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A geometric approach to generalized Stokes conjectures

by

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Dedicated to John Toland on the occasion of his 60th birthday.

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1. Introduction

Consider a 2-dimensional inviscid incompressible fluid acted on by gravity and with a free surface. If we denote by $D(t)\subset\mathbf{R}^2$ the domain occupied by the fluid at time t, then the dynamics of the fluid is described by the Euler equations for the vector velocity field

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$(u(t,\cdot),v(t,\cdot)): D(t)\to\mathbf{R}^2$ and the scalar pressure field $P(t,\cdot): D(t)\to\mathbf{R}$:

$$\begin{aligned} u_t+uu_x+vu_y &= -P_x && \text{in } D(t),\\ v_t+uv_x+vv_y &= -P_y-g && \text{in } D(t),\\ u_x+v_y &= 0 && \text{in } D(t), \end{aligned}$$

where subscripts denote partial derivatives and g is the gravity constant. The boundary $\partial D(t)$ of the fluid domain contains a part, denoted by $\partial_a D(t)$, which is free and in contact with the air region. The equations of motion are supplemented by the standard kinematic boundary condition

$$V=(u,v)\cdot\nu \quad \text{on } \partial_a D(t),$$

where V is the normal speed of $\partial_a D(t)$ and ν is the outer normal vector, and the dynamic boundary condition

$$P \text{ is locally constant on } \partial_a D(t).$$

We further assume that the flow is irrotational:

$$u_y-v_x=0 \quad \text{in } D(t).$$

While recent years have seen great progress in the study of the initial-value problem (see [40] for large-time well-posedness for small data, and the references therein for short-time well-posedness for arbitrary data), in the present paper we confine ourselves to traveling-wave solutions of the above problem, for which there exists $D\subset\mathbf{R}^2$, $c\in\mathbf{R}$, $(\tilde u,\tilde v): D\to\mathbf{R}^2$ and $\widetilde P: D\to\mathbf{R}$ such that

$$D(t)=D+ct(1,0) \quad \text{for all } t\in\mathbf{R},$$

and for all $t\in\mathbf{R}$ and $(x,y)\in D(t)$,

$$u(x,y,t)=\tilde u(x-ct,y)+c, \quad v(x,y,t)=\tilde v(x-ct,y) \quad \text{and} \quad P(x,y,t)=\widetilde P(x-ct,y).$$

Consequently the following equations are satisfied:

$$\begin{aligned} \tilde u\tilde u_x+\tilde v\tilde u_y &= -\widetilde P_x && \text{in } D,\\ \tilde u\tilde v_x+\tilde v\tilde v_y &= -\widetilde P_y-g && \text{in } D,\\ \tilde u_x+\tilde v_y &= 0 && \text{in } D,\\ \tilde u_y-\tilde v_x &= 0 && \text{in } D,\\ (\tilde u,\tilde v)\cdot\nu &= 0 && \text{on } \partial_a D,\\ \widetilde P \text{ is locally constant} & && \text{on } \partial_a D. \end{aligned}$$

The above problem describes both *water waves*, in which case we would add homogeneous Neumann boundary conditions on a flat horizontal bottom $y=-d$ combined with periodicity in the x-direction or some condition at $x=\pm\infty$, and the equally physical problem of the equilibrium state of a fluid when pumping in water from one lateral boundary and sucking it out at the other lateral boundary. In the latter setting we would consider a bounded domain with an inhomogeneous Neumann boundary condition at the lateral boundary, and the bottom could be a non-flat surface.

In both cases, the incompressibility and the kinematic boundary condition imply that there exists a *stream function* ψ in D, defined up to a constant by

$$\psi_x = -\tilde{v} \text{ and } \psi_y = \tilde{u} \quad \text{in } D.$$

It follows that

$$\psi \text{ is locally constant on } \partial_a D.$$

The irrotationality condition shows that

$$\psi \text{ is a harmonic function in } D,$$

and then Bernoulli's principle gives that

$$\widetilde{P} + \tfrac{1}{2}|\nabla\psi|^2 + gy \quad \text{is constant in } D.$$

The dynamic boundary condition implies therefore the *Bernoulli condition*

$$|\nabla\psi|^2 + 2gy \quad \text{is locally constant on } \partial_a D.$$

A *stagnation point* is one at which the relative velocity field $(\tilde{u}, \tilde{v})$ is zero, and a wave with stagnation points on the free surface will be referred to as an *extreme wave*. Consideration of extreme waves goes back to Stokes, who in 1880 made the famous conjecture that the free surface of an extreme wave is not smooth at a stagnation point, but has symmetric lateral tangents forming an angle of 120°. Stokes [27] gave a formal argument in support of his conjecture, which can be found at the end of this introduction, but a rigorous proof has not been given until 1982, when Amick, Fraenkel and Toland [3] and Plotnikov [20] proved the conjecture independently in brilliant papers. These proofs use an equivalent formulation of the problem as a non-linear singular integral equation due to Nekrasov (derived via conformal mapping), and are based on rather formidable estimates for this equation. In addition, Plotnikov's proof uses ordinary differential equations in the complex plane. Moreover, Plotnikov and Toland proved convexity of the two branches of the free surface [21]. Prior to these works on the Stokes conjecture, the existence of extreme periodic waves, of finite and infinite depth, had been established by Toland [28] and McLeod [18], building on earlier existence results for large-amplitude smooth waves by Krasovskiĭ [17] and by Keady and Norbury [16]. Also, the existence of large-amplitude smooth solitary waves and of extreme solitary waves had been shown by Amick and Toland [4].

In the present paper we confine ourselves to the case when

$$\begin{aligned} \Delta\psi &= 0 && \text{in } D, \\ \psi &= 0 && \text{on } \partial_a D, \\ |\nabla\psi(x,y)|^2 &= -y && \text{on } \partial_a D, \end{aligned}$$

and we investigate the shape of the free surface $\partial_a D$ close to stagnation points for extreme waves which a priori satisfy minimal regularity assumptions. Note that, since $(\tilde{u},\tilde{v})=(\psi_y,-\psi_x)$, the Bernoulli condition implies that the free surface is contained in the lower half-plane and that the stagnation points on the free surface necessarily lie on the real axis and are points of maximal height.

Weak solutions of the above free-boundary problem have been studied by Shargorodsky and Toland [25] and Varvaruca [31], who consider solutions for which the free surface $\partial_a D$ is a locally rectifiable curve, $\psi\in C^2(D)\cap C^0(\overline{D})$ is harmonic and satisfies the zero Dirichlet boundary condition in the classical sense, while the Bernoulli condition is satisfied almost everywhere with respect to the 1-dimensional Hausdorff measure by the non-tangential limits of $\nabla\psi$. They prove that the set S of stagnation points on the free surface is a set of zero 1-dimensional Hausdorff measure, that $\partial_a D\setminus S$ is a union of real-analytic arcs, and that ψ has a harmonic extension across $\partial_a D\setminus S$ which satisfies all free-boundary conditions in the classical sense outside stagnation points.

The main objectives of the present paper are to give affirmative answers to the following two questions:

(i) Does the set S consist only of isolated points?

(ii) Is the Stokes conjecture valid at each point of S?

Prior to our work, Question (i) has been completely open, while the answer to Question (ii) has been known only partially: from [3] and [20] which have recently been simplified in [30] and [32], we know (ii) to be true at those points of S which satisfy the following conditions in a neighborhood of the stagnation point: the stagnation point is isolated, the free surface is symmetric with respect to the vertical line passing through the stagnation point, it is a monotone graph on each side of that point, and ψ is strictly decreasing in the y-direction in D. All of these conditions are essential for the proofs in the cited results. Let us mention that from the point of view of applications, the requirement of symmetry is most inconvenient, as numerical results indicate the existence of non-symmetric extreme waves [7], [29], [41]. Also, for waves with non-zero vorticity, ψ need not be monotone in the y-direction [10], [34].

Similarly to [25] and [31], we consider weak solutions which are roughly speaking solutions in the sense of distributions. The precise notion will be given in Definition 3.2. We assume that $\psi>0$ in D, and we extend ψ by the value 0 to the air region so that the fluid domain can be identified with the set $\{(x,y):\psi(x,y)>0\}$ (in short, $\{\psi>0\}$). Since our arguments are local, we work in a bounded domain Ω which has a non-empty intersection with the real axis and on which is defined a continuous function ψ such that, within Ω, $\{\psi>0\}$ corresponds to the fluid region and $\{\psi=0\}$ to the air region, the part of Ω in the upper half-plane being occupied by air.

In the case of only a finite number of connected components of the air region, we recover the Stokes conjecture by geometric methods (Theorem 11.2), without assuming isolatedness, symmetry or any monotonicity.

Theorem A. *Let ψ be a weak solution of*

$$\begin{aligned} \Delta\psi &= 0 \qquad \textit{in } \Omega\cap\{\psi>0\},\\ |\nabla\psi|^2 &= -y \quad \textit{on } \Omega\cap\partial\{\psi>0\}, \end{aligned}$$

and suppose that

$$|\nabla\psi|^2 \leqslant -y \quad \textit{in } \Omega\cap\{\psi>0\}.$$

Suppose moreover that $\{\psi=0\}$ has locally only finitely many connected components. Then the set S of stagnation points is locally in Ω a finite set. At each stagnation point (x^0,y^0) the scaled solution converges to the Stokes corner flow, that is,

$$\frac{\psi((x^0,y^0)+r(x,y))}{r^{3/2}} \to \frac{\sqrt{2}}{3}\varrho^{3/2}\cos\left(\frac{3}{2}\left(\min\left\{\max\left\{\theta,-\frac{5\pi}{6}\right\},-\frac{\pi}{6}\right\}+\frac{\pi}{2}\right)\right) \quad \textit{as } r\searrow 0,$$

strongly in $W^{1,2}_{\mathrm{loc}}(\mathbf{R}^2)$ and locally uniformly on $\mathbf{R}^2$, where $(x,y)=(\varrho\cos\theta,\varrho\sin\theta)$, and in an open neighborhood of (x^0,y^0) the topological free boundary $\partial\{\psi>0\}$ is the union of two C^1-graphs with right and left tangents at (x^0,y^0).

Let us remark that the assumption

$$|\nabla\psi|^2 \leqslant -y \quad \text{in } \{\psi>0\}$$

has been verified in [31, Proof of Theorem 3.6] for weak solutions, in the sense of [25] and [31] described earlier, of the water-wave problem in all its classical versions: periodic and solitary waves of finite depth (in which the fluid domain has a fixed flat bottom $y=-d$, at which ψ is constant), and periodic waves of infinite depth (in which the fluid domain extends to $y=-\infty$ and the condition $\lim_{y\to-\infty}\nabla\psi(x,y)=(0,-c)$ holds, where c is the speed of the wave). The proof is merely an extension of that of Spielvogel [26, Proof of Theorem 3b] for classical solutions, which is based on the Bernstein technique.

In the case of an infinite number of connected components of the air region, we obtain the following result (cf. Theorem 11.1).

THEOREM B. *Let ψ be a weak solution of*

$$\begin{aligned} \Delta\psi &= 0 && \text{in } \Omega\cap\{\psi>0\},\\ |\nabla\psi|^2 &= -y && \text{on } \Omega\cap\partial\{\psi>0\}, \end{aligned}$$

and suppose that

$$|\nabla\psi|^2 \leqslant -y \quad \text{in } \Omega\cap\{\psi>0\}.$$

Then the set S of stagnation points is a finite or countable set. Each accumulation point of S is a point of the locally finite set Σ described in more detail in the following lines.

At each point (x^0,y^0) of $S\setminus\Sigma$,

$$\frac{\psi((x^0,y^0)+r(x,y))}{r^{3/2}} \to \frac{\sqrt{2}}{3}\varrho^{3/2}\cos\left(\frac{3}{2}\left(\min\left\{\max\left\{\theta,-\frac{5\pi}{6}\right\},-\frac{\pi}{6}\right\}+\frac{\pi}{2}\right)\right) \quad \text{as } r\searrow 0,$$

strongly in $W^{1,2}_{\mathrm{loc}}(\mathbf{R}^2)$ and locally uniformly on $\mathbf{R}^2$, where $(x,y)=(\varrho\cos\theta,\varrho\sin\theta)$. The scaled free surface converges to that of the Stokes corner flow in the sense that, as $r\searrow 0$,

$$\mathcal{L}^2\left(B_1\cap\left(\{(x,y):\psi((x^0,y^0)+r(x,y))>0\}\triangle\left\{(x,y):-\frac{5\pi}{6}<\theta<-\frac{\pi}{6}\right\}\right)\right)\to 0.$$

At each point (x^0,y^0) of Σ there exists an integer $N=N(x^0,y^0)\geqslant 2$ such that

$$\frac{\psi((x^0,y^0)+r(x,y))}{r^\beta}\to 0 \quad \text{as } r\searrow 0,$$

strongly in $L^2_{\mathrm{loc}}(\mathbf{R}^2)$ for each $\beta\in[0,N)$, and

$$\frac{\psi((x^0,y^0)+r(x,y))}{\sqrt{r^{-1}\int_{\partial B_r((x^0,y^0))}\psi^2\,d\mathcal{H}^1}} \to \frac{\varrho^N|\sin(N\min\{\max\{\theta,-\pi\},0\})|}{\sqrt{\int_{-\pi}^0\sin^2(N\theta)\,d\theta}} \quad \text{as } r\searrow 0,$$

strongly in $W^{1,2}_{\mathrm{loc}}(B_1\setminus\{0\})$ and weakly in $W^{1,2}(B_1)$, where $(x,y)=(\varrho\cos\theta,\varrho\sin\theta)$.

Although the new dynamics suggested by Theorem B at degenerate points cannot happen in the case of a finite number of air components, there seems to be no obvious reason precluding the scenario in Figure 1 with an infinite number of air components, and the situation is even less clear in the case of inhomogeneous Neumann boundary

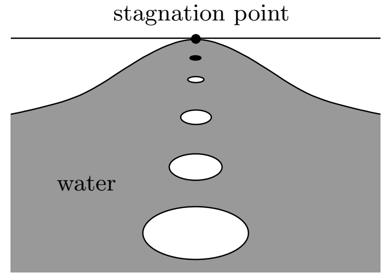

Figure 1. A degenerate point.

conditions. Note that multiple air components without surface tension have previously been considered in [13]. It is noteworthy that while the water-wave problem has a variational structure, the solutions of interest are *not* minimizers of the energy functional. Consequently, standard methods in free-boundary problems based on non-degeneracy, which would in the present case be the estimate

$$\int_{\partial B_1((0,0))} \frac{\psi((x^0,y^0)+r(x,y))}{r^{3/2}}\, d\mathcal{H}^1 \geqslant c_1 > 0 \quad \text{for all } r \in (0, r_0),$$

do not apply.

As far as the water-wave problem is concerned, the new perspective of our approach is that we work with the original variables $(\tilde{u}, \tilde{v})$ and use geometric methods, as for example a *blow-up analysis*, in order to show that the scaled solution is close to a homogeneous function. This part of the blow-up analysis works in n dimensions and does not require ad hoc methods previously applied to classify global solutions (see for example [32]). This also means that we do not require isolated singularities, symmetry or monotonicity, which had been assumed in all previous results. Original tools in the present paper include the new *frequency formula* (Theorem 7.1) which allows a blow-up analysis at degenerate points, where the scaling of the solution is different from the invariant scaling of the equation, and leads in combination with the result [12] by Evans and Müller to concentration compactness (Theorem 10.1).

