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There are no C^1-stable intersections of regular Cantor sets

by

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1. Introduction

The existence of stable intersections of regular Cantor sets is a fundamental tool to provide persistent examples of non-hyperbolic C^2 diffeomorphisms of surfaces, as did Newhouse [N], by means of the concept of thickness of a Cantor set. The thickness is a fractal invariant, which is continuous and positive in the C^2 (or even in the $C^{1+\alpha}$, $0<\alpha<1$) topology, such that, if the product of the thicknesses of two regular Cantor sets is larger than 1 and their support intervals intersect in a non-trivial way, then they have stable intersection (see [PT]). However Ures [U] showed that, in the C^1 topology, the thickness of regular Cantor sets is terribly discontinuous. Indeed, generic C^1-regular Cantor sets have zero thickness. He also showed that two regular Cantor sets whose support intervals touch at one point cannot have extremal stable intersection (in the sense of [M]) in the C^1 topology, and he used these results to show that C^1-generic first homoclinic bifurcations present full upper density of hyperbolicity at the initial bifurcation parameter.

However, despite the discontinuity of the thickness in the C^1 topology, the Hausdorff dimension of regular Cantor sets is continuous and positive in the C^1 topology (and coincides with the limit capacity). On the other hand, it was showed in [MY] that generic pairs of regular Cantor sets in the C^2 (or $C^{1+\alpha}$) topology whose sum of Hausdorff dimensions is larger than 1 have translations which have stable intersection. Moreover, they have translations whose intersections have stably positive Hausdorff dimensions. This poses a more difficult problem: is it always possible to destroy intersections of regular Cantor sets by performing arbitrarily small C^1 perturbations of them? The situation is particularly delicate when the intersection between the Cantor sets has positive Hausdorff dimension, which is a typical situation in the C^2 topology as seen before. We solve this problem in the following theorem.

THEOREM 1. *Given any pair* (K,K') *of regular Cantor sets, we can find, arbitrarily close to it in the* C^1 *topology, pairs* $(\widetilde{K},\widetilde{K}')$ *of regular Cantor sets with* $\widetilde{K}\cap\widetilde{K}'=\varnothing$.

Moreover, for generic pairs (K,K') *of* C^1*-regular Cantor sets, the arithmetic difference* $K-K'=\{x-y:x\in K \text{ and } y\in K'\}=\{t\in\mathbb{R}:K\cap(K'+t)\neq\varnothing\}$ *has empty interior (and so is a Cantor set).*

This answers a question by Christian Bonatti.

Since stable intersections of Cantor sets are the main known obstructions to density of hyperbolicity for diffeomorphisms of surfaces, the previous result gives some hope of proving density of hyperbolicity in the C^1 topology for diffeomorphisms of surfaces. In particular it is used in a forthcoming joint work with Carlos Matheus and Enrique Pujals on a family of 2-dimensional maps (the so-called Benedicks–Carleson *toy model* for Hénon dynamics) which present stable homoclinic tangencies (Newhouse's phenomenon) in the C^2 topology but whose elements can be arbitrarily well approximated in the C^1 topology by hyperbolic maps.

The main technical difference between the C^1 case and the C^2 (or even $C^{1+\alpha}$) cases is the lack of bounded distortion of the iterates of ψ in the C^1 case, and this fact will be fundamental for the proof of the previous result.

The previous result may be used to show that there are no C^1-robust tangencies between leaves of the stable and unstable foliations of respectively two given hyperbolic horseshoes Λ_1 and Λ_2 of a diffeomorphism of a surface. This is also very different from the situation in the C^∞ topology—for instance, in [MY2] it is proved that, in the unfolding of a homoclinic or heteroclinic tangency associated with two horseshoes, when the sum of the corresponding stable and unstable Hausdorff dimensions is larger than 1, there are generically stable tangencies associated with these two horseshoes.

THEOREM 2. *Given a* C^1 *diffeomorphism* ψ *of a surface* M *having two (not necessarily disjoint) horseshoes* Λ_1 *and* Λ_2, *we can find, arbitrarily close to it in the* C^1 *topology, a diffeomorphism* $\tilde{\psi}$ *of the surface for which the horseshoes* Λ_1 *and* Λ_2 *have hyperbolic continuations* $\tilde{\Lambda}_1$ *and* $\tilde{\Lambda}_2$, *and there are no tangencies between leaves of the stable and unstable foliations of* $\tilde{\Lambda}_1$ *and* $\tilde{\Lambda}_2$, *respectively. Moreover, there is a generic set* $\mathcal{R}$ *of* C^1 *diffeomorphisms of* M *such that, for every* $\check{\psi}\in\mathcal{R}$, *there are no tangencies between leaves of the stable and unstable foliations of* Λ_1 *and* Λ_2, *for any horseshoes* Λ_1 *and* Λ_2 *of* $\check{\psi}$.

Remark. In the previous statement, by *horseshoe* we mean a compact, locally maximal, hyperbolic invariant set of saddle type.

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2. Proofs of the results

We recall that K is a C^k-*regular Cantor set*, $k\geqslant 1$, if the following properties hold:

(i) there are disjoint compact intervals $I_1, I_2, ..., I_r$ such that $K\subset I_1\cup...\cup I_r$ and the boundary of each I_j is contained in K;

(ii) there is a C^k expanding map ψ defined in a neighborhood of $I_1\cup I_2\cup...\cup I_r$ such that $\psi(I_j)$ is the convex hull of a finite union of some intervals I_s satisfying

(ii.1) for each j, $1\leqslant j\leqslant r$, and n sufficiently big, $\psi^n(K\cap I_j)=K$;

(ii.2) $K=\bigcap_{n\in\mathbb{N}}\psi^{-n}(I_1\cup I_2\cup...\cup I_r)$.

