Acta Math., 206 (2011), 245–310
DOI: 10.1007/s11511-011-0063-1

Generalized Bäcklund–Darboux transformations for Coxeter–Toda flows from a cluster algebra perspective

by

Michael Gekhtman

University of Notre Dame
Notre Dame, IN, U.S.A.

Michael Shapiro

Michigan State University
East Lansing, MI, U.S.A.

Alek Vainshtein

University of Haifa
Haifa, Mount Carmel, Israel

1. Introduction

This is the third in the series of papers in which we investigate Poisson geometry of directed networks. In [21] and [22], we studied Poisson structures associated with weighted directed networks in a disk and in an annulus. The study was motivated in part by Poisson properties of cluster algebras. In fact, it was shown in [21] that if a universal Poisson bracket on the space of edge weights of a directed network in a disk satisfy an analogue of the Poisson–Lie property with respect to concatenation, then the Poisson structure induced by this bracket on the corresponding Grassmannian is compatible with the cluster algebra structure in the homogeneous coordinate ring of the Grassmannian. In this paper we deal with an example that ties together objects and concepts from the theory of cluster algebras and directed networks with the theory of integrable systems.

Integrable systems in question are the *Toda flows* on GL_n. These are commuting Hamiltonian flows generated by conjugation-invariant functions on GL_n with respect to the standard Poisson–Lie structure. Toda flows (also known as *characteristic Hamiltonian systems* [30]) are defined for an arbitrary standard semisimple Poisson–Lie group, but we will concentrate on the GL_n case, where as a maximal algebraically independent family of conjugation-invariant functions one can choose $F_k\colon \mathrm{GL}_n \ni X \mapsto (1/k)\operatorname{tr} X^k$, $k=1,\dots,n-1$. The equation of motion generated by F_k has the *Lax form*

$$\frac{d}{dt}X = \left[X, -\frac{1}{2}(\pi_+(X^k) - \pi_-(X^k))\right], \tag{1.1}$$

where $\pi_+(A)$ and $\pi_-(A)$ denote strictly upper and lower parts of a matrix A.

Any double Bruhat cell $G^{u,v}$, $u,v\in S_n$, is a regular Poisson submanifold in GL_n invariant under the right and left multiplication by elements of the maximal torus (the subgroup of diagonal matrices) $\mathbf{H}\subset\mathrm{GL}_n$. In particular, $G^{u,v}$ is invariant under the conjugation by elements of $\mathbf{H}$. The standard Poisson–Lie structure is also invariant under the conjugation action of $\mathbf{H}$ on GL_n. This means that Toda flows defined by (1.1) induce commuting Hamiltonian flows on $G^{u,v}/\mathbf{H}$, where $\mathbf{H}$ acts on $G^{u,v}$ by conjugation. In the case when $v=u^{-1}=(n\ 1\ 2\ \ldots\ n-1)$, $G^{u,v}$ consists of tridiagonal matrices with non-zero off-diagonal entries, and $G^{u,v}/\mathbf{H}$ can be conveniently described as the set Jac of *Jacobi matrices* of the form

$$L=\begin{pmatrix} b_1 & 1 & 0 & \ldots & 0\\ a_1 & b_2 & 1 & \ldots & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & \ldots & a_{n-2} & b_{n-1} & 1\\ 0 & \ldots & 0 & a_{n-1} & b_n \end{pmatrix}, \quad a_1\ldots a_{n-1}\neq 0, \quad \det L\neq 0. \tag{1.2}$$

The Lax equations (1.1) then become the equations of the *finite non-periodic Toda hierarchy*

$$\frac{d}{dt}L=[L,\pi_-(L^k)],$$

the first of which, corresponding to $k=1$, is the celebrated *Toda lattice*

$$\begin{aligned}\frac{d}{dt}a_j&=a_j(b_{j+1}-b_j), \quad j=1,\ldots,n-1,\\ \frac{d}{dt}b_j&=(a_j-a_{j-1}), \quad j=1,\ldots,n,\end{aligned}$$

with the boundary conditions $a_0=a_n=0$. Recall that $\det L$ is a Casimir function for the standard Poisson–Lie bracket. The level sets of the function $\det L$ foliate Jac into $2(n-1)$-dimensional symplectic manifolds, and the Toda hierarchy defines a completely integrable system on every symplectic leaf. Note that although Toda flows on an arbitrary double Bruhat cell $G^{u,v}$ can be exactly solved via the so-called *factorization method* (see, e.g. [31]), in most cases the dimension of symplectic leaves in $G^{u,v}/\mathbf{H}$ exceeds $2(n-1)$, which means that conjugation-invariant functions do not form a Poisson commuting family rich enough to ensure Liouville complete integrability.

An important role in the study of Toda flows is played by the *Weyl function*

$$m(\lambda)=m(\lambda;X)=((\lambda\mathbf{1}-X)^{-1}e_1,e_1)=\frac{q(\lambda)}{p(\lambda)}, \tag{1.3}$$

where $p(\lambda)$ is the characteristic polynomial of X and $q(\lambda)$ is the characteristic polynomial of the $(n-1)\times(n-1)$ submatrix of X formed by deleting the first row and column (see, e.g., [5], [7] and [28]). Differential equations that describe the evolution of $m(\lambda;X)$ induced by Toda flows do not depend on the initial value $X(0)$ and are easy to solve: though non-linear, they are also induced by *linear differential equations with constant coefficients* on the space

$$\left\{M(\lambda)=\frac{Q(\lambda)}{P(\lambda)}:\deg P=n,\ \deg Q=n-1,\ P \text{ and } Q \text{ are coprime},\ P(0)\neq 0\right\} \tag{1.4}$$

by the map $M(\lambda)\mapsto m(\lambda)=-M(-\lambda)/H_0$, where $H_0=\lim_{\lambda\to\infty}\lambda M(\lambda)\neq 0$.

It is easy to see that $m(\lambda;X)$ is invariant under the action of $\mathbf{H}$ on $G^{u,v}$ by conjugation. Thus we have a map from $G^{u,v}/\mathbf{H}$ into the space

$$\mathcal{W}_n=\left\{m(\lambda)=\frac{q(\lambda)}{p(\lambda)}:\deg p=n,\ \deg q=n-1,\ p \text{ and } q \text{ are monic and coprime},\ p(0)\neq 0\right\}.$$

In the tridiagonal case, this map, sometimes called the *Moser map*, is invertible: it is a classical result in the theory of moment problems that matrix entries of an element in Jac can be restored from its Weyl function $m(\lambda;X)$ via determinantal formulas for matrix entries of X in terms of Hankel determinants built from the coefficients of the Laurent expansion of $m(\lambda;X)$. These formulas go back to the work of Stieltjes on continuous fractions [33] (see, e.g. [1] for details).

In this paper, we study double Bruhat cells $G^{u,v}$ that share common features with the tridiagonal case:

(i) the Toda hierarchy defines a completely integrable system on level sets of the determinant in $G^{u,v}/\mathbf{H}$, and

(ii) the Moser map $m_{u,v}\colon G^{u,v}/\mathbf{H}\to\mathcal{W}_n$ defined in the same way as in the tridiagonal case is invertible.

We will see that double Bruhat cells $G^{u,v}$ associated with any pair of Coxeter elements $u,v\in S_n$ enjoy these properties. (Recall that a Coxeter element in S_n is a product of $n-1$ distinct elementary transpositions.) Double Bruhat cells of this kind has previously appeared (for an arbitrary simple Lie group) in [23] in the context of integrable systems, and in [3] and [36] in connection with cluster algebras of finite type. We will call any such double Bruhat cell a *Coxeter double Bruhat cell*. Integrable equations induced on $G^{u,v}/\mathbf{H}$ by Toda flows will be called *Coxeter–Toda lattices*. This term was first used in [23] in the case $u=v$ for an arbitrary simple Lie group, which generalizes the *relativistic Toda lattice* that corresponds to the choice $u=v=s_{n-1}\dots s_1$ in GL_n. In [12] and [13], the corresponding integrable systems for $v=s_{n-1}\dots s_1$ and an arbitrary Coxeter element u were called *elementary Toda lattices*. In the latter case, $G^{u,v}/\mathbf{H}$ can be described as a

subset of Hessenberg matrices subject to certain rank conditions on submatrices. The tridiagonal case corresponds to the choice $v=s_{n-1}\dots s_1$ and $u=s_1\dots s_{n-1}$.

Since Coxeter–Toda flows associated with different choices of (u,v) lead to the same evolution of the Weyl function, and the corresponding Moser maps are invertible, one can construct transformations between different $G^{u,v}/\mathbf{H}$ that preserve the corresponding Coxeter–Toda flows and thus serve as *generalized Bäcklund–Darboux transformations* between them.

Our goal is to describe these transformations from the cluster algebra point of view. To this end, we construct a cluster algebra of rank $2n-2$ associated with an extension of the space (1.4)

$$\mathcal{R}_n=\left\{\frac{Q(\lambda)}{P(\lambda)}:\deg P=n,\ \deg Q<n,\ P \text{ and } Q \text{ are coprime},\ P(0)\neq 0\right\}.$$

(Note that $\mathcal{W}_n$ is embedded into $\mathcal{R}_n$ as a codimension-1 subspace.) Distinguished clusters $\mathbf{x}_{u,v}$ in this algebra correspond to Coxeter double Bruhat cells, and are formed by certain collections of Hankel determinants built out of coefficients of the Laurent expansion of an element in $\mathcal{R}_n$. Sequences of cluster transformations connecting these distinguished clusters are then used as the main ingredient in the construction of generalized Bäcklund–Darboux transformations.

The insight necessary to implement this construction is drawn from two sources:

(i) the procedure for the inversion of the Moser map, which can be viewed as a generalization of the inverse moment problem, and

(ii) interpretation of functions in $\mathcal{R}_n$ as boundary measurement functions associated with a particular kind of networks in an annulus.

Before discussing the organization of the paper, we would like to make two remarks. First, birational transformations between $G^{u,s_{n-1}\dots s_1}/\mathbf{H}$ and $G^{u',s_{n-1}\dots s_1}/\mathbf{H}$ for two different Coxeter elements u and u' which serve as generalized Bäcklund–Darboux transformations between the corresponding elementary Toda lattices were first studied in [12]. Second, a cluster algebra closely related to the one we considered here recently appeared in [25] and was subject of a detailed combinatorial study in [10], where cluster mutations along the edges of a certain subgraph of its exchange graph were shown to describe an evolution of an *A_n-type Q-system*—a discrete evolution arising in the analysis of the XXX-model, which is an example of a *quantum* integrable model. In [10], solutions of the Q-system are represented as Hankel determinants built from coefficients of a certain generating function, which turns out to be rational and can be represented as a matrix element of a resolvent of an appropriate linear operator.

The paper is organized as follows.

In §2 we go over the necessary background information on double Bruhat cells, Toda flows, cluster algebras, networks and associated Poisson structures. We then proceed, in §3, to describe a parametrization of a Coxeter double Bruhat cell. This is a particular case of the Berenstein–Fomin–Zelevinsky parametrization [2], [15]: for a generic element X in $G^{u,v}$, we consider a factorization of X into elementary bidiagonal factors consistent with the Gauss factorization of X, that is $X=X_-X_0X_+$, where X_0 is the diagonal matrix $\operatorname{diag}(d_1,...,d_n)$, X_+ is the product of $n-1$ elementary upper bidiagonal factors $E_i^+(c_i^+)$, $i=1,...,n-1$, with the order of factors in the product prescribed by v, and X_- is the product of $n-1$ elementary lower bidiagonal factors $E_i^-(c_i^-)$, $i=1,...,n-1$, with the order of factors in the product prescribed by u. We also give an intrinsic characterization of a double Bruhat cell.

Elements $G^{u,v}/\mathbf{H}$ are parametrized by d_i and $c_i=c_i^+c_i^-$, $i=1,...,n-1$. In §4 we show that these parameters can be restored as monomial expressions in terms of an appropriately chosen collection of Hankel determinants built from the coefficients of the Laurent expansion of the Weyl function $m(\lambda)$. (In [14], a similar inverse problem was solved for the case $v=s_{n-1}\dots s_1$, u arbitrary.) Both the choice of Hankel determinants and exponents entering monomial expressions for d_i and c_i are uniquely determined by the pair (u,v).

In §5, the map $X\mapsto m(\lambda;X)$ is given a combinatorial interpretation in terms of weighted directed planar networks. To an elementary bidiagonal factorization of $X\in G^{u,v}$ there corresponds a network $N_{u,v}$ in a square (disk) with n sources located on one side of the square and n sinks located at the opposite side, both numbered bottom to top (see, e.g. [11], [15] and [16]). By gluing opposite sides of the square containing sinks and sources in such a way that each sink is glued to the corresponding source and adding two additional edges, one incoming and one outgoing, one obtains a weighted directed network in an annulus (the outer and inner boundary circles of the annulus are formed by the remaining two sides of the square). Networks in an annulus were studied in [22]. The network we just described, $N^\circ_{u,v}$, has one sink and one source on the outer boundary of an annulus and, according to [22], the boundary measurement corresponding to this network is a rational function $M(\lambda)$ in an auxiliary parameter λ. We show that $-M(-\lambda)$ is equal to $m(\lambda;X)$ times the product of weights of the incoming and outgoing edges in $N^\circ_{u,v}$.

The determinantal formulas for the inverse of the Moser map are homogeneous of degree zero with respect to the coefficients of the Laurent expansion, and therefore the same formulas applied to $M(\lambda)$ also recover c_i and d_i. Thus, we can define a map $\varrho_{u,v}\colon(\mathbb{C}^*)^{2n}\to G^{u,v}/\mathbf{H}$ in such a way that the through map

$$G^{u,v}/\mathbf{H}\xrightarrow{m_{u,v}}\mathcal{W}_n\hookrightarrow\mathcal{R}_n\xrightarrow{\mathbf{x}_{u,v}}(\mathbb{C}^*)^{2n}\xrightarrow{\varrho_{u,v}}G^{u,v}/\mathbf{H}$$

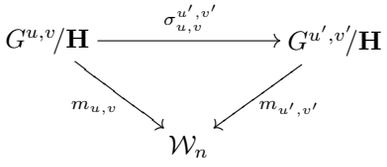
is the identity map.

