrm{top}}(\mathbf{R})$$

is easily computed to be

$$d-1-\delta-|\alpha|+m+|\alpha|+n+d+g-1 = g+2d-2+n+m-\delta.$$

Since the cycle dimension of the virtual class is $2g+2d-2+n+m$, the push-forward $I(g,d,\alpha)$ has cycle dimension

$$2g+2d-2+n+m-(g+2d-2+n+m-\delta) = g+\delta = 3g-3+n-d.$$

Equivalently, $I(g,d,\alpha) \in A^d_{\mathbb{C}^*}(\overline{M}_{g,n})$. As the class ϱ_{n+1} appears with exponent

$$d-\delta-|\alpha| \geqslant 2,$$

$I(g,d,\alpha)$ vanishes in the non-equivariant limit.

8.3.4. Localization terms

The virtual localization formula of [12] calculates $I(g,d,\alpha)$ in terms of tautological classes on the moduli space $\overline{M}_{g,n}$. To prove Theorem 8.1, we will calculate the restriction of the localization formula to $M^c_{g,n}$.

The localization formula expresses $I(g,d,\alpha)$ as a sum over connected decorated graphs Γ indexing the $\mathbb{C}^*$-fixed loci of $\overline{M}_{g,n+m}(\mathbb{P}(V),d)$. The vertices of the graphs lie over the fixed points $\mathsf{p}_1,\mathsf{p}_2\in\mathbb{P}(V)$ and are labeled with genera (which sum over the graph to $g-h^1(\Gamma)$). The edges of the graphs lie over $\mathbb{P}^1$ and are labeled with degrees (which sum over the graph to d). Finally, the graphs carry $n+m$ markings on the vertices. The valence $\mathrm{val}(v)$ of a vertex $v\in\Gamma$ counts both the incident edges and markings. The edge valence of v counts only the incident edges.

Only a very restricted subset of graphs will yield non-vanishing contributions to $I(g,d,\alpha)$ in the non-equivariant limit. If a graph Γ contains a vertex lying over p_1 of edge valence greater than 1, then the contribution of Γ to $I(g,d,\alpha)$ vanishes by our choice of linearization on the bundle $\mathbf{R}$. A vertex over p_1 of edge valence greater than 1 yields a trivial Chern root of $\mathbf{R}$ (with trivial weight zero) in the numerator of the localization formula to force the vanishing.

By the above vanishing, only *comb* graphs Γ contribute to $I(g,d,\alpha)$. Comb graphs contain $l\leqslant d$ vertices lying over p_1, each connected by a distinct edge to a unique vertex lying over p_2.

If Γ contains a vertex over p_1 of positive genus, then the restriction to $M^c_{g,n}$ of the contribution of Γ to $I(g,d,\alpha)$ vanishes by the following argument. Let v be a vertex of genus $g(v)>0$ lying over p_1. The integrand term $c_{\mathrm{top}}(\mathbf{R})$ yields a factor $c_{g(v)}(\mathbb{E}^*)$ with trivial $\mathbb{C}^*$-weight on the moduli space of genus $g(v)$ corresponding to the vertex v. As

$$\lambda_{g(v)}|_{M^c_{g(v),\mathrm{val}(v)}}=0$$

by [9], the required vanishing holds.

The linearizations of the classes ϱ_i and $\tilde{\varrho}_j$ place restrictions on the marking distribution. Since the class $\tilde{\varrho}_j$ is obtained from $\mathcal{O}_{\mathbb{P}(V)}(1)$ with linearization $[0,-1]$, the first n markings must lie on the unique vertex over p_2. As the class ϱ_i is obtained from $\mathcal{O}_{\mathbb{P}(V)}(1)$ with linearization $[1,0]$, the last m markings must lie on vertices over p_1.

Finally, we claim that the last m markings of Γ must lie on distinct vertices over p_1 for non-vanishing contribution to $I(g,d,\alpha)$. Let v be a vertex over p_1 (with $g(v)=0$). If v carries at least two markings, the fixed locus corresponding to Γ contains a product factor $\overline{M}_{0,r+1}$, where r is the number of markings incident to v. The classes $\psi^{\alpha_i}_{n+i}$ carry trivial $\mathbb{C}^*$-weight. Moreover, as each $\alpha_i>0$ for $i>1$, we see that the sum of the α_i, as i

ranges over the set of markings incident to v, is at least $r-1$. Since the sum exceeds the dimension of $\overline{M}_{0,r+1}$, the graph contribution to $I(g,d,\alpha)$ vanishes.

The proof of the main result about the localization terms for $I(g,d,\alpha)$ is now complete.

PROPOSITION 8.2. *The restriction of* $I(g,d,\alpha)$ *to* $M^c_{g,n}$ *is expressed via the virtual localization formula as a sum over genus-*g*, degree-*d*, marked comb graphs* Γ *satisfying*:

(i) *all vertices over* p_1 *are of genus zero*;

(ii) *the unique vertex over* p_2 *carries all of the first* n *markings*;

(iii) *the last* m *markings all lie over* p_1;

(iv) *each vertex over* p_1 *carries at most one of the last* m *markings.*

8.3.5. Formulas

The precise contributions of allowable graphs Γ to the non-equivariant limit of $I(g,d,\alpha)$ are now calculated.