Large parts of the paper are written down for the non-physical but mathematically interesting free-boundary problem in n dimensions; see for example the partial regularity result Proposition 5.8 showing that non-degenerate stagnation points form a set of dimension less than or equal to $n-2$.

Our methods can still be applied when dropping the condition of irrotationality of the flow (see [32], the forthcoming papers [33] and [23], and [8] and [9] for a background on water waves with vorticity). Part of the methods extend even to water waves with surface tension (see the forthcoming paper [39]).

It is interesting to observe that in his formal proof of the conjecture, Stokes worked with the original variables $(\tilde{u}, \tilde{v})$ and approximated the velocity potential (the harmonic conjugate of $-\psi$) by a *homogeneous function*. This is very close in spirit to what we do on a rigorous level in the monotonicity formula (Theorem 3.5) and the frequency formula (Theorem 7.1), so let us close our introduction with a quotation taken from [27, pp. 226–227]:

> Reduce the wave motion to steady motion by superposing a velocity equal and opposite to that of propagation. Then a particle at the surface may be thought of as gliding along a fixed smooth curve: this follows directly from physical considerations, or from the ordinary equation of steady motion. On arriving at a crest the particle must be momentarily at rest, and on passing it must be ultimately in the condition of a particle starting from rest down an inclined or vertical plane. Hence the velocity must vary ultimately as the square root of the distance from the crest.
>
> Hitherto the motion has been rotational or not, let us now confine ourselves to the case of irrotational motion. Place the origin at the crest, refer the function ϕ to polar coordinates r and θ; θ being measured from the vertical, and consider the value of ϕ very near the origin, where ϕ may be supposed to vanish, as the arbitrary constant may be omitted. In general ϕ will be of the form $\sum A_n r^n \sin n\theta + \sum B_n r^n \cos n\theta$. In the present case ϕ must contain sines only on account of the symmetry of the motion, as already shown (p. 212), so that retaining only the most important term we may take $\phi = Ar^n \sin n\theta$. Now for a point in the section of the profile we must have $d\phi/d\theta = 0$, and $d\phi/d\theta$ varying ultimately as $r^{1/2}$. This requires $n = \frac{3}{2}$, and for the profile that $\frac{3}{2}\theta = \frac{1}{2}\pi$, so that the two branches are inclined at angles of $\pm 60^\circ$ to the vertical, and at an angle of 120° to each other, not of 90° as supposed by Rankine.

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2. Notation

We denote by χ_A the characteristic function of the set A, and by $A \triangle B$ the set $(A \setminus B) \cup (B \setminus A)$. For any real number a, the notation a^+ stands for $\max\{a, 0\}$. We denote by $x \cdot y$ the Euclidean inner product in $\mathbf{R}^n \times \mathbf{R}^n$, by $|x|$ the Euclidean norm in $\mathbf{R}^n$ and by $B_r(x^0) := \{x \in \mathbf{R}^n : |x - x^0| < r\}$ the ball of center x^0 and radius r. We will use the notation

B_r for $B_r(0)$, and denote by ω_n the n-dimensional volume of B_1. Also, $\mathcal{L}^n$ shall denote the n-dimensional Lebesgue measure and $\mathcal{H}^s$ the s-dimensional Hausdorff measure. By ν we will always refer to the outer normal on a given surface. We will use functions of bounded variation $\mathrm{BV}(U)$, i.e. functions $f\in L^1(U)$ for which the distributional derivative is a vector-valued Radon measure. Here $|\nabla f|$ denotes the total variation measure (cf. [15]). Note that for a smooth open set $E\subset\mathbf{R}^n$, $|\nabla\chi_E|$ coincides with the surface measure on ∂E. Last, we will use the notation $r\searrow 0$ for $r\to 0^+$ and $r\nearrow 0$ for $r\to 0^-$.

3. Notion of solution and monotonicity formula

Throughout the rest of the paper we work with an n-dimensional generalization of the problem described in the introduction. Let Ω be a bounded domain in $\mathbf{R}^n$ which has a non-empty intersection with the hyperplane $\{x_n=0\}$, in which to consider the combined problem for fluid and air. We study solutions u, in a sense to be specified, of the problem

$$\begin{aligned} \Delta u &= 0 && \text{in } \Omega\cap\{u>0\}, \\ |\nabla u|^2 &= x_n && \text{on } \Omega\cap\partial\{u>0\}. \end{aligned} \tag{3.1}$$

(Note that, compared with the introduction, we have switched notation from ψ to u and we have "reflected" the problem at the hyperplane $\{x_n=0\}$.) Since our results are completely local, we do not specify boundary conditions on $\partial\Omega$.

We begin by introducing our notion of a *variational solution* of the problem (3.1).

Definition 3.1. (Variational solution) We define $u\in W^{1,2}_{\mathrm{loc}}(\Omega)$ to be a *variational solution* of (3.1) if $u\in C^0(\Omega)\cap C^2(\Omega\cap\{u>0\})$, $u\geqslant 0$ in Ω, $u\equiv 0$ in $\Omega\cap\{x_n\leqslant 0\}$, and the first variation with respect to domain variations of the functional

$$J(v):=\int_\Omega (|\nabla v|^2+x_n\chi_{\{v>0\}})\,dx$$

vanishes at $v=u$, i.e.

$$\begin{aligned} 0 &= -\frac{d}{d\varepsilon}J(u(x+\varepsilon\phi(x)))\Big|_{\varepsilon=0} \\ &= \int_\Omega (|\nabla u|^2\operatorname{div}\phi-2\nabla u D\phi\nabla u+x_n\chi_{\{u>0\}}\operatorname{div}\phi+\chi_{\{u>0\}}\phi_n)\,dx \end{aligned}$$

for any $\phi\in C^1_0(\Omega;\mathbf{R}^n)$.

The assumption $u\in C^0(\Omega)\cap C^2(\Omega\cap\{u>0\})$ is necessary in that it cannot be deduced from the other assumptions in Definition 3.1 by regularity theory, but it is rather mild in the sense that it can be verified without effort for "reasonable" solutions, for example solutions obtained by a diffuse interface approximation. Also we like to emphasize that regularity properties of the free boundary, like for example finite perimeter, are not required at all. Note for future reference that the fact that u is continuous and non-negative in Ω, as well as harmonic in $\{u>0\}$, implies that Δu is a non-negative Radon measure in Ω with support on $\Omega\cap\partial\{u>0\}$.

We will also use *weak solutions* of (3.1), i.e. solutions in the sense of distributions. For a comparison of variational and weak solutions see Lemma 3.4.

Definition 3.2. (Weak solution) We define $u\in W^{1,2}_{\mathrm{loc}}(\Omega)$ to be a *weak solution* of (3.1) if the following are satisfied: $u\in C^0(\Omega)$, $u\geqslant 0$ in Ω, $u\equiv 0$ in $\Omega\cap\{x_n\leqslant 0\}$, u is harmonic in $\{u>0\}\cap\Omega$ and, for every $\tau>0$, the topological free boundary $\partial\{u>0\}\cap\Omega\cap\{x_n>\tau\}$ can be locally decomposed into an $(n-1)$-dimensional $C^{2,\alpha}$-surface, relatively open to $\partial\{u>0\}$ and denoted by $\partial_{\mathrm{red}}\{u>0\}$, and a singular set of vanishing $\mathcal{H}^{n-1}$-measure; for an open neighborhood V of each point $x^0\in\Omega\cap\{x_n>\tau\}$ of $\partial_{\mathrm{red}}\{u>0\}$, $u\in C^1(V\cap\overline{\{u>0\}})$ satisfies

$$|\nabla u(x)|^2=x_n \quad \text{on } V\cap\partial_{\mathrm{red}}\{u>0\}.$$

Remark 3.3. (i) By [2, Theorem 8.4], the weak solutions in [2] with $Q(x)=x_n^+$ satisfy Definition 3.2.

(ii) By [31, Theorem 3.5], the weak solutions in [25] and [31] satisfy Definition 3.2.

LEMMA 3.4. *Any weak solution of* (3.1) *such that*

$$|\nabla u|^2\leqslant Cx_n^+ \quad \textit{locally in } \Omega,$$

is a variational solution of (3.1). *Moreover*, $\chi_{\{u>0\}}$ *is locally in* $\{x_n>0\}$ *a function of bounded variation, and the total variation measure* $|\nabla\chi_{\{u>0\}}|$ *satisfies*

$$r^{1/2-n}\int_{B_r(y)}\sqrt{x_n}\,|\nabla\chi_{\{u>0\}}|\,dx\leqslant C_0$$

for all $B_r(y)\Subset\Omega$ *such that* $y_n=0$.

The proof follows [35, Theorem 5.1] and will be given in the appendix.

A first tool in our analysis is an extension of the monotonicity formula in [36] and [35, Theorem 3.1] to the boundary case. The roots of those monotonicity formulas are harmonic mappings ([22], [24]) and blow-up ([19]).

THEOREM 3.5. (Monotonicity formula) *Let u be a variational solution of* (3.1), *let* $x^0\in\Omega$ *and let* $\delta:=\frac{1}{2}\operatorname{dist}(x^0,\partial\Omega)$.

(i) Interior case $x^0_n\geqslant 0$. *The function*

$$\Phi^{\mathrm{int}}_{x^0,u}(r):=r^{-n}\int_{B_r(x^0)}(|\nabla u|^2+x_n\chi_{\{u>0\}})\,dx-r^{-n-1}\int_{\partial B_r(x^0)}u^2\,d\mathcal{H}^{n-1},$$

defined in $(0,\delta)$, *satisfies the formula*

$$\Phi^{\mathrm{int}}_{x^0,u}(\sigma)-\Phi^{\mathrm{int}}_{x^0,u}(\varrho)=\int_\varrho^\sigma r^{-n}\int_{\partial B_r(x^0)}2\Big(\nabla u\cdot\nu-\frac{u}{r}\Big)^2\,d\mathcal{H}^{n-1}\,dr$$
$$+\int_\varrho^\sigma r^{-n-1}\int_{B_r(x^0)}(x_n-x_n^0)\chi_{\{u>0\}}\,dx\,dr$$

for any $0<\varrho<\sigma<\delta$. *The absolute value of the second term in the right-hand side is estimated by* $\sigma-\varrho$ *and is therefore* $O(\sigma)$.

(ii) Boundary case $x_n^0=0$. *The function*

$$\Phi^{\mathrm{bound}}_{x^0,u}(r):=r^{-n-1}\int_{B_r(x^0)}(|\nabla u|^2+x_n\chi_{\{u>0\}})\,dx-\frac{3}{2}r^{-n-2}\int_{\partial B_r(x^0)}u^2\,d\mathcal{H}^{n-1},$$

defined in $(0,\delta)$, *satisfies the formula*

$$\Phi^{\mathrm{bound}}_{x^0,u}(\sigma)-\Phi^{\mathrm{bound}}_{x^0,u}(\varrho)=\int_\varrho^\sigma r^{-n-1}\int_{\partial B_r(x^0)}2\left(\nabla u\cdot\nu-\frac{3}{2}\frac{u}{r}\right)^2 d\mathcal{H}^{n-1}\,dr$$

for any $0<\varrho<\sigma<\delta$.

Remark 3.6. Let us assume that $x^0=0$. Then the integrand on the right-hand side of the monotonicity formula is a scalar multiple of $\big(\nabla u(x)\cdot x-\frac{3}{2}u(x)\big)^2$, and therefore vanishes if and only if u is a homogeneous function of degree $\frac{3}{2}$.

Proof. We start with a general observation: for any $u\in W^{1,2}_{\mathrm{loc}}(\Omega)$ and $\alpha\in\mathbf{R}$, the following identity holds a.e. on $(0,\delta)$, where $w_r(x)=u(x^0+rx)$,

$$\begin{aligned}
&\frac{d}{dr}\left(r^\alpha\int_{\partial B_r(x^0)}u^2\,d\mathcal{H}^{n-1}\right)\\
&\quad=\frac{d}{dr}\left(r^{\alpha+n-1}\int_{\partial B_1}w_r^2\,d\mathcal{H}^{n-1}\right)\\
&\quad=(\alpha+n-1)r^{\alpha-1}\int_{\partial B_r(x^0)}u^2\,d\mathcal{H}^{n-1}+r^{\alpha+n-1}\int_{\partial B_1}2w_r\nabla u(x^0+rx)\cdot x\,d\mathcal{H}^{n-1}\\
&\quad=(\alpha+n-1)r^{\alpha-1}\int_{\partial B_r(x^0)}u^2\,d\mathcal{H}^{n-1}+r^\alpha\int_{\partial B_r(x^0)}2u\nabla u\cdot\nu\,d\mathcal{H}^{n-1}.
\end{aligned}\tag{3.2}$$

Suppose now that u is a variational solution of (3.1). For small positive τ and $\eta_\tau(t):=\max\{0,\min\{1,(r-t)/\tau\}\}$, we take after approximation

$$\phi_\tau(x):=\eta_\tau(|x-x^0|)(x-x^0)$$

as a test function in the definition of a variational solution. We obtain

$$\begin{aligned}0=&\int_\Omega(|\nabla u|^2+x_n\chi_{\{u>0\}})(n\eta_\tau(|x-x^0|)+\eta_\tau'(|x-x^0|)|x-x^0|)\,dx\\&-2\int_\Omega\left(|\nabla u|^2\eta_\tau(|x-x^0|)+\nabla u\cdot\frac{x-x^0}{|x-x^0|}\nabla u\cdot\frac{x-x^0}{|x-x^0|}\eta'(|x-x^0|)|x-x^0|\right)dx\\&+\int_\Omega\eta_\tau(|x-x^0|)(x_n-x_n^0)\chi_{\{u>0\}}\,dx.\end{aligned}$$

Passing to the limit as $\tau\to 0$, we obtain, for a.e. $r\in(0,\delta)$,

$$\begin{aligned}0=&\,n\int_{B_r(x^0)}(|\nabla u|^2+x_n\chi_{\{u>0\}})\,dx-r\int_{\partial B_r(x^0)}(|\nabla u|^2+x_n\chi_{\{u>0\}})\,d\mathcal{H}^{n-1}\\&+2r\int_{\partial B_r(x^0)}(\nabla u\cdot\nu)^2\,d\mathcal{H}^{n-1}-2\int_{B_r(x^0)}|\nabla u|^2\,d\mathcal{H}^{n-1}+\int_{B_r(x^0)}(x_n-x_n^0)\chi_{\{u>0\}}\,dx.\end{aligned}\tag{3.3}$$

Observe that letting $\varepsilon\to 0$ in

$$\int_{B_r(x^0)}\nabla u\cdot\nabla\max\{u-\varepsilon,0\}^{1+\varepsilon}\,dx=\int_{\partial B_r(x^0)}\max\{u-\varepsilon,0\}^{1+\varepsilon}\nabla u\cdot\nu\,d\mathcal{H}^{n-1}$$

for a.e. $r\in(0,\delta)$, we obtain the integration by parts formula

$$\int_{B_r(x^0)}|\nabla u|^2\,dx=\int_{\partial B_r(x^0)}u\nabla u\cdot\nu\;d\mathcal{H}^{n-1}\tag{3.4}$$

for a.e. $r\in(0,\delta)$.