In the remainder of §5, we use the combinatorial data determined by the pair (u, v) (or, in a more transparent way, by the corresponding network $N^{\circ}_{u,v}$) to construct a cluster algebra $\mathcal{A}{=}\mathcal{A}_{u,v}$, with the (slightly modified) collection $\mathbf{x}_{u,v}$ serving as the initial cluster. The matrix $B_{u,v}$ that determines cluster transformations for the initial cluster is closely related to the incidence matrix of the graph dual to $N^{\circ}_{u,v}$. To construct $\mathcal{A}_{u,v}$, we start with the Poisson structure induced on boundary measurement functions by a so-called *standard Poisson bracket* on the space of face weights associated with $N^{\circ}_{u,v}$ (this bracket is a particular case of the general construction for networks in the annulus given in [22]). Initial cluster variables, viewed as functions on $\mathcal{R}_n$, form a coordinate system in which this Poisson structure takes a particular simple form: the Poisson bracket of logarithms of any two functions in the family is constant. This allows us to follow the strategy from [20] to construct $\mathcal{A}_{u,v}$ as a cluster algebra compatible with this Poisson bracket. We then show that $\mathcal{A}_{u,v}$ does not depend on the choice of Coxeter elements u and v, that is, that for any (u', v') the initial seed of $\mathcal{A}_{u',v'}$ is a seed in the cluster algebra $\mathcal{A}_{u,v}$. Therefore, the change of coordinates $T^{u',v'}_{u,v}\colon \mathbf{x}_{u,v}\mapsto\mathbf{x}_{u',v'}$ is accomplished by a sequence of cluster transformations. Moreover, the ring of regular functions on $\mathcal{R}_n$ coincides with the localization of the complex form of $\mathcal{A}$ with respect to the stable variables. We complete §5 with the discussion of the interplay between our results and those of [10]. In particular, we provide an alternative proof for one of the main results of [10] concerning the Laurent positivity of the solutions of Q-systems.

In the final section, we interpret generalized Bäcklund–Darboux transformations between Coxeter–Toda lattices corresponding to different pairs of Coxeter elements in terms of the cluster algebra $\mathcal{A}$ by observing that the map

$$\sigma^{u',v'}_{u,v} = \varrho_{u',v'}\circ T^{u',v'}_{u,v}\circ\tau_{u,v}\colon G^{u,v}/\mathbf{H}\longrightarrow G^{u',v'}/\mathbf{H}, \tag{1.5}$$

with $\tau_{u,v}$ being the right inverse of $\varrho_{u,v}$, preserves flows generated by conjugation-invariant functions and makes the diagram



commutative. We obtain explicit formulas for $\sigma^{u',v'}_{u,v}$ and, as a nice application, present formulas that transform solutions of the usual Toda lattice into solutions of the relativistic one. Besides, we explain how one represents generalized Bäcklund–Darboux transformations as equivalent transformations of the network $N^{\circ}_{u,v}$. Finally we show that classical Darboux transformations are also related to cluster algebra transformations via a formula similar to (1.5).

2. Preliminaries

In this section we collect the necessary background information on double Bruhat cells, Toda flows and directed networks on surfaces. Though notions and results that we will need on the first two subjects can be as easily stated for an arbitrary semisimple group, we will limit ourselves to the GL_n case.

2.1. Double Bruhat cells

Let $\mathfrak{b}_+$, $\mathfrak{n}_+$, $\mathfrak{b}_-$ and $\mathfrak{n}_-$ be the algebras of upper triangular, strictly upper triangular, lower triangular and strictly lower triangular matrices, respectively.

The connected subgroups that correspond to $\mathfrak{b}_+$, $\mathfrak{b}_-$, $\mathfrak{n}_+$ and $\mathfrak{n}_-$ will be denoted by $\mathbf{B}_+$, $\mathbf{B}_-$, $\mathbf{N}_+$ and $\mathbf{N}_-$, respectively. We denote by $\mathbf{H}$ the maximal torus (the subgroup of diagonal matrices) in GL_n.

Every $\xi \in \mathfrak{gl}_n$ can be uniquely decomposed into

$$\xi = \xi_- + \xi_0 + \xi_+,$$

where $\xi_+ \in \mathfrak{n}_+$, $\xi_- \in \mathfrak{n}_-$ and ξ_0 is diagonal. Consequently, for every X in an open Zariski dense subset of GL_n there exists a unique *Gauss factorization*

$$X = X_- X_0 X_+, \quad X_+ \in \mathbf{N}_+, X_- \in \mathbf{N}_-, X_0 \in \mathbf{H}.$$

Let s_i, $i \in [1, n-1]$, denote the elementary transposition $(i, i+1)$ in the symmetric group S_n. A *reduced decomposition* of an element $w \in S_n$ is a representation of w as a product $w = s_{i_1} \dots s_{i_l}$ of the smallest possible length. A reduced decomposition is not unique, but the number l depends only on w and is called the *length* of w and denoted by $l(w)$. The sequence of indices $\mathbf{i} = (i_1, \dots, i_l)$ corresponding to a given reduced decomposition of w is called a *reduced word* for w. The unique element of S_n of maximal length (also called *the longest element* of S_n) is denoted by w_0.

We will also need need a notion of a *reduced word* for an ordered pair (u, v) of elements in S_n. It is defined as follows: if $(i_1, \dots, i_{l(u)})$ is a reduced word for u and $(i'_1, \dots, i'_{l(v)})$ is a reduced word for v, then any shuffle of the sequences $(i_1, \dots, i_{l(u)})$ and $(-i'_1, \dots, -i'_{l(v)})$ is called a reduced word for (u, v).

Let us fix an embedding of S_n into GL_n and denote the representative of $w \in S_n$ in GL_n by the same letter w. The *Bruhat decompositions* of GL_n with respect to $\mathbf{B}_+$ and $\mathbf{B}_-$ are defined, respectively, by

$$\mathrm{GL}_n = \bigcup_{u \in S_n} \mathbf{B}_+ u \mathbf{B}_+ \quad \text{and} \quad \mathrm{GL}_n = \bigcup_{v \in S_n} \mathbf{B}_- v \mathbf{B}_-.$$

For any $u, v \in S_n$, the *double Bruhat cell* is defined as

$$G^{u,v} = \mathbf{B}_+ u \mathbf{B}_+ \cap \mathbf{B}_- v \mathbf{B}_-.$$

According to [15], the variety $G^{u,v}$ is biregularly isomorphic to a Zariski open subset of $\mathbb{C}^{l(u)+l(v)+n}$. A corresponding birational map from $\mathbb{C}^{l(u)+l(v)+n}$ to $G^{u,v}$ can be constructed quite explicitly, though not in a unique way. Namely, fix a reduced word $\mathbf{i}$ for the pair (u, v) and consider, in addition, a sequence $\mathbf{k}=(k_1, ..., k_n)$ obtained as an arbitrary rearrangement of the numbers $\sqrt{-1}, 2\sqrt{-1}, ..., n\sqrt{-1}$. Let $\mathbf{j}=(j_1, ..., j_{l(u)+l(v)+n})$ be a shuffle of $\mathbf{k}$ and $\mathbf{i}$; we set

$$\theta(j_l) = \begin{cases} +, & \text{if } j_l > 0, \\ -, & \text{if } j_l < 0, \\ 0, & \text{if } j_l \in \mathbf{k}. \end{cases}$$

Denote by e_{ij} an elementary $n \times n$ matrix $(\delta_{i\alpha}\delta_{j\beta})_{\alpha,\beta=1}^n$. For $t \in \mathbb{C}$, $i, j \in [1, n-1]$ and $k \in [1, n]$, let

$$E_i^-(t) = \mathbf{1} + te_{i+1\,i}, \quad E_j^+(t) = \mathbf{1} + te_{j\,j+1} \quad \text{and} \quad E_k^0(t) = \mathbf{1} + (t-1)e_{kk}. \tag{2.1}$$

Then the map $X_{\mathbf{j}}: \mathbb{C}^{l(u)+l(v)+n} \to G^{u,v}$ can be defined by

$$X_{\mathbf{j}}(\mathbf{t}) = \prod_{q=1}^{l(u)+l(v)+n} E_{|j_q|}^{\theta(j_q)}(t_q). \tag{2.2}$$

The parameters $t_1, ..., t_{l(u)+l(v)+n}$ constituting $\mathbf{t}$ are called *factorization parameters*. Explicit formulas for the inverse of the map (2.2) in terms of the so-called *twisted generalized minors* were found in [15].

2.2. Toda flows

Next, we review the basic facts about the Toda flows on GL_n.

Recall that the standard Poisson–Lie structure on GL_n is given by

$$\{f_1, f_2\}_{\mathrm{SL}_n}(X) = \tfrac{1}{2}(R(\nabla f_1(X)X), \nabla f_2(X)X) - \tfrac{1}{2}(R(X\nabla f_1(X)), X\nabla f_2(X)),$$

where $(\cdot, \cdot)$ denotes the *trace-form*, ∇ is the gradient defined with respect to the trace-form, and $R: \mathfrak{gl}_n \to \mathfrak{gl}_n$ is the standard R-matrix given by

$$R(\xi) = \xi_+ - \xi_- = (\mathrm{sign}(j-i)\xi_{ij})_{i,j=1}^n.$$

Double Bruhat cells are regular Poisson submanifolds of GL_n equipped with the standard Poisson–Lie structure (see [26], [30] and [35]). Furthermore,

(i) any symplectic leaf of GL_n is of the form $S^{u,v}a$, where $S^{u,v}\subset G^{u,v}$ is a certain distinguished symplectic leaf and a is an element of the Cartan subgroup, and

(ii) the dimension of symplectic leaves in $G^{u,v}$ equals $l(u)+l(v)+\mathrm{corank}(uv^{-1}-\mathrm{Id})$, see [26] and [30].

Conjugation-invariant functions on GL_n form a Poisson-commuting family (see, e.g., [31]). Any such function F generates a Hamiltonian flow described by the *Lax equation*

$$\frac{dX}{dt}=\left[X,-\frac{1}{2}R(X\nabla F(X))\right]. \tag{2.3}$$

The resulting family of equations is called *the hierarchy of Toda flows* (in [30], the term *characteristic Hamiltonian systems* is used). If one chooses $F(X)=F_k(X)=(1/k)\,\mathrm{tr}\,X^k$, then equation (2.3) becomes (1.1). The functions $F_1,...,F_{n-1}$ form a maximal family of algebraically independent conjugation-invariant functions on GL_n.

For an element $h\in\mathrm{GL}_n$, denote by C_h the action of h on GL_n by conjugation: $C_h(X)=hXh^{-1}$. For any smooth function f on GL_n we have

$$\nabla(f\circ C_h)=\mathrm{Ad}_{h^{-1}}(\nabla f).$$

Furthermore, if h belongs to $\mathbf{H}$, then it is easy to see that

$$R(\mathrm{Ad}_{h^{-1}}(\xi))=\mathrm{Ad}_{h^{-1}}(R(\xi))$$

for any $\xi\in\mathfrak{gl}_n$. Together, these observations imply that for any $h\in\mathbf{H}$ and any pair of smooth functions f_1 and f_2 on GL_n,

$$\{f_1\circ C_h,f_2\circ C_h\}=\{f_1,f_2\}\circ C_h.$$

In other words, the action of $\mathbf{H}$ on GL_n by conjugation is Poisson with respect to the standard Poisson–Lie structure. Since the action preserves double Bruhat cells, the standard Poisson–Lie structure induces a Poisson structure on $G^{u,v}/\mathbf{H}$, and the Toda hierarchy induces the family of commuting Hamiltonian flows on $G^{u,v}/\mathbf{H}$.

Remark 2.1. (i) The Lax equation (2.3) can be solved explicitly via the *factorization method* [31], which we will not review here.

(ii) Written in terms of matrix entries, equations (2.3) have exactly the same form as equations of the Toda hierarchy on $\mathfrak{gl}_n$, where the relevant Poisson structure is the Lie–Poisson structure associated with the R-matrix Lie bracket

$$[\xi,\eta]_R=\tfrac{1}{2}([R(\xi),\eta]+[\xi,R(\eta)]).$$

In fact, viewed as equations on the algebra of $n\times n$ matrices, the Toda hierarchy becomes a family of biHamiltonian flows with compatible linear and quadratic Poisson brackets given by Lie–Poisson and the extension of the Poisson–Lie brackets, respectively. However, we will not need the linear Poisson structure in the current paper.

2.3. Cluster algebras and compatible Poisson brackets

First, we recall the basics of cluster algebras of geometric type. The definition that we present below is not the most general one, see, e.g., [3] and [17] for a detailed exposition.

The *coefficient group* $\mathbf{P}$ is a free multiplicative abelian group of finite rank m with generators $g_1, ..., g_m$. An *ambient field* is the field $\mathfrak{F}$ of rational functions in n independent variables with coefficients in the field of fractions of the integer group ring $\mathbb{Z}\mathbf{P}=\mathbb{Z}[g_1^{\pm 1}, ..., g_m^{\pm 1}]$ (here we write $x^{\pm 1}$ instead of x, x^{-1}). It is convenient to think of $\mathfrak{F}$ as of the field of rational functions in $n+m$ independent variables with rational coefficients.

A *seed* (of *geometric type*) in $\mathfrak{F}$ is a pair $\Sigma=(\mathbf{x}, B)$, where $\mathbf{x}=(x_1, ..., x_{n+m})$, $x_1, ..., x_n$ is a transcendence basis of $\mathfrak{F}$ over the field of fractions of $\mathbb{Z}\mathbf{P}$, $x_{n+i}=g_i$ for $i\in[1, m]$, and B is an $n\times(n+m)$ integer matrix whose principal part (that is, the $n\times n$ submatrix formed by the columns $1, ..., n$) is skew-symmetric. The $(n+m)$-tuple $\mathbf{x}$ is called a *cluster*, its elements $x_1, ..., x_n$ are called *cluster variables*, and $x_{n+1}, ..., x_{n+m}$ are *stable variables*.