We say that $\{I_1, I_2, ..., I_r\}$ is a *Markov partition* for K and that K is *defined* by ψ.

An *interval of the construction of the regular Cantor set* K is a connected component of $\psi^{-n}(I_j)$ for some $n\in\mathbb{N}$, $j\leqslant r$.

Remark. We gave the definition of regular Cantor set from [PT]. There are alternative definitions of regular Cantor sets. For instance, in [MY] they are defined as follows:

Let $\mathbb{A}$ be a finite alphabet, $\mathcal{B}$ a subset of $\mathbb{A}^2$, and Σ the subshift of finite type of $\mathbb{A}^{\mathbb{Z}}$ with allowed transitions $\mathcal{B}$ which is topologically mixing, and such that every letter in $\mathbb{A}$ occurs in Σ. An *expansive map of type* Σ is a map g with the following properties:

(i) the domain of g is a disjoint union $\bigcup_{\mathcal{B}} I(a,b)$, where, for each (a,b), $I(a,b)$ is a compact subinterval of $I(a):=[0,1]\times\{a\}$;

(ii) for each $(a,b)\in\mathcal{B}$, the restriction of g to $I(a,b)$ is a smooth diffeomorphism onto $I(b)$ satisfying $|Dg(t)|>1$ for all t.

The regular Cantor set associated with g was defined as the maximal invariant set

$$K=\bigcap_{n\geqslant 0} g^{-n}\biggl(\bigcup_{\mathcal{B}} I(a,b)\biggr).$$

These two definitions are equivalent. On one hand, we may, in the first definition, take $I(i):=I_i$ for each $i\leqslant r$, and, for each pair i,j such that $\psi(I_i)\supset I_j$, take $I(i,j)=I_i\cap\psi^{-1}(I_j)$. Conversely, in the second definition, we could consider an abstract line containing all intervals $I(a)$ as subintervals, and $\{I(a,b),(a,b)\in\mathcal{B}\}$ as the Markov partition.

Given $s\in[1,k]$ and another regular Cantor set $\widetilde{K}$, we say that $\widetilde{K}$ is close to K in the C^s topology if $\widetilde{K}$ has a Markov partition $\{\tilde{I}_1, \tilde{I}_2, ..., \tilde{I}_r\}$ such that the interval $\tilde{I}_j$ has endpoints close to the endpoints of I_j, for $1\leqslant j\leqslant r$, and $\widetilde{K}$ is defined by a C^s map $\tilde{\psi}$ which is close to ψ in the C^s topology.

Given a C^1-regular Cantor set K, we define the parameter

$$\lambda(K)=\max\left\{|\psi'(x)|:x\in\bigcup_{j=1}^{r}I_j\right\}>0,$$

which depends continuously on K in the C^1 topology.

We may associate with each $j\leqslant r$ a gap $U_j\subset I_j$ of K, which is determined by the combinatorics of (K,ψ) in the following way: we take the smallest $m_j\geqslant 1$ such that $\psi^{m_j}(I_j)$ is the convex hull of the union of more than one interval of the Markov partition of K, say of $I_{s_j}\cup I_{s_j+1}\cup\ldots\cup I_{s_j+l_j}$, and, if V_j is the gap between I_{s_j} and I_{s_j+1}, we set $U_j:=\psi^{-m_j}(V_j)$. We define a parameter $b(K)$ in the following way: given $n\in\mathbb{N}$, $j\leqslant r$ and a connected component $\tilde{J}$ of $\psi^{-n}(I_j)$, we define $b(\tilde{J})=|U|/|\tilde{J}|$, where $U\subset\tilde{J}$ is the gap of K such that $\psi^n(U)=U_j$, and we define

$$b(K)=\inf\{b(\tilde{J}):\tilde{J}\text{ is a connected component of }\psi^{-n}(I_j)\text{ for some }n\in\mathbb{N}\text{ and }j\leqslant r\}.$$

We also define a similar parameter $g(K)$ as follows: given $n\in\mathbb{N}$, $j\leqslant r$ and a connected component $\tilde{J}$ of $\psi^{-n}(I_j)$, we define $L_{\tilde{J}}$ and $R_{\tilde{J}}$ as the gaps of K attached to the left and right endpoints of $\tilde{J}$, respectively. We set $g(\tilde{J})=\min\{|L_{\tilde{J}}|/|\tilde{J}|,|R_{\tilde{J}}|/|\tilde{J}|\}$, and we define

$$g(K)=\inf\{g(\tilde{J}):\tilde{J}\text{ is a connected component of }\psi^{-n}(I_j)\text{ for some }n\in\mathbb{N}\text{ and }j\leqslant r\}.$$

Note that $b(K)>0$ and $g(K)>0$. Finally, we define $a(K)=\min\{b(K),g(K)\}$. Notice that $b(K)$, $g(K)$ and $a(K)$ depend continuously on K in the $C^{3/2}$ topology. This follows (as the continuity of other fractal invariants, like the thickness) from the *distortion lemma* for regular Cantor sets, which is the following result (see for instance [PT]).