Now let for all $r\in(0,\delta)$,

$$\begin{aligned}U_{\text{int}}(r)&:=r^{-n}\int_{B_r(x^0)}(|\nabla u|^2+x_n\chi_{\{u>0\}})\,dx,\\W_{\text{int}}(r)&:=r^{-n-1}\int_{\partial B_r(x^0)}u^2\,d\mathcal{H}^{n-1},\end{aligned}$$

so that $\Phi^{\text{int}}_{x^0,u}=U_{\text{int}}-W_{\text{int}}$. Note that, for a.e. $r\in(0,\delta)$,

$$\begin{aligned}U_{\text{int}}'(r)=&-nr^{-n-1}\int_{B_r(x^0)}(|\nabla u|^2+x_n\chi_{\{u>0\}})\,dx\\&+r^{-n}\int_{\partial B_r(x^0)}(|\nabla u|^2+x_n\chi_{\{u>0\}})\,d\mathcal{H}^{n-1}.\end{aligned}$$

It follows, using (3.3) and (3.4), that for a.e. $r\in(0,\delta)$,

$$\begin{aligned}U_{\text{int}}'(r)=&\,2r^{-n}\int_{\partial B_r(x^0)}(\nabla u\cdot\nu)^2\,d\mathcal{H}^{n-1}-2r^{-n-1}\int_{\partial B_r(x^0)}u\nabla u\cdot\nu\,d\mathcal{H}^{n-1}\\&+r^{-n-1}\int_{B_r(x^0)}(x_n-x_n^0)\chi_{\{u>0\}}\,dx.\end{aligned}\tag{3.5}$$

On the other hand, plugging $\alpha := -n-1$ into (3.2), we obtain that for a.e. $r\in(0,\delta)$,

$$W'_{\rm int}(r) = 2r^{-n-1}\int_{\partial B_r(x^0)} u\nabla u\cdot\nu\, d\mathcal{H}^{n-1} - 2r^{-n-2}\int_{\partial B_r(x^0)} u^2\, d\mathcal{H}^{n-1}. \tag{3.6}$$

Combining (3.5) and (3.6) yields (i).

Next, let for all $r\in(0,\delta)$,

$$U_{\rm bound}(r) := r^{-n-1}\int_{B_r(x^0)} (|\nabla u|^2 + x_n\chi_{\{u>0\}})\, dx,$$
$$W_{\rm bound}(r) := r^{-n-2}\int_{\partial B_r(x^0)} u^2\, d\mathcal{H}^{n-1},$$

so that $\Phi^{\rm bound}_{x^0,u} = U_{\rm bound} - \frac{3}{2}W_{\rm bound}$. Now observe that, in the case when $x^0_n = 0$, formula (3.3) means that

$$\begin{aligned}0 = (n+1)\int_{B_r(x^0)} (|\nabla u|^2 + x_n\chi_{\{u>0\}})\, dx - r\int_{\partial B_r(x^0)} (|\nabla u|^2 + x_n\chi_{\{u>0\}})\, d\mathcal{H}^{n-1}\\ + 2r\int_{\partial B_r(x^0)} (\nabla u\cdot\nu)^2\, d\mathcal{H}^{n-1} - 3\int_{B_r(x^0)} |\nabla u|^2\, dx.\end{aligned} \tag{3.7}$$

Also, for a.e. $r\in(0,\delta)$,

$$\begin{aligned}U'_{\rm bound}(r) = -(n+1)r^{-n-2}\int_{B_r(x^0)} (|\nabla u|^2 + x_n\chi_{\{u>0\}})\, dx\\ + r^{-n-1}\int_{\partial B_r(x^0)} (|\nabla u|^2 + x_n\chi_{\{u>0\}})\, d\mathcal{H}^{n-1}.\end{aligned}$$

It follows, using (3.7) and (3.4), that for a.e. $r\in(0,\delta)$,

$$U'_{\rm bound}(r) = 2r^{-n-1}\int_{\partial B_r(x^0)} (\nabla u\cdot\nu)^2\, d\mathcal{H}^{n-1} - 3r^{-n-2}\int_{\partial B_r(x^0)} u\nabla u\cdot\nu\, d\mathcal{H}^{n-1}. \tag{3.8}$$

On the other hand, plugging $\alpha := -n-2$ into (3.2), we obtain that for a.e. $r\in(0,\delta)$,

$$W'_{\rm bound}(r) = 2r^{-n-2}\int_{\partial B_r(x^0)} u\nabla u\cdot\nu\, d\mathcal{H}^{n-1} - 3r^{-n-3}\int_{\partial B_r(x^0)} u^2\, d\mathcal{H}^{n-1}. \tag{3.9}$$

Combining (3.8) and (3.9) yields (ii). □

4. Densities

From Theorem 3.5 we infer that the functions $\Phi^{\rm int}_{x^0,u}$ and $\Phi^{\rm bound}_{x^0,u}$ have right limits

$$\Phi^{\rm int}_{x^0,u}(0^+) = \lim_{r\searrow 0}\Phi^{\rm int}_{x^0,u}(r) \in [-\infty,\infty) \quad\text{and}\quad \Phi^{\rm bound}_{x^0,u}(0^+) = \lim_{r\searrow 0}\Phi^{\rm bound}_{x^0,u}(r) \in [-\infty,\infty).$$

In this section we derive structural properties of these "densities"

$$\Phi^{\rm int}_{x^0,u}(0^+) \quad \text{and} \quad \Phi^{\rm bound}_{x^0,u}(0^+).$$

The term "density" is justified somewhat by Lemma 4.2 (i) and (ii).

Note that most of the statements concerning $\Phi^{\rm int}_{x^0,u}$ will not be used in subsequent sections but serve to illustrate differences between the boundary and the interior case.

Lemma 4.1. *Let u be a variational solution of* (3.1) *and suppose that*

$$|\nabla u|^2 \leqslant C x_n^+ \quad \textit{locally in } \Omega.$$

(i) *Let $x^0\in\Omega$ be such that $x_n^0>0$. Then $\Phi^{\rm int}_{x^0,u}(0^+)$ is finite if $u(x^0)=0$, and is $-\infty$ otherwise.*

(ii) *Let $x^0\in\Omega$ be such that $x_n^0=0$. Then $\Phi^{\rm bound}_{x^0,u}(0^+)$ is finite.* (*Note that $u=0$ in $\{x_n=0\}$ by assumption.*)

(iii) *Let $x^0\in\Omega$ be such that $x_n^0>0$ and $u(x^0)=0$, and let $0<r_m\searrow 0$ as $m\to\infty$ be a sequence such that the* blow-up *sequence*

$$u_m(x) := \frac{u(x^0+r_m x)}{r_m}$$

converges weakly in $W^{1,2}_{\rm loc}(\mathbf{R}^n)$ to a blow-up limit u_0. Then u_0 is a homogeneous function of degree 1, i.e. $u_0(\lambda x)=\lambda u_0(x)$.

(iv) *Let $x^0\in\Omega$ be such that $x_n^0=0$, and let $0<r_m\searrow 0$ as $m\to\infty$ be a sequence such that the* blow-up *sequence*

$$u_m(x) := \frac{u(x^0+r_m x)}{r_m^{3/2}}$$

converges weakly in $W^{1,2}_{\rm loc}(\mathbf{R}^n)$ to a blow-up limit u_0. Then u_0 is a homogeneous function of degree $\frac{3}{2}$, i.e. $u_0(\lambda x)=\lambda^{3/2}u_0(x)$.

(v) *Let u_m be a converging sequence of* (iii) *or* (iv). *Then u_m converges* strongly *in $W^{1,2}_{\rm loc}(\mathbf{R}^n)$.*

Proof. (i), (ii) If $u(x^0)=0$, the finiteness claims follow directly from the growth assumption $|\nabla u|^2\leqslant Cx_n^+$. If $x_n^0>0$ and $u(x^0)>0$, then, since $|\nabla u|^2\leqslant Cx_n^+$ by assumption, we obtain that $\Phi^{\rm int}_{x^0,u}(r)\leqslant C_1-C_2r^{-2}$ for $r\leqslant r_0$, implying that $\Phi^{\rm int}_{x^0,u}(0^+)=-\infty$.

(iii), (iv) For each $0<\sigma<\infty$ the sequence u_m is by assumption bounded in $C^{0,1}(B_\sigma)$. From the monotonicity formula (Theorem 3.5) we infer therefore, setting $\alpha=1$ in the interior case and $\alpha=\frac{3}{2}$ in the boundary case, that for all $0<\varrho<\sigma<\infty$,

$$\int_\varrho^\sigma \int_{\partial B_r} (\nabla u_m(x)\cdot x-\alpha u_m(x))^2 \, d\mathcal{H}^{n-1}\, dr \to 0 \quad \text{as } m\to\infty,$$

which yields the desired homogeneity of u_0.

(v) The proof follows [6, Lemma 7.2]. In order to show strong convergence of u_m in $W^{1,2}_{\rm loc}(\mathbf{R}^n)$, it is sufficient, in view of the weak L^2-convergence of ∇u_m, to show that

$$\limsup_{m\to\infty}\int_{\mathbf{R}^n}|\nabla u_m|^2\eta\,dx\leqslant\int_{\mathbf{R}^n}|\nabla u_0|^2\eta\,dx$$

for each $\eta\in C^1_0(\mathbf{R}^n)$. Using the uniform convergence, the continuity of u_0, as well as the fact that u_0 is harmonic in $\{u_0>0\}$, we obtain as in the proof of (3.4) that

$$\int_{\mathbf{R}^n}|\nabla u_m|^2\eta\,dx=-\int_{\mathbf{R}^n}u_m\nabla u_m\cdot\nabla\eta\,dx\to-\int_{\mathbf{R}^n}u_0\nabla u_0\cdot\nabla\eta\,dx=\int_{\mathbf{R}^n}|\nabla u_0|^2\eta\,dx$$

as $m\to\infty$. It follows that u_m converges to u_0 strongly in $W^{1,2}_{\rm loc}(\mathbf{R}^n)$ as $m\to\infty$. □

LEMMA 4.2. *Let u be a variational solution of* (3.1) *and suppose that*

$$|\nabla u|^2\leqslant Cx_n^+\quad\textit{locally in }\Omega.$$

(i) *Let $x^0\in\Omega$ be such that $x^0_n>0$ and $u(x^0)=0$. Then*

$$\Phi^{\rm int}_{x^0,u}(0^+)=x^0_n\lim_{r\searrow0}r^{-n}\int_{B_r(x^0)}\chi_{\{u>0\}}\,dx,$$

and in particular $\Phi^{\rm int}_{x^0,u}(0^+)\in[0,\infty)$. Moreover, $\Phi^{\rm int}_{x^0,u}(0^+)=0$ implies that $u_0=0$ in $\mathbf{R}^n$ for each blow-up limit u_0 of Lemma 4.1 (iii).

(ii) *Let $x^0\in\Omega$ be such that $x^0_n=0$. Then*

$$\Phi^{\rm bound}_{x^0,u}(0^+)=\lim_{r\searrow0}r^{-n-1}\int_{B_r(x^0)}x_n^+\chi_{\{u>0\}}\,dx,$$

and in particular $\Phi^{\rm bound}_{x^0,u}(0^+)\in[0,\infty)$. Moreover, $\Phi^{\rm bound}_{x^0,u}(0^+)=0$ implies that $u_0=0$ in $\mathbf{R}^n$ for each blow-up limit u_0 of Lemma 4.1 (iv).

(iii) *The function $x\mapsto\Phi^{\rm int}_{x,u}(0^+)$ is upper semicontinuous in $\{x_n>0\}$.*

(iv) *The function $x\mapsto\Phi^{\rm bound}_{x,u}(0^+)$ is upper semicontinuous in $\{x_n=0\}$.*

(v) *Let u_m be a sequence of variational solutions of* (3.1) *which converges strongly to u_0 in $W^{1,2}_{\rm loc}(\mathbf{R}^n)$ and such that $\chi_{\{u_m>0\}}$ converges weakly in $L^2_{\rm loc}(\mathbf{R}^n)$ to χ_0. Then u_0 is a variational solution of* (3.1) *and satisfies the monotonicity formula, but with $\chi_{\{u_0>0\}}$ replaced by χ_0. Moreover, for each $x^0\in\Omega$, and all instances of $\chi_{\{u_0>0\}}$ replaced by χ_0,*

$$\Phi^{\rm int}_{x^0,u_0}(0^+)\geqslant\limsup_{m\to\infty}\Phi^{\rm int}_{x^0,u_m}(0^+)$$

in the interior case $x^0_n>0$, and

$$\Phi^{\rm bound}_{x^0,u_0}(0^+)\geqslant\limsup_{m\to\infty}\Phi^{\rm bound}_{x^0,u_m}(0^+)$$

in the boundary case $x^0_n=0$.

Proof. (i), (ii) Take a sequence $r_m \searrow 0$ such that u_m defined in Lemma 4.1 (iii) and (iv) converges weakly in $W^{1,2}_{\mathrm{loc}}(\mathbf{R}^n)$ to a function u_0. Using Lemma 4.1 (v) and the homogeneity of u_0, in the interior case we obtain that

$$\begin{aligned}\lim_{m\to\infty} \Phi^{\mathrm{int}}_{x^0,u}(r_m) &= \int_{B_1} |\nabla u_0|^2\,dx - \int_{\partial B_1} u_0^2\,d\mathcal{H}^{n-1} + x_n^0 \lim_{r\searrow 0} r^{-n} \int_{B_r(x^0)} \chi_{\{u>0\}}\,dx \\ &= x_n^0 \lim_{r\searrow 0} r^{-n} \int_{B_r(x^0)} \chi_{\{u>0\}}\,dx,\end{aligned}$$

(the limit here exists because $\lim_{r\searrow 0} \Phi^{\mathrm{int}}_{x^0,u}(r)$ exists), while in the boundary case we obtain that

$$\begin{aligned}\lim_{m\to\infty} \Phi^{\mathrm{bound}}_{x^0,u}(r_m) &= \int_{B_1} |\nabla u_0|^2\,dx - \frac{3}{2}\int_{\partial B_1} u_0^2\,d\mathcal{H}^{n-1} + \lim_{r\searrow 0} r^{-n-1} \int_{B_r(x^0)} x_n^+ \chi_{\{u>0\}}\,dx \\ &= \lim_{r\searrow 0} r^{-n-1} \int_{B_r(x^0)} x_n^+ \chi_{\{u>0\}}\,dx.\end{aligned}$$

Thus $\Phi^{\mathrm{int}}_{x^0,u}(0^+) \geqslant 0$ in the interior case, $\Phi^{\mathrm{bound}}_{x^0,u}(0^+) \geqslant 0$ in the boundary case, and equality in either case implies that for each $\tau > 0$, u_m converges to 0 in measure in the set $\{x_n > \tau\}$ as $m \to \infty$, and consequently $u_0 = 0$ in $\mathbf{R}^n$.