Given a seed as above, the *cluster transformation* in direction $k\in[1, n]$ is defined by

$$\mathbf{x}\longmapsto \mathbf{x}_k=(\mathbf{x}\setminus\{x_k\})\cup\{\bar{x}_k\},$$

where the new cluster variable $\bar{x}_k$ is given by the *exchange relation*

$$x_k\bar{x}_k=\prod_{\substack{1\leqslant i\leqslant n+m\\ b_{ki}>0}} x_i^{b_{ki}}+\prod_{\substack{1\leqslant i\leqslant n+m\\ b_{ki}<0}} x_i^{-b_{ki}}; \tag{2.4}$$

here, as usual, the product over the empty set is assumed to be equal to 1.

We say that $\overline{B}$ is obtained from B by a *matrix mutation* in direction k if

$$\bar{b}_{ij}=\begin{cases} -b_{ij}, & \text{if } i=k \text{ or } j=k,\\ b_{ij}+\frac{1}{2}(|b_{ik}|b_{kj}+b_{ik}|b_{kj}|), & \text{otherwise.}\end{cases}$$

Given a seed $\Sigma=(\mathbf{x}, B)$, we say that a seed $\overline{\Sigma}=(\bar{\mathbf{x}}, \overline{B})$ is *adjacent* to Σ (in direction k) if $\bar{\mathbf{x}}$ is obtained from $\mathbf{x}$ and $\overline{B}$ is obtained from B by a cluster transformation and a matrix mutation, respectively, in direction k. Two seeds are *mutation equivalent* if they can be connected by a sequence of pairwise adjacent seeds. The *cluster algebra* (of *geometric*

type) $\mathcal{A}=\mathcal{A}(B)$ associated with Σ is the $\mathbb{Z}\mathbf{P}$-subalgebra of $\mathfrak{F}$ generated by all cluster variables in all seeds mutation equivalent to Σ. The *complex form* of $\mathcal{A}$ is defined as $\mathcal{A}$ tensored by $\mathbb{C}$ and is denoted by $\mathcal{A}_{\mathbb{C}}$.

Let V be a Zariski open subset in $\mathbb{C}^{n+m}$, and $\mathcal{A}_{\mathbb{C}}$ be the complex form of a cluster algebra of geometric type. We assume that the variables in some extended cluster are identified with a set of algebraically independent rational functions on V. This allows us to identify cluster variables in any cluster with rational functions on V as well, and thus to consider $\mathcal{A}_{\mathbb{C}}$ as a subalgebra of the field $\mathbb{C}(V)$ of rational functions on V. Finally, we denote by $\mathcal{A}_{\mathbb{C}}^V$ the localization of $\mathcal{A}_{\mathbb{C}}$ with respect to the stable variables that do not vanish on V.

PROPOSITION 2.1. *Let V and $\mathcal{A}$ as above satisfy the following conditions*:

(i) *each regular function on V belongs to $\mathcal{A}_{\mathbb{C}}^V$*;

(ii) *there exists a cluster $\mathbf{x}=(x_1,...,x_{n+m})$ in $\mathcal{A}_{\mathbb{C}}$ consisting of algebraically independent functions regular on V*;

(iii) *any cluster variable $\bar{x}_k$, $k\in[1,n]$, obtained by the cluster transformation* (2.4) *applied to $\mathbf{x}$ is regular on V.*

Then $\mathcal{A}_{\mathbb{C}}^V$ is isomorphic to the ring $\mathcal{O}(V)$ of regular functions on V.

Proof. All we have to prove is that any element in $\mathcal{A}_{\mathbb{C}}^V$ is a regular function on V. The proof follows the proof of a similar statement for double Bruhat cells in [37] and consists of three steps.

LEMMA 2.1. *Let $\mathbf{z}=(z_1,...,z_{n+m})$ be an arbitrary cluster in $\mathcal{A}_{\mathbb{C}}$. If a Laurent monomial $M=z_1^{d_1}...z_{n+m}^{d_{n+m}}$ is regular on V then $d_i\geqslant 0$ for $i\in[1,n]$.*

Proof. Indeed, assume that $d_k<0$ for some $k\in[1,n]$ and consider the cluster $\mathbf{z}_k$. By (2.4), M can be rewritten as $M=M_1\bar{z}_k^{-d_k}/P^{-d_k}$, where M_1 is a Laurent monomial in the common variables of $\mathbf{z}$ and $\mathbf{z}_k$, and P is the binomial (in the same variables) that appears in the right-hand side of (2.4). By condition (i) and the Laurent phenomenon (Theorem 3.1 in [17]), M can be written as a Laurent polynomial in the variables of $\mathbf{z}_k$. Equating two expressions for M, we see that P^{-d_k} times a polynomial in the variables of $\mathbf{z}_k$ equals a Laurent monomial in the same variables. This contradicts the algebraic independence of the variables in $\mathbf{z}_k$, which follows from the algebraic independence of the variables in $\mathbf{x}$. □

LEMMA 2.2. *Let z be a cluster variable in an arbitrary cluster $\mathbf{z}$, and assume that z is a regular function on V. Then z is irreducible in the ring of regular functions on V.*

Proof. Without loss of generality, assume that $z_{n+1}=x_{n+1},...,z_{n+m'}=x_{n+m'}$ do not vanish on V, and $z_{n+m'+1}=x_{n+m'+1},...,z_{n+m}=x_{n+m}$ may vanish on V. Moreover, as-

sume on the contrary that $z=fg$, where f and g are non-invertible regular functions on V. By condition (i) and Proposition 11.2 of [18], both f and g are Laurent polynomials in $z_1, ..., z_{n+m'}$ whose coefficients are polynomials in $z_{n+m'+1}, ..., z_{n+m}$. Applying the same argument as in the proof of Lemma 2.1, we see that both f and g are, in fact, Laurent monomials in $z_1, ..., z_{n+m}$ and that $z_{n+1}, ..., z_{n+m'}$ enter both f and g with a non-negative degree. Moreover, by Lemma 2.1, each cluster variable $z_1, ..., z_n$ enters both f and g with a non-negative degree. This can only happen if one of f and g is invertible in $\mathcal{O}(V)$, a contradiction. □

Denote by $U_0 \subset V$ the locus of all $t \in V$ such that $x_i(t) \neq 0$ for all $i \in [1, n]$. Besides, denote by $U_k \subset V$ the locus of all $t \in V$ such that $x_i(t) \neq 0$ for all $i \in [1, n] \setminus k$ and $\bar{x}_k(t) \neq 0$.

LEMMA 2.3. *Let* $U = \bigcup_{i=0}^{n} U_i$. *Then* $\operatorname{codim} V \setminus U \geqslant 2$.

Proof. This follows immediately from Lemma 2.2 and conditions (ii) and (iii). □

Assume that there exists $f \in \mathcal{A}_{\mathbb{C}}^{V}$ which is not regular on V. Recall that V is the complement of a finite union of irreducible hypersurfaces D_i in $\mathbb{C}^{n+m}$. Therefore, the divisor of the poles of f has codimension 1 in $\mathbb{C}^{n+m}$. Since f is not regular on V, this latter divisor does not lie entirely in the union of D_i, and hence its intersection with V has codimension 1 in V. Therefore, by Lemma 2.3, it intersects U non-trivially. To complete the proof, note that by Proposition 11.2 of [18], any function in $\mathcal{A}_{\mathbb{C}}^{V}$ is regular on U, a contradiction. □

Let $\{\cdot, \cdot\}$ be a Poisson bracket on the ambient field $\mathfrak{F}$. We say that it is *compatible* with the cluster algebra $\mathcal{A}(B)$ if, for any cluster $\mathbf{x} = (x_1, ..., x_{n+m})$, one has

$$\{x_i, x_j\} = \omega_{ij} x_i x_j,$$

where $\omega_{ij} \in \mathbb{Z}$ are constants for all $i, j \in [1, n+m]$. The matrix $\Omega^{\mathbf{x}} = (\omega_{ij})$ is called the *coefficient matrix* of $\{\cdot, \cdot\}$ (in the basis $\mathbf{x}$); clearly, $\Omega^{\mathbf{x}}$ is skew-symmetric. A complete description of Poisson brackets compatible with $\mathcal{A}(B)$ in the case where $\operatorname{rank} B = n$ is given in [20].

2.4. Networks on surfaces with boundaries

Let S be a disk with $c \geqslant 0$ holes, so that its boundary ∂S has $c+1$ connected components, and let $G = (V, E)$ be a directed graph embedded in S with the vertex set V and the edge set E. Exactly r of its vertices are located on the boundary ∂S. They are denoted $b_1, ..., b_r$ and called *boundary vertices*. Each boundary vertex is labeled as a source or a sink. A *source* is a vertex with exactly one outgoing edge and no incoming edges. *Sinks*

are defined in the same way, with the direction of the single edge reversed. The number of sources is denoted by n and the number of sinks by $m=r-n$. All the internal vertices of G have degree 3 and are of two types: either they have exactly one incoming edge, or exactly one outgoing edge. The vertices of the first type are called (and shown on figures) *white*, those of the second type, *black*.

A pair (v,e), $v\in V$, $e\in E$, is called a *flag* if v is an endpoint of e. To each flag (v,e) we assign an independent variable $x_{v,e}$. Let u and v be two endpoints of e. The *edge weight* w_e is defined by $w_e=x_{v,e}x_{u,e}$. A *perfect network* $N=(G,w,\varrho_1,...,\varrho_c)$ is obtained from G, weighted as above, by adding c non-intersecting oriented curves ϱ_i (called *cuts*) in such a way that cutting S along all ϱ_i makes it into a disk (note that the endpoints of each cut belong to distinct connected components of ∂S). The points of the *space of edge weights* $\mathcal{E}_N=(\mathbb{R}\setminus\{0\})^{|E|}$ (or $(\mathbb{C}\setminus\{0\})^{|E|}$) can be considered as copies of the graph G with edges weighted by non-zero numbers obtained by specializing the variables $x_{v,e}$ to non-zero values.

Assign an independent variable λ_i to each cut ϱ_i. The *weight* of a path P between two boundary vertices is defined as the product of the weights of all edges constituting the path times a Laurent monomial in λ_i. Each intersection point of P with ϱ_i contributes to this monomial λ_i if the oriented tangents to P and ϱ_i at this point form a positively oriented basis, and λ_i^{-1} otherwise (assuming that all intersection points are transversal). Besides, the sign of the monomial is defined via the rotation number of a certain closed curve built from P itself, and cuts and arcs of ∂S. For a detailed description of the corresponding constructions, see [21] and [29] in the case $c=0$ (networks in a disk, no cuts needed, the path weight is a signed product of the edge weights) and [22] in the case $c=1$ (networks in an annulus, one cut ϱ and one additional independent variable λ involved, the path weight is a signed product of the edge weights times an integer power of λ). The *boundary measurement* between a source b_i and a sink b_j is then defined as the sum of path weights over all (not necessary simple) paths from b_i to b_j. It is proved in the above cited papers that a boundary measurement is a rational function in the weights of edges (in the case of the disk) or in the weights of edges and λ (in the case of the annulus).

Boundary measurements are organized in the *boundary measurement matrix*, and thus give rise to the *boundary measurement map* from $\mathcal{E}_N$ to the space of $n\times m$ matrices (for $c=0$), or the space of $n\times m$ rational matrix functions (for $c=1$). The gauge group acts on $\mathcal{E}_N$ as follows: for any internal vertex v of N and any Laurent monomial L in the weights w_e of N, the weights of all edges leaving v are multiplied by L, and the weights of all edges entering v are multiplied by L^{-1}. Clearly, the weights of paths between boundary vertices, and hence boundary measurements, are preserved under this action.

Therefore, the boundary measurement map can be factorized through the space $\mathcal{F}_N$ defined as the quotient of $\mathcal{E}_N$ by the action of the gauge group. In [22] we explained that $\mathcal{F}_N$ can be identified with the relative cohomology group $H^1(G, G\cap\partial S)$ with coefficients in the multiplicative group of non-zero real numbers. This gives rise to the representation

$$\mathcal{F}_N = H^1(G\cup\partial S)/H^1(\partial S)\oplus H^0(\partial S)/H^0(G\cup\partial S) = \mathcal{F}_N^f \oplus \mathcal{F}_N^t.$$

The space $\mathcal{F}_N^f$ can be described as follows. The graph G divides S into a finite number of connected components called *faces*. The boundary of each face consists of edges of G and, possibly, of several arcs of ∂S. A face is called *bounded* if its boundary contains only edges of G and *unbounded* otherwise. Given a face f, we define its *face weight* y_f as the function on $\mathcal{E}_N$ which assigns to the edge weights w_e, $e\in E$, the value

$$y_f = \prod_{e\in\partial f} w_e^{\gamma_e},$$

where $\gamma_e=1$ if the direction of e is compatible with the counterclockwise orientation of the boundary ∂f and $\gamma_e=-1$ otherwise. Face weights are invariant under the gauge group action, and hence are functions on $\mathcal{F}_N^f$, and, moreover, form a basis in the space of such functions.

In [21] and [22] we studied the ways to turn the space of edge weights into a Poisson manifold by considering Poisson brackets on the space of flag variables satisfying certain natural conditions. We proved that all such Poisson brackets on $\mathcal{E}_N$ form a 6-parameter family, and that this family gives rise to a 2-parameter family of Poisson brackets on $\mathcal{F}_N$. In what follows we are interested in a specific member of the latter family (obtained by setting $\alpha=\frac{1}{2}$ and $\beta=-\frac{1}{2}$ in the notation of [21] and [22]). For reasons that will be explained later, we call this bracket the *standard* Poisson bracket on $\mathcal{F}_N$. The corresponding 4-parameter family of Poisson brackets on $\mathcal{E}_N$ is called *standard* as well.