Let Γ be a genus-g, degree-d comb graph with $n+m$ markings satisfying conditions (i)–(iv) of Proposition 8.2. By condition (iv), Γ must have $l\geqslant m$ edges. Γ may be described uniquely by the data

$$(p_1,...,p_m)\cup\{p_{m+1},...,p_l\}, \tag{30}$$

satisfying

$$p_i>0 \quad \text{and} \quad \sum_{i=1}^{l} p_i=d.$$

The elements of the ordered m-tuple $(p_1,...,p_m)$ correspond to the degree assignments of the edges incident to the vertices marked by the last m markings. The elements of the unordered partition $\{p_{m+1},...,p_l\}$ correspond to the degrees of the edges incident to the unmarked vertices over p_1. The group of graph automorphisms is

$$\mathrm{Aut}(\Gamma)=\mathrm{Aut}(\{p_{m+1},...,p_l\}).$$

By a direct application of the virtual localization formula of [12], we find the contribution of the graph (30) to the normalized([16]) push-forward

$$(-1)^{g+1+|\alpha|+n+m}I(g,d,\alpha)$$

([16]) The parallel equation on page 106 of [7] has a sign error in the normalization. Instead of $(-1)^{g+1}I(g,d,\alpha)$ there, the normalization should be $(-1)^{g+1+|\alpha|+l(\alpha)}I(g,d,\alpha)$. The sign change makes no difference.

equals

$$\frac{1}{|\mathrm{Aut}(\Gamma)|}\left(\prod_{i=1}^{m} p_i^{-\alpha_i}\right)\left(\prod_{i=m+1}^{l}\frac{1}{-p_i}\right)\left(\prod_{i=1}^{l}\frac{p_i^{p_i}}{p_i!}\right)\langle p_1,...,p_l\rangle.$$

Hence, the vanishing of $I(g,d,\alpha)$ yields the relation

$$\sum_{\Gamma}\frac{1}{|\mathrm{Aut}(\Gamma)|}\left(\prod_{i=1}^{m} p_i^{-\alpha_i}\right)\left(\prod_{i=m+1}^{l}\frac{1}{-p_i}\right)\left(\prod_{i=1}^{l}\frac{p_i^{p_i}}{p_i!}\right)\langle p_1,...,p_l\rangle = 0,$$

where the sum is over all graphs (30).

9. Proof of Theorem 1.5

9.1. Matrix of relations

Theorem 8.1 yields relations in $\varkappa^d(M_{g,n}^c)$, indexed by $\alpha=(\alpha_1,...,\alpha_m)$ satisfying conditions (i) and (ii) of §8.2.1 with

$$\delta = 2g-3+n-d \geqslant 0.$$

We rewrite the relation obtained from the vanishing of $I(g,d,\alpha)$ as

$$\sum_{\mathbf{p}\in P(d)} \mathsf{C}_\alpha^{\mathbf{p}}\langle\mathbf{p}\rangle = 0. \tag{31}$$

The coefficients are

$$\mathsf{C}_\alpha^{\mathbf{p}} = \frac{1}{|\mathrm{Aut}(\mathbf{p})|}\left(\prod_{i=1}^{l}\frac{p_i^{p_i}}{p_i!}\right)\sum_{\phi}\left(\prod_{i=1}^{m} p_{\phi(i)}^{-\alpha_i}\right)\prod_{j\in\mathrm{Im}(\phi)^c}\frac{1}{-p_j},$$

where the sum is over all injections $\phi\colon\{1,...,m\}\to\{1,...,l\}$ and $\mathrm{Im}(\phi)^c\subset\{1,...,l\}$ is the complement of the image of ϕ.

In order to prove Proposition 7.4, we will show that the system (31) has rank at least $|P(d)|-|P(d,\delta+1)|$. The claim is empty unless $0\leqslant\delta\leqslant d-2$.

9.2. Ordering

For $0\leqslant\delta\leqslant d-2$, define the subset $P_\delta(d)\subset P(d)$ by removing partitions of length at most $\delta+1$, that is

$$P_\delta(d) = P(d)\setminus P(d,\delta+1).$$

We order $P_\delta(d)$ by the following rules:

- longer partitions appear before shorter partitions;

• for partitions of the same length, we use the lexicographic ordering with larger parts([17]) appearing before smaller parts.

For example, the ordered list of the 10 elements of $P_0(6)$ is

$$(1^6),\ (2,1^4),\ (3,1^3),\ (2^2,1^2),\ (4,1^2),\ (3,2,1),\ (2^3),\ (5,1),\ (4,2),\ (3,3).$$

Given a partition $\mathbf{p}\in P(d)$, let $\widehat{\mathbf{p}}$ be the partition obtained after removing all parts equal to 1. For example,

$$\widehat{(1^6)}=\varnothing \quad \text{and} \quad \widehat{(3,2,1)}=(3,2).$$

Let $\mathbf{p}^-$ be the partition obtained by lowering all the parts of $\mathbf{p}$ by 1. For example,

$$(1^6)^-=\varnothing \quad \text{and} \quad (3,2,1)^-=(2,1).$$

If $\mathbf{p}$ has length l, then $\mathbf{p}^-\in P(d-l)$.

With each partition $\mathbf{p}\in P_\delta(d)$, we associate data $\alpha[\mathbf{p}]$ satisfying conditions (i) and (ii) with respect to δ by the following rules. The special designation

$$\alpha[(1^d)]=(0)$$

is given. Otherwise

$$\alpha[\mathbf{p}]=\mathbf{p}^-.$$

We note that condition (i) of §8.2.1,

$$|\alpha[\mathbf{p}]|\leqslant d-2-\delta,$$

is satisfied in all cases.