(iii), (iv) For each $\delta > 0$ and $K < \infty$ we obtain from the monotonicity formula (Theorem 3.5) that in the interior case

$$\Phi^{\mathrm{int}}_{x,u}(0^+) \leqslant \Phi^{\mathrm{int}}_{x,u}(r) \leqslant \Phi^{\mathrm{int}}_{x^0,u}(r) + \frac{\delta}{2} \leqslant \begin{cases} \Phi^{\mathrm{int}}_{x^0,u}(0^+) + \delta, & \text{if } \Phi^{\mathrm{int}}_{x^0,u}(0^+) > -\infty, \\ -K, & \text{if } \Phi^{\mathrm{int}}_{x^0,u}(0^+) = -\infty, \end{cases}$$

and in the boundary case

$$\Phi^{\mathrm{bound}}_{x,u}(0^+) \leqslant \Phi^{\mathrm{bound}}_{x,u}(r) \leqslant \Phi^{\mathrm{bound}}_{x^0,u}(r) + \frac{\delta}{2} \leqslant \Phi^{\mathrm{bound}}_{x^0,u}(0^+) + \delta,$$

if we choose for fixed x^0 first $r > 0$ and then $|x - x^0|$ small enough.

(v) The fact that u_0 is a variational solution of (3.1) and satisfies the monotonicity formula in the sense indicated follows directly from the convergence assumption. The proof of the rest of the claim follows by the same argument as in (iii) and (iv). □

Lemma 4.3. *Let u be a variational solution of* (3.1) *and suppose that*

$$|\nabla u|^2 \leqslant C x_n^+ \quad \textit{locally in } \Omega.$$

Then $\Phi^{\mathrm{int}}_{x^0,u}(0^+) = 0$ implies that $u \equiv 0$ in some open n-dimensional ball containing x^0.

Proof. By the upper semicontinuity (Lemma 4.2 (iii)), $\Phi^{\text{int}}_{x,u}(0^+)\leqslant\varepsilon$ in $B_\delta(x^0)\subset\Omega$ for some $\delta\in(0,x^0_n)$. Suppose towards a contradiction that $u\not\equiv0$ in $B_\delta(x^0)$. Then there exist a ball $A\subset\{u>0\}\cap B_\delta(x^0)$ and $z\in\partial A\cap\{u=0\}$. It follows that

$$\Phi^{\text{int}}_{z,u}(0^+)=z_n\lim_{r\searrow0}r^{-n}\int_{B_r(z)}\chi_{\{u>0\}}\,dx\geqslant z_n\frac{\omega_n}{2},$$

a contradiction for sufficiently small ε. □

Unfortunately, a boundary version of Lemma 4.3, stating that boundary density 0 at x^0 implies the solution being 0 in an open n-dimensional ball with center x^0, cannot be obtained in the same way. Instead we prove the following result in the 2-dimensional case.

LEMMA 4.4. *Let $n=2$, let u be a weak solution of* (3.1) *and suppose that*

$$|\nabla u|^2\leqslant x_2^+\quad\textit{in }\Omega.$$

Then $\Phi^{\text{bound}}_{x^0,u}(0^+)=0$ implies that $u\equiv0$ in some open 2-dimensional ball containing x^0.

Proof. Suppose towards a contradiction that $x^0\in\partial\{u>0\}$, and let us take a blow-up sequence

$$u_m(x):=\frac{u(x^0+r_mx)}{r_m^{3/2}}$$

converging weakly in $W^{1,2}_{\text{loc}}(\mathbf{R}^n)$ to a blow-up limit u_0. Lemma 4.2 (iv) shows that $u_0=0$ in $\mathbf{R}^2$. Consequently,

$$0\leftarrow\Delta u_m(B_2)\geqslant\int_{B_2\cap\partial_{\text{red}}\{u_m>0\}}\sqrt{x_2}\,d\mathcal{H}^1\quad\text{as } m\to\infty. \tag{4.1}$$

(Recall that Δu is a non-negative Radon measure in Ω.) On the other hand, there is at least one connected component V_m of $\{u_m>0\}$ touching the origin and containing, by the maximum principle, a point $x^m\in\partial A$, where $A=(-1,1)\times(0,1)$. If

$$\max\{x_2:x\in V_m\cap\partial A\}\not\to0\quad\text{as } m\to\infty,$$

we immediately obtain a contradiction to (4.1). If

$$\max\{x_2:x\in V_m\cap\partial A\}\to0,$$

we use the free-boundary condition as well as $|\nabla u|^2\leqslant x_2^+$ to obtain

$$0=\Delta u_m(V_m\cap A)\leqslant\int_{V_m\cap\partial A}\sqrt{x_2}\,d\mathcal{H}^1-\int_{A\cap\partial_{\text{red}}V_m}\sqrt{x_2}\,d\mathcal{H}^1.$$

However $\int_{V_m\cap\partial A}\sqrt{x_2}\,d\mathcal{H}^1$ is the unique minimizer of $\int_{\partial D}\sqrt{x_2}\,d\mathcal{H}^1$ with respect to all open sets D with $D=V_m$ on ∂A. So V_m cannot touch the origin, a contradiction. □

Remark 4.5. Note that we have not really used the full information contained in the weak formulation. What we have used is the *inequality* $\Delta u \geqslant \sqrt{x_2}\,\mathcal{H}|_{\partial_{\mathrm{red}}\{u>0\}}$ (which is true for any limit of the singular perturbation considered in [37]) and the fact that we can locate a non-empty portion of $\partial_{\mathrm{red}}\{u>0\}$ touching x^0.

In higher dimensions it is not so clear whether *cusps* can be excluded. Of course that does not happen for Lipschitz free boundaries.

LEMMA 4.6. *Let u be a variational solution of* (3.1) *and suppose that*

$$|\nabla u|^2 \leqslant C x_n^+ \quad \textit{locally in } \Omega,$$

and that $\{u>0\}$ is locally a Lipschitz set. Then $\Phi^{\mathrm{bound}}_{x^0,u}(0^+)=0$ implies that $u\equiv 0$ in some open n-dimensional ball containing x^0.

Proof. This is an immediate consequence of Lemma 4.2 (ii) and the Lipschitz continuity. □

PROPOSITION 4.7. (2-dimensional case) *Let $n=2$, let u be a variational solution of* (3.1), *and suppose that*

$$|\nabla u|^2 \leqslant C x_2^+ \quad \textit{locally in } \Omega.$$

Let $x^0 \in \Omega$ be such that $u(x^0)=0$, and suppose that

$$r^{-1}\int_{B_r(x^0)} |\nabla \chi_{\{u>0\}}|\,dx \leqslant C_0$$

for all $r>0$ such that $B_r(x^0) \Subset \Omega$ in the interior case, and that

$$r^{-3/2}\int_{B_r(x^0)} \sqrt{x_2}\,|\nabla \chi_{\{u>0\}}|\,dx \leqslant C_0$$

for all $r>0$ such that $B_r(x^0) \Subset \Omega$ in the boundary case.

(i) Interior case $x_2^0>0$. *The only possible blow-up limits are*

$$u_0(x) = \sqrt{x_2^0}\max\{x\cdot e, 0\} \quad \textit{and} \quad u_0(x) = \gamma|x\cdot e|,$$

where e is a unit vector and γ is a non-negative constant. If $u_0(x)=\sqrt{x_2^0}\max\{x\cdot e,0\}$ then the corresponding density value is $\frac{1}{2}\omega_2$, if $u_0(x)=\gamma|x\cdot e|$ with $\gamma>0$ then the density is ω_2, while if $u_0=0$ the density may be either 0 or ω_2.

(ii) Boundary case $x_2^0=0$. *The only possible blow-up limits are*

$$u_0(\varrho,\theta) = \frac{\sqrt{2}}{3}\varrho^{3/2}\cos\left(\frac{3}{2}\left(\min\left\{\max\left\{\theta,\frac{\pi}{6}\right\},\frac{5\pi}{6}\right\}-\frac{\pi}{2}\right)\right),$$

with the corresponding density

$$\int_{B_1} x_2^+ \chi_{\{x:\pi/6<\theta<5\pi/6\}}\, dx,$$

and $u_0(x)=0$, *with possible values of the density*

$$\int_{B_1} x_2^+\, dx \quad \textit{and} \quad 0.$$

Proof. Consider a blow-up sequence u_m as in Lemma 4.1, where $r_m \searrow 0$, with blow-up limit u_0. Because of the strong convergence of u_m to u_0 in $W^{1,2}_{\mathrm{loc}}(\mathbf{R}^2)$ and the compact embedding from BV into L^1, u_0 is a homogeneous solution of

$$0=\int_{\mathbf{R}^2}(|\nabla u_0|^2 \operatorname{div}\phi - 2\nabla u_0 D\phi \nabla u_0)\, dx + x_2^0 \int_{\mathbf{R}^2} \chi_0 \operatorname{div}\phi\, dx \tag{4.2}$$

for any $\phi \in C^1_0(\mathbf{R}^2;\mathbf{R}^2)$ in the interior case, and of

$$0=\int_{\mathbf{R}^2}(|\nabla u_0|^2 \operatorname{div}\phi - 2\nabla u_0 D\phi \nabla u_0)\, dx + \int_{\mathbf{R}^2}(x_2 \chi_0 \operatorname{div}\phi + \chi_0 \phi_2)\, dx \tag{4.3}$$

for any $\phi \in C^1_0(\mathbf{R}^2;\mathbf{R}^2)$ in the boundary case, where χ_0 is the strong L^1_{loc}-limit of $\chi_{\{u_m>0\}}$ along a subsequence. The values of the function χ_0 are almost everywhere in $\{0,1\}$, and the locally uniform convergence of u_m to u_0 implies that $\chi_0=1$ in $\{u_0>0\}$. The homogeneity of u_0 and its harmonicity in $\{u_0>0\}$ show that each connected component of $\{u_0>0\}$ is a half-plane passing trough the origin in the interior case, and a cone with vertex at the origin and of opening angle 120° in the boundary case. Also, (4.2) and (4.3) imply that χ_0 is constant in the connected set $\{u_0=0\}^\circ$, i.e. the interior of $\{u_0=0\}$.

Consider first the case when $\{u_0>0\}$ has exactly one connected component. Let z be an arbitrary point in $\partial\{u_0=0\}\setminus\{0\}$. Note that the normal to $\partial\{u_0=0\}$ has the constant value $\nu(z)$ in $B_\delta(z)$ for some $\delta>0$. Plugging in $\phi(x):=\eta(x)\nu(z)$ into (4.2) and (4.3), where $\eta \in C^1_0(B_\delta(z))$ is arbitrary, and integrating by parts, it follows that

$$0=\int_{\partial\{u_0>0\}}(-|\nabla u_0|^2 + x_2^0(1-\bar{\chi}_0))\eta\, d\mathcal{H}^1 \tag{4.4}$$

in the interior case, and that

$$0=\int_{\partial\{u_0>0\}}(-|\nabla u_0|^2 + x_2(1-\bar{\chi}_0))\eta\, d\mathcal{H}^1 \tag{4.5}$$

in the boundary case. Here $\bar{\chi}_0$ denotes the constant value of χ_0 in $\{u_0=0\}^\circ$. Note that by Hopf's principle, $\nabla u_0 \cdot \nu \neq 0$ on $B_\delta(z)\cap\partial\{u_0>0\}$. In both the interior and boundary case it follows therefore that $\bar{\chi}_0 \neq 1$, and hence necessarily $\bar{\chi}_0=0$. We deduce from (4.4) and (4.5) that $|\nabla u_0|^2 = x_2^0$ on $\partial\{u_0>0\}$ in the interior case, and that $|\nabla u_0|^2=x_2$ on $\partial\{u_0>0\}$ in the boundary case. Computing the solution u_0 of the ordinary differential equation on ∂B_1 yields the statement of the proposition in the case under consideration.

Consider now the case $u_0=0$. In the interior case, (4.2) shows that χ_0 is constant on $\mathbb{R}^2$, with value either 0 or 1. In the boundary case, (4.3) shows that χ_0 is constant in the upper half-plane, with value either 0 or 1, and that χ_0 is constant with the value 0 in the lower half-plane.

Last, consider the situation when, in the interior case, the set $\{u_0>0\}$ has two connected components. The argument for (4.4) now yields that the constant values of $|\nabla u_0|^2$ on either side of $\partial\{u_0>0\}$ are equal. This completes the proof. □

5. Partial regularity of non-degenerate solutions

Definition 5.1. (Stagnation points) Let u be a variational solution of (3.1). We call $S^u := \{x\in\Omega : x_n=0 \text{ and } x\in\partial\{u>0\}\}$ the set of *stagnation points*.

Definition 5.2. (Non-degeneracy and density condition) Let u be a variational solution of (3.1).

(i) We say that a point $x^0\in\Omega\cap\partial\{u>0\}\cap\{x_n=0\}$ satisfies *property* (N) if

$$\liminf_{r\searrow 0} r^{-n-3}\int_{B_r(x^0)} u^2\,dx > 0.$$

Moreover we define for each $\tau>0$ and $\varsigma>0$ the set

$$N^u_{\varsigma,\tau} := \left\{x^0\in\Omega\cap\partial\{u>0\}\cap\{x_n=0\} : r^{-n-3}\int_{B_r(x^0)} u^2\,dx \geqslant \tau \text{ for } r\in(0,\varsigma]\right\}.$$

(ii) We say that a point $x^0\in\Omega\cap\partial\{u>0\}\cap\{x_n=0\}$ satisfies *property* (D) if

$$0<\liminf_{r\searrow 0} r^{-n-1}\int_{B_r(x^0)} x_n^+\chi_{\{u>0\}}\,dx \leqslant \limsup_{r\searrow 0} r^{-n-1}\int_{B_r(x^0)} x_n^+\chi_{\{u>0\}}\,dx < \int_{B_1} x_n^+\,dx.$$

Note that $\bigcup_{\varsigma,\tau} N^u_{\varsigma,\tau}$ is the set of all points satisfying property (N).

Lemma 5.3. *Let u be a variational solution of* (3.1) *and suppose that*

$$|\nabla u|^2 \leqslant C x_n^+ \quad \textit{locally in } \Omega$$

and that

$$r^{1/2-n}\int_{B_r(y)} \sqrt{x_n}\,|\nabla\chi_{\{u>0\}}|\,dx \leqslant C_0$$

for all $B_r(y)\Subset\Omega$ such that $y_n=0$. Then properties (N) *and* (D) *are equivalent.*

Proof. (D) ⇒ (N) Consider a blow-up limit u_0 of the sequence

$$u_m(x) := \frac{u(x^0+r_m x)}{r_m^{3/2}},$$

where $r_m \searrow 0$, and suppose towards a contradiction that $u_0=0$. Passing to the limit in the domain variation equation we obtain

$$0 = \int_{\mathbf{R}^n} (|\nabla u_0|^2 \operatorname{div}\phi - 2\nabla u_0 D\phi \nabla u_0 + x_n\chi_0 \operatorname{div}\phi + \chi_0\phi_n)\,dx = \int_{\mathbf{R}^n} (x_n\chi_0 \operatorname{div}\phi + \chi_0\phi_n)\,dx$$

for any $\phi \in C_0^1(\mathbf{R}^n;\mathbf{R}^n)$, where χ_0 is the limit of $\chi_{\{u_m>0\}}$ with respect to a subsequence. This implies that χ_0 is a constant function. On the other hand, the condition on $|\nabla\chi_{\{u>0\}}|$ implies that the values of χ_0 are almost everywhere in $\{0,1\}$, and then condition (D) shows that the function χ_0 is not constant, a contradiction.