DISTORTION LEMMA. *Let $K\subset\mathbb{R}$ be a dynamically defined Cantor set with expanding map ψ of class $C^{1+\varepsilon}$, for some $\varepsilon>0$. Then, given $\delta>0$, there is $c(\delta)>0$, which converges to 0 when $\delta\to 0$, such that, for all q, $\tilde{q}$ and $n\geqslant 1$ with $|\psi^n(q)-\psi^n(\tilde{q})|\leqslant\delta$ for which the interval with endpoints $\psi^j(q)$ and $\psi^j(\tilde{q})$ is contained in the domain of ψ for $0\leqslant j\leqslant n-1$, we have $\bigl|\log|(\psi^n)'(q)|-\log|(\psi^n)'(\tilde{q})|\bigr|\leqslant c(\delta)$.*

In the next lemma we will exploit the lack of bounded distortion (using the idea of the construction of Ures in [U], in a more global way) in order to produce a very distorted geometry (with very large gaps) near some subsets of a C^2-regular Cantor set, by a small perturbation in the C^1 topology.

LEMMA 1. *Let K be a C^2-regular Cantor set. Let $c(K)=2\lambda(K)/a(K)$. Then, given $\varepsilon>0$, let $n_0=\lceil c(K)\varepsilon^{-1}\log\varepsilon^{-1}\rceil$. Suppose that $X\subset K$ is a compact set satisfying $\psi^i(X)\cap\psi^j(X)=\varnothing$ for $0\leqslant i<j\leqslant n_0$. Then, for any $\delta>0$, we can find a covering of K formed by intervals J_i of its construction which have size smaller than δ satisfying the following properties*:

Let D be the union of the intervals J_i for all i and the intervals $\psi^j(J_i)$, $1\leqslant j\leqslant n_0$, for the intervals J_i which intersect X. There is a Cantor set $\widetilde{K}$ in the ε-neighborhood of K in the C^1 topology, with $a(\widetilde{K})\geqslant a(K)$, such that all connected components of D are still intervals of the construction of $\widetilde{K}$ and such that $\widetilde{K}\cap J_i$ has a gap V_i with $|V_i|\geqslant(1-\varepsilon)|J_i|$ whenever $J_i\cap X\neq\varnothing$.

Proof. Let $A=\{i:J_i\cap X\neq\varnothing\}$, where $\{J_i\}_i$ is the convering of K by the connected components of

$$\psi^{-N}\left(\bigcup_{j=1}^{r} I_j\right),$$

for some large N, so that, in particular, $|J_i|<\delta$ for all i.

Since X is compact, if N is large enough, then we have, by the hypothesis of the lemma, that $\psi^i(\widetilde{X})\cap\psi^j(\widetilde{X})=\varnothing$ for $0\leqslant i<j\leqslant n_0$, where

$$\widetilde{X}:=\bigcup_{i\in A} J_i.$$

We may perform, as in [M, Lemma II.2.1], a small change in ψ in the $C^{3/2}$ topology in such a way that the restrictions of ψ to the intervals $\psi^j(J_i)$ with $i\in A$, $0\leqslant j<n_0$, become affine; we change ψ just in these intervals and in the gaps attached to them.

Now we will make small C^1 perturbations on the restriction of ψ to the intervals $\psi^j(J_i)$ with $i\in A$, $0\leqslant j<n_0$. We will begin changing ψ in the intervals $\psi^{n_0-1}(J_i)$, then in the intervals $\psi^{n_0-2}(J_i)$ and so on, in order to make the proposition of a gap in each of these intervals grow in such a may that the size of each of the two remaining intervals is multiplied by $1-2a\varepsilon/3\lambda$, where $\lambda=\lambda(K)$ and $a=a(K)$.

More precisely, if $\psi^j(J_i)=[r,s]$, $i\in A$, $0\leqslant j<n_0$, is some of these intervals and m is such that $\psi^m(J_i)=I_l$, let $\widetilde{U}\subset J_i$ be such that $\psi^m(\widetilde{U})=U_l$ and let $\psi^j(\widetilde{U})=(u,v)\subset[r,s]$. Writing $\psi|_{[r,s]}(x)=\tilde{\lambda}x+t$, we consider the affine map $\tilde{\psi}|_{[r,s]}$ given by

$$\tilde{\psi}|_{[r,s]}(x)=\begin{cases}\tilde{\lambda}\left(1-\dfrac{2a\varepsilon}{3\lambda}\right)^{-1}(x-r)+\tilde{\lambda}r+t, & \text{if } x\in\left[r,r+\left(1-\dfrac{2a\varepsilon}{3\lambda}\right)(u-r)\right],\\ \tilde{\lambda}\left(1-\dfrac{2a\varepsilon}{3\lambda}\right)^{-1}(x-s)+\tilde{\lambda}s+t, & \text{if } x\in\left[s-\left(1-\dfrac{2a\varepsilon}{3\lambda}\right)(s-v),s\right],\end{cases}$$

and we extend $\tilde{\psi}$ to $[r,s]$ in such a way that $\tilde{\psi}|_{[r,s]}$ is a C^1 function. Notice that the image

$$\tilde{\psi}\left(\left(r+\left(1-\frac{2a\varepsilon}{3\lambda}\right)(u-r), s-\left(1-\frac{2a\varepsilon}{3\lambda}\right)(s-v)\right)\right)=\psi((u,v))$$

of the gap remains the same. The size of the new gap in $[r,s]$ is

$$v-u+\frac{2a\varepsilon}{3\lambda}(s-v+u-r)<v-u+\frac{2a\varepsilon}{3\lambda}(s-r)<\left(1+\frac{2\varepsilon}{3\lambda}\right)(v-u).$$

In particular, it is not difficult to see that we may construct such a function $\tilde{\psi}$ with $\|\tilde{\psi}-\psi\|_{C^1}<\varepsilon$.