Let $\mathsf{M}_\delta(d)$ be the square matrix indexed by the ordered set $P_\delta(d)$ with elements

$$M_\delta(d)[\mathbf{p},\mathbf{q}]=\mathsf{C}^{\mathbf{q}}_{\alpha[\mathbf{p}]}.$$

The rank of the system (31) is at least

$$|P_\delta(d)|=|P(d)|-|P(d,\delta+1)|,$$

by the following non-singularity result proven in §§9.3–9.6 below.

PROPOSITION 9.1. *For* $0\leqslant\delta\leqslant d-2$, *the matrix* $\mathsf{M}_\delta(d)$ *is non-singular.*

Proposition 9.1 implies Proposition 7.4 and thus Theorem 1.5. Moreover, Proposition 9.1 provides a new approach to [7].

([17]) Remember that the parts of $\mathbf{p}=(p_1,...,p_l)$ are ordered by $p_1\geqslant...\geqslant p_l$.

9.3. Scaling

Let $\mathsf{X}_\delta(d)$ be the square matrix indexed by the ordered set $P_\delta(d)$ with elements

$$\mathsf{X}_\delta(d)[\mathbf{p},\mathbf{q}] = \begin{cases} (-1)^{l(\mathbf{q})-1} d, & \text{if } \mathbf{p} = (1^d), \\ \sum_\phi (-1)^{l(\mathbf{q})-l(\widehat{\mathbf{p}})} \prod_{i=1}^{l(\widehat{\mathbf{p}})} q_{\phi(i)}^{-\hat{p}_i+2}, & \text{if } \mathbf{p} \neq (1^d), \end{cases}$$

where the sum is over all injections

$$\phi\colon \{1,\ldots,l(\widehat{\mathbf{p}})\} \longrightarrow \{1,\ldots,l(\mathbf{q})\}.$$

For example, $\mathsf{X}_0(6)$ is

$$\begin{pmatrix} -6 & 6 & -6 & -6 & 6 & 6 & 6 & -6 & -6 & -6 \\ -6 & 5 & -4 & -4 & 3 & 3 & 3 & -2 & -2 & -2 \\ -6 & \frac{9}{2} & -\frac{10}{3} & -3 & \frac{9}{4} & \frac{11}{6} & \frac{3}{2} & -\frac{6}{5} & -\frac{3}{4} & -\frac{2}{3} \\ 30 & -20 & 12 & 12 & -6 & -6 & -6 & 2 & 2 & 2 \\ -6 & \frac{17}{4} & -\frac{28}{9} & -\frac{5}{2} & \frac{33}{16} & \frac{49}{36} & \frac{3}{4} & -\frac{26}{25} & -\frac{5}{16} & -\frac{2}{9} \\ 30 & -18 & 10 & 9 & -\frac{9}{2} & -\frac{11}{3} & -3 & \frac{6}{5} & \frac{3}{4} & \frac{2}{3} \\ -120 & 60 & -24 & -24 & 6 & 6 & 6 & 0 & 0 & 0 \\ -6 & \frac{33}{8} & -\frac{82}{27} & -\frac{9}{4} & \frac{129}{64} & \frac{251}{216} & \frac{3}{8} & -\frac{126}{125} & -\frac{9}{64} & -\frac{2}{27} \\ 30 & -17 & \frac{28}{3} & \frac{15}{2} & -\frac{33}{8} & -\frac{49}{18} & -\frac{3}{2} & \frac{26}{25} & \frac{5}{16} & \frac{2}{9} \\ 30 & -16 & 8 & \frac{13}{2} & -3 & -2 & -\frac{3}{2} & \frac{2}{5} & \frac{1}{4} & \frac{2}{9} \end{pmatrix}.$$

The matrix $\mathsf{X}_\delta(d)$ is obtained from $\mathsf{M}_\delta(d)$ by dividing each column corresponding to $\mathbf{q}$ by

$$\frac{1}{|\mathrm{Aut}(\mathbf{q})|} \prod_{i=1}^{l(\mathbf{q})} \frac{q_i^{q_i-1}}{q_i!}.$$

Hence, $\mathsf{X}_\delta(d)$ is non-singular if and only if $\mathsf{M}_\delta(d)$ is non-singular.

9.4. Elimination

Our strategy for proving Proposition 9.1 is to find an upper-triangular square matrix $\mathsf{Y}_0(d)$ for which the product

$$\mathsf{X}_0(d)\mathsf{Y}_0(d) \tag{32}$$

is lower-triangular with ± 1's on the diagonal. Since $\mathsf{X}_\delta(d)$, for $0 \leqslant \delta \leqslant d-2$, occurs as an upper-left minor of $\mathsf{X}_0(d)$, the lower-triangularity of the product (32) will establish Proposition 9.1 for the full range of δ values.