(N) ⇒ (D) The proof draws on [37, Proof of Proposition 9.1]. Let us again consider a blow-up limit u_0 of the sequence

$$u_m(x) := \frac{u(x^0+r_m x)}{r_m^{3/2}},$$

and suppose towards a contradiction that $\chi_0 := \lim_{m\to\infty} \chi_{\{u_m>0\}} \equiv 1$. By the monotonicity formula (which holds for u_0 with $\chi_{\{u_0>0\}}$ replaced by χ_0) and the growth estimate we obtain for each point x such that $x_n=0$,

$$\begin{aligned} 0 \leftarrow \Phi^{\text{bound}}_{x,u_0}(\sigma) - \Phi^{\text{bound}}_{0,u_0}(\sigma) &= \Phi^{\text{bound}}_{x,u_0}(\sigma) - \Phi^{\text{bound}}_{0,u_0}(0^+) \\ = \Phi^{\text{bound}}_{x,u_0}(\sigma) - \Phi^{\text{bound}}_{x,u_0}(0^+) &= \int_0^\sigma r^{-n-1} \int_{\partial B_r(x)} 2\left(\nabla u\cdot\nu - \frac{3}{2}\frac{u}{r}\right)^2 d\mathcal{H}^{n-1}\,dr \end{aligned}$$

as $\sigma\to\infty$. But this means that u_0 is homogeneous of degree $\frac{3}{2}$ with respect to each point x such that $x_n=0$. It follows that u_0 depends only on the x_n-variable. Thus $u_0(x)=\alpha(x_n^+)^{3/2}$ for some $\alpha \geqslant 0$, a contradiction to the definition of variational solution unless $\alpha=0$. □

PROPOSITION 5.4. (2-dimensional case) *Let $n=2$, let u be a variational solution of* (3.1), *and suppose that*

$$|\nabla u|^2 \leqslant C x_2^+ \quad \textit{locally in } \Omega$$

and that

$$r^{-3/2} \int_{B_r(y)} \sqrt{x_2}\,|\nabla\chi_{\{u>0\}}|\,dx \leqslant C_0$$

for all $B_r(y) \Subset \Omega$ such that $y_n=0$. At each non-degenerate stagnation point x^0, the density $\Phi^{\text{bound}}_{x^0,u}(0^+)$ has the value

$$\int_{B_1} x_2^+ \chi_{\{x:\pi/6<\theta<5\pi/6\}}\,dx$$

and

$$\frac{u(x^0+rx)}{r^{3/2}} \to \frac{\sqrt{2}}{3}\varrho^{3/2}\cos\left(\frac{3}{2}\left(\min\left\{\max\left\{\theta,\frac{\pi}{6}\right\},\frac{5\pi}{6}\right\}-\frac{\pi}{2}\right)\right) \quad \text{as } r\searrow 0,$$

strongly in $W^{1,2}_{\mathrm{loc}}(\mathbf{R}^2)$ *and locally uniformly on* $\mathbf{R}^2$, *where* $x=(\varrho\cos\theta,\varrho\sin\theta)$. *Moreover*,

$$\mathcal{L}^2\left(B_1\cap\left(\{x:u(x^0+rx)>0\}\triangle\left\{x:\frac{\pi}{6}<\theta<\frac{5\pi}{6}\right\}\right)\right)\to 0 \quad \text{as } r\searrow 0,$$

and, for each $\delta>0$,

$$r^{-3/2}\Delta u\left((x^0+B_r)\setminus\left\{x:\min\left\{\left|\theta-\frac{\pi}{6}\right|,\left|\theta-\frac{5\pi}{6}\right|\right\}<\delta\right\}\right)\to 0 \quad \text{as } r\searrow 0.$$

(*Recall that* Δu *is a non-negative Radon measure in* Ω.)

Proof. The value of the density and the uniqueness of the blow-up limit follow directly from Proposition 4.7 (ii) and the non-degeneracy assumption.

Let $r_m\searrow 0$ be an arbitrary sequence, let us consider once more the blow-up sequence u_m defined in Lemma 4.1 (iv), and let

$$u_0(\varrho,\theta)=\frac{\sqrt{2}}{3}\varrho^{3/2}\cos\left(\frac{3}{2}\left(\min\left\{\max\left\{\theta,\frac{\pi}{6}\right\},\frac{5\pi}{6}\right\}-\frac{\pi}{2}\right)\right).$$

By the proof of Proposition 4.7, $\chi_{\{u_m>0\}}$ converges strongly in $L^1(B_1)$ to $\chi_{\{u_0>0\}}$ along a subsequence. Since this is true for *all* sequences $r_m\searrow 0$, it follows that

$$\chi_{\{x:u(x^0+rx)>0\}}\to\chi_{\{u_0>0\}} \quad \text{strongly in } L^1(B_1) \quad \text{as } r\searrow 0,$$

which is exactly the first measure estimate. The convergence of u_m to u_0 implies the weak convergence of the sequence of non-negative Radon measures Δu_m to Δu_0. As u_0 is harmonic in

$$B_1\setminus\left\{x:\min\left\{\left|\theta-\frac{\pi}{6}\right|,\left|\theta-\frac{5\pi}{6}\right|\right\}<\frac{\delta}{2}\right\},$$

it follows that

$$\Delta u_m\left(B_1\setminus\left\{x:\min\left\{\left|\theta-\frac{\pi}{6}\right|,\left|\theta-\frac{5\pi}{6}\right|\right\}<\delta\right\}\right)\to 0$$

as $m\to\infty$. Since this is true for all sequences $r_m\searrow 0$, the second measure estimate follows. □

PROPOSITION 5.5. (Partial regularity in two dimensions) *Let* $n=2$, *let* u *be a variational solution of* (3.1), *and suppose that*

$$|\nabla u|^2\leqslant Cx_2^+ \quad \text{locally in } \Omega$$

and that

$$r^{-3/2}\int_{B_r(y)}\sqrt{x_2}\,|\nabla\chi_{\{u>0\}}|\,dx\leqslant C_0$$

for all $B_r(y)\Subset\Omega$ *such that* $y_2=0$. *Let* $x^0\in S^u$ *be a non-degenerate point. Then in some open neighborhood*, x^0 *is the* only *non-degenerate stagnation point.*

Proof. Suppose towards a contradiction that there exists a sequence x^m of non-degenerate points converging to x^0, with $x^m\neq x^0$ for all m. Choosing $r_m:=|x^m-x^0|$, there is no loss of generality in assuming that the sequence $(x^m-x^0)/r^m$ is constant, with value $z\in\{(-1,0),(1,0)\}$. Consider the blow-up sequence

$$u_m(x)=\frac{u(x^0+r_mx)}{r_m^{3/2}}.$$

Since x^m is a non-degenerate point for u, it follows that z is a non-degenerate point for u_m, and therefore Proposition 5.4 shows that

$$\Phi^{\mathrm{bound}}_{z,u^m}(0^+)=\int_{B_1}x_2^+\chi_{\{x:\pi/6<\theta<5\pi/6\}}\,dx.$$

By Lemma 4.1 (v) and the proof of Proposition 4.7 (ii), the sequence u_m converges strongly in $W^{1,2}_{\mathrm{loc}}(\mathbf{R}^2)$ to the homogeneous solution

$$u_0(\varrho,\theta)=\frac{\sqrt{2}}{3}\varrho^{3/2}\cos\biggl(\frac{3}{2}\biggl(\min\biggl\{\max\biggl\{\theta,\frac{\pi}{6}\biggr\},\frac{5\pi}{6}\biggr\}-\frac{\pi}{2}\biggr)\biggr),$$

where $x=(\varrho\cos\theta,\varrho\sin\theta)$, while $\chi_{\{u_m>0\}}$ converges strongly in $L^1_{\mathrm{loc}}(\mathbf{R}^2)$ to $\chi_{\{u_0>0\}}$. It follows from Lemma 4.2 (v) that

$$\Phi^{\mathrm{bound}}_{z,u^0}(0^+)\geqslant\limsup_{m\to\infty}\Phi^{\mathrm{bound}}_{z,u^m}(0^+)=\int_{B_1}x_2^+\chi_{\{x:\pi/6<\theta<5\pi/6\}}\,dx$$

contradicting the fact that

$$\Phi^{\mathrm{bound}}_{z,u^0}(0^+)=0.$$ □

Remark 5.6. It follows that in two dimensions S^u can be decomposed into a countable set of "Stokes points" with the asymptotics as in Proposition 5.4, accumulating (if at all) only at "degenerate stagnation points", and a set of "degenerate stagnation points" which will be analyzed in the following sections.

The following lemma will be used in order to prove the partial regularity result (Proposition 5.8).

Lemma 5.7. *Let u be a variational solution of* (3.1), *and suppose that*

$$|\nabla u|^2 \leqslant C x_n^+ \quad \textit{locally in } \Omega,$$

and that

$$r^{1/2-n} \int_{B_r(y)} \sqrt{x_n}\, |\nabla \chi_{\{u>0\}}|\, dx \leqslant C_0$$

for all $B_r(y) \Subset \Omega$ such that $y_n=0$. Suppose that $x^0 \in S^u$ and let u_0 be a blow-up limit of the sequence

$$u_m(x) := \frac{u(x^0+r_m x)}{r_m^{3/2}}.$$

Then for each compact set $K \subset \mathbf{R}^n$ and each open set $U \supset K \cap N^{u_0}_{\varsigma,\tau}$ there exists $m_0 < \infty$ such that $N^{u_m}_{\varsigma,\tau} \cap K \subset U$ for $m \geqslant m_0$.

Proof. Suppose towards a contradiction that $N^{u_m}_{\varsigma,\tau} \cap (K \setminus U)$ contains a sequence x^m converging to $\bar{x}$ as $m \to \infty$. Then $\bar{x}_n = 0$, and by the locally uniform Lipschitz continuity of u_m, $\bar{x} \in \{u_0 = 0\} \cap (K \setminus U)$. But this contradicts the assumption $U \supset K \cap N^{u_0}_{\varsigma,\tau}$ by the uniform convergence of u_m. □

Proposition 5.8. (Partial regularity in higher dimensions) *Let u be a variational solution of* (3.1) *and suppose that*

$$|\nabla u|^2 \leqslant C x_n^+ \quad \textit{locally in } \Omega,$$

and that

$$r^{1/2-n} \int_{B_r(y)} \sqrt{x_n}\, |\nabla \chi_{\{u>0\}}|\, dx \leqslant C_0$$

for all $B_r(y) \Subset \Omega$ such that $y_n=0$. Then the Hausdorff dimension of the set $\bigcup_{\varsigma,\tau} N^u_{\varsigma,\tau}$ of all non-degenerate points is less than or equal to $n-2$.

The proof uses standard tools of geometric measure theory and will be given in the appendix.

Remark 5.9. It follows that the Hausdorff dimension of the set of non-degenerate stagnation points is less than or equal to $n-2$. From Lemma 5.3 we infer that the set of stagnation points satisfying the density condition also has dimension at most $n-2$.

6. Degenerate points

Definition 6.1. Let u be a variational solution of (3.1). We define

$$\Sigma^u := \left\{ x^0 \in S^u : \Phi^{\text{bound}}_{x^0,u}(0^+) = \int_{B_1} x_n^+ \, dx \right\}.$$

Remark 6.2. The set Σ^u is closed, as a consequence of the upper semicontinuity Lemma 4.2 (iv).

Remark 6.3. In the case of two dimensions and a weak solution u, we infer from Lemmas 5.3 and 4.4 that the set $S^u \setminus \Sigma^u$ equals the set of non-degenerate stagnation points and is, according to Proposition 5.4, a finite or countable set.

The following lemma is drawn from [37, Theorem 11.1].

LEMMA 6.4. *Let u be a variational solution of* (3.1), *let $x^0 \in \Sigma^u$ and let*

$$\delta := \tfrac{1}{2}\operatorname{dist}(x^0, \partial\Omega).$$

(i) *The mean frequency satisfies, for all $r \in (0,\delta)$,*

$$r\frac{\int_{B_r(x^0)} |\nabla u|^2\,dx}{\int_{\partial B_r(x^0)} u^2\,d\mathcal{H}^{n-1}} - \frac{3}{2} \geqslant r\frac{\int_{B_r(x^0)} x_n^+(1-\chi_{\{u>0\}})\,dx}{\int_{\partial B_r(x^0)} u^2\,d\mathcal{H}^{n-1}} \geqslant 0.$$

(ii) *The function*

$$r \longmapsto r^{-n-2}\int_{\partial B_r(x^0)} u^2\,d\mathcal{H}^{n-1} \tag{6.1}$$

is non-decreasing on $(0,\delta)$ *and has the right limit* 0 *at* 0.

(iii) *The function*

$$r \longmapsto r^{-n-2}\int_{B_r(x^0)} x_n^+(1-\chi_{\{u>0\}})\,dx \tag{6.2}$$

is integrable on $(0,\delta)$.

Proof. (i) The inequality

$$\Phi^{\mathrm{bound}}_{x^0,u}(0^+) \leqslant \Phi^{\mathrm{bound}}_{x^0,u}(r)$$

can be rearranged into

$$r^{-n-1}\int_{B_r(x^0)} |\nabla u|^2\,dx - \frac{3}{2}r^{-n-2}\int_{\partial B_r(x^0)} u^2\,d\mathcal{H}^{n-1} \geqslant r^{-n-1}\int_{B_r(x^0)} x_n^+(1-\chi_{\{u>0\}})\,dx,$$

and the right-hand side is clearly non-negative.

(ii) Plugging in $\alpha := -n-2$ into (3.2) and using (3.4), it follows that

$$\begin{aligned}\frac{d}{dr}&\left(r^{-n-2}\int_{\partial B_r(x^0)} u^2\,d\mathcal{H}^{n-1}\right)\\ &= \frac{2}{r}\left(r^{-n-1}\int_{B_r(x^0)} |\nabla u|^2\,dx - \frac{3}{2}r^{-n-2}\int_{\partial B_r(x^0)} u^2\,d\mathcal{H}^{n-1}\right)\\ &\geqslant 2r^{-n-2}\int_{B_r(x^0)} x_n^+(1-\chi_{\{u>0\}})\,dx.\end{aligned}$$

Hence the function (6.1) is non-decreasing on $(0,\delta)$. Using Lemma 5.3 we obtain that its right limit at 0 is 0.

(iii) The above inequality implies that the function (6.2) is in $L^1(0,\delta)$. □

7. The frequency formula

THEOREM 7.1. (Frequency formula) *Let u be a variational solution of* (3.1), *let x^0 be a point of the closed set Σ^u and let $\delta:=\frac{1}{2}\operatorname{dist}(x^0,\partial\Omega)$. The function*

$$F_{x^0,u}(r):=r\frac{\int_{B_r(x^0)}(|\nabla u|^2+x_n^+(\chi_{\{u>0\}}-1))\,dx}{\int_{\partial B_r(x^0)}u^2\,d\mathcal{H}^{n-1}}$$

satisfies, for a.e. $r\in(0,\delta)$, the identity

$$\begin{aligned}\frac{d}{dr}F_{x^0,u}(r)=\frac{2}{r}\left(\int_{\partial B_r(x^0)}u^2\,d\mathcal{H}^{n-1}\right)^{-2}\Bigg[\int_{\partial B_r(x^0)}(\nabla u\cdot(x-x^0))^2\,d\mathcal{H}^{n-1}\int_{\partial B_r(x^0)}u^2\,d\mathcal{H}^{n-1}\\ -\left(\int_{\partial B_r(x^0)}u\nabla u\cdot(x-x^0)\,d\mathcal{H}^{n-1}\right)^2\Bigg]\\ +2\frac{\int_{B_r(x^0)}x_n^+(1-\chi_{\{u>0\}})\,dx}{\int_{\partial B_r(x^0)}u^2\,d\mathcal{H}^{n-1}}\left(r\frac{\int_{B_r(x^0)}|\nabla u|^2\,dx}{\int_{\partial B_r(x^0)}u^2\,d\mathcal{H}^{n-1}}-\frac{3}{2}\right).\end{aligned}$$

The function $r\mapsto F_{x^0,u}(r)$ is non-decreasing on $(0,\delta)$ and the following limit exists

$$F_{x^0,u}(0^+):=\lim_{r\searrow 0}F_{x^0,u}(r)\in\left[\tfrac{3}{2},\infty\right).$$

Remark 7.2. This formula is based on an analogous formula in the interior case derived by the second author for a more general class of semilinear elliptic equations ([38]). The root is the classical frequency formula of F. Almgren for Q-valued harmonic functions [1]. Almgren's formula has subsequently been extended to various perturbations (see [14] for a recent extension). Note however that while our formula may look like a perturbation of the "linear" formula for Q-valued harmonic functions, it is in fact a truly non-linear formula. This fact will be become more obvious in the paper [38] for more general semilinearities.