Finally, the total proportion of the complement of the new gap $V_i=\widetilde{U}$ for the modified ψ (indeed, $\tilde{\psi}|_{J_i}^{-n_0}(\psi^{n_0}(\widetilde{U}))$) is at most

$$\left(1-\frac{2a\varepsilon}{3\lambda}\right)^{n_0}(1-a)\leqslant\left(1-\frac{2a\varepsilon}{3\lambda}\right)^{(2\lambda/a\varepsilon)\log\varepsilon^{-1}}(1-a)<\varepsilon^{4/3}<\varepsilon.$$

It is not difficult to see that after these perturbations we will also have $a(\widetilde{K})\geqslant a(K)$ (indeed in the non-affine part of the dynamics we are only increasing the proportion of some gaps, and in the affine and local part of the dynamics the proportion of the gaps is preserved), which concludes the proof of the lemma. □

LEMMA 2. *Given a C^2-regular Cantor set K', for a residual set of C^2-regular Cantor sets K, if $k=\lfloor(1-\dim_{\mathrm{H}}(K'))^{-1}\rfloor+1$ then $\bigcap_{j=1}^k A_j=\varnothing$, where $A_j=F_j((K\cap K')\cap P_j)$, $1\leqslant j\leqslant k$, and (F_j,P_j), $1\leqslant j\leqslant k$, are distinct elements of*

$$\{(\psi^r|_I, I): r\in\mathbb{N} \text{ and } I \text{ is a maximal interval of the construction of } K \text{ where } \psi^r \text{ is injective}\}.$$

Here $\dim_{\mathrm{H}}$ denotes the Hausdorff dimension.

Proof. It is enough to show that, for generic C^2-regular Cantor sets K, given distinct sets $A_j=\psi^{r_j}((K\cap K')\cap P_j), 1\leqslant j\leqslant k$, where P_j is a maximal interval of the construction of K such that $\psi^{r_j}|_{P_j}$ is injective, we have $\bigcap_{j=1}^k\psi^{r_j}(K\cap K'\cap P_j)=\varnothing$. So we will, from now on, fix the branches $\psi^{r_j}|_{P_j}$ that we will consider. This is possible since the perturbations of the maps which define K that we will perform are all topologically conjugate, and so the continuations of the branches are well defined.

Typically, $K\cap K'$ does not contain any preperiodic point of ψ (indeed, there is only a countable number of them in K, and so almost all translations of K' do not intersect them). In this case, given a point $x\in\bigcap_{j=1}^k A_j$, with $x=\psi^{r_j}(y_j)$, $1\leqslant j\leqslant k$, with $y_j\in K\cap K'\cap P_j$ for all $j\leqslant k$, we have that the points y_j are all distinct (indeed, if $y_i=y_j$

with $i\neq j$ then $r_i\neq r_j$, since A_i and A_j are distinct, so x is periodic and the points $y_i, y_j\in K\cap K'$ are preperiodic). We may define a partial order of the points y_j in the following way: we say that $y_i\ll y_j$ if there is s with $0<s<r_i$ and $\psi^s(y_i)=y_j$. Since the points y_j are not periodic, this order has no cycles, and so can be extended to a total order, which will still be denoted by $\ll$. Assume without loss of generality that $y_1\ll y_2\ll ...\ll y_k$. We may find disjoint intervals $Q_j(x)\subset P_j$ of the construction of K with $y_j\in Q_j(x), 1\leqslant j\leqslant k$, such that, if $i<j$ and $0<s<r_j$, then $\psi^s(Q_j(x))\cap Q_i(x)=\varnothing$. We may now take $\varepsilon_x>0$ and a neighborhood $\mathcal{V}_x$ of K in the C^1 topology such that, for $\widetilde{K}\in\mathcal{V}_x$, if $\widetilde{P}_j$, $\widetilde{Q}_j(x)$ and $\tilde{\psi}$ denote the continuations of the intervals P_j, $Q_j(x)$ and of the map ψ which defines K, respectively, we have

$$\bigcap_{j=1}^{k}\tilde{\psi}^{r_j}((\widetilde{K}\cap K')\cap\widetilde{P}_j)\cap(x-\varepsilon_x, x+\varepsilon_x)\subset\bigcap_{j=1}^{k}\tilde{\psi}^{r_j}((\widetilde{K}\cap K')\cap\widetilde{Q}_j(x)),$$

and, if $i<j$ and $0<s<r_j$, then $\tilde{\psi}^s(\widetilde{Q}_j(x))\cap\widetilde{Q}_i(x)=\varnothing$. We may take a finite covering of the compact set $\bigcap_{j=1}^k A_j$ by intervals $(x_i-\varepsilon_{x_i}, x_i+\varepsilon_{x_i})$, $1\leqslant i\leqslant m$. We denote by $\widetilde{\mathcal{V}}$ the neighborhood $\bigcap_{i=1}^m\mathcal{V}_{x_i}$ of K in the C^1 topology. We will assume (by reducing $\widetilde{\mathcal{V}}$, if necessary) that, for any $\widetilde{K}\in\widetilde{\mathcal{V}}$,

$$\bigcap_{j=1}^{k}\tilde{\psi}^{r_j}((\widetilde{K}\cap K')\cap\widetilde{P}_j)\subset\bigcup_{i=1}^{m}(x_i-\varepsilon_{x_i}, x_i+\varepsilon_{x_i}),$$

so

$$\bigcap_{j=1}^{k}\tilde{\psi}^{r_j}((\widetilde{K}\cap K')\cap\widetilde{P}_j)\subset\bigcup_{i=1}^{m}\bigcap_{j=1}^{k}\tilde{\psi}^{r_j}((\widetilde{K}\cap K')\cap\widetilde{Q}_j(x_i)).$$