Given a perfect network N as above, define the *directed dual network* $N^*=(G^*, w^*)$ as follows. Vertices of G^* are the faces of N. Edges of G^* correspond to the edges of N which connect either two internal vertices of different colors, or an internal vertex with a boundary vertex; note that there might be several edges between the same pair of vertices in G^*. An edge e^* of G^* corresponding to e is directed in such a way that the white endpoint of e (if it exists) lies to the left of e^* and the black endpoint of e (if it exists) lies to the right of e. The weight $w^*(e^*)$ equals 1 if both endpoints of e are internal vertices, and $\frac{1}{2}$ if one of the endpoints of e is a boundary vertex.

PROPOSITION 2.2. *The restriction of the standard Poisson bracket on $\mathcal{F}_N$ to the space $\mathcal{F}_N^f$ is given by*

$$\{y_f, y_{f'}\} = \Bigg(\sum_{e^*: f\to f'} w^*(e^*) - \sum_{e^*: f'\to f} w^*(e^*)\Bigg) y_f y_{f'}.$$

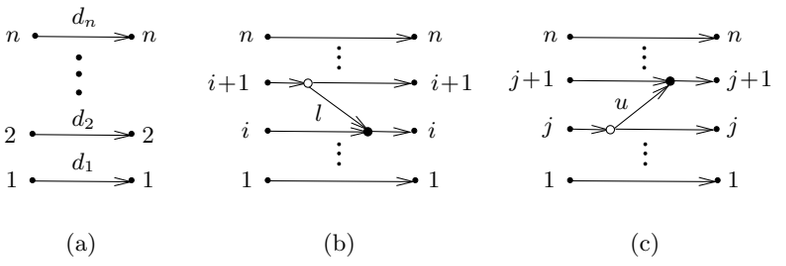


Figure 1. Three building blocks used in matrix factorization.

For networks in a disk, the above proposition is a special case of [21, Lemma 5.3]. For other surfaces the proof is literally the same.

In what follows, we will deal with networks of two kinds: acyclic networks in a disk with the same number of non-alternating sources and sinks, and networks in an annulus obtained from the networks of the first kind by a certain construction, to be described below.

In the former case we assume that n sources are numbered clockwise and are followed by n sinks numbered counterclockwise. The weight of a path in this case is exactly the product of edge weights involved. The boundary measurements are organized into a $n\times n$ matrix X in such a way that X_{ij} is the boundary measurement between the ith source and the jth sink. One can concatenate two networks of this kind by gluing the sinks of the former to the sources of the latter. If X_1 and X_2 are the matrices associated with the two networks, then the matrix associated with their concatenation is X_1X_2. This fact can be used to visualize parametrization (2.2). Indeed, an $n\times n$ diagonal matrix $\operatorname{diag}(d_1,...,d_n)$ and elementary bidiagonal matrices $E_i^-(l)$ and $E_j^+(u)$ defined by (2.1) correspond to building blocks shown on Figure 1 (a), (b) and (c), respectively; all weights not shown explicitly are equal to 1. Note that building blocks themselves are not networks, since their edge weights do not comply with the rules introduced above. However, as we will see below, objects glued from building blocks comply with all the rules.

The concatenation of $n(n-1)$ building blocks of the second and the third types and one building block of the first type, in an appropriately chosen order and with each building block having its own non-trivial weights, describes a generic element of GL_n (see, e.g. [11]). The structure of the obtained network is given by Figure 2. Here and in what follows we use the gauge group action to decrease the number of parameters of networks in question. In particular, this network has $2n(n-1)$ internal vertices, and therefore, one can use the gauge group action to change the weights of $2n(n-1)$ edges to 1. It is convenient to choose these edges to be all the horizontal edges except for one middle edge in each horizontal chain. The weights on the remaining edges are Laurent

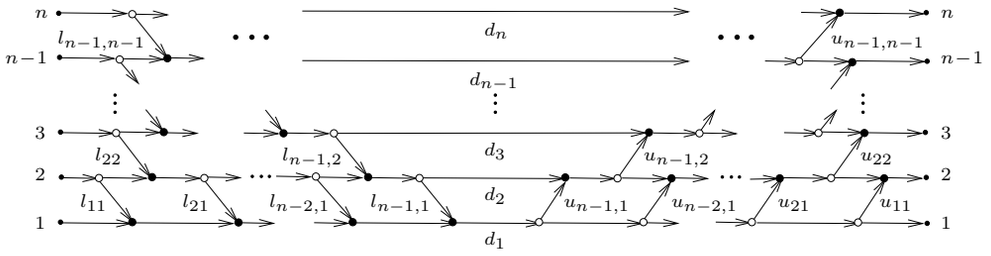


Figure 2. A generic planar network; the weights of edges are Laurent monomials of the initial weights.

monomials in the initial weights of the network. For example, if the endpoints of an edge e belong to levels i and $i+1$, then $u_e=w_e w_{P_{i+1}}/w_{P_i}$, where P_i and P_{i+1} are the horizontal paths from the endpoints of e to the sinks i and $i+1$, respectively.

The following result, which is a special case of Theorem 4.1 from [21], explains why we call the bracket in consideration standard.

THEOREM 2.1. *For any network N as above with n sources and n sinks, the map from $\mathcal{E}_N$ to the space of $n\times n$ matrices given by the boundary measurement matrix is Poisson with respect to any standard Poisson bracket on $\mathcal{E}_N$ and the standard Sklyanin bracket on* GL_n.

Remark 2.2. Note that the definition of the R-matrix $R_{\alpha,\beta}$ in [21] contains a superfluous factor $\frac{1}{2}$.

Networks in an annulus that we study in this paper are obtained from the above described networks in a disk by a gluing procedure described in detail in §5. These networks have one source and one sink, both lying on the same connected component of the boundary. The other connected component of the boundary does not carry boundary vertices, and hence for our networks $H^0(\partial S)=H^0(G\cup\partial S)$, which implies that $\mathcal{F}_N=\mathcal{F}_N^f$. Therefore, the standard Poisson bracket on $\mathcal{F}_N$ is completely described by Proposition 2.2.

3. Coxeter double Bruhat cells

We start this section with describing a particular instance of the Berenstein–Fomin–Zelevinsky parametrization [2], [15] in the case of Coxeter double Bruhat cells in GL_n.

Let $s_{[p,q]}=s_p s_{p+1}\dots s_{q-1}$ for $1\leqslant p<q\leqslant n$ and recall that every Coxeter element $v\in S_n$ can be written in the form

$$v = s_{[i_{k-1},i_k]}\dots s_{[i_1,i_2]}s_{[1,i_1]} \tag{3.1}$$

for some subset $I=\{1=i_0<i_1<\dots<i_k=n\}\subseteq[1,n]$. Besides, define $L=\{1=l_0<l_1<\dots<l_{n-k}=n\}$ by $\{l_1<\dots<l_{n-k-1}\}=[1,n]\setminus I$.

LEMMA 3.1. *Let v be given by* (3.1). *Then*

$$v^{-1} = s_{[l_{n-k-1},l_{n-k}]} \cdots s_{[l_1,l_2]} s_{[1,l_1]}.$$

Proof. We use induction on n. Denote the right-hand side of the above relation by $\bar{v}$. The index $n-1$ belongs either to I or to L. In the latter case $l_{n-k-1}=n-1$, and we have

$$v = s_{[i_{k-1},n]} \cdots s_{[i_1,i_2]} s_{[1,i_1]} = s_{[i_{k-1},n-1]} \cdots s_{[i_1,i_2]} s_{[1,i_1]} s_{n-1}$$

and

$$\bar{v} = s_{n-1} s_{[l_{n-k-2},l_{n-k-1}]} \cdots s_{[l_1,l_2]} s_{[1,l_1]}.$$

Then $v=v's_{n-1}$ and $\bar{v}=s_{n-1}\bar{v}'$, where v' and $\bar{v}'$ are the Coxeter elements in S_{n-1} corresponding to the index sets $I\setminus\{n\}\cup\{n-1\}$ and $L\setminus\{n\}$, respectively, and hence $v\bar{v}=v'\bar{v}'=1$ by the induction hypothesis. Otherwise, if $n-1$ belongs to I, we interchange the roles of v and $\bar{v}$ and use the same argument. □

LEMMA 3.2. *The permutation matrix corresponding to a Coxeter element v is*

$$\tilde{v} = \sum_{j=1}^{k} e_{i_{j-1}i_j} + \sum_{j=1}^{n-k} e_{l_j l_{j-1}}.$$

Proof. We use the same inductive argument as in the proof of Lemma 3.1. Assuming that $n-1\in L$, the relation $v=v's_{n-1}$ and the induction hypothesis imply that

$$\begin{aligned}\tilde{v} &= (e_{1i_1}+\ldots+e_{i_{k-1}\,n-1}+e_{l_1 1}+\ldots+e_{n-1\,l_{n-k-2}}+e_{nn})(e_{11}+\ldots+e_{n-2\,n-2}+e_{n\,n-1}+e_{n-1\,n})\\ &= e_{1i_1}+\ldots+e_{i_{k-1}n}+e_{l_1 1}+\ldots+e_{n-1\,l_{n-k-2}}+e_{n\,n-1}\end{aligned}$$

as claimed. □

Let now (u,v) be a pair of Coxeter elements and let

$$\begin{aligned} I^+ &= \{1=i_0^+<i_1^+<\ldots<i_{k^+}^+=n\},\\ I^- &= \{1=i_0^-<i_1^-<\ldots<i_{k^-}^-=n\},\\ L^+ &= \{1=l_0^+<l_1^+<\ldots<l_{n-k^+-1}^+<l_{n-k^+}^+=n\},\\ L^- &= \{1=l_0^-<l_1^-<\ldots<l_{n-k^--1}^-<l_{n-k^-}^-=n\}\end{aligned} \tag{3.2}$$

be subsets of $[1,n]$ corresponding to v and u^{-1} in the way just described. For a set of complex parameters $c_1^-,\ldots,c_{n-1}^-;c_1^+,\ldots,c_{n-1}^+;d_1,\ldots,d_n$, define matrices $D=\mathrm{diag}(d_1,\ldots,d_n)$,

$$\begin{aligned} C_j^+ &= \sum_{\alpha=i_{j-1}^+}^{i_j^+-1} c_\alpha^+ e_{\alpha,\alpha+1},\ j\in[1,k^+], & C_j^- &= \sum_{\alpha=i_{j-1}^-}^{i_j^--1} c_\alpha^- e_{\alpha+1,\alpha},\ j\in[1,k^-],\\ \overline{C}_j^+ &= \sum_{\alpha=l_{j-1}^+}^{l_j^+-1} c_\alpha^+ e_{\alpha,\alpha+1},\ j\in[1,n-k^+], & \overline{C}_j^- &= \sum_{\alpha=l_{j-1}^-}^{l_j^--1} c_\alpha^- e_{\alpha+1,\alpha},\ j\in[1,n-k^-].\end{aligned} \tag{3.3}$$

LEMMA 3.3. *A generic element $X \in G^{u,v}$ can be written as*

$$X = (\mathbf{1}-C_1^-)^{-1} \dots (\mathbf{1}-C_{k^-}^-)^{-1} D (\mathbf{1}-C_{k^+}^+)^{-1} \dots (\mathbf{1}-C_1^+)^{-1}, \tag{3.4}$$

and its inverse can be factored as

$$X^{-1} = (\mathbf{1}+\overline{C}_{n-k^+}^+)^{-1} \dots (\mathbf{1}+\overline{C}_1^+)^{-1} D^{-1} (\mathbf{1}+\overline{C}_1^-)^{-1} \dots (\mathbf{1}+\overline{C}_{k^-}^-)^{-1}. \tag{3.5}$$

Proof. It is easy to see that

$$\begin{aligned}
(\mathbf{1}-C_j^+)^{-1} &= E^+_{i^+_{j-1}}(c^+_{i^+_{j-1}}) \dots E^+_{i^+_j-1}(c^+_{i^+_j-1}),\\
(\mathbf{1}-C_j^-)^{-1} &= E^-_{i^-_j-1}(c^-_{i^-_j-1}) \dots E^-_{i^-_{j-1}}(c^-_{i^-_{j-1}}),\\
(\mathbf{1}+\overline{C}_j^+)^{-1} &= E^+_{l^+_{j-1}}(-c^+_{l^+_{j-1}}) \dots E^+_{l^+_j-1}(-c^+_{l^+_j-1}),\\
(\mathbf{1}+\overline{C}_j^-)^{-1} &= E^-_{l^-_j-1}(-c^-_{l^-_j-1}) \dots E^-_{l^-_{j-1}}(-c^-_{l^-_{j-1}}).
\end{aligned} \tag{3.6}$$

Then, by (2.2) and (3.1), a generic $X \in G^{u,v}$ can be written as in (3.4). Next, the same reasoning as in the proof of Lemma 3.1 implies that

$$\begin{aligned}
(\mathbf{1}-C_1^+) \dots (\mathbf{1}-C_{k^+}^+) &= \Bigg(\Bigg(\prod_{s=i^+_{k-1}}^{i^+_k-1} E^+_s(c^+_s) \Bigg) \dots \Bigg(\prod_{s=1}^{i^+_1-1} E^+_s(c^+_s) \Bigg) \Bigg)^{-1}\\
&= \Bigg(\prod_{s=l^+_{n-k-1}}^{l^+_{n-k}-1} E^+_s(-c^+_s) \Bigg) \dots \Bigg(\prod_{s=1}^{l^+_1-1} E^+_s(-c^+_s) \Bigg)\\
&= (\mathbf{1}+\overline{C}_{n-k^+}^+)^{-1} \dots (\mathbf{1}+\overline{C}_1^+)^{-1},
\end{aligned}$$

and, similarly,

$$(\mathbf{1}-C_{k^-}^-) \dots (\mathbf{1}-C_1^-) = (\mathbf{1}+\overline{C}_1^-)^{-1} \dots (\mathbf{1}+\overline{C}_{k^-}^-)^{-1}.$$

Therefore,

$$\begin{aligned}
X^{-1} &= (\mathbf{1}-C_1^+) \dots (\mathbf{1}-C_{k^+}^+) D^{-1} (\mathbf{1}-C_{k^-}^-) \dots (\mathbf{1}-C_1^-)\\
&= (\mathbf{1}+\overline{C}_{n-k^+}^+)^{-1} \dots (\mathbf{1}+\overline{C}_1^+)^{-1} D^{-1} (\mathbf{1}+\overline{C}_1^-)^{-1} \dots (\mathbf{1}+\overline{C}_{k^-}^-)^{-1}.
\end{aligned}$$

□

The network $N_{u,v}$ corresponding to the factorization (3.4) is obtained by the concatenation (left to right) of $2n-1$ building blocks (as depicted in Figure 1) corresponding to the elementary matrices

$$\begin{aligned}
&E^-_{i^-_2-1}(c^-_{i^-_2-1}), \dots, E^-_1(c^-_1), E^-_{i^-_3-1}(c^-_{i^-_3-1}), \dots, E^-_{i^-_2}(c^-_{i^-_2}), \dots,\\
&E^-_{n-1}(c^-_{n-1}), \dots, E^-_{i^-_{k^--1}}(c^-_{i^-_{k^--1}}), D, E^+_{i^+_{k^+-1}}(c^+_{i^+_{k^+-1}}), \dots,\\
&E^+_{n-1}(c^+_{n-1}), \dots, E^+_{i^+_2}(c^+_{i^+_2}), \dots, E^+_{i^+_3-1}(c^+_{i^+_3-1}), E^+_1(c^+_1), \dots, E^+_{i^+_2-1}(c^+_{i^+_2-1}).
\end{aligned}$$

This network has $4(n-1)$ internal vertices and $5n-4$ horizontal edges. Similarly to the case of generic networks discussed in §2.4, one can use the gauge group action to change the weights of all horizontal edges, except for those belonging to block D, to 1.