Proof. Assuming the validity of the claimed identity, the monotonicity of $F_{x^0,u}$ follows from combining the Cauchy–Schwarz inequality

$$\int_{\partial B_r(x^0)}(\nabla u\cdot(x-x^0))^2\,d\mathcal{H}^{n-1}\int_{\partial B_r(x^0)}u^2\,d\mathcal{H}^{n-1}\geqslant\left(\int_{\partial B_r(x^0)}u\nabla u\cdot(x-x^0)\,d\mathcal{H}^{n-1}\right)^2$$

with Lemma 6.4 (i). The same lemma also shows that $r\mapsto F_{x^0,u}(r)$ is bounded below by $\frac{3}{2}$. Thus it remains to prove the claimed identity.

Note that

$$F_{x^0,u}(r)=\frac{U(r)-\int_{B_1}x_n^+\,dx}{W(r)},$$

where $U{:=}U_{\text{bound}}$ and $W{:=}W_{\text{bound}}$ are the functions in the proof of Theorem 3.5. Hence

$$\frac{d}{dr}F_{x^0,u}(r)=\frac{U'(r)W(r)-W'(r)\big(U(r)-r^{-n-1}\int_{B_r(x^0)}x_n^+\,dx\big)}{W^2(r)}.$$

Using (3.8) and (3.9), it follows that

$$\begin{aligned}\frac{d}{dr}F_{x^0,u}(r)=&\left(r^{-n-2}\int_{\partial B_r(x^0)}u^2\,d\mathcal{H}^{n-1}\right)^{-2}\\ &\times\Bigg[\left(2r^{-n-1}\int_{\partial B_r(x^0)}(\nabla u\cdot\nu)^2\,d\mathcal{H}^{n-1}-3r^{-n-2}\int_{\partial B_r(x^0)}u\nabla u\cdot\nu\,d\mathcal{H}^{n-1}\right)\\ &\quad\times\left(r^{-n-2}\int_{\partial B_r(x^0)}u^2\,d\mathcal{H}^{n-1}\right)\\ &\quad-\left(r^{-n-1}\int_{B_r(x^0)}(|\nabla u|^2+x_n^+(\chi_{\{u>0\}}-1))\,dx\right)\\ &\quad\times\left(2r^{-n-2}\int_{\partial B_r(x^0)}u\nabla u\cdot\nu\,d\mathcal{H}^{n-1}-3r^{-n-3}\int_{\partial B_r(x^0)}u^2\,d\mathcal{H}^{n-1}\right)\Bigg].\end{aligned}$$

Using (3.4), we obtain

$$\begin{aligned}\frac{d}{dr}F_{x^0,u}(r)=&\left(\int_{\partial B_r(x^0)}u^2\,d\mathcal{H}^{n-1}\right)^{-2}\\ &\times\Bigg[2r\int_{\partial B_r(x^0)}(\nabla u\cdot\nu)^2\;d\mathcal{H}^{n-1}\int_{\partial B_r(x^0)}u^2\,d\mathcal{H}^{n-1}\\ &\quad-2r\left(\int_{\partial B_r(x^0)}u\nabla u\cdot\nu\,d\mathcal{H}^{n-1}\right)^2+\left(\int_{B_r(x^0)}x_n^+(1-\chi_{\{u>0\}})\,dx\right)\\ &\quad\times\left(2r\int_{\partial B_r(x^0)}u\nabla u\cdot\nu\,d\mathcal{H}^{n-1}-3\int_{\partial B_r(x^0)}u^2\,d\mathcal{H}^{n-1}\right)\Bigg],\end{aligned}$$

which, upon rearranging and using again (3.4) (this time in the reverse direction), gives the required result. □

COROLLARY 7.3. *Let u be a variational solution of* (3.1), *let x^0 be a point of the closed set Σ^u, and let $\delta{:=}\frac{1}{2}\operatorname{dist}(x,\partial\Omega)$. Let us consider, for $r\in(0,\delta)$, the functions*

$$D(r):=r\frac{\int_{B_r(x^0)}|\nabla u|^2\,dx}{\int_{\partial B_r(x^0)}u^2\,d\mathcal{H}^{n-1}}\quad\text{and}\quad V(r):=r\frac{\int_{B_r(x^0)}x_n^+(1-\chi_{\{u>0\}})\,dx}{\int_{\partial B_r(x^0)}u^2\,d\mathcal{H}^{n-1}},$$

so that $F_{x_0,u}(r)=D(r)-V(r)$.

(i) *For every $r\in(0,\delta)$, the following inequalities hold*

$$(D-V)'(r)\geqslant\frac{2}{r}V(r)\left(D(r)-\frac{3}{2}\right)\geqslant\frac{2}{r}V^2(r).$$

(ii) *The function $r\mapsto 2V^2(r)/r$ is integrable on $(0,\delta)$.*

Proof. The inequalities follow from Lemma 6.4 and Theorem 7.1. The integrability of $r \mapsto 2V^2(r)/r$ is a consequence of the inequalities. □

COROLLARY 7.4. (Density) *Let u be a variational solution of* (3.1). *The function*

$$x \longmapsto F_{x,u}(0^+)$$

is upper semicontinous on the closed set Σ^u.

Proof. For each $\delta > 0$, we have that

$$F_{x,u}(0^+) \leqslant F_{x,u}(r) \leqslant F_{x^0,u}(r) + \tfrac{1}{2}\delta \leqslant F_{x^0,u}(0^+) + \delta,$$

if we choose for fixed $x^0 \in \Sigma^u$ first $r > 0$ and then $|x - x^0|$ small enough. □

The next result is an improvement of Lemma 6.4 at those points of Σ^u at which the frequency is greater than $\frac{3}{2}$.

LEMMA 7.5. *Let u be a variational solution of* (3.1), *let $x^0 \in \Sigma^u$ and let*

$$\delta := \tfrac{1}{2} \operatorname{dist}(x^0, \partial\Omega).$$

Suppose that $F_{x^0,u}(0^+) > \frac{3}{2}$ and let $\gamma := F_{x^0,u}(0^+)$.

(i) *For all $r \in (0, \delta)$,*

$$r \frac{\int_{B_r(x^0)} |\nabla u|^2 \, dx}{\int_{\partial B_r(x^0)} u^2 \, d\mathcal{H}^{n-1}} - \gamma \geqslant r \frac{\int_{B_r(x^0)} x_n^+ (1 - \chi_{\{u>0\}}) \, dx}{\int_{\partial B_r(x^0)} u^2 \, d\mathcal{H}^{n-1}} \geqslant 0.$$

(ii) *The function $r \mapsto r^{1-n-2\gamma} \int_{\partial B_r(x^0)} u^2 \, d\mathcal{H}^{n-1}$ is non-decreasing on $(0, \delta)$.*

(iii) *The function $r \mapsto r^{1-n-2\gamma} \int_{B_r(x^0)} x_n^+ (1 - \chi_{\{u>0\}}) \, dx$ is integrable on $(0, \delta)$.*

(iv) *For each $\beta \in [0, \gamma)$,*

$$\frac{u(x^0 + rx)}{r^\beta} \to 0 \quad \textit{strongly in } L^2_{\mathrm{loc}}(\mathbf{R}^n) \quad \textit{as } r \searrow 0.$$

Proof. Part (i) follows from the fact that $F_{x_0,u}(r) \geqslant \gamma$ for all $r \in (0, \delta)$. Parts (ii) and (iii) follow by the same arguments as for the corresponding statements in Lemma 6.4. It is a consequence of part (ii) that $r \mapsto r^{-n-2\gamma} \int_{B_r(x^0)} u^2 \, dx$ is non-decreasing on $(0, \delta)$, and therefore, for each $\beta \in [0, \gamma)$,

$$r^{-n-2\beta} \int_{B_r(x^0)} u^2 \to 0 \quad \text{as } r \searrow 0.$$

This implies part (iv) of the lemma. □

8. Blow-up limits

The frequency formula allows passing to blow-up limits.

PROPOSITION 8.1. *Let u be a variational solution of* (3.1) *and let $x^0\in\Sigma^u$.*

(i) *The limits* $\lim_{r\searrow 0} V(r)=0$ *and* $\lim_{r\searrow 0} D(r)=F_{x^0,u}(0^+)$ *exist.*

(ii) *For any sequence $r_m\searrow 0$ as $m\to\infty$, the sequence*

$$v_m(x):=\frac{u(x^0+r_m x)}{\sqrt{r_m^{1-n}\int_{\partial B_{r_m}(x^0)} u^2\,d\mathcal{H}^{n-1}}} \tag{8.1}$$

is bounded in $W^{1,2}(B_1)$.

(iii) *For each sequence $r_m\searrow 0$ as $m\to\infty$ such that the sequence v_m in* (8.1) *converges weakly in $W^{1,2}(B_1)$ to a blow-up limit v_0, the function v_0 is continuous and homogeneous of degree $F_{x^0,u}(0^+)$ in B_1, and satisfies $v_0\geqslant 0$ in B_1, $v_0\equiv 0$ in $B_1\cap\{x_n\leqslant 0\}$ and $\int_{\partial B_1} v_0^2\,d\mathcal{H}^{n-1}=1$.*

Proof. The key step in the proof is statement (8.5), which we prove first. We start by writing the right-hand side of the frequency formula in a more convenient form, using a simple algebraic identity. For any real inner-product space $(H,\langle\cdot,\cdot\rangle)$ with norm $\|\cdot\|$, any vectors u and v and any scalar α,

$$\frac{1}{\|u\|^4}(\|v\|^2\|u\|^2-\langle v,u\rangle^2)=\left\|\frac{v}{\|u\|}-\frac{\langle v,u\rangle}{\|u\|^2}\frac{u}{\|u\|}\right\|^2=\left\|\frac{v}{\|u\|}-\frac{\langle v,u\rangle}{\|u\|^2}\frac{u}{\|u\|}+\alpha\frac{u}{\|u\|}\right\|^2-\alpha^2,$$

where we have used a cancellation due to orthogonality. Using the notation introduced in Corollary 7.3, we apply the above identity in the space $L^2(\partial B_r(x^0))$, with $u:=u$, $v:=r(\nabla u\cdot\nu)$ and $\alpha:=V(r)$, after also taking into account (3.4) in the form

$$\frac{\langle v,u\rangle}{\|u\|^2}=D(r),$$

to obtain from the frequency formula that

$$\frac{d}{dr}F_{x^0,u}(r)=\frac{2}{r}\int_{\partial B_r(x^0)}\left(\frac{v}{\|u\|}-D(r)\frac{u}{\|u\|}+V(r)\frac{u}{\|u\|}\right)^2 d\mathcal{H}^{n-1}$$
$$-\frac{2}{r}V^2(r)+\frac{2}{r}V(r)\left(D(r)-\frac{3}{2}\right).$$

This formula can be rewritten as

$$\begin{aligned}\frac{d}{dr}&F_{x^0,u}(r)\\ &=\frac{2}{r}\int_{\partial B_r(x^0)}\left[\frac{r(\nabla u\cdot\nu)}{\left(\int_{\partial B_r(x^0)}u^2\,d\mathcal{H}^{n-1}\right)^{1/2}}-F_{x^0,u}(r)\frac{u}{\left(\int_{\partial B_r(x^0)}u^2\,d\mathcal{H}^{n-1}\right)^{1/2}}\right]^2 d\mathcal{H}^{n-1}\\ &\quad+\frac{2}{r}V(r)\left(F_{x^0,u}(r)-\frac{3}{2}\right).\end{aligned}\tag{8.2}$$

Since $F_{x^0,u}(r)\geqslant\frac{3}{2}$ for all $r\in(0,\delta)$, we obtain therefore that for all $0<\varrho<\sigma<\delta$,

$$\int_\varrho^\sigma \frac{2}{r}\int_{\partial B_r(x^0)}\left[\frac{r(\nabla u\cdot\nu)}{\left(\int_{\partial B_r(x^0)} u^2\,d\mathcal{H}^{n-1}\right)^{1/2}}-F_{x^0,u}(r)\frac{u}{\left(\int_{\partial B_r(x^0)} u^2\,d\mathcal{H}^{n-1}\right)^{1/2}}\right]^2 d\mathcal{H}^{n-1}\,dr$$
$$\leqslant F_{x^0,u}(\sigma)-F_{x^0,u}(\varrho). \tag{8.3}$$

Let us consider now an arbitrary sequence r_m such that $r_m\searrow 0$ as $m\to\infty$, and let v_m be the sequence defined in (8.1). It follows by scaling from (8.3) that, for every m such that $r_m\delta<1$ and for every $0<\varrho<\sigma<1$,

$$\int_\varrho^\sigma \frac{2}{r}\int_{\partial B_r}\left[\frac{r(\nabla v_m\cdot\nu)}{\left(\int_{\partial B_r} v_m^2\,d\mathcal{H}^{n-1}\right)^{1/2}}-F_{x^0,u}(r_m r)\frac{v_m}{\left(\int_{\partial B_r} v_m^2\,d\mathcal{H}^{n-1}\right)^{1/2}}\right]^2 d\mathcal{H}^{n-1}\,dr$$
$$\leqslant F_{x^0,u}(r_m\sigma)-F_{x^0,u}(r_m\varrho)\to 0 \quad \text{as } m\to\infty,$$

since $F_{x^0,u}$ has a finite limit at 0. The above implies that

$$\int_\varrho^\sigma \frac{2}{r}\int_{\partial B_r}\left[\frac{r(\nabla v_m\cdot\nu)}{\left(\int_{\partial B_r} v_m^2\,d\mathcal{H}^{n-1}\right)^{1/2}}-F_{x^0,u}(0^+)\frac{v_m}{\left(\int_{\partial B_r} v_m^2\,d\mathcal{H}^{n-1}\right)^{1/2}}\right]^2 d\mathcal{H}^{n-1}\,dr\to 0 \tag{8.4}$$

as $m\to\infty$. Now note that, for every $r\in(\varrho,\sigma)\subset(0,1)$ and all m as before, it follows by Lemma 6.4 that

$$\int_{\partial B_r} v_m^2\,d\mathcal{H}^{n-1}=\frac{\int_{\partial B_{r_m r}(x_0)} u^2\,d\mathcal{H}^{n-1}}{\int_{\partial B_{r_m}(x_0)} u^2\,d\mathcal{H}^{n-1}}\leqslant r^{n+2}\leqslant 1.$$

Therefore (8.4) implies that

$$\int_{B_\sigma\setminus B_\varrho}|x|^{-n-3}[\nabla v_m(x)\cdot x-F_{x^0,u}(0^+)v_m(x)]^2\,dx\to 0 \quad \text{as } m\to\infty. \tag{8.5}$$

We can now prove all parts of the proposition.