It is enough to show that, for each $i\leqslant m$, there is an open and dense set $\widetilde{\mathcal{V}}_i\subset\widetilde{\mathcal{V}}$ such that, for any $\widetilde{K}\in\widetilde{\mathcal{V}}_i$, we have

$$\bigcap_{j=1}^{k}\tilde{\psi}^{r_j}((\widetilde{K}\cap K')\cap\widetilde{Q}_j(x_i))=\varnothing.$$

Since the sets $\psi^{r_j}((\widetilde{K}\cap K')\cap\widetilde{Q}_j(x_i))$ are compact, the above condition is clearly open, so it is enough to prove that it is dense in $\widetilde{\mathcal{V}}$.

Now, since the intervals $\widetilde{Q}_j(x_i)$, $1\leqslant j\leqslant k$, are disjoint, and, if $i<j$ and $0<s<r_j$, then $\tilde{\psi}^s(\widetilde{Q}_j(x))\cap\widetilde{Q}_i(x)=\varnothing$, we may consider families of perturbations $\widetilde{K}_{t_1,...,t_k}\in\widetilde{\mathcal{V}}$ of K defined by maps $\tilde{\psi}_{t_1,...,t_k}$ which form a family of perturbations of $\tilde{\psi}$ depending on k small parameters $t_1, t_2, ..., t_k\in(-\delta,\delta)$, for some (small) $\delta>0$ for which

$$\tilde{\psi}_{t_1,...,t_k}^{r_j}|_{\widetilde{Q}_j(x_i)}=\tilde{\psi}^{r_j}|_{\widetilde{Q}_j}+t_j \quad \text{for all } t_1, t_2, ..., t_k\in(-\delta,\delta)$$

(it is enough to make suitable perturbations of $\tilde{\psi}$ in the disjoint intervals $\widetilde{Q}_j(x_i)$, $1\leqslant j\leqslant k$: we first perturb it in $\widetilde{Q}_k(x_i)$ in order to have

$$\tilde{\psi}^{r_k}_{t_k}|_{\widetilde{Q}_k(x_i)} = \tilde{\psi}^{r_k}|_{\widetilde{Q}_k} + t_k \quad \text{for all } t_k \in (-\delta, \delta),$$

then we perturb it in $\widetilde{Q}_{k-1}(x_i)$ in order to have

$$\tilde{\psi}^{r_{k-1}}_{t_{k-1},t_k}|_{\widetilde{Q}_{k-1}}(x_i) = \tilde{\psi}^{r_{k-1}}|_{\widetilde{Q}_{k-1}} + t_{k-1} \quad \text{for all } t_{k-1}, t_k \in (-\delta, \delta),$$

and so on).

Since the limit capacities (box dimensions) of regular Cantor sets coincide with their Hausdorff dimensions (see [PT, Chapter 4]), the limit capacities of the sets $\tilde{\psi}^{r_j}(K'\cap\widetilde{P}_j)$ are bounded by $\dim_{\mathrm{H}}(K')$, i.e., for any $d>\dim_{\mathrm{H}}(K')$, and any small $\eta>0$, it is possible to cover each of these sets by at most η^{-d} intervals of size η, and so

$$\prod_{j=1}^{k} \tilde{\psi}^{r_j}(K'\cap\widetilde{P}_j)$$

has limit capacity bounded by $k\dim_{\mathrm{H}}(K') < k-1$: given $D>k\dim_{\mathrm{H}}(K')$, say

$$D = \tfrac{1}{2}(k\dim_{\mathrm{H}}(K') + k - 1) < k-1,$$

we may, for any small $\eta>0$, cover this cartesian product by at most η^{-D} cubes of side η, and so linear projections of it in $\mathbb{R}^{k-1}$ have zero Lebesgue measure. Since

$$\left\{(t_1, ..., t_k) \in \mathbb{R}^k : \bigcap_{j=1}^{k} (\psi^{r_j}(K'\cap\widetilde{P}_j) + t_j) \neq \varnothing\right\} \\ = \left\{(t, t+s_1, ..., t+s_{k-1}) : (s_1, ..., s_{k-1}) = D_k\left(\prod_{j=1}^{k} \psi^{r_j}(K'\cap\widetilde{P}_j)\right)\right\},$$

where D_k is the linear map given by $D_k(x_1, ..., x_k) := (x_2-x_1, x_3-x_1, ..., x_k-x_1)$, for almost all $(t_1, ..., t_k)$ the intersection is empty, which implies the result. □

Lemma 3. *Let (K, K') be a pair of C^2-regular Cantor sets and*

$$B = \{(\psi^r|_I, I) : r \in \mathbb{N} \text{ and } I \text{ is a maximal interval of the construction of } K \text{ where } \psi^r \text{ is injective}\}.$$

For any $m\geqslant 1$, for any fixed distinct elements $(\psi^{r_j}|_{P_j}, P_j)$, $1\leqslant j\leqslant m$, of B and for any $\eta>0$, there is a C^2-regular Cantor set $\widetilde{K}$ at a distance smaller than η from K in the C^1 topology and with $a(\widetilde{K})>a(K)-\eta$ such that $\bigcap_{j=1}^{m} \tilde{\psi}^{r_j}(\widetilde{K}\cap K'\cap P_j) = \varnothing$, where $\tilde{\psi}$ is the map which defines $\widetilde{K}$.