One can use the network $N_{u,v}$ to derive expressions for the factorization parameters d_i, c_i^+ and c_i^- in terms of the matrix entries of X. These formulas are a simple particular case of the general formulas by Berenstein–Fomin–Zelevinsky for restoring factorization parameters in double Bruhat cells [2], [15].

For a matrix A, denote by $A^{p_1,\dots,p_k}_{r_1,\dots,r_k}$ its minor formed by rows numbered $r_1<\dots<r_k$ and columns $p_1<\dots<p_k$.

LEMMA 3.4. *For any $i\in[1,n]$, we have*

$$d_i=\frac{X^{[1,i]}_{[1,i]}}{X^{[1,i-1]}_{[1,i-1]}};$$

for any $i\in[1,n-1]$, we have

$$c_i^-=\frac{X^{i_0^-,\dots,i_j^-}_{i_1^-,\dots,i_j^-,i+1}}{X^{i_0^-,\dots,i_j^-}_{i_1^-,\dots,i_j^-,i}},\ i_j^-<i<i_{j+1}^-,\quad c^-_{i_j^-}=\frac{X^{i_0^-,\dots,i_j^-}_{i_1^-,\dots,i_j^-,i_j^-+1}X^{[1,i_j^--1]}_{[1,i_j^--1]}}{X^{i_0^-,\dots,i_{j-1}^-}_{i_1^-,\dots,i_j^-}X^{[1,i_j^-]}_{[1,i_j^-]}}$$

$$c_i^+=\frac{X^{i_1^+,\dots,i_j^+,i+1}_{i_0^+,\dots,i_j^+}}{X^{i_1^+,\dots,i_j^+,i}_{i_0^+,\dots,i_j^+}},\ i_j^+<i<i_{j+1}^+,\quad c^+_{i_j^+}=\frac{X^{i_1^+,\dots,i_j^+,i_j^++1}_{i_0^+,\dots,i_j^+}X^{[1,i_j^+-1]}_{[1,i_j^+-1]}}{X^{i_1^+,\dots,i_j^+}_{i_0^+,\dots,i_{j-1}^+}X^{[1,i_j^+]}_{[1,i_j^+]}}.$$

Proof. Since by (3.4), $X^{[1,i]}_{[1,i]}=D^{[1,i]}_{[1,i]}=d_1\dots d_i$, the first formula follows easily. Next, note that, for $i_j^-<i\leqslant i_{j+1}^-$, there is exactly one directed path in $N_{u,v}$ joining the ith source with the i_j^-th sink. The weight of this path equals $c^-_{i-1}\dots c^-_{i_j}d_{i_j^-}$. Thus, there is a unique collection of vertex-disjoint paths in $N_{u,v}$ joining sources $i_1^-,\dots,i_j^-,i$ with sinks $i_0^-,\dots,i_j^-$. By Lindström's lemma [24], [27], this implies that $X^{i_1^-,\dots,i_j^-,i}_{i_0^-,\dots,i_j^-}$ is equal to the product of the weights of these paths. Clearly, for $i_j^-<i<i_{j+1}^-$, we have

$$X^{i_1^-,\dots,i_j^-,i+1}_{i_0^-,\dots,i_j^-}=c_i^-X^{i_1^-,\dots,i_j^-,i}_{i_0^-,\dots,i_j^-}.$$

Also,

$$X^{i_1^-,\dots,i_j^-,i_j^-+1}_{i_0^-,\dots,i_j^-}=c^-_{i_j^-}d^-_{ij}X^{i_1^-,\dots,i_j^-}_{i_0^-,\dots,i_{j-1}^-}.$$

Formulas for c_i^- follow from these relations. Formulas for c_i^+ are derived in a similar way. □

Let us now introduce some combinatorial data that will be useful in the following sections.

Let us fix a pair (u,v) of Coxeter elements, and hence, fix the sets $I^{\pm}$ given by (3.2). For any $i\in[1,n]$ define integers $\varepsilon_i^{\pm}$ and $\zeta_i^{\pm}$ by setting

$$\varepsilon_i^{\pm}=\begin{cases}0, & \text{if } i=i_j^{\pm} \text{ for some } 0<j\leqslant k_{\pm},\\ 1, & \text{otherwise,}\end{cases} \tag{3.7}$$

and

$$\zeta_i^{\pm}=i(1-\varepsilon_i^{\pm})-\sum_{\beta=1}^{i-1}\varepsilon_{\beta}^{\pm}. \tag{3.8}$$

Note that, by definition, $\varepsilon_1^{\pm}=1$ and $\zeta_1^{\pm}=0$. (Here and in what follows a relation involving variables with superscripts $\pm$ is a shorthand for two similar relations: the one obtained by simultaneously replacing each $\pm$ by $+$, and the other, by $-$.) Further, put

$$M_i^{\pm}=\{\zeta_{\alpha}^{\pm}:\alpha=1,...,i\}$$

and

$$k_i^{\pm}=\max\{j:i_j^{\pm}\leqslant i\}. \tag{3.9}$$

Finally, put

$$\varepsilon_i=\varepsilon_i^{+}+\varepsilon_i^{-} \tag{3.10}$$

and

$$\varkappa_i=i+1-\sum_{\beta=1}^{i}\varepsilon_{\beta}. \tag{3.11}$$

Remark 3.1. It is easy to see that there exist distinct pairs (u,v) and (u',v') which produce the same n-tuple ε. The ambiguity occurs when $\varepsilon_i=1$ for some $i\in[2,n-1]$. By (3.10), this situation corresponds either to $\varepsilon_i^{+}=1$ and $\varepsilon_i^{-}=0$, or to $\varepsilon_i^{+}=0$ and $\varepsilon_i^{-}=1$. Consequently, the number of pairs (u,v) with the identical n-tuple ε equals 2^a, where a is the number of times ε_i takes the value 1.

LEMMA 3.5. (i) *The n-tuples $\varepsilon^{\pm}=(\varepsilon_i^{\pm})_{i=1}^n$ and $\zeta^{\pm}=(\zeta_i^{\pm})_{i=1}^n$ uniquely determine each other.*

(ii) *For any $i\in[1,n]$,*

$$\zeta_i^{\pm}=\begin{cases}j, & \text{if } i=i_j^{\pm} \text{ for some } 0<j\leqslant k_{\pm},\\ -\sum_{\beta=1}^{i-1}\varepsilon_{\beta}^{\pm}, & \text{otherwise.}\end{cases}$$

(iii) *For any $i\in[1,n]$,*

$$k_i^{\pm}=i-\sum_{\beta=1}^{i}\varepsilon_{\beta}^{\pm}.$$

(iv) *For any* $i\in[1,n]$,

$$M_i^{\pm}=[k_i^{\pm}-i+1,k_i^{\pm}]=\left[1-\sum_{\beta=1}^{i}\varepsilon_\beta^{\pm},i-\sum_{\beta=1}^{i}\varepsilon_\beta^{\pm}\right].$$

Proof. (i) This follows form the fact that the transformation $\varepsilon^{\pm}\mapsto\zeta^{\pm}$ defined by (3.8) is given by a lower–triangular matrix with a non-zero diagonal.

(ii) By (3.8), the first equality is equivalent to

$$i_j^{\pm}=j+\sum_{\beta=1}^{i_j^{\pm}-1}\varepsilon_\beta^{\pm}. \tag{3.12}$$

By (3.7), the latter can be interpreted as counting the first $i_j^{\pm}$ elements of $(\varepsilon_i^{\pm})_{i=1}^n$: exactly j of them are equal to 0, and all the other are equal to 1.

The second equality follows trivially from (3.7) and (3.8).

(iii) For $i=i_j^{\pm}$, this follows immediately from (3.9) and (3.12). For $i\neq i_j^{\pm}$, the counting argument used in (3.12) gives

$$i=k_i^{\pm}+\sum_{\beta=1}^{i}\varepsilon_\beta^{\pm}.$$

(iv) This follows from parts (ii) and (iii). □

Remark 3.2. (i) If $v=s_{n-1}\dots s_1$, then X is a lower Hessenberg matrix, and if $u=s_1\dots s_{n-1}$, then X is an upper Hessenberg matrix.

(ii) If $v=s_{n-1}\dots s_1$ and $u=s_1\dots s_{n-1}$, then $G^{u,v}$ consists of tridiagonal matrices with non-zero off-diagonal entries (*Jacobi matrices*). In this case $I^+=I^-=[1,n]$, $\varepsilon_1^{\pm}=1$ and $\varepsilon_i^{\pm}=0$ for $i=2,\dots,n$.

(iii) If $u=v=s_{n-1}\dots s_1$ (which leads to $I^+=[1,n]$ and $I^-=\{1,n\}$), then elements of $G^{u,v}$ have a structure of recursion operators arising in the theory of orthogonal polynomials on the unit circle (see, e.g. [32]).

(iv) The choice $u=v=(s_1s_3\dots)(s_2s_4\dots)$ (the so-called *bipartite Coxeter element*) gives rise to a special kind of pentadiagonal matrices X (called *CMV matrices*, after Cantero, Moral and Velázquez), which serve as an alternative version of recursion operators for orthogonal polynomials on the unit circle, see [6] and [32].

Example 3.1. Let $n=5$, $v=s_4s_3s_1s_2$ and $u=s_3s_2s_1s_4$. The network $N_{u,v}$ corresponding to the factorization (3.4) is shown in Figure 3.

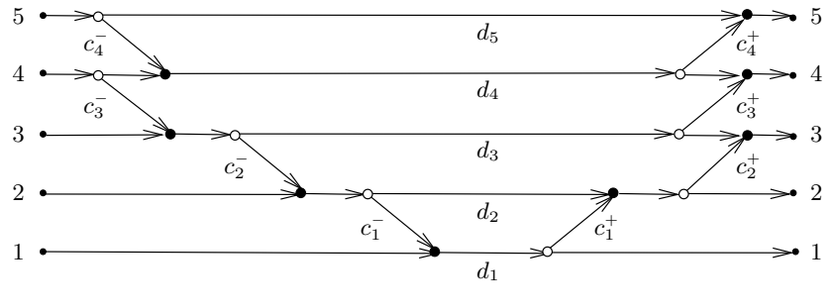


Figure 3. Network representation for elements in $G^{s_3s_2s_1s_4,s_4s_3s_1s_2}$.

A generic element $X\in G^{u,v}$ has the form

$$X=(x_{ij})_{i,j=1}^5=\begin{pmatrix} d_1 & x_{11}c_1^+ & x_{12}c_2^+ & 0 & 0\\ c_1^-x_{11} & d_2+c_1^-x_{12} & x_{22}c_2^+ & 0 & 0\\ c_2^-x_{21} & c_2^-x_{22} & d_3+c_2^-x_{23} & d_3c_3^+ & 0\\ c_3^-x_{31} & c_3^-x_{32} & c_3^-x_{33} & d_4+c_3^-x_{34} & d_4c_4^+\\ 0 & 0 & 0 & c_4^-d_4 & d_5+c_4^-x_{45}\end{pmatrix}.$$

One finds, by direct observation, that $k^+=3$ and $I^+=\{i_0^+,i_1^+,i_2^+,i_3^+\}=\{1,3,4,5\}$, and hence $L^+=\{l_0^+,l_1^+,l_2^+\}=\{1,2,5\}$. Next, $u^{-1}=s_4s_1s_2s_3$, and therefore, $k^-=2$ and $I^-=\{i_0^-,i_1^-,i_2^-\}=\{1,4,5\}$, and hence $L^-=\{l_0^-,l_1^-,l_2^-,l_3^-\}=\{1,2,3,5\}$. Further,

$$\varepsilon^+=(1,1,0,0,0)\quad\text{and}\quad\varepsilon^-=(1,1,1,0,0),$$

and hence

$$\zeta^+=(0,-1,1,2,3)\quad\text{and}\quad\zeta^-=(0,-1,-2,1,2).$$

Therefore,

$$(k_i^+)_{i=1}^5=(0,0,1,2,3)\quad\text{and}\quad(k_i^-)_{i=1}^5=(0,0,0,1,2),$$

and hence

$$(M_i^+)_{i=1}^5=([0,0],[-1,0],[-1,1],[-1,2],[-1,3]),$$
$$(M_i^-)_{i=1}^5=([0,0],[-1,0],[-2,0],[-2,1],[-2,2]).$$

Finally, $\varepsilon=(2,2,1,0,0)$ and $\varkappa=(0,-1,-1,0,1)$. By Remark 3.1, there is one more pair of Coxeter elements producing the same 5-tuples: $v'=s_4s_1s_2s_3$ and $u'=s_2s_1s_3s_4$.