We define $\mathsf{Y}_0(d)$ to be the square matrix indexed by the ordered set $P_0(d)$ given by the following rules. The upper-left corner is

$$\mathsf{Y}_0(d)[(1^d),(1^d)] = \frac{1}{d}.$$

If at least one of $\{\mathbf{p},\mathbf{q}\}$ is not equal to (1^d), then the matrix elements are

$$\mathsf{Y}_0(d)[\mathbf{p},\mathbf{q}] = \frac{1}{|\mathrm{Aut}(\mathbf{p})|}\frac{1}{|\mathrm{Aut}(\widehat{\mathbf{q}})|}\sum_\theta \prod_{i=1}^{l(\mathbf{q})} \binom{q_i}{p_{i[1]},\ldots,p_{i[l_i]}} q_i^{l_i-2}\prod_{j=1}^{l_i} p_{ij}^{p_{ij}-1},$$

where the sum is over all functions $\theta\colon \{1,\ldots,l(\mathbf{p})\} \to \{1,\ldots,l(\mathbf{q})\}$ with

$$\theta^{-1}(i) = \{i[1],\ldots,i[l_i]\}$$

satisfying

$$q_i = \sum_{j=1}^{l_i} p_{i[j]}.$$

For example, $\mathsf{Y}_0(6)$ is

$$\begin{pmatrix}
\frac16 & 1 & 3 & \frac12 & 16 & 3 & \frac16 & 125 & 16 & \frac92\\
0 & 1 & 6 & 1 & 48 & 9 & \frac12 & 500 & 64 & 18\\
0 & 0 & 3 & 0 & 36 & 3 & 0 & 450 & 36 & 9\\
0 & 0 & 0 & \frac12 & 12 & 6 & \frac12 & 300 & 60 & 18\\
0 & 0 & 0 & 0 & 16 & 0 & 0 & 320 & 16 & 0\\
0 & 0 & 0 & 0 & 0 & 3 & 0 & 180 & 36 & 18\\
0 & 0 & 0 & 0 & 0 & 0 & \frac16 & 0 & 12 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 125 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac92
\end{pmatrix}.$$

By the conditions on θ in the definition, $\mathsf{Y}_0(d)$ is easily seen to be upper-triangular.

9.5. Generating functions

Let $\mathbb{Q}[t]$ denote the polynomial ring in infinitely many variables

$$t=\{t_1,t_2,t_3,...\}.$$

Define a $\mathbb{Q}$-linear function

$$\langle\cdot\rangle:\mathbb{Q}[t]\longrightarrow\mathbb{Q}$$

by the equations $\langle 1\rangle=1$ and

$$\langle t_{d_1}t_{d_2}...t_{d_k}\rangle=(d_1+d_2+...+d_k)^{k-3}.$$

We may extend $\langle\cdot\rangle$ uniquely to define an x-linear function

$$\langle\cdot\rangle:\mathbb{Q}[t][[x]]\longrightarrow\mathbb{Q}[[x]].$$

For each non-negative integer i, let

$$Z_i(t,x)=\sum_{j=1}^{\infty}x^jt_j\frac{j^{j-i}}{j!}\in\mathbb{Q}[t][[x]].$$

Applying the bracket, we define

$$\mathsf{F}_{\alpha_1,...,\alpha_m}=\langle e^{-Z_1}Z_{\alpha_1}...Z_{\alpha_m}\rangle\in\mathbb{Q}[[x]].$$

LEMMA 9.2. *Let $\alpha=(\alpha_1,...,\alpha_n)$ be a non-empty sequence of non-negative integers satisfying $\alpha_i>0$ for $i>1$. The series $\mathsf{F}_{\alpha_1,...,\alpha_m}\in\mathbb{Q}[[x]]$ is a* polynomial *in x of degree at most $1+\sum_{i=1}^m\alpha_i$.*

LEMMA 9.3. *Let $\alpha_1\geqslant 0$. Then,*

$$\mathsf{F}_{\alpha_1}=\frac{(-1)^{\alpha_1}}{(1+\alpha_1)(1+\alpha_1)!}x^{1+\alpha_1}+...,$$

where the dots stand for lower-order terms.

Lemma 9.2 can be proven by various methods. A proof via localization on moduli space is given in [7, §1.7]. Lemma 9.3 is more interesting.

Proof of Lemma 9.3. The integral

$$J_{1+\alpha_1}=\int_{\overline{M}_{0,1}(\mathbb{P}^1,1+\alpha_1)}\varrho_1\psi_1^{\alpha_1}c_{\mathrm{top}}(\mathbf{R})\tag{33}$$

can be evaluated by exactly following([18]) the localization analysis of §8.3. We find that

$$J_{1+\alpha_1} = (-1)^{\alpha_1} \sum_{\Gamma} \frac{1}{|\mathrm{Aut}(\Gamma)|} p_1^{-\alpha_1} \left(\prod_{i=2}^{l} \frac{1}{-p_i} \right) \left(\prod_{i=1}^{l} \frac{p_i^{p_i}}{p_i!} \right) (1+\alpha_1)^{l-3},$$

where the sum is over all 1-pointed comb graphs (30) of total degree $1+\alpha_1$. We conclude that $J_{1+\alpha_1}$ equals, up to the factor of $(-1)^{\alpha_1}$, the leading $x^{1+\alpha_1}$ coefficient of

$$\langle e^{-Z_1} Z_{\alpha_1} \rangle.$$

To calculate the integral (33), we use well-known equations in Gromov–Witten theory. Certainly

$$J_1 = 1. \tag{34}$$

By two applications of the divisor equation,

$$k^2 J_k = \int_{\overline{M}_{0,3}(\mathbb{P}^1,k)} \varrho_1 \psi_1^{k-1} \varrho_2 \varrho_3 c_{\mathrm{top}}(\mathbf{R}).$$

By the topological recursion relation [3] applied to the right-hand side,

$$k^2 J_k = \int_{\overline{M}_{0,2}(\mathbb{P}^1,k-1)} \varrho_1 \psi_1^{k-2} \varrho_2 c_{\mathrm{top}}(\mathbf{R}) \int_{\overline{M}_{0,3}(\mathbb{P}^1,1)} \varrho_1 \varrho_2 \varrho_3 c_{\mathrm{top}}(\mathbf{R}).$$

We obtain the recursion

$$k^2 J_k = (k-1) J_{k-1} J_1 = (k-1) J_{k-1},$$

which we can easily solve

$$J_k = \frac{1}{k \cdot k!}$$

starting with the initial condition (34). □

The case where the α data is empty will arise naturally. We define

$$\mathsf{F}_{\varnothing} = \langle e^{-Z_1} \rangle.$$

The following result is derived from Lemma 9.2 by the relation

$$x \frac{d}{dx} \mathsf{F}_{\varnothing} = -\mathsf{F}_0.$$

LEMMA 9.4. *$\mathsf{F}_{\varnothing} = 1 - x$.*

([18]) The equivariant lifts are taken just as in §8.3.2.