(i) Suppose towards a contradiction that (i) is not true. Let $s_m\searrow 0$ be a sequence such that $V(s_m)$ is bounded away from 0. From the integrability of $r\mapsto 2V^2(r)/r$, we obtain that

$$\min_{r\in[s_m,2s_m]} V(r)\to 0 \quad \text{as } m\to\infty.$$

Let $t_m\in[s_m,2s_m]$ be such that $V(t_m)\to 0$ as $m\to\infty$. For the choice $r_m:=t_m$ for each m, the sequence v_m given by (8.1) satisfies (8.5). The fact that $V(r_m)\to 0$ implies that $D(r_m)$ is bounded, and hence v_m is bounded in $W^{1,2}(B_1)$. Let v_0 be any weak limit of v_m along a subsequence. Note that v_0 has norm 1 on $L^2(\partial B_1)$, since this is true for v_m

for all m. It follows from (8.5) that v_0 is homogeneous of degree $F_{x^0,u}(0^+)$. Note that, by Lemma 6.4 (ii),

$$\begin{aligned} V(s_m) &= \frac{s_m^{-n-1}\int_{B_{s_m}(x^0)} x_n^+(1-\chi_{\{u>0\}})\,dx}{s_m^{-n-2}\int_{\partial B_{s_m}(x^0)} u^2\,d\mathcal{H}^{n-1}} \leqslant \frac{s_m^{-n-1}\int_{B_{r_m}(x^0)} x_n^+(1-\chi_{\{u>0\}})\,dx}{(r_m/2)^{-n-2}\int_{\partial B_{r_m/2}(x^0)} u^2\,d\mathcal{H}^{n-1}} \\ &\leqslant \frac{1}{2}\frac{\int_{\partial B_{r_m}(x^0)} u^2\,d\mathcal{H}^{n-1}}{\int_{\partial B_{r_m/2}(x^0)} u^2\,d\mathcal{H}^{n-1}} V(r_m) = \frac{1}{2\int_{\partial B_{1/2}} v_m^2\,d\mathcal{H}^{n-1}} V(r_m). \end{aligned} \tag{8.6}$$

Since, at least along a subsequence,

$$\int_{\partial B_{1/2}} v_m^2\,d\mathcal{H}^{n-1} \to \int_{\partial B_{1/2}} v_0^2\,d\mathcal{H}^{n-1} > 0,$$

(8.6) leads to a contradiction. It follows that indeed $V(r)\to 0$ as $r\searrow 0$. This implies that $D(r)\to F_{x^0,u}(0^+)$.

(ii) Let r_m be an arbitrary sequence with $r_m\searrow 0$. The boundedness of the sequence v_m in $W^{1,2}(B_1)$ is equivalent to the boundedness of $D(r_m)$, which is true by (i).

(iii) Let $r_m\searrow 0$ be an arbitrary sequence such that v_m converges weakly to v_0. The homogeneity of degree $F_{x^0,u}(0^+)$ of v_0 follows directly from (8.5). The homogeneity of v_0, together with the fact that v_0 belongs to $W^{1,2}(B_1)$, imply that v_0 is continuous. The fact that $\int_{\partial B_1} v_0^2\,d\mathcal{H}^{n-1}=1$ is a consequence of $\int_{\partial B_1} v_m^2\,d\mathcal{H}^{n-1}=1$ for all m, and the remaining claims of the proposition are obvious. □

9. Concentration compactness in two dimensions

In the 2-dimensional case we prove concentration compactness which allows us to preserve variational solutions in the blow-up limit at degenerate points and excludes concentration. In order to do so, we combine the concentration compactness result of Evans and Müller [12] with information gained by our frequency formula. In addition, we obtain strong convergence of our blow-up sequence which is necessary in order to prove our main theorems. The question whether the following theorem holds in any dimension seems to be a hard one.

THEOREM 9.1. *Let $n=2$, let u be a variational solution of* (3.1) *and let $x^0\in\Sigma^u$. Let $r_m\searrow 0$ be such that the sequence v_m given by* (8.1) *converges weakly to v_0 in $W^{1,2}(B_1)$. Then v_m converges to v_0 strongly in $W^{1,2}_{\mathrm{loc}}(B_1\setminus\{0\})$, and v_0 satisfies $v_0\Delta v_0=0$ in the sense of Radon measures on B_1.*

Proof. Note first that, since v_0 is by Proposition 8.1 a non-negative continuous function, $v_0\Delta v_0$ is well defined as a non-negative Radon measure on B_1.

Let σ and ϱ with $0<\varrho<\sigma<1$ be arbitrary. We know that

$$\Delta v_m \geqslant 0 \text{ and } \Delta v_m(B_{(\sigma+1)/2}) \leqslant C_1 \quad \text{for all } m.$$

In order to apply the concentrated compactness result [12], we regularize each v_m to

$$\tilde{v}_m := v_m * \phi_m \in C^\infty(B_1),$$

where ϕ_m is a standard mollifier such that

$$\Delta \tilde{v}_m \geqslant 0 \text{ and } \int_{B_\sigma} \Delta \tilde{v}_m \, dx \leqslant C_2 < \infty \quad \text{for all } m,$$

and

$$\|v_m - \tilde{v}_m\|_{W^{1,2}(B_\sigma)} \to 0 \quad \text{as } m \to \infty.$$

From [11, Chapter 4, Theorem 3] we know that $\nabla \tilde{v}_m$ converges a.e. to the weak limit ∇v_0, and the only possible problem is concentration of $|\nabla \tilde{v}_m|^2$. By [12, Theorems 1.1 and 3.1], we obtain that

$$\partial_1 \tilde{v}_m \partial_2 \tilde{v}_m \to \partial_1 v_0 \partial_2 v_0$$

and

$$(\partial_1 \tilde{v}_m)^2 - (\partial_2 \tilde{v}_m)^2 \to (\partial_1 v_0)^2 - (\partial_2 v_0)^2$$

in the sense of distributions on B_σ as $m \to \infty$. It follows that

$$\partial_1 v_m \partial_2 v_m \to \partial_1 v_0 \partial_2 v_0 \tag{9.1}$$

and

$$(\partial_1 v_m)^2 - (\partial_2 v_m)^2 \to (\partial_1 v_0)^2 - (\partial_2 v_0)^2$$

in the sense of distributions on B_σ as $m \to \infty$. Let us remark that this alone would allow us to pass to the limit in the domain variation formula for v_m in the set $\{x_2 > 0\}$.

Observe now that (8.5) shows that

$$\nabla v_m(x) \cdot x - F_{x^0,u}(0^+) v_m(x) \to 0$$

strongly in $L^2(B_\sigma \setminus B_\varrho)$ as $m \to \infty$. It follows that

$$\partial_1 v_m x_1 + \partial_2 v_m x_2 \to \partial_1 v_0 x_1 + \partial_2 v_0 x_2$$

strongly in $L^2(B_\sigma \setminus B_\varrho)$ as $m \to \infty$. But then

$$\int_{B_\sigma \setminus B_\varrho} (\partial_1 v_m \partial_1 v_m x_1 + \partial_1 v_m \partial_2 v_m x_2) \eta \, dx \to \int_{B_\sigma \setminus B_\varrho} (\partial_1 v_0 \partial_1 v_0 x_1 + \partial_1 v_0 \partial_2 v_0 x_2) \eta \, dx$$

for each $\eta\in C_0^0(B_\sigma\setminus\overline{B}_\varrho)$ as $m\to\infty$. Using (9.1), we obtain that

$$\int_{B_\sigma\setminus B_\varrho}(\partial_1 v_m)^2 x_1\eta\,dx\to\int_{B_\sigma\setminus B_\varrho}(\partial_1 v_0)^2 x_1\eta\,dx$$

for each $0\leqslant\eta\in C_0^0((B_\sigma\setminus\overline{B}_\varrho)\cap\{x_1>0\})$ and each $0\geqslant\eta\in C_0^0((B_\sigma\setminus\overline{B}_\varrho)\cap\{x_1<0\})$ as $m\to\infty$. Repeating the above procedure three times for rotated sequences of solutions (by 45°) yields that ∇v_m converges strongly in $L^2_{\text{loc}}(B_\sigma\setminus\overline{B}_\varrho)$. Since σ and ϱ with $0<\varrho<\sigma<1$ were arbitrary, it follows that ∇v_m converges to ∇v_0 strongly in $L^2_{\text{loc}}(B_1\setminus\{0\})$.

As a consequence of the strong convergence, we see that

$$\int_{B_1}\nabla(\eta v_0)\cdot\nabla v_0\,dx=0\quad\text{for all }\eta\in C_0^1(B_1\setminus\{0\}).$$

Combined with the fact that $v_0=0$ in $B_1\cap\{x_2\leqslant 0\}$, this proves that $v_0\Delta v_0=0$ in the sense of Radon measures on B_1. □

10. Degenerate points in two dimensions

Theorem 10.1. *Let $n=2$ and let u be a variational solution of* (3.1). *Then at each point x^0 of the set Σ^u there exists an integer $N(x^0)\geqslant 2$ such that*

$$F_{x^0,u}(0^+)=N(x^0)$$

and

$$\frac{u(x^0+rx)}{\sqrt{r^{-1}\int_{\partial B_r(x^0)}u^2\,d\mathcal{H}^1}}\to\frac{\varrho^{N(x^0)}|\sin(N(x^0)\min\{\max\{\theta,0\},\pi\})|}{\sqrt{\int_0^\pi\sin^2(N(x^0)\theta)\,d\theta}}\quad\text{as } r\searrow 0,$$

strongly in $W^{1,2}_{\text{loc}}(B_1\setminus\{0\})$ and weakly in $W^{1,2}(B_1)$, where $x=(\varrho\cos\theta,\varrho\sin\theta)$.

Proof. Let $r_m\searrow 0$ be an arbitrary sequence such that the sequence v_m given by (8.1) converges weakly in $W^{1,2}(B_1)$ to a limit v_0. By Proposition 8.1 (iii) and Theorem 9.1, $v_0\not\equiv 0$, v_0 is homogeneous of degree $F_{x^0,u}(0^+)\geqslant\frac{3}{2}$, v_0 is continuous, $v_0\geqslant 0$, $v_0\equiv 0$ in $\{x_2\leqslant 0\}$, $v_0\Delta v_0=0$ in B_1 as a Radon measure, and the convergence of v_m to v_0 is strong in $W^{1,2}_{\text{loc}}(B_1\setminus\{0\})$. Moreover, the strong convergence of v_m and the fact proved in Proposition 8.1 (i) that $V(r_m)\to 0$ as $m\to\infty$ imply that

$$0=\int_{B_1}(|\nabla v_0|^2\operatorname{div}\phi-2\nabla v_0 D\phi\nabla v_0)\,dx$$

for every $\phi\in C_0^1(B_1\cap\{x_2>0\};\mathbf{R}^2)$. It follows that at each polar coordinate point $(1,\theta)\in\partial B_1\cap\partial\{v_0>0\}$,

$$\lim_{\tau\searrow\theta}\partial_\theta v_0(1,\tau)=-\lim_{\tau\nearrow\theta}\partial_\theta v_0(1,\tau).$$

Computing the solution of the ordinary differential equation on ∂B_1, using the homogeneity of degree $F_{x^0,u}(0^+)$ of v_0 and the fact that $\int_{\partial B_1} v_0^2\,d\mathcal{H}^1=1$, yields that $F_{x^0,u}(0^+)$ must be an *integer* $N(x^0)\geqslant 2$ and that

$$v_0(\varrho,\theta)=\frac{\varrho^{N(x^0)}|\sin(N(x^0)\min\{\max\{\theta,0\},\pi\})|}{\sqrt{\int_0^\pi \sin^2(N(x^0)\theta)\,d\theta}}. \tag{10.1}$$

The desired conclusion follows from Proposition 8.1 (ii). □

Theorem 10.2. *Let $n=2$ and let u be a variational solution of* (3.1). *Then the set Σ^u is locally in Ω a finite set.*

Proof. Suppose towards a contradiction that there is a sequence of points $x^m\in\Sigma^u$ converging to $x^0\in\Omega$, with $x^m\neq x^0$ for all m. The upper semicontinuity (Lemma 4.2 (iv)) implies that $x^0\in\Sigma^u$. Choosing $r_m:=2|x^m-x^0|$, there is no loss of generality in assuming that the sequence $(x^m-x^0)/r_m$ is constant, with value $z\in\{(-\frac{1}{2},0),(\frac{1}{2},0)\}$. Consider the blow-up sequence v_m given by (8.1), and also the sequence

$$u_m(x)=\frac{u(x^0+r_m x)}{r_m^{3/2}}.$$

Note that each u_m is a variational solution of (3.1), and v_m is a scalar multiple of u_m. Since $x^m\in\Sigma^u$, it follows that $z\in\Sigma^{u_m}$. Therefore, Lemma 6.4 shows that, for each m,

$$r\int_{B_r(z)}|\nabla v_m|^2\,dx\geqslant\frac{3}{2}\int_{\partial B_r(z)} v_m^2\,d\mathcal{H}^1\quad\text{for all } r\in\left(0,\frac{1}{2}\right).$$

Theorem 10.1 implies that the sequence v_m converges strongly in $W^{1,2}(B_{1/4}(z))$ to v_0 given by (10.1), and hence

$$r\int_{B_r(z)}|\nabla v_0|^2\,dx\geqslant\frac{3}{2}\int_{\partial B_r(z)} v_0^2\,d\mathcal{H}^1\quad\text{for all } r\in\left(0,\frac{1}{4}\right).$$

But this contradicts the fact, which can be checked directly, that

$$\lim_{r\searrow 0} r\frac{\int_{B_r(z)}|\nabla v_0|^2\,dx}{\int_{\partial B_r(z)} v_0^2\,d\mathcal{H}^1}=1.$$ □

11. Conclusion

THEOREM 11.1. *Let* $n=2$, *let* u *be a weak solution of* (3.1), *and suppose that*

$$|\nabla u|^2 \leqslant x_2^+ \quad \textit{in } \Omega.$$

Then the set S^u *of stagnation points is a finite or countable set. Each accumulation point of* S^u *is a point of the locally finite set* Σ^u.