To illustrate Lemma 3.4, we find c_3^- and c_4^-. First, $1=i_0^-<3<i_1^-=4$, so the corresponding formula in Lemma 3.4 gives $c_3^-=X_4^1/X_3^1$. Second, $4=i_1^-$, so the other formula in Lemma 3.4 gives $c_4^-=X_{4,5}^{1,4}X_{[1,3]}^{[1,3]}/X_4^1X_{[1.4]}^{[1,4]}$. It is easy to check that the right-hand sides of both formulas indeed produce correct answers. We will use this example as our running example in the next section.

We conclude this section with a proposition explaining how to recognize an element of a Coxeter double Bruhat cell in GL_n. For any two subsets $R, P \subseteq [1,n]$ and a matrix $X \in \mathrm{GL}_n$, we denote by $X(R,P)$ the submatrix of X formed by rows $r \in R$ and columns $p \in P$.

PROPOSITION 3.1. *An element $X \in \mathrm{GL}_n$ belongs to a Coxeter double Bruhat cell if and only if all the following conditions hold for any $l \in [1,n-1]$:*

(i_+) $\operatorname{rank} X([l+1,n],[1,l])=1$;

(i_-) $\operatorname{rank} X([1,l],[l+1,n])=1$;

(ii_+) $\operatorname{rank} X([1,n],[1,l])>1$ *implies that* $X([l+1,n],[1,l-1])=0$;

(ii_-) $\operatorname{rank} X([1,l],[1,n])>1$ *implies that* $X([1,l-1],[l+1,n])=0$.

Proof. Let $X \in G^{u,v}$. Note that the rank of the submatrix $X([r,n],[1,p])$, for any $p,r \in [1,n]$, does not change under right and left multiplication of X by elements of $\mathbf{B}_+$. Since $G^{u,v} \subset \mathbf{B}_+ u \mathbf{B}_+$, this means that we only need to check conditions (i_+) and (ii_+) for the permutation matrix $\tilde{u}$, for which they are clearly true in view of Lemma 3.2. Similarly, conditions (i_-) and (ii_-) reduce to considering $\tilde{v}$.

On the other hand, let X satisfy condition (i_+) for any $l \in [1,n-1]$, and let i_1^- be the largest index such that $x_{i_1^- 1} \neq 0$. Condition (i_+) for $l=1$ implies that $i_1^- > 1$. Further, condition (i_+) for $l=i_1^- -1$ implies that $X([i_1^-+1,n],[1,i_1^- -1])=0$. Similarly, we can define i_2^- to be the largest index such that $x_{i_2^- i_1^-} \neq 0$ and conclude from condition (i_+) for $l=i_1^-$ that $i_2^- > i_1^-$, and from condition (i_+) for $l=i_2^- -1$ that $X([i_2^-+1,n],[i_1^-,i_2^- -1])=0$. Continuing in this manner, we construct a sequence $I_-=\{1=i_0^-<i_1^-<\ldots<i_{k^-}^-=n\}$ such that

$$x_{i_s^- i_{s-1}^-} \neq 0 \quad \text{and} \quad X([i_s^-+1,n],[i_{s-1}^-,i_s^- -1]) = 0 \tag{3.13}$$

for $s \in [1,k^- -1]$. Multiplying X on the right and on the left by appropriate elements of $\mathbf{B}_+$, we can reduce it to a matrix $X'=(x'_{ij})$ satisfying (3.13) and such that $x'_{i i_{s-1}^-}=\delta_{i i_s^-}$ and $x'_{i_s^- j}=\delta_{i_{s-1}^- j}$. Moreover, condition ($\mathrm{ii}_+$) implies that $x'_{ij}=0$ for $i_{s-1}<j<i<i_s$. To summarize, the lower triangular part of X' is $e_{i_1^- 1}+e_{i_2^- i_1^-}+\ldots+e_{i_{k^- -1}^- n}$.

Now, let s be the smallest index such that $i_s^- > s+1$. Then $i_1^-=2,\ldots,i_{s-1}^-=s$. Consider the $(s+1)$-st column of X'. Entries $x'_{i\,s+1}$ are zero for $i>i_s^-$ due to (3.13) and for $1<i\leqslant i_s^-$ due to the properties of X' described above. Since X' is invertible, this means that $x_{1\,s+1} \neq 0$. Then the right multiplication by an invertible upper triangular matrix reduces X' to a matrix X'' such that $x''_{1\,s+1}=1$ and the rest of the first row entries are equal to zero, while the lower triangular part of X'' and the entries in the strictly upper triangular part made zero by previous reductions are left unchanged. Comparing with Lemma 3.2, we see that the lower triangular part and the first s rows of X'' coincide with those of a permutation matrix corresponding to some Coxeter element v of S_n. Contin-

uing in the same fashion, we can eventually reduce X'' through right multiplication by upper triangular matrices to the permutation matrix $\tilde{u}$, thus showing that $X \in \mathbf{B}_+ u \mathbf{B}_+$.

The same argument can be used to show that $X \in \mathbf{B}_- v \mathbf{B}_-$ for some Coxeter element v, based on conditions (i$_-$) and (ii$_-$). This completes the proof. □

4. Inverse problem

4.1. In this section we show how an element X of a Coxeter double Bruhat cell $G^{u,v}$ which admits the factorization (3.4) can be restored from its Weyl function (1.3) up to a conjugation by a diagonal matrix.

Recall the useful representations for the Weyl function

$$m(\lambda; X) = ((\lambda \mathbf{1} - X)^{-1} e_1, e_1) = \frac{q(\lambda)}{p(\lambda)} = \sum_{j=0}^{\infty} \frac{h_j(X)}{\lambda^{j+1}}. \tag{4.1}$$

Here e_i denotes the vector $(\delta_{i\alpha})_{\alpha=1}^n$ of the standard basis in $\mathbb{C}^n$, $(\cdot\,,\cdot)$ is the standard inner product, $p(\lambda)$ is the characteristic polynomial of X, $q(\lambda)$ is the characteristic polynomial of the $(n-1)\times(n-1)$ submatrix of X formed by deleting the first row and column, and

$$h_j(X) = (X^j)_{11} = (X^j e_1, e_1), \quad j \in \mathbb{Z},$$

is the *j-th moment* of X. (Only moments with non-negative indices are present in (4.1), however, $h_j(X)$ for $j<0$, which we will need below, are also well defined, since X is invertible.) In what follows, when it does not lead to confusion, we occasionally omit the argument and write h_j instead of $h_j(X)$.

To solve the inverse problem, we generalize the approach of [14], where only the cases of symmetric or Hessenberg X were treated. The main idea stems from the classical moments problem [1]: one considers the space $\mathbb{C}[\lambda, \lambda^{-1}]/\det(\lambda - X)$ equipped with the so-called *moment functional*—a bilinear functional $\langle \cdot\,, \cdot \rangle$ on Laurent polynomials in one variable, uniquely defined by the property

$$\langle \lambda^i, \lambda^j \rangle = h_{i+j}. \tag{4.2}$$

X is then realized as a matrix of the operator of multiplication by λ relative to appropriately selected bases $(p_i^+(\lambda))_{i=0}^{n-1}$ and $(p_i^-(\lambda))_{i=0}^{n-1}$ biorthogonal with respect to the moment functional:

$$\langle p_i^-(\lambda), p_j^+(\lambda) \rangle = \delta_{ij}.$$

For example, the classical tridiagonal case corresponds to the orthogonalization of the sequence $1, \lambda, ..., \lambda^{n-1}$. Elements of $G^{s_{n-1}\dots s_1, s_{n-1}\dots s_1}$ (cf. Remark 3.2 (iii)) result from the

biorthogonalization of the sequences $1, \lambda, ..., \lambda^{n-1}$ and $\lambda^{-1}, ..., \lambda^{1-n}$, while CMV matrices (Remark 3.2 (iv)) correspond to the biorthogonalization of the sequences $1, \lambda, \lambda^{-1}, \lambda^2, ...$ and $1, \lambda^{-1}, \lambda, \lambda^{-2}, ...$.

For any $l \in \mathbb{Z}$, $i \in \mathbb{N}$ define Hankel matrices

$$\mathcal{H}_i^{(l)} = (h_{\alpha+\beta+l-i-1})_{\alpha,\beta=1}^{i},$$

and Hankel determinants

$$\Delta_i^{(l)} = \det \mathcal{H}_i^{(l)}; \tag{4.3}$$

we assume that $\Delta_0^{(l)} = 1$ for any $l \in \mathbb{Z}$.

Remark 4.1. (i) Let X be an $n \times n$ matrix, then it follows from the Cayley–Hamilton theorem that, for $i > n$, the columns of $\mathcal{H}_i^{(l)}$ are linearly dependent and so $\Delta_i^{(l)} = 0$.

(ii) In what follows we will frequently use the identity

$$\Delta_{i+1}^{(l)} \Delta_{i-1}^{(l)} = \Delta_i^{(l-1)} \Delta_i^{(l+1)} - (\Delta_i^{(l)})^2, \tag{4.4}$$

which is a particular case of Jacobi's determinantal identity. In particular, for $i = n$, (4.4) and the first part of the remark imply that $\Delta_n^{(l-1)} \Delta_n^{(l+1)} = (\Delta_n^{(l)})^2$ for any l.

The following is the main result of this section.

THEOREM 4.1. *If $X \in G^{u,v}$ admits the factorization* (3.4), *then*

$$\begin{aligned} d_i &= \frac{\Delta_i^{(\varkappa_i+1)} \Delta_{i-1}^{(\varkappa_{i-1})}}{\Delta_i^{(\varkappa_i)} \Delta_{i-1}^{(\varkappa_{i-1}+1)}}, \\ c_i^+ c_i^- &= \frac{\Delta_{i-1}^{(\varkappa_{i-1})} \Delta_{i+1}^{(\varkappa_{i+1})}}{(\Delta_i^{(\varkappa_i+1)})^2} \left(\frac{\Delta_{i+1}^{(\varkappa_{i+1}+1)}}{\Delta_{i+1}^{(\varkappa_{i+1})}} \right)^{\varepsilon_{i+1}} \left(\frac{\Delta_{i-1}^{(\varkappa_{i-1}+1)}}{\Delta_{i-1}^{(\varkappa_{i-1})}} \right)^{2-\varepsilon_i} \end{aligned} \tag{4.5}$$

for any $i \in [1, n]$.

Remark 4.2. Formulas (4.5) allow us to restore an element $X \in G^{u,v}$ from its Weyl function $m(\lambda; X)$ only modulo the diagonal conjugation. Indeed, it is clear from (1.3) that $m(\lambda; X) = m(\lambda; TXT^{-1})$ for any invertible diagonal matrix $T = \operatorname{diag}(t_1, ..., t_n)$. On the other hand, under the action $X \mapsto TXT^{-1}$, the factorization parameters d_i and $c_i^{\pm}$ in (3.4) are transformed by $d_i \mapsto d_i$ and $c_i^{\pm} \mapsto (t_i/t_{i+1})^{\pm 1} c_i^{\pm}$, thus leaving the left-hand sides in (4.5) unchanged.

4.2. The rest of the section is devoted to the proof of Theorem 4.1. The proof relies on properties of polynomials of the form

$$\mathcal{P}_i^{(l)}(\lambda)=\det\begin{pmatrix} h_{l-i+1} & h_{l-i+2} & \dots & h_l & h_{l+1}\\ h_{l-i+2} & h_{l-i+3} & \dots & h_{l+1} & h_{l+2}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ h_l & h_{l+1} & \dots & h_{l+i-1} & h_{l+i}\\ 1 & \lambda & \dots & \lambda^{i-1} & \lambda^i \end{pmatrix}. \tag{4.6}$$

To prove the first equality in (4.5) we need two auxiliary lemmas.

LEMMA 4.1. *Let $m\in[1,n-1]$ and let X_m be the $m\times m$ submatrix of $X\in G^{u,v}$ obtained by deleting the $n-m$ last rows and columns. Then*

$$h_\alpha(X_m)=h_\alpha(X) \tag{4.7}$$

for $\alpha\in[\varkappa_m-m+1,\varkappa_m+m]$.

Proof. It is enough to prove the claim for $X\in G^{u,v}$ which admits the factorization (3.4). It is clear that X_m does not depend on the parameters $c_m^\pm,\dots,c_{n-1}^\pm$ and $d_{m+1},\dots,d_n$. Moreover, $X_m\in G^{u_m,v_m}$, where u_m and v_m are obtained from u and v, respectively, by deleting all transpositions s_i with $i\geqslant m$. Consequently, the network N_{u_m,v_m} can be obtained from the network $N_{u,v}$ by deleting all the edges above the horizontal line joining the mth source with the mth sink. Note also that if $\alpha>0$ then $h_\alpha(X)$ is the sum of path weights over all paths from the first source to the first sink in the network obtained by the concatenation of α copies $N_{u,v}$. Thus, $h_\alpha(X_m)=h_\alpha(X)$ as long as none of the paths involved reaches above the mth horizontal level. The smallest positive power of X such that in the corresponding network there is a path joining the first source to the first sink and reaching above the mth horizontal level is $r=r^+ +r^-$, where

$$r^\pm=\min\{j: i_j^\pm\geqslant m+1\}.$$

By (3.9), $r^\pm=k_m^\pm+1$. Therefore, (4.7) holds for $\alpha\in[0,k_m^+ +k_m^- +1]$. By Lemma 3.5 (iii), (3.10) and (3.11), the latter interval coincides with $[0,\varkappa_m+m]$.

Next, consider the network $\overline{N}_{u^{-1},v^{-1}}$ which represents X^{-1} corresponding to the factorization (3.5). Note that this network differs from $N_{u^{-1},v^{-1}}$. In particular, in $\overline{N}$ all "north-east" edges are to the left of any "south-east" edge. Once again, the network $\overline{N}_{u_m^{-1},v_m^{-1}}$ is obtained from the network $\overline{N}_{u^{-1},v^{-1}}$ by deleting all the edges above the horizontal line joining the mth sink with the mth source. The smallest positive power of X^{-1} such that in the corresponding network obtained by concatenation of copies of

$\overline{N}_{u^{-1},v^{-1}}$ there is a path joining the first source to the first sink and reaching above the mth horizontal level is $\bar{r}=\bar{r}^{+}+\bar{r}^{-}-1$, where

$$\bar{r}^{\pm}=\min\{j : l_j^{\pm} \geqslant m+1\}.$$

The difference in the formulas for r and $\bar{r}$ stems from the difference in the structure of the networks N and $\overline{N}$: the latter already contains paths from the first source to the first sink which reach above the first horizontal level. Consequently, it is possible that $\bar{r}=1$ for some $m>1$, whereas $r>1$ for any $m>1$.