9.6. Product

We will now prove the basic identity

$$\mathsf{X}_0(d)\mathsf{Y}_0(d)=\mathsf{L}_0(d), \tag{35}$$

where $\mathsf{L}_0(d)$ is lower triangular with all diagonal entries ± 1.

We first address the special upper-left corner. The product on the left-hand side of (35) is

$$\mathsf{L}_0(d)[(1^d),(1^d)]=(-1)^{d-1}d\frac{1}{d}=(-1)^{d-1},$$

a diagonal entry of the specified form.

Next assume that $\mathbf{p}\neq(1^d)$. Then, the matrix elements are

$$\mathsf{L}_0(d)[\mathbf{p},\mathbf{q}]=\frac{1}{|\mathrm{Aut}(\widehat{\mathbf{q}})|}\sum_{\gamma}\prod_{i=1}^{l(\mathbf{q})}\mathrm{Coeff}(F_{\gamma^{-1}(i)},x^{q_i})q_i\cdot q_i!, \tag{36}$$

where the sum is over all functions

$$\gamma:\{1,...,l(\hat{\mathbf{p}})\}\longrightarrow\{1,...,l(\mathbf{q})\}.$$

In case $\gamma^{-1}(i)=\{i[1],...,i[l_i]\}$ is non-empty, we define

$$\mathsf{F}_{\gamma^{-1}(i)}=\mathsf{F}_{\hat{p}_{i[1]}-1,\ldots,\hat{p}_{i[l_i]}-1}.$$

If $\gamma^{-1}(i)=\varnothing$, then

$$\mathsf{F}_{\varnothing}=\langle e^{-Z_1}\rangle=1-x.$$

Equation (36) is obtained from a simple unravelling of the definitions.

If $q_i>1$, $\mathrm{Coeff}(F_{\gamma^{-1}(i)},x^{q_i})$ vanishes unless $\gamma^{-1}(i)$ is non-empty by Lemma 9.4 and unless

$$q_i\leqslant 1-l_i+\sum_{j=1}^{l_i}\hat{p}_{i[j]} \tag{37}$$

by Lemma 9.2. Inequality (37) for all parts $q_i>1$ implies that $l(\mathbf{q})\geqslant l(\mathbf{p})$. Moreover, if equality of length holds, then inequality (37) implies that either $\mathbf{q}$ precedes $\mathbf{p}$ in the ordering of $P_0(d)$ or $\mathbf{q}=\mathbf{p}$.

We conclude that the matrix $\mathsf{L}_0(d)$ is lower-triangular when the first coordinate $\mathbf{p}$ is not (1^d). The diagonal elements for $\mathbf{p}\neq(1^d)$ are

$$\mathsf{L}_0(d)[\mathbf{p},\mathbf{p}]=(-1)^{l(\mathbf{p})-l(\widehat{\mathbf{p}})}\prod_{i=1}^{l(\widehat{\mathbf{p}})}(-1)^{\hat{p}_i-1}$$

by Lemmas 9.3 and 9.4.

To complete the proof of the lower-triangularity of $\mathsf{L}_0(d)$, we must show the vanishing of $\mathsf{L}_0(d)[(1^d),\mathbf{q}]$, $\mathbf{q}\neq(1^d)$. The matrix elements are

$$\mathsf{L}_0(d)[(1^d),\mathbf{q}]=\frac{1}{|\mathrm{Aut}(\widehat{\mathbf{q}})|}\sum_{\widetilde{\gamma}}\prod_{i=1}^{l(\mathbf{q})}\mathrm{Coeff}(\widetilde{F}_{\widetilde{\gamma}^{-1}(i)},x^{q_i})q_i\cdot q_i!,\quad \mathbf{q}\neq(1^d),$$

where the sum is over all functions

$$\widetilde{\gamma}:\{1\}\longrightarrow\{1,...,l(\mathbf{q})\}.$$

In case $\widetilde{\gamma}^{-1}(i)=\{1\}$, we define

$$\widetilde{\mathsf{F}}_{\widetilde{\gamma}^{-1}(i)}=\mathsf{F}_0.$$

If $\widetilde{\gamma}^{-1}(i)=\varnothing$, then

$$\widetilde{\mathsf{F}}_{\varnothing}=\langle e^{-Z_1}\rangle=1-x.$$

Let $q_1>1$ be the largest part of $\mathbf{q}$. Then

$$\mathrm{Coeff}(\widetilde{F}_{\widetilde{\gamma}^{-1}(1)},x^{q_1})=0$$

by Lemmas 9.2 and 9.4. Hence,

$$\mathsf{L}_0(d)[(1^d),\mathbf{q}]=0,\quad \mathbf{q}\neq(1^d),$$

and the lower-triangularity of $\mathsf{L}_0(d)$ is fully proven.