Proof. Since the quantity $a(K)$ depends continuously on K in the $C^{3/2}$ topology (and, in particular, in the C^2 topology), Lemma 2 implies that the above statement is true for $m\geqslant k=\lfloor(1-\dim_{\mathrm{H}}(K'))^{-1}\rfloor+1$. We will argue by backward induction on m: we will show that, if $q\geqslant 1$ and the statement of Lemma 3 is true for every $m\geqslant q+1$, then it is true for q.

Let (K,K') be a pair of C^2-regular Cantor sets, $\eta\in(0,1)$ and $(\psi^{r_j}|_{P_j},P_j)$, $1\leqslant j\leqslant q$, be fixed distinct elements of B. Let

$$X=\bigcap_{j=1}^{q}\psi^{r_j}(K\cap K'\cap P_j).$$

Since K' is a C^2-regular Cantor set, it has bounded geometry; in particular, there is $\bar\lambda>0$ such that, for every interval J_1' of the construction of K', there is an interval J_0' of a previous step of the construction of K' which strictly contains J_1' such that $|J_0'|/|J_1'|<\bar\lambda$. Let $\varepsilon=a(K)a(K')\eta/5\bar\lambda$ and let $N_0=\lceil 2c(K)\varepsilon^{-1}\log\varepsilon^{-1}\rceil$, where $c(K)=2\lambda(K)/a(K)$. Since two intervals of the construction of K are disjoint or one of them contains the other, for any $(\psi^{r_j}|_{P_j},P_j)\in B$ and any $i\in\mathbb{N}$, $\psi^{i+r_j}(K\cap P_j)$ can be decomposed as a disjoint finite union of sets of the form $\psi^{i+r_j}(K\cap I)$, where $I\subset P_j$ and $(\psi^{i+r_j}|_I,I)\in B$. So, for each pair (i,j) with $0\leqslant i<j\leqslant N_0$, the intersection $\psi^i(X)\cap\psi^j(X)$ can be written as a finite union of intersections of at least $q+1$ sets of the form $\psi^r(K\cap K'\cap I)$, where $(\psi^r|_I,I)\in B$. By the induction hypothesis (applied several times to make a sequence of small perturbations of the first Cantor set, one for each one of the intersections mentioned above), we may approximate K by a C^2-regular Cantor set $\breve K$ at a distance smaller than $\frac12\eta$ from K in the C^1 topology with

$$a(\breve K)>\max\bigl\{a(K)-\tfrac12\eta,\,\tfrac12 a(K)\bigr\}$$

and $c(\breve K)<4\lambda(K)/a(K)$, such that, if

$$\breve X=\bigcap_{j=1}^{q}\breve\psi^{r_j}(\breve K\cap K'\cap\breve P_j),$$

where $\breve\psi$ is the map which defines $\breve K$, then the sets $\breve\psi^j(\breve X)$, $0\leqslant j\leqslant N_0$, are pairwise disjoint. So, if

$$\breve Y:=(\breve\psi^{r_1}|_{\breve P_1})^{-1}(\breve X)=(\breve K\cap K'\cap\breve P_1)\cap\bigcap_{j=2}^{q}(\breve\psi^{r_1}|_{\breve P_1})^{-1}(\breve\psi^{r_j}(\breve K\cap K'\cap\breve P_j)),$$

then the sets $\breve\psi^j(\breve Y)$, $0\leqslant j\leqslant N_0$, are pairwise disjoint. Indeed, if $\breve\psi^i(\breve Y)\cap\breve\psi^j(\breve Y)\neq\varnothing$, with $0\leqslant i<j\leqslant N_0$, we would have

$$\varnothing\neq\breve\psi^{r_1}(\breve\psi^i(\breve Y))\cap\breve\psi^{r_1}(\breve\psi^j(\breve Y))=\breve\psi^i(\breve\psi^{r_1}(\breve Y))\cap\breve\psi^j(\breve\psi^{r_1}(\breve Y))=\breve\psi^i(\breve X)\cap\breve\psi^j(\breve X)=\varnothing,$$

a contradiction. This implies, by compactness, that we may find a covering of $\breve{Y}$ by (small) intervals $\check{J}_i^*$ of the construction of $\breve{K}$ such that, if $Y^*\supset\breve{Y}$ is the intersection of $\breve{K}$ with the union of the intervals $\check{J}_i^*$, we still have that the sets $\check{\psi}^j(Y^*)$, $0\leqslant j\leqslant N_0$, are pairwise disjoint.