One can define combinatorial parameters $\bar{\varepsilon}_i^{\pm}$ and $\bar{k}_i^{\pm}$ similarly to (3.7) and (3.9) based on the sets $L^{\pm}$ rather than on $I^{\pm}$ (cf. (3.2)). It follows immediately from the definitions that $\bar{\varepsilon}_i^{\pm}=1-\varepsilon_i^{\pm}$ for $i\in[2,n-1]$ and $\bar{\varepsilon}_1^{\pm}=\varepsilon_1^{\pm}=1$. One can prove, similarly to Lemma 3.5 (iii), that $\bar{k}_i^{\pm}=i-\sum_{\beta=1}^{i}\bar{\varepsilon}_\beta^{\pm}$, which translates to $\bar{k}_i^{\pm}=\sum_{\beta=1}^{i}\varepsilon_\beta^{\pm}-1$. Since $\bar{r}^{\pm}=\bar{k}_m^{\pm}+1$, we get $\bar{r}=\sum_{\beta=1}^{m}\varepsilon_\beta-1$, and hence, by (3.11), $\bar{r}=m-\varkappa_m$. If $\bar{r}=1$, then $\varkappa_m-m+1=0$, and the intervals $[0,\varkappa_m+m]$ coincides with $[\varkappa_m-m+1,\varkappa_m+m]$. Otherwise we can concatenate up to $\bar{r}-1=m-1-\varkappa_m$ networks $\overline{N}_{u^{-1},v^{-1}}$, and therefore (4.7) holds additionally for $\alpha\in[\varkappa_m-m+1,-1]$. □

LEMMA 4.2. *Let $m\in[1,n-1]$. Then*

$$\det(\lambda-X_m)=\frac{1}{\Delta_m^{(\varkappa_m)}}\mathcal{P}_m^{(\varkappa_m)}(\lambda). \tag{4.8}$$

In particular,

$$d_1\dots d_m=\frac{\Delta_m^{(\varkappa_m+1)}}{\Delta_m^{(\varkappa_m)}}. \tag{4.9}$$

Proof. Let

$$\det(\lambda-X_m)=\lambda^m+\sum_{i=0}^{m-1}a_{mi}\lambda^i.$$

Then the Cayley–Hamilton theorem implies that

$$h_{\alpha+m}(X_m)+\sum_{i=0}^{m-1}a_{mi}h_{\alpha+i}(X_m)=0$$

for any $\alpha\in\mathbb{Z}$. By Lemma 4.1, this relation remains valid if we replace $h_{\alpha+i}(X_m)$ with $h_{\alpha+i}=h_{\alpha+i}(X)$ for $i=0,\dots,m$, as long as $\varkappa_m-m+1\leqslant\alpha\leqslant\varkappa_m$. This means that, after right multiplication of the matrix used in the definition (4.6) of $\mathcal{P}_m^{(\varkappa_m)}$ by the unipotent matrix $\mathbf{1}+\sum_{\beta=0}^{m-1}a_{m\beta}e_{\beta+1,m+1}$, one gets a matrix of the form

$$\begin{pmatrix} \mathcal{H}_m^{(\varkappa_m)} & 0 \\ 1\ \lambda\ \dots\ \lambda^{m-1} & \det(\lambda-X_m)\end{pmatrix},$$

and (4.8) follows. Since $\det X_m=d_1\dots d_m$, (4.9) drops out immediately from (4.8) and (4.6) after the substitution $\lambda=0$. □

Remark 4.3. Combining Remark 4.1 (ii) with (4.9) for $m{=}n$ and taking into account that $\det X{=}d_1 \dots d_n$, we see that for any l,

$$\frac{\Delta_n^{(l+1)}}{\Delta_n^{(l)}} = \det X,$$

which implies that for any l,

$$\Delta_n^{(l)} = \Delta_n^{(n-1)} \det X^{l+1-n}. \tag{4.10}$$

Now, the first formula in (4.5) is an easy consequence of (4.9). To be in a position to prove the second formula in (4.5), we first need the following statement. For any $i{\in}[1,n]$ define the subspaces

$$\mathcal{L}_i^+ = \operatorname{span}\{e_1^T, \dots, e_i^T\} \quad \text{and} \quad \mathcal{L}_i^- = \operatorname{span}\{e_1, \dots, e_i\}.$$

Besides, put

$$\gamma_i^{\pm} = (-1)^{(i-1)\varepsilon_i^{\pm}} d_i^{-\varepsilon_i^{\pm}} \prod_{j=1}^{i-1} c_j^{\pm} d_j^{\bar{\varepsilon}_j^{\pm} - \varepsilon_i^{\pm}}, \quad i \in [2,n], \tag{4.11}$$

where $\bar{\varepsilon}_j^{\pm}$ were defined in the proof of Lemma 4.1 and $\gamma_1^{\pm}{=}1$.

LEMMA 4.3. *For any $i{\in}[1,n]$, one has*

$$\gamma_i^+ e_i^T = e_1^T X^{\zeta_i^+} \bmod \mathcal{L}_{i-1}^+ \tag{4.12}$$

and

$$\gamma_i^- e_i = X^{\zeta_i^-} e_1 \bmod \mathcal{L}_{i-1}^-. \tag{4.13}$$

In particular,

$$\mathcal{L}_i^+ = \operatorname{span}\{e_1^T X^{\zeta_1^+}, \dots, e_1^T X^{\zeta_i^+}\} \quad \textit{and} \quad \mathcal{L}_i^- = \operatorname{span}\{X^{\zeta_1^-} e_1, \dots, X^{\zeta_i^-} e_1\}.$$

Proof. A proof for (4.12) was given in [14]. We present it here in order to keep the paper self-contained. The case of (4.13) can be treated similarly.

For any X given by (3.4), consider an upper triangular matrix

$$V = D(\mathbf{1}{-}C_{k^+}^+)^{-1}(\mathbf{1}{-}C_{k^+-1}^+)^{-1} \dots (\mathbf{1}{-}C_1^+)^{-1}.$$

Note that V is the upper triangular factor in the Gauss factorization of X.

By (3.3), $e_r^T C_j^+{=}0$ for $r{<}i_{j-1}^+$ and $r{\geqslant}i_j^+$, and hence

$$e_r^T(1{-}C_j^+)^{-1} = \begin{cases} e_r^T, & \text{if } r < i_{j-1}^+, \\ e_r^T \bmod \mathcal{L}_{r-1}^+, & \text{if } r \geqslant i_j^+. \end{cases}$$

Thus, for $j\in[1,k^+]$,

$$\begin{aligned} e^T_{i^+_{j-1}}V &= d_{i^+_{j-1}}e^T_{i^+_{j-1}}(\mathbf{1}-C^+_{k^+})^{-1}\dots(\mathbf{1}-C^+_1)^{-1}\\ &= d_{i^+_{j-1}}e^T_{i^+_{j-1}}(\mathbf{1}-C^+_j)^{-1} \bmod \mathcal{L}^+_{i^+_j-1}\\ &= d_{i^+_{j-1}}c^+_{i^+_{j-1}}\dots c^+_{i^+_j-1}e^T_{i^+_j} \bmod \mathcal{L}^+_{i^+_j-1}. \end{aligned}$$

A similar argument shows that $e^T_rV\in\mathcal{L}^+_{i^+_j-1}$ for $r<i^+_{j-1}$. This implies that

$$e^T_1V^j=\Bigg(\prod_{\beta=0}^{j-1}d_{i^+_\beta}c^+_{i^+_\beta}\dots c^+_{i^+_{\beta+1}-1}\Bigg)e^T_{i^+_j} \bmod \mathcal{L}^+_{i^+_j-1}=\Bigg(\prod_{r=1}^{i^+_j-1}c^+_r d_r^{\bar\varepsilon^+_r}\Bigg)e^T_{i^+_j} \bmod \mathcal{L}^+_{i^+_j-1}.$$

Besides, $e^T_{i^+_{j-1}}XV^{-1}=e^T_{i^+_{j-1}} \bmod \mathcal{L}^+_{i^+_{j-1}}$, and hence the above relation can be rewritten as

$$e^T_1X^j=\Bigg(\prod_{r=1}^{i^+_j-1}c^+_r d_r^{\bar\varepsilon^+_r}\Bigg)e^T_{i^+_j} \bmod \mathcal{L}^+_{i^+_j-1}. \tag{4.14}$$

On the other hand, for $l\in[i^+_{j-1},i^+_j-1]$, define $m\geqslant 0$, so that $l+m+1$ is the smallest index greater than l belonging to the index set L^+. Then

$$\begin{aligned} e^T_lV^{-1} &= e^T_l(\mathbf{1}-C^+_j)\dots(\mathbf{1}-C^+_{k^+})D^{-1} \bmod \mathcal{L}^+_{l+m}\\ &= ((-1)^{m+1}c^+_l\dots c^+_{l+m}d^{-1}_{l+m+1})e^T_{l+m+1} \bmod \mathcal{L}^+_{l+m}. \end{aligned}$$

The latter equality implies that

$$e^T_1V^{-\alpha}=((-1)^{l^+_\alpha-1}c^+_1\dots c^+_{l^+_\alpha-1}d^{-1}_{l^+_1}\dots d^{-1}_{l^+_\alpha})e^T_{l^+_\alpha} \bmod \mathcal{L}^+_{l^+_\alpha-1} \tag{4.15}$$

for any $l^+_\alpha\in L^+$ distinct from 1 and n. Note now that $l_\alpha=i$ if and only if

$$\alpha=\sum_{\beta=1}^{i-1}\varepsilon^+_\beta$$

(this can be considered as an analog of (3.12)). Furthermore,

$$d^{-1}_{l^+_1}\dots d^{-1}_{l^+_\alpha}=\prod_{j=2}^{i}d_j^{-\varepsilon^+_j}.$$

Thus, one can rewrite (4.15) as

$$e^T_1V^{-\sum_{j=1}^{i-1}\varepsilon^+_j}=(-1)^{i-1}\prod_{j=1}^{i-1}c^+_jd^{-\varepsilon^+_{j+1}}_{j+1}e^T_i \bmod \mathcal{L}^+_{i-1}.$$

Together with $e^T_lVX^{-1}=e^T_l \bmod \mathcal{L}^+_{l+m}$ this leads to

$$e^T_1X^{-\sum_{j=1}^{i-1}\varepsilon^+_j}=(-1)^{i-1}\prod_{j=1}^{i-1}c^+_jd^{-\varepsilon^+_{j+1}}_{j+1}e^T_i \bmod \mathcal{L}^+_{i-1}.$$

Combining this relation with (4.14) and Lemma 3.5 (ii), one gets (4.12). □

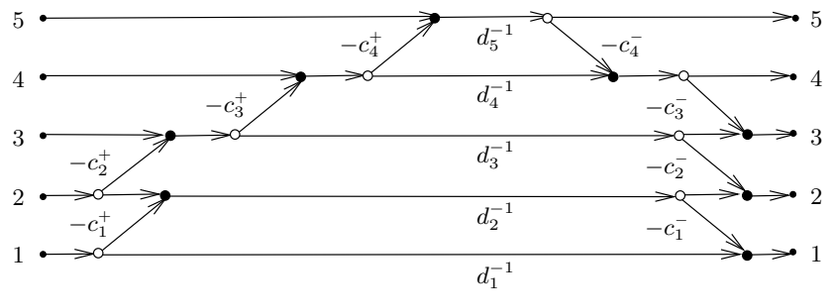


Figure 4. The network $\overline{N}_{u^{-1},v^{-1}}$ for the double Bruhat cell $G^{u,v}$ from Example 3.1.

Example 4.1. We illustrate (4.12) using Example 3.1 and Figure 3. If $j>0$ then to find i such that $e_1^T X^j = \gamma_i^+ e_i^T \bmod \mathcal{L}_{i-1}^+$ it is enough to find the highest sink which can be reached by a path starting from the source 1 in the network obtained by concatenation of j copies of $N_{u,v}$. Thus we conclude from Figure 3 that

$$\begin{aligned} e_1^T X &= d_1 c_1^+ c_2^+ e_3^T \bmod \mathcal{L}_2^+, \\ e_1^T X^2 &= d_1 c_1^+ c_2^+ d_3 c_3^+ e_4^T \bmod \mathcal{L}_3^+, \\ e_1^T X^3 &= d_1 c_1^+ c_2^+ d_3 c_3^+ d_4 c_4^+ e_5^T \bmod \mathcal{L}_4^+. \end{aligned}$$

Similarly, using the network $\overline{N}_{u^{-1},v^{-1}}$ shown in Figure 4, one observes that

$$e_1^T X^{-1} = -c_1^+ d_2^{-1} e_2^T \bmod \mathcal{L}_1^+.$$

These relations are in agreement with (4.12).