The proof of Proposition 9.1 is complete. Following the implications back, the proof of Theorem 1.5 is also complete.

Since we know explicitly the diagonal elements of the triangular matrices $\mathsf{Y}_0(d)$ and $\mathsf{L}_0(d)$, the product

$$\mathsf{X}_0(d)\mathsf{Y}_0(d)=\mathsf{L}_0(d)$$

yields a simple formula for the determinant, namely

$$\det(\mathsf{X}_{0,d})=(-1)^{d-1}\prod_{\mathbf{p}\in P_0(d)\setminus\{(1^d)\}}\frac{|\mathrm{Aut}(\widehat{\mathbf{p}})|}{\prod_{i=1}^{l(\mathbf{p})}p_i^{p_i-2}}(-1)^{l(\mathbf{p})}\prod_{i=1}^{l(\widehat{\mathbf{p}})}(-1)^{\hat{p}_i}.$$

10. Proof of Theorem 1.8

10.1. Bound

By Theorem 1.5, we have a surjection

$$\varkappa^d(M^c_{0,2g+n}) \xrightarrow{\iota_{g,n}} \varkappa^d(M^c_{g,n}) \longrightarrow 0.$$

By Theorem 1.6, to prove that $\iota_{g,n}$ is an isomorphism, we only need to establish

$$\dim_{\mathbb{Q}} \varkappa^d(M^c_{g,n}) \geqslant |P(d, 2g-2+n-d)|$$

for $n>0$. We will obtain the bound by refining the argument for Theorem 1.2.

10.2. Dual graph types

A dual graph of type $A(g_1, ..., g_r)$ with $g_i \geqslant 1$ is a chain of r vertices of genera $g_1, ..., g_r$ with 2 markings on the ends. The corresponding curves are of the form

$$C^*_{g_1} \text{ —— } C_{g_2} \text{ —— } \dots \text{ —— } C^*_{g_r}.$$

If $r=1$, the unique vertex carries both markings.

A dual graph of type $B(g_1, ..., g_r | h_1, ..., h_{r-1})$ with $g_i, h_j \geqslant 1$ is a comb of $2r-1$ vertices with 1 marking. The corresponding curves are of the form

$$\begin{array}{ccccccccc} C^*_{g_1} & \text{——} & C_{g_2} & \text{——} & \dots & \text{——} & C_{g_{r-1}} & \text{——} & C_{g_r} \\ \Big| & & \Big| & & & & \Big| & & \\ C_{h_1} & & C_{h_2} & & \dots & & C_{h_{r-1}}. & & \end{array}$$

There are $r-1$ vertices of valence 3 and r vertices of valence 1. The marking is included in the valence count.

10.3. Case $n=1$

Let $\mathbf{p} \in P(d)$ be a partition of length $l=a+b$ with parts([19])

$$(p_1, ..., p_a, p'_1, ..., p'_b),$$

where the p_i are odd and the p'_j are even. We see that

$$d+l = b \quad \text{mod } 2.$$

([19]) All parts of $\mathbf{p}$ here are positive.

If $d+l$ is odd, then $b=2r-1$ for $r>0$. Let $\Gamma_{\mathbf{p}}$ be the dual graph obtained by the following construction:

$$\begin{array}{l} \Gamma_{\mathbf{p}} = A\left(\frac{p_1+1}{2}, ..., \frac{p_a+1}{2}\right) \\ \qquad\Big| \\ \qquad B\left(\frac{p'_1}{2}, ..., \frac{p'_{r-1}}{2}, \frac{p'_r}{2}+1 \,\Big|\, \frac{p'_{r+1}}{2}+1, ..., \frac{p'_{2r-1}}{2}+1\right), \end{array}$$

where the graphs are attached at the first marking of A and the unique marking of B. The graph $\Gamma_{\mathbf{p}}$ has a unique marking (obtained from the second marking of A). The genus of $\Gamma_{\mathbf{p}}$ is easily calculated:

$$2g(\Gamma_{\mathbf{p}})-1 = d+a+2r-1 = d+l. \tag{38}$$

If $a=0$, then $\Gamma_{\mathbf{p}}$ consists just of B, but the genus and marking results are the same.

The dual graph $\Gamma_{\mathbf{p}}$ determines a stratum in $M^c_{g(\Gamma_{\mathbf{p}}),1}$ which is a product of the moduli spaces,

$$\prod_{v\in \mathrm{Vert}(\Gamma_{\mathbf{p}})} M^c_{g(v),\mathrm{val}(v)} \longrightarrow M^c_{g(\Gamma_{\mathbf{p}}),1}.$$

The socle dimensions of $M^c_{g(v),\mathrm{val}\,v}$ for $v\in \mathrm{Vert}(\Gamma_{\mathbf{p}})$ are exactly the parts of d.

If $d+l$ is even, then b must be even. If $b>0$, then $b=2r-1+1$ for $r>0$. Let

$$\begin{array}{l} \Gamma_{\mathbf{p}} = A\left(\frac{p_1+1}{2}, ..., \frac{p_a+1}{2}\right) — C^*_{p'_{2r}/2} — E \\ \qquad\Big| \\ \qquad B\left(\frac{p'_1}{2}, ..., \frac{p'_{r-1}}{2}, \frac{p'_r}{2}+1 \,\Big|\, \frac{p'_{r+1}}{2}+1, ..., \frac{p'_{2r-1}}{2}+1\right), \end{array}$$

where the graphs A and B are attached at the markings. The graph $\Gamma_{\mathbf{p}}$ has a unique marking (on $C^*_{p'_{2r}/2}$) and an elliptic tail E. The genus of $\Gamma_{\mathbf{p}}$ is

$$2g(\Gamma_{\mathbf{p}})-1 = d+a+2r+2-1 = d+l+1. \tag{39}$$

If $a=0$, then A is empty, but the genus and marking results are the same. The socle dimensions of $M^c_{g(v),\mathrm{val}\,v}$ for $v\in \mathrm{Vert}(\Gamma_{\mathbf{p}})$ are exactly the parts of d together with zero for the elliptic tail.