So, Y^* satisfies the hypothesis of Lemma 1 for ε, since

$$N_0=\lceil 2c(K)\varepsilon^{-1}\log\varepsilon^{-1}\rceil\geqslant\lceil c(\breve{K})\varepsilon^{-1}\log\varepsilon^{-1}\rceil,$$

and thus the conclusion of Lemma 1 holds: for any $\delta>0$, we can find a covering of $\breve{K}$ formed by intervals $\check{J}_i$ of its construction which have size smaller than δ satisfying the following properties: let D be the union of the intervals $\check{J}_i$ for all i and the intervals $\check{\psi}^j(\check{J}_i)$, $1\leqslant j\leqslant n_0$, for the intervals $\check{J}_i$ which intersect Y^*. There is a Cantor set $\overline{K}$ in the ε-neighborhood of $\breve{K}$ in the C^1 topology with $a(\overline{K})\geqslant a(\breve{K})$ such that all connected components of D are still intervals of the construction of $\overline{K}$ and such that $\overline{K}\cap\check{J}_i$ has a gap V_i with $|V_i|\geqslant(1-\varepsilon)|\check{J}_i|$ whenever $\check{J}_i\cap Y^*\neq\varnothing$. If δ is small enough, the intervals $\check{J}_i^*$ are still intervals of the construction of $\overline{K}$, and

$$(\overline{K}\cap K'\cap\overline{P}_1)\cap\bigcap_{j=2}^{q}(\bar{\psi}^{r_1}|_{\overline{P}_1})^{-1}(\bar{\psi}^{r_j}(\overline{K}\cap K'\cap\overline{P}_j))$$

is contained in the union of the intervals $\check{J}_i^*$, and so is contained in the union of the intervals $\check{J}_i$ which intersect Y^*.

Now, we may find a C^1 diffeomorphism of $\mathbb{R}$, $\left(\frac{1}{2}\eta\right)$-close to the identity in the C^1 topology, such that the image of its restriction to $\overline{K}$ is disjoint from K'. Indeed, let $(\check{J}_i)^{(1)}$ and $(\check{J}_i)^{(2)}$ be the connected components of $\check{J}_i\setminus V_i$. We will make small independent translations of these intervals (if they do intersect K') in the following way: if such an interval $(\check{J}_i)^{(s)}$ intersects K', take an interval J' of the construction of K' intersecting it whose size belongs to the interval $(|(\check{J}_i)^{(s)}|/a(K'),\bar{\lambda}|(\check{J}_i)^{(s)}|/a(K')]$; the gaps attached to the ends of this interval have size larger than $|(\check{J}_i)^{(s)}|$ so we can apply a translation of it of size at most $\bar{\lambda}|(\check{J}_i)^{(s)}|/a(K')$ whose image is contained in one of these gaps. These translations can be performed all together (for all i and s) by a diffeomorphism at a C^1 distance to the identity smaller than $\frac{1}{2}\eta$, since the gaps attached to the intervals $(\check{J}_i)^{(s)}$ have size at least $a(\breve{K})|J_i|$ and the size of the translations is at most

$$\frac{\bar{\lambda}|(\check{J}_i)^{(s)}|}{a(K')}<\frac{\varepsilon\bar{\lambda}|\check{J}_i|}{a(K')}=\frac{a(\breve{K})|\check{J}_i|\eta}{5}.$$

If we denote by $\widetilde{K}$ the image of $\overline{K}$ by this diffeomorphism (which is a regular Cantor set defined by a map $\tilde{\psi}$ conjugated to $\bar{\psi}$ by the diffeomorphism that we applied to $\overline{K}$) we will have $(\widetilde{K}\cap\check{J}_i)\cap K'=\varnothing$ for all i, so

$$\widetilde{Y}:=(\widetilde{K}\cap K'\cap\widetilde{P}_1)\cap\bigcap_{j=2}^{q}(\tilde{\psi}^{r_1}|_{\widetilde{P}_1})^{-1}\tilde{\psi}^{r_j}(\widetilde{K}\cap K'\cap\widetilde{P}_j)=\varnothing,$$

and, applying $\tilde{\psi}^{r_1}$, we get

$$\widetilde{X}:=\bigcap_{j=1}^{q}\tilde{\psi}^{r_j}(\widetilde{K}\cap K'\cap\widetilde{P}_j)=\tilde{\psi}^{r_1}(\widetilde{Y})=\varnothing,$$

and we are done. □

Proof of Theorem 1. Lemma 3 for $m=1$ implies that generically $K\cap K'=\varnothing$. It follows, as in [M, Theorem I.1], that, for each $r\in\mathbb{Q}$,

$$\{(K,K'):r\notin K-K'\}=\{(K,K'):K\cap(K'+r)=\varnothing\}$$

is residual, and thus $\{(K,K'):(K-K')\cap\mathbb{Q}=\varnothing\}$ is residual. So, generically, for a pair (K,K') of C^1-regular Cantor sets, $K-K'$ has empty interior, and so is a Cantor set. □

Proof of Theorem 2. Take a C^1 diffeomorphism ψ having a horseshoe Λ. Approximate it by a C^∞ diffeomorphism φ. Now the foliations of the horseshoe are $C^{1+\varepsilon}$, and can be extended to a neighborhood of it as $C^{1+\varepsilon}$ invariant foliations (see [PT]). There is a Markov partition of the horseshoe such that in each piece P_i of the partition, in the coordinates given by the stable and unstable foliations the diffeomorphism has the form $\psi(x,y)=(f_i(x),g_i(y))$. Now we replace the foliations by C^∞ foliations, and change the diffeomorphism in order to have the new foliations invariant in a neighborhood of the horseshoe, defining it by the C^∞ formulas $(\tilde{f}_i(x),\tilde{g}_i(y))$ in the pieces P_i, where $(\tilde{f}_i(x),\tilde{g}_i(y))$ is C^1 close to $(f_i(x),g_i(y))$.