At each point x^0 *of* $S^u \setminus \Sigma^u$,

$$\frac{u(x^0+rx)}{r^{3/2}} \to \frac{\sqrt{2}}{3} \varrho^{3/2} \cos\left(\frac{3}{2}\left(\min\left\{\max\left\{\theta, \frac{\pi}{6}\right\}, \frac{5\pi}{6}\right\} - \frac{\pi}{2}\right)\right) \quad \textit{as } r \searrow 0,$$

strongly in $W^{1,2}_{\mathrm{loc}}(\mathbf{R}^2)$ *and locally uniformly on* $\mathbf{R}^2$, *where* $x=(\varrho\cos\theta, \varrho\sin\theta)$. *Moreover,*

$$\mathcal{L}^2\left(B_1 \cap \left(\{x : u(x^0+rx) > 0\} \triangle \left\{x : \frac{\pi}{6} < \theta < \frac{5\pi}{6}\right\}\right)\right) \to 0 \quad \textit{as } r \searrow 0,$$

and, for each $\delta>0$,

$$r^{-3/2} \Delta u\left((x^0+B_r) \setminus \left\{x : \min\left\{\left|\theta - \frac{\pi}{6}\right|, \left|\theta - \frac{5\pi}{6}\right|\right\} < \delta\right\}\right) \to 0 \quad \textit{as } r \searrow 0.$$

At each point x^0 *of* Σ^u *there exists an integer* $N(x^0) \geqslant 2$ *such that*

$$\frac{u(x^0+rx)}{r^\beta} \to 0 \quad \textit{as } r \searrow 0,$$

strongly in $L^2_{\mathrm{loc}}(\mathbf{R}^2)$ *for each* $\beta \in [0, N(x^0))$, *and*

$$\frac{u(x^0+rx)}{\sqrt{r^{-1}\int_{\partial B_r(x^0)} u^2 \, d\mathcal{H}^1}} \to \frac{\varrho^{N(x^0)} |\sin(N(x^0) \min\{\max\{\theta, 0\}, \pi\})|}{\sqrt{\int_0^\pi \sin^2(N(x^0)\theta) \, d\theta}} \quad \textit{as } r \searrow 0,$$

strongly in $W^{1,2}_{\mathrm{loc}}(B_1 \setminus \{0\})$ *and weakly in* $W^{1,2}(B_1)$, *where* $x=(\varrho\cos\theta, \varrho\sin\theta)$.

Proof. By Lemma 3.4, u is a variational solution of (3.1) and satisfies

$$r^{-3/2} \int_{B_r(y)} \sqrt{x_2}\, |\nabla \chi_{\{u>0\}}| \, dx \leqslant C_0$$

for all $B_r(y) \Subset \Omega$ such that $y_n=0$. Combining Proposition 5.4, Lemmas 5.3 and 4.4, Proposition 5.5, Lemma 7.5 and Theorems 10.2 and 10.1, we obtain that the set S^u is a finite or countable set with asymptotics as in the statement, and that the only possible accumulation points are elements of Σ^u. □

THEOREM 11.2. *Let $n=2$, let u be a weak solution of* (3.1) *and suppose that*

$$|\nabla u|^2 \leqslant x_2^+ \quad in\ \Omega.$$

Suppose moreover that $\{u=0\}$ has locally only finitely many connected components. Then the set S^u of stagnation points is locally in Ω a finite set. At each stagnation point x^0,

$$\frac{u(x^0+rx)}{r^{3/2}} \to \frac{\sqrt{2}}{3}\varrho^{3/2}\cos\left(\frac{3}{2}\left(\min\left\{\max\left\{\theta,\frac{\pi}{6}\right\},\frac{5\pi}{6}\right\}-\frac{\pi}{2}\right)\right) \quad as\ r\searrow 0,$$

strongly in $W^{1,2}_{\mathrm{loc}}(\mathbf{R}^2)$ and locally uniformly on $\mathbf{R}^2$, where $x=(\varrho\cos\theta,\varrho\sin\theta)$, and in an open neighborhood of x^0 the topological free boundary $\partial\{u>0\}$ is the union of two C^1-graphs with right and left tangents at x^0.

Proof. We first show that the set Σ^u is empty. Suppose towards a contradiction that there exists $x^0\in\Sigma^u$. From Theorem 10.1 we infer that there exists an integer $N(x^0)\geqslant 2$ such that

$$\frac{u(x^0+rx)}{\sqrt{r^{-1}\int_{\partial B_r(x^0)} u^2\,d\mathcal{H}^1}} \to \frac{|\sin(N(x^0)\min\{\max\{\theta,0\},\pi\})|}{\sqrt{\int_0^\pi \sin^2(N(x^0)\theta)\,d\theta}} \quad \text{as } r\searrow 0,$$

strongly in $W^{1,2}_{\mathrm{loc}}(B_1\setminus\{0\})$ and weakly in $W^{1,2}(B_1)$, where $x=(\varrho\cos\theta,\varrho\sin\theta)$. But then the assumption about connected components implies that $\partial_{\mathrm{red}}\{x:u(x^0+rx)>0\}$ contains the image of a continuous curve converging, as $r\searrow 0$, locally in $\{x_2>0\}$ to a half-line $\{\alpha z:\alpha>0\}$ where $z_2>0$. It follows that

$$\mathcal{H}^1\left(\left\{x_2>\tfrac{1}{2}\right\}\cap\partial_{\mathrm{red}}\{x:u(x^0+rx)>0\}\right)\geqslant c_1>0,$$

contradicting

$$0\leftarrow \Delta\frac{u(x^0+rx)}{r^{3/2}}(B_1)=\int_{B_1\cap\partial_{\mathrm{red}}\{x:u(x^0+rx)>0\}}\sqrt{x_2}\,d\mathcal{H}^1.$$

Hence Σ^u is indeed empty.

Let $x^0\in S^u$. Theorem 11.1 shows that

$$\frac{u(x^0+rx)}{r^{3/2}} \to \frac{\sqrt{2}}{3}\varrho^{3/2}\cos\left(\frac{3}{2}\left(\min\left\{\max\left\{\theta,\frac{\pi}{6}\right\},\frac{5\pi}{6}\right\}-\frac{\pi}{2}\right)\right) \quad \text{as } r\searrow 0,$$

strongly in $W^{1,2}_{\mathrm{loc}}(\mathbf{R}^2)$ and locally uniformly on $\mathbf{R}^2$, where $x=(\varrho\cos\theta,\varrho\sin\theta)$. To prove the last statement we use flatness-implies-regularity results in the vein of [2, Theorem 8.1]. More precisely, for each $\sigma\leqslant\sigma_0$ and $y^0\in B_\delta(x^0)\cap\partial\{u>0\}\cap\{y^0_1<x^0_1\}$, $u\in F(\sigma,0;\sigma_0\sigma^2)$ in

$B_{r/2}(y^0)$ in the direction $\eta = \left(-\frac{1}{2}, -\frac{1}{2}\sqrt{3}\right)$ (cf. [5, Definition 4.1]) provided that δ has been chosen small enough, meaning that u is a weak solution and satisfies

$$u(x) = 0 \quad \text{in } \{x \in B_r(y^0) : x \cdot \eta \geqslant \sigma r\}$$

and

$$|\nabla u| \leqslant \sqrt{y_2^0}(1+\sigma_0\sigma^2) \quad \text{in } B_r(y^0).$$

From the proof of [5, Theorem 8.4] (with the proviso that the parabolic monotonicity formula in [5] is replaced by the local elliptic formula in Theorem 3.5 (i) and the solution has been extended to a constant function of the time variable) we infer that

$$B_{r/2}(x^0) \cap \partial\{u > 0\} \cap \{y_1^0 < x_1^0\}$$

is the graph of a $C^{1,\alpha}$-function and that the outer normal ν satisfies $|\nu(y^0) - \eta| \leqslant \sigma$. It follows that $B_\delta(x^0) \cap \partial\{u>0\} \cap \{y_1^0 \leqslant x_1^0\}$ is the graph of a C^1-function. The same holds for $B_\delta(x^0) \cap \partial\{u>0\} \cap \{y_1^0 \geqslant x_1^0\}$. □

12. Appendix

Proof of Lemma 3.4. For any $\phi \in C_0^1(\Omega \cap \{x_n > \tau\}; \mathbf{R}^n)$ and a small positive δ we find a covering

$$\bigcup_{i=1}^{\infty} B_{r_i}(x^i) \supset \operatorname{supp} \phi \cap (\partial\{u > 0\} \setminus \partial_{\text{red}}\{u > 0\})$$

satisfying $\sum_{i=1}^{\infty} r_i^{n-1} \leqslant \delta$, and by the fact that $\operatorname{supp} \phi \cap (\partial\{u>0\} \setminus \partial_{\text{red}}\{u>0\})$ is a compact set we may pass to a finite subcovering

$$\bigcup_{i=1}^{N_\delta} B_{r_i}(x^i) \supset \operatorname{supp} \phi \cap (\partial\{u > 0\} \setminus \partial_{\text{red}}\{u > 0\})$$

satisfying $\sum_{i=1}^{N_\delta} r_i^{n-1} \leqslant \delta$.

We also know that $u \in C^1\left(\overline{\{u>0\}} \cap \left(\operatorname{supp} \phi \setminus \bigcup_{i=1}^{N_\delta} B_{r_i}(x^i)\right)\right)$ and that u satisfies the free-boundary condition

$$|\nabla u|^2 = x_n \quad \text{on } \partial_{\text{red}}\{u > 0\} \cap \left(\operatorname{supp} \phi \setminus \bigcup_{i=1}^{N_\delta} B_{r_i}(x^i)\right).$$

Formally integrating by parts in $\{u>0\} \setminus \bigcup_{i=1}^{N_\delta} B_{r_i}(x^i)$ (this can be justified rigorously approximating $\bigcup_{i=1}^{N_\delta} B_{r_i}(x^i)$ from above by A_ε such that $\partial(\{u>0\} \setminus A_\varepsilon)$ is locally in supp ϕ

a C^3-surface) we therefore obtain

$$\begin{aligned}
&\left|\int_\Omega (|\nabla u|^2 \operatorname{div}\phi - 2\nabla u D\phi \nabla u + x_n \chi_{\{u>0\}} \operatorname{div}\phi + \chi_{\{u>0\}}\phi_n)\,dx\right| \\
&\quad \leqslant \left|\int_{\bigcup_{i=1}^{N_\delta} B_{r_i}(x^i)} (|\nabla u|^2 \operatorname{div}\phi - 2\nabla u D\phi \nabla u + x_n \chi_{\{u>0\}} \operatorname{div}\phi + \chi_{\{u>0\}}\phi_n)\,dx\right| \\
&\qquad + \left|\int_{\partial\left(\bigcup_{i=1}^{N_\delta} B_{r_i}(x^i)\right)\cap\{u>0\}} (|\nabla u|^2 \phi\cdot\nu - 2\nabla u\cdot\nu \nabla u\cdot\phi + x_n \phi\cdot\nu)\,d\mathcal{H}^{n-1}\right| \\
&\qquad + \left|\int_{\partial\{u>0\}\setminus\bigcup_{i=1}^{N_\delta} B_{r_i}(x^i)} (x_n - |\nabla u|^2)\phi\cdot\nu\,d\mathcal{H}^{n-1}\right| \\
&\quad \leqslant C_1 \sum_{i=1}^{N_\delta} {r_i}^n + C_2 \sum_{i=1}^{N_\delta} {r_i}^{n-1} + 0,
\end{aligned}$$

and letting $\delta\to 0$, we realize that u is a variational solution of (3.1) in the set $\Omega\cap\{x_n>\tau\}$. Note that the above extends to Lipschitz functions ϕ. Next, let us take $\phi\in C_0^1(\Omega;\mathbf{R}^n)$ and $\eta:=\min\{1, x_n/\tau\}$, plug in the product $\eta\phi$ into the already obtained result, and use the assumption $|\nabla u|^2 \leqslant C x_n^+$,

$$\begin{aligned}
0 &= \int_\Omega \eta(|\nabla u|^2 \operatorname{div}\phi - 2\nabla u D\phi \nabla u + x_n \chi_{\{u>0\}} \operatorname{div}\phi + \chi_{\{u>0\}}\phi_n)\,dx \\
&\quad + \frac{1}{\tau}\int_{\Omega\cap\{0<x_n<\tau\}} \phi\cdot(|\nabla u|^2 e_n - 2\partial_n u \nabla u + x_n \chi_{\{u>0\}} e_n)\,dx \\
&= o(1) + \int_\Omega (|\nabla u|^2 \operatorname{div}\phi - 2\nabla u D\phi \nabla u + x_n \chi_{\{u>0\}} \operatorname{div}\phi + \chi_{\{u>0\}}\phi_n)\,dx \quad \text{as } \tau\to 0.
\end{aligned}$$

Last, let us prove that

$$r^{1/2-n}\int_{B_r(y)} \sqrt{x_n}\,|\nabla \chi_{\{u>0\}}|\,dx \leqslant C_0$$

for all $B_r(y)\Subset\Omega$ such that $y_n=0$. Let us consider such a y, and the family of scaled functions

$$u_r(x) := \frac{u(y+rx)}{r^{3/2}}.$$

Using the assumption

$$|\nabla u|^2 \leqslant C x_n^+ \quad \text{locally in } \Omega$$

and the weak solution property, it follows that

$$\begin{aligned}
C_0 &\geqslant \int_{\partial B_1} \nabla u_r(x)\cdot x\,d\mathcal{H}^{n-1} = \Delta u_r(B_1) \\
&= \int_{B_1\cap\partial_{\mathrm{red}}\{u_r>0\}} \sqrt{x_n}\,d\mathcal{H}^{n-1} = r^{1/2-n}\int_{B_r(y)\cap\partial_{\mathrm{red}}\{u>0\}} \sqrt{x_n}\,d\mathcal{H}^{n-1},
\end{aligned}$$

as required. □

Proof of Proposition 5.8. The proof is a standard dimension reduction argument following [15, §11]. In each step, blowing up once transforms the free boundary into a cone, blowing up a second time at a point different from the origin transforms the free boundary into a cylinder, and passing to a codimension-1 cylinder section reduces the dimension of the whole problem by 1.

Let us do this in some more detail: Suppose that there exists $s>n-2$, $\varsigma>0$ and $\tau>0$ such that $\mathcal{H}^s(N^u_{\varsigma,\tau})>0$. Then we may use [15, Proposition 11.3], Lemma 5.7 as well as [15, Lemma 11.5] at $\mathcal{H}^s$-a.e. point of $N^u_{\varsigma,\tau}$ to obtain a blow-up limit u_0 satisfying $\mathcal{H}^{s,\infty}(N^{u_0}_{\varsigma,\tau})>0$. According to Lemma 4.1, u_0 is a homogeneous variational solution on $\mathbf{R}^n$, where $\chi_{\{u>0\}}$ has to be replaced by $\chi_0:=\lim_{m\to\infty}\chi_{\{u_m>0\}}$ in Definition 3.1. We proceed with the dimension reduction: By [15, Lemma 11.2] we find a point $\bar{x}\in N^{u_0}_{\varsigma,\tau}\setminus\{0\}$ at which the density in [15, Proposition 11.3] is estimated from below. Now each blow-up limit u_{00} with respect to $\bar{x}$ (and with respect to a subsequence $m\to\infty$ such that the limit superior in [15, Proposition 11.3] becomes a limit) again satisfies the assumptions of Lemma 4.1. In addition, we obtain from the homogeneity of u_{00} as in [36, Lemma 3.1] that the rotated u_{00} is constant in the direction of the nth unit vector. Defining $\bar{u}$ as the restriction of this rotated solution to $\mathbf{R}^{n-1}$, it follows therefore that $\mathcal{H}^{s-1}(N^{\bar{u}}_{\varsigma,\tau})>0$. Repeating the whole procedure $n-2$ times, we obtain a non-trivial homogeneous solution $u^\star$ in $\mathbf{R}^2$, satisfying $\mathcal{H}^{s-(n-2)}(N^{u^\star}_{\varsigma,\tau})>0$, by Proposition 4.7 a contradiction. □

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