If $d+l$ is even and $b=0$, let

$$\Gamma_{\mathbf{p}} = A\left(\frac{p_1+1}{2}, ..., \frac{p_a+1}{2}\right) — E.$$

The graph $\Gamma_{\mathbf{p}}$ has a unique marking (obtained from the first marking of A) and ends in the elliptic tail E. The genus of $\Gamma_{\mathbf{p}}$ is

$$2g(\Gamma_{\mathbf{p}})-1=d+a+2-1=d+l+1. \tag{40}$$

The socle dimensions of $M^c_{g(v),\mathrm{val}\,v}$ for $v\in\mathrm{Vert}(\Gamma_{\mathbf{p}})$ are exactly the parts of d together with zero for the elliptic tail.

We now turn to the proof of Theorem 1.8 in the $n=1$ case. We will prove that

$$\dim_{\mathbb{Q}} \varkappa^d(M^c_{g,1})\geqslant |P(d,2g-1-d)|, \tag{41}$$

by intersecting $\varkappa$ monomials with tautological classes.

Let $\mathbf{p}\in P(d,2g-1-d)$ be a partition of length l. Let $\Gamma_{\mathbf{p}}$ be the dual graph of genus $g(\Gamma_{\mathbf{p}})$ obtained by the above constructions. Since $2g-1\geqslant d+l$, equations (38)–(40) imply that

$$g-g(\Gamma_{\mathbf{p}})=\delta\geqslant 0.$$

We associate with $\mathbf{p}$ a class $w_{\mathbf{p}}\in R^{2g-2-d}(M^c_{g,1})$ by the following construction. Let $v^*\in\mathrm{Vert}(\Gamma_{\mathbf{p}})$ be the vertex which carries the marking. Increase the genus of v^* by δ. The resulting graph determines a stratum $W_{\mathbf{p}}\subset M^c_{g,1}$ of codimension $2g(\Gamma_{\mathbf{p}})-2-d$. Let

$$w_{\mathbf{p}}=\psi_1^{2\delta}[W_{\mathbf{p}}]\in R^{2g-2-d}(M^c_{g,1}).$$

The pairing on $P(d,2g-1-d)$ given by

$$(\mathbf{p},\mathbf{q})\longmapsto\int_{\overline{M}_{g,1}}\varkappa_{\mathbf{p}}w_{\mathbf{q}} \tag{42}$$

is upper-triangular. The diagonal elements are non-vanishing because

$$\int_{\overline{M}_h}\varkappa_{2h-3}\lambda_h=\frac{2^{2h-1}-1}{2^{2h-1}}\frac{|B_{2h}|}{(2h)!}\neq 0,$$
$$\int_{\overline{M}_{h,1}}\psi_1^k\varkappa_{2h-2-k}\lambda_h=\binom{2h-1}{k}\int_{\overline{M}_h}\varkappa_{2h-3}\lambda_h\neq 0,$$

by [7]. Here, B_{2h} is a Bernoulli number. Hence, the pairing (42) is non-singular and the bound (41) is established.

10.4. Case $n=2$

We will need an additional dual graph type. A dual graph of type $\widetilde{B}(g_1, ..., g_r \,|\, h_1, ..., h_{r-1})$ with $g_i, h_j \geqslant 1$ is a comb of $2r-1$ vertices with three markings. The corresponding curves are of the form

$$\begin{array}{ccccccccc} C^*_{g_1} & — & C_{g_2} & — & ... & — & C_{g_{r-1}} & — & C^{**}_{g_r} \\ | & & | & & & & | & & \\ C_{h_1} & & C_{h_2} & & ... & & C_{h_{r-1}}. & & \end{array}$$

There are r vertices of valence 3 and $r-1$ vertices of valence 1. The markings are included in the valence count.

As before, let $\mathbf{p} \in P(d)$ be a partition of length $l=a+b$ with parts

$$(p_1, ..., p_a, p'_1, ..., p'_b),$$

where the p_i are odd and the p'_j are even.

If $d+l$ is even, then b must be even. If $b>0$, then $b=2r-1+1$ for $r>0$. Let

$$\begin{array}{ll} \widetilde{\Gamma}_{\mathbf{p}} = A\left(\dfrac{p_1+1}{2}, ..., \dfrac{p_a+1}{2}\right) & — \; C_{p'_{2r}/2+1} \\ \qquad\;\; | & \\ \qquad\;\; \widetilde{B}\left(\dfrac{p'_1}{2}, ..., \dfrac{p'_{r-1}}{2}, \dfrac{p'_r}{2} \,\middle|\, \dfrac{p'_{r+1}}{2}+1, ..., \dfrac{p'_{2r-1}}{2}+1\right), & \end{array}$$

where the graphs A and $\widetilde{B}$ are attached at the initial markings. The graph $\widetilde{\Gamma}_{\mathbf{p}}$ has two markings (on the extremal component of $\widetilde{B}$). The genus of $\widetilde{\Gamma}_{\mathbf{p}}$ is

$$2g(\widetilde{\Gamma}_{\mathbf{p}}) = d+a+2r = d+l. \tag{43}$$

If $a=0$, then A is empty, but the genus and marking results are the same. The socle dimensions of $M^c_{g(v),\operatorname{val} v}$ for $v \in \operatorname{Vert}(\widetilde{\Gamma}_{\mathbf{p}})$ are exactly the parts of d.