We do this for Λ_1 and Λ_2, fix compact parts of the foliations and make a new (generic) C^∞ perturbation of the diffeomorphism outside the horseshoe in order to make all the tangencies between the two foliations quadratic, except for a discrete (and thus finite) set of points which typically do not belong to the foliations of the horseshoe. The last claim can be proved using Morse–Sard's theorem in the following way: locally, we may choose C^∞ coordinate systems for which the two foliations are given by the level curves of the function y and of a C^∞ function $h(x,y)$. The two foliations are tangent at a point (x_0,y_0) if and only if $(\partial h/\partial x)(x_0,y_0)=0$, and the tangency is quadratic provided $(\partial^2 h/\partial x^2)(x_0,y_0)\neq 0$. So, if (x_0,y_0) is a point of non-quadratic tangency between the two foliations, then $\mathcal{G}(x,y)=(0,0)$, where

$$\mathcal{G}(x,y):=\left(\frac{\partial h}{\partial x}(x,y)-x\frac{\partial^2 h}{\partial x^2}(x,y),\frac{\partial^2 h}{\partial x^2}(x,y)\right).$$

Perturbing the second foliation by replacing $h(x,y)$ by $h_{\varepsilon,\delta}(x,y)=h(x,y)-\frac{1}{2}\varepsilon x^2-\delta x$, the corresponding map $\mathcal{G}_{\varepsilon,\delta}(x,y)$ will be given by $\mathcal{G}(x,y)-(\delta,\varepsilon)$, and since, by Morse–Sard's theorem, generic (and almost all) pairs (δ,ε) are regular values of $\mathcal{G}$, for generic (and almost all) pairs (δ,ε), all tangencies between the first foliation and the perturbed second foliation in the open set under consideration are quadratic, except for a discrete set of points. Usual arguments of differential topology (using partitions of the unity) conclude the proof of the claim.

In the above situation, there exist lines of tangencies, which are C^∞ curves (given locally, in the notation of the preceding discussion, implicitely by $(\partial h/\partial x)(x,y)=0$) where the foliations are tangent. This reduces the study of tangencies between the two foliations to the study of intersections of Cantor sets, images of the stable and unstable Cantor sets of the horseshoe by the holonomy maps, in the lines of tangencies. Arbitrary C^1-small perturbations of these Cantor sets can be done by replacing the expressions $(\tilde{f}_i(x),\tilde{g}_i(y))$ by C^1 close expressions $(\check{f}_i(x),\check{g}_i(y))$, keeping the foliations unchanged. Since C^1-stable intersections of regular Cantor sets do not exist, it is possible to eliminate, by arbitrarily small C^1 perturbations, all tangencies between the compact parts of the stable and unstable foliations of the horseshoe we are considering. Baire's theorem implies that, residually in a neighborhood $\mathcal{U}$ of ψ in which Λ_1 and Λ_2 have hyperbolic continuations, there are no tangencies between leaves of the stable and unstable foliations of the continuations of Λ_1 and Λ_2.

If a diffeomorphism φ of M has a horseshoe $\check{\Lambda}$, we say that an open set $U\subset M$ is *good* for $\check{\Lambda}$ if $\check{\Lambda}\subset U$ and $\check{\Lambda}$ is the maximal invariant of φ in $\overline{U}$ (this is equivalent to the existence of another open set V such that $\overline{U}\subset V$ and $\check{\Lambda}$ is the maximal invariant of φ in V, and so it is a C^1-open condition on φ). Fixing a countable basis of open sets of M, if a diffeomorphism φ of M has a horseshoe $\check{\Lambda}$, then there is a good open set U for $\check{\Lambda}$ which is a finite union of open sets of the basis. If we fix two such open sets U_1 and U_2, for a C^1-generic set of diffeomorphisms φ of M, if the maximal invariant sets of φ in U_1 and U_2 are horseshoes, and U_1 and U_2 are good for them, then there are no tangencies between leaves of the stable and unstable foliations of the horseshoes. Since there are only a countable number of open sets which are finite unions of elements of the basis, another application of Baire's theorem finishes the proof of Theorem 2. □

References

[M] Moreira, C. G., Stable intersections of Cantor sets and homoclinic bifurcations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 13 (1996), 741–781.

[MY] Moreira, C. G. & Yoccoz, J. C., Stable intersections of regular Cantor sets with large Hausdorff dimensions. *Ann. of Math.*, 154 (2001), 45–96.

[MY2] — Tangences homoclines stables pour des ensembles hyperboliques de grande dimension fractale. *Ann. Sci. Éc. Norm. Supér.*, 43 (2010), 1–68.

[N] Newhouse, S. E., The abundance of wild hyperbolic sets and nonsmooth stable sets for diffeomorphisms. *Inst. Hautes Études Sci. Publ. Math.*, (1979), 101–151.

[PT] Palis, J. & Takens, F., *Hyperbolicity and Sensitive Chaotic Dynamics at Homoclinic Bifurcations*. Cambridge Studies in Advanced Mathematics, 35. Cambridge University Press, Cambridge, 1993.

[U] Ures, R., Abundance of hyperbolicity in the C^1 topology. *Ann. Sci. École Norm. Sup.*, 28 (1995), 747–760.

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