If $d+l$ is even and $b=0$, let

$$\widetilde{\Gamma}_{\mathbf{p}} = A\left(\frac{p_1+1}{2}, ..., \frac{p_a+1}{2}\right).$$

The graph $\widetilde{\Gamma}_{\mathbf{p}}$ has two markings. The genus of $\widetilde{\Gamma}_{\mathbf{p}}$ is

$$2g(\Gamma_{\mathbf{p}}) = d+a = d+l. \tag{44}$$

The socle dimensions of $M^c_{g(v),\operatorname{val} v}$ for $v \in \operatorname{Vert}(\widetilde{\Gamma}_{\mathbf{p}})$ are exactly the parts of d.

If $d+l$ is odd, then $b=2r-1$ for $r>0$. Let

$$\begin{array}{ll} \widetilde{\Gamma}_{\mathbf{p}} = A\left(\frac{p_1+1}{2}, ..., \frac{p_a+1}{2}\right) \text{——} \; E \\ \qquad\;\; \Big| \\ \qquad\;\; \widetilde{B}\left(\frac{p_1'}{2}, ..., \frac{p_{r-1}'}{2}, \frac{p_r'}{2} \;\Big|\; \frac{p_{r+1}'}{2}+1, ..., \frac{p_{2r-1}'}{2}+1\right), \end{array}$$

where the graphs A and $\widetilde{B}$ are attached at the initial markings. The graph $\widetilde{\Gamma}_{\mathbf{p}}$ has two markings (on the extremal component of $\widetilde{B}$). The genus of $\widetilde{\Gamma}_{\mathbf{p}}$ is

$$2g(\widetilde{\Gamma}_{\mathbf{p}}) = d+a+2(r-1)+2 = d+l+1. \tag{45}$$

If $a=0$, then A is empty, but the genus and marking results are the same. The socle dimensions of $M^c_{g(v),\mathrm{val}\,v}$ for $v\in\mathrm{Vert}(\Gamma_{\mathbf{p}})$ are exactly the parts of d together with zero for the elliptic tail.

The proof of Theorem 1.8 now follows the $n=1$ case. Let $\mathbf{p}\in P(d,2g-d)$ be a partition of length l. Let $\widetilde{\Gamma}_{\mathbf{p}}$ be the dual graph of genus $g(\widetilde{\Gamma}_{\mathbf{p}})$ obtained by the above constructions. Since $2g\geqslant d+l$, we see that

$$g-g(\widetilde{\Gamma}_{\mathbf{p}}) = \delta \geqslant 0.$$

We associate with $\mathbf{p}$ a class $\widetilde{w}_{\mathbf{p}}\in R^{2g-1-d}(M^c_{g,2})$ by the following construction. Let $v^*\in\mathrm{Vert}(\widetilde{\Gamma}_{\mathbf{p}})$ be the vertex which carries the first marking. Increase the genus of v^* by δ. The resulting graph determines a stratum

$$\widetilde{W}_{\mathbf{p}} \subset M^c_{g,2}$$

of codimension $2g(\widetilde{\Gamma}_{\mathbf{p}})-1-d$. Let

$$\widetilde{w}_{\mathbf{p}} = \psi_1^{2\delta}[\widetilde{W}_{\mathbf{p}}] \in R^{2g-1-d}(M^c_{g,2}).$$

The pairing on $P(d,2g-d)$ given by

$$(\mathbf{p},\mathbf{q}) \longmapsto \int_{\overline{M}_{g,2}} \varkappa_{\mathbf{p}} \widetilde{w}_{\mathbf{q}}$$

is upper-triangular and non-singular as before. Hence,

$$\dim_{\mathbb{Q}} \varkappa^d(M^c_{g,2}) \geqslant |P(d,2g-d)|,$$

which is the required bound.

10.5. Case $n\geqslant 3$

The higher-pointed cases are easily reduced to the 1-pointed or 2-pointed cases depending on the parity of n. The trading of genera for markings follows the proof of Theorem 1.2 in §5.4. We leave the details to the reader.

11. Gorenstein conjecture

11.1. Proof of Theorem 1.10

If $n>0$, the pairing

$$\varkappa^d(M^c_{g,n})\times R^{2g-3+n-d}(M^c_{g,n})\longrightarrow\mathbb{Q}$$

is shown to have rank at least $|P(d,2g-2+n-d)|$ in §10. Since

$$\dim_{\mathbb{Q}}\varkappa^d(M^c_{g,n})=|P(d,2g-2+n-d)|$$

by Theorem 1.8, Theorem 1.10 follows.

11.2. Further directions

Perhaps the universality of Theorem 1.5 extends to larger subrings of $R^*(M^c_{g,n})$. A natural place to start is the ring

$$S^*(M^c_{g,n})\subset R^*(M^c_{g,n})$$

generated by all the $\varkappa$ and ψ classes.

Question 11.1. Is $S^*(M^c_{g,n})$ canonically a subring of $S^*(M^c_{0,2g+n})$?

At least the condition $n>0$ must be imposed in Question 11.1. How to include the strata classes in a universality statement is not clear.

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