Define the Laurent polynomials

$$p_i^{\pm}(\lambda) = \frac{(-1)^{(i-1)\varepsilon_i^{\pm}}}{\gamma_i^{\pm} \Delta_{i-1}^{(\varkappa_{i-1})}} \lambda^{k_i^{\pm}-i+1} \mathcal{P}_{i-1}^{(\varkappa_{i-1}-\varepsilon_i^{\pm})}(\lambda), \quad i \in [1,n].$$

Corollary 4.1. (i) *One has*

$$e_1^T p_i^+(X) = e_i^T \quad \textit{and} \quad p_i^-(X) e_1 = e_i, \quad i \in [1,n].$$

(ii) *For any eigenvalue λ of X, the column-vector $(p_i^+(\lambda))_{i=1}^n$ and the row-vector $(p_i^-(\lambda))_{i=1}^n$ are right and left eigenvectors of X corresponding to λ, respectively.*

Proof. (i) We will only give a proof for $p_i^+(\lambda)$. By Lemma 4.3, $e_1^T X^{\zeta_\alpha^+}$, $\alpha=1,\dots,i-1$, form a basis of $\mathcal{L}_{i-1}^+$. Hence, taking into account (4.12), we get

$$\gamma_i^+ e_i^T = e_1^T \left(X^{\zeta_i^+} + \sum_{\alpha=1}^{i-1} \pi_\alpha X^{\zeta_\alpha^+} \right)$$

for some coefficients π_α. By Lemma 3.5 (iv), this can be rewritten as

$$\gamma_i^+ e_i^T = e_1^T X^{k_i^+ - i + 1} \sum_{\alpha=1}^{i} \tilde{\pi}_\alpha X^{\alpha-1},$$

where either $\tilde{\pi}_i = 1$ (if $\varepsilon_i^+ = 0$), or $\tilde{\pi}_1 = 1$ (if $\varepsilon_i^+ = 1$). Define a polynomial

$$p(\lambda) = \sum_{\alpha=1}^{i} \tilde{\pi}_\alpha \lambda^{\alpha-1}.$$

By Lemma 4.3, the vectors $X^\alpha e_1$, $\alpha \in M_{i-1}^-$, span the subspace $\mathcal{L}_{i-1}^-$. Therefore, by Lemma 3.5 (iv), $(X^{k_i^+ - i+1} p(X) X^\alpha e_1, e_1) = 0$ for $\alpha \in [k_{i-1}^- - i + 2, k_{i-1}^-]$. This system of linear equations determines $p(\lambda)$ uniquely as

$$p(\lambda) = \frac{(-1)^{(i-1)\varepsilon_i^+}}{\Delta_{i-1}^{(k_i^+ + k_{i-1}^- - i + 1 + \varepsilon_i^+)}} \mathcal{P}_{i-1}^{(k_i^+ + k_{i-1}^- - i + 1)}(\lambda),$$

which, by (3.11) and Lemma 3.5 (iii), gives

$$p(\lambda) = \frac{(-1)^{(i-1)\varepsilon_i^+}}{\Delta_{i-1}^{(\varkappa_{i-1})}} \mathcal{P}_{i-1}^{(\varkappa_{i-1} - \varepsilon_i^+)}(\lambda) = \gamma_i^+ \lambda^{-k_i^+ + i - 1} p_i^+(\lambda).$$

It remains to notice that $\gamma_i^+ e_i^T = e_1^T X^{k_i^+ - i + 1} p(X)$, and the result follows.

(ii) Let $z^\lambda = (z_i^\lambda)_{i=1}^n$ be a right eigenvector of X corresponding to an eigenvalue λ. Then $e_1^T p_i^+(X) z^\lambda = e_i^T z^\lambda$, which means that $z_i^\lambda = p_i^+(\lambda) e_1^T z^\lambda = p_i^+(\lambda) z_1^\lambda$. Therefore, $z_1^\lambda \neq 0$, $p_i^+(\lambda) = z_i^\lambda / z_1^\lambda$ and $(p_i^+(\lambda))_{i=1}^n$ is a right eigenvector of X. The case of $(p_i^-(\lambda))_{i=1}^n$ can be treated in the same way. □

Remark 4.4. Statement (i) of Corollary 4.1 is equivalent to saying that the Laurent polynomials $p_i^\pm(\lambda)$ form a biorthonormal family with respect to the moment functional (4.2) obtained by the Gram process applied to the sequences $1, \lambda^{\zeta_1^+}, \lambda^{\zeta_2^+}, \ldots$ and $1, \lambda^{\zeta_1^-}, \lambda^{\zeta_2^-}, \ldots$. Indeed,

$$\langle p_i^+(\lambda), p_j^-(\lambda) \rangle = e_1^T p_i^+(X) p_j^-(X) e_1 = (e_i, e_j) = \delta_{ij}.$$

For $l \in [2, n]$ define

$$\Gamma_l = \prod_{i=2}^{l} d_i^{-\varepsilon_i} \prod_{j=1}^{i-1} c_j d_j^{\bar{\varepsilon}_j - \varepsilon_i}, \tag{4.16}$$

where $\bar{\varepsilon}_j = \bar{\varepsilon}_j^+ + \bar{\varepsilon}_j^-$, $j \in [1, n]$, similarly to (3.10).

COROLLARY 4.2. *For any* $k\in\mathbb{Z}$,

$$\Delta_l^{(\varkappa_l+k)} = \Gamma_l (X^k)_{[1,l]}^{[1,l]}. \tag{4.17}$$

Proof. By Lemma 4.3,

$$(X^k)_{[1,l]}^{[1,l]} = \det(e_i^T X^k e_j)_{i,j=1}^l = \det\left(\frac{1}{\gamma_i^+\gamma_j^-} e_1^T X^{k+\zeta_i+\zeta_j} e_1\right)_{i,j=1}^l$$
$$= \left(\prod_{i=2}^l \frac{1}{\gamma_i^+\gamma_i^-}\right) \det(e_1^T X^{k+\zeta_i+\zeta_j} e_1)_{i,j=1}^l.$$

By Lemma 3.5 (iii), (iv) and (3.11), the determinant in the last expression above is equal, up to the sign, to $\Delta_l^{(\varkappa_l+k)}$. By Lemma 3.5 (ii), the sign is determined as

$$\prod_{i=2}^l (-1)^{(i-1)\varepsilon_i^+} \prod_{j=2}^l (-1)^{(j-1)\varepsilon_j^-} = (-1)^{\sum_{i=2}^l (i-1)\varepsilon_i}.$$

On the other hand, (4.11) implies that

$$\prod_{i=2}^l \gamma_i^+\gamma_i^- = (-1)^{\sum_{i=2}^l (i-1)\varepsilon_i} \Gamma_l,$$

and (4.17) follows. □

Finally, we can complete the proof of Theorem 4.1. To prove the second relation in (4.5), observe that by Lemma 4.3 and Corollary 4.1 (i),

$$\gamma_i^+ = (X^{\zeta_i^+} p_i^-(X) e_1, e_1) = \frac{(-1)^{(i-1)\varepsilon_i^-}}{\gamma_i^- \Delta_{i-1}^{(\varkappa_{i-1})}} (X^{\zeta_i^+ + k_i^- - i+1} \mathcal{P}_{i-1}^{(\varkappa_{i-1}-\varepsilon_i^-)}(X) e_1, e_1).$$

Since by (3.8), (3.10), (3.11) and Lemma 3.5 (iii), $\zeta_i^+ + k_i^- - i + 1 = \varkappa_i - (i-1)\varepsilon_i^+$, the above equality gives

$$\gamma_i^+ = \frac{(-1)^{(i-1)\varepsilon_i^-}}{\gamma_i^- \Delta_{i-1}^{(\varkappa_{i-1})}} (-1)^{(i-1)\varepsilon_i^+} \Delta_i^{(\varkappa_i)},$$

and so

$$\gamma_i^+\gamma_i^- = (-1)^{(i-1)\varepsilon_i} \frac{\Delta_i^{(\varkappa_i)}}{\Delta_{i-1}^{(\varkappa_{i-1})}}.$$

Consider the ratio

$$\frac{\gamma_{i+1}^+\gamma_{i+1}^-}{\gamma_i^+\gamma_i^-} = (-1)^{i\varepsilon_{i+1} - (i-1)\varepsilon_i} \frac{\Delta_{i-1}^{(\varkappa_{i-1})} \Delta_{i+1}^{(\varkappa_{i+1})}}{(\Delta_i^{(\varkappa_i)})^2}.$$

Taking into account (4.11), we obtain

$$c_i^+ c_i^- = \frac{\Delta_{i-1}^{(\varkappa_{i-1})} \Delta_{i+1}^{(\varkappa_{i+1})}}{(\Delta_i^{(\varkappa_i)})^2} \frac{d_{i+1}^{\varepsilon_{i+1}}}{d_i^{2-\varepsilon_i}} (d_1 d_2 \dots d_{i-1})^{\varepsilon_i - \varepsilon_{i+1}},$$

which together with the first relation in (4.5) gives the second one.

5. Cluster algebra

5.1. Let $N_{u,v}$ be the network associated with $X \in \mathrm{GL}_n$ and the factorization scheme (2.2). We will now construct a network $N^\circ_{u,v}$ in an annulus as follows:

(i) For each $i \in [1,n]$, add an edge which is directed from the ith sink on the right to the ith source on the left in such a way that moving from the ith source to the ith sink in $N_{u,v}$ and then returning to the ith source along the new edge, one traverses a closed contour in the counterclockwise direction. These n new edges do not intersect and to each of them we assign weight 1.

(ii) Place the resulting network in the interior of an annulus in such a way that the cut (as defined in §2.4) intersects n new edges, and the inner boundary of the annulus is inside the domain bounded by the top horizontal path in $N_{u,v}$ and the nth new edge.

(iii) Place one source and one sink on the outer boundary of the annulus, the former slightly to the right and the latter slightly to the left of the cut. Split the first (the outermost) new edge into three similarly directed edges by adding two vertices, a black one slightly to the right and a white one slightly to the left of the cut. Add an edge with weight w_{in} directed from the source to the new black vertex and another edge with weight w_{out} directed from the new white vertex to the sink.

It is important to note that the gauge group is rich enough to assure the possibility of assigning weights as described above, with unit weights at prescribed edges.

Example 5.1. The network $N^\circ_{u,v}$ corresponding to $N_{u,v}$ discussed in Example 3.1 is shown in Figure 5.

Now let X be an element in a Coxeter double Bruhat cell $G^{u,v}$, $N_{u,v}$ be the network corresponding to the factorization (3.4) and $N^\circ_{u,v}$ be the corresponding network in an annulus. Then $N^\circ_{u,v}$ has $2(n-1)$ bounded faces f_{0i}, f_{1i}, $i \in [1,n-1]$, which we enumerate as follows: each face f_{0i} contains a piece of the cut and each face f_{1i} does not, and the value of i is assigned according to the natural bottom to top order inherited from $N_{u,v}$. There are also three unbounded faces: two of them, adjacent to the outer boundary of the annulus, will be denoted f_{00} and f_{10}, where the first index is determined using the same convention as for bounded faces. The third unbounded face is adjacent to the inner boundary. It will be denoted by f_{0n}.

Recall that the faces of $N^\circ_{u,v}$ correspond to the vertices of the directed dual network $(N^\circ_{u,v})^*$ (as defined in §2.4). To describe adjacency properties of $(N^\circ_{u,v})^*$, let us first consider inner vertices of $N^\circ_{u,v}$. There are altogether $4n-2$ inner vertices. For every $i \in [1,n-1]$, the ith level contains two black and two white vertices. One of the black vertices is an endpoint of an edge directed from the $(i+1)$-st level; it is denoted $v_b^-(i)$. The other one is an endpoint of an edge directed from the $(i-1)$-st level (or from the

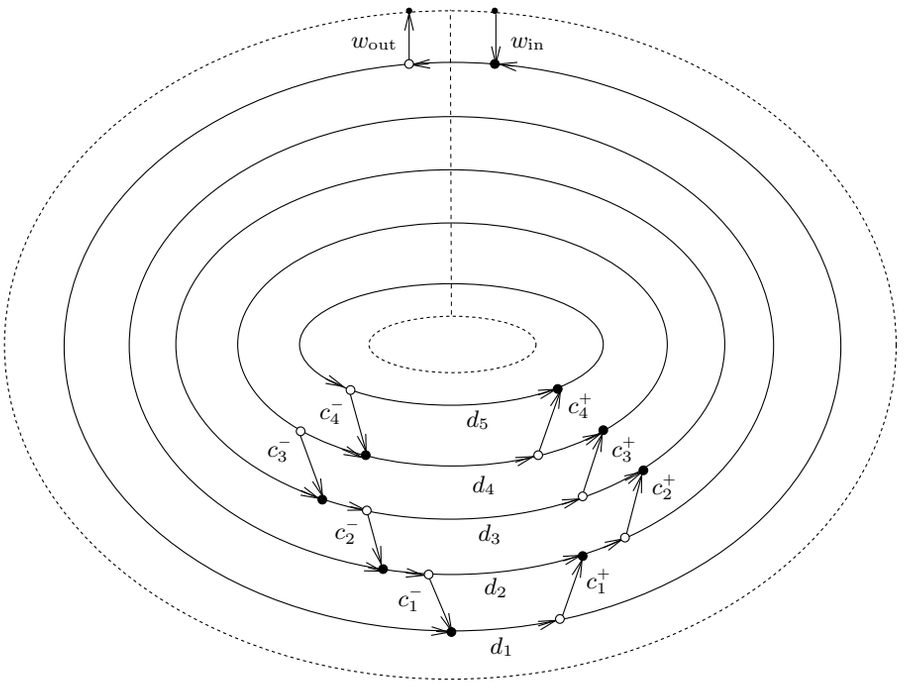


Figure 5. The network $N^\circ_{u,v}$ corresponding to $N_{u,v}$ from Example 3.1.

source, for $i{=}1$); it is denoted $v_b^+(i)$. Similarly, white vertices are starting points of the edges directed towards the $(i{+}1)$-st and the $(i{-}1)$-st levels (or towards the sink, for $i{=}1$); they are denoted $v_w^+(i)$ and $v_w^-(i)$, respectively. The nth level contains only $v_b^+(n)$ and $v_w^-(n)$.

Now we can describe the faces f_{0i}, $i{\in}[0,n]$, and f_{1i}, $i{\in}[0,n{-}1]$, by listing their vertices in counterclockwise order. Below we use the following convention: if a vertex appears in the description of a face with the exponent 0, it means that this vertex does not belong to the boundary of the face. With this in mind, we obtain

$$f_{1i} = (v_b^-(i)v_w^-(i)^{\varepsilon_i^-}\, v_b^+(i)^{\varepsilon_i^+}\, v_w^+(i)v_b^+(i{+}1)v_w^+(i{+}1)^{\bar\varepsilon_{i+1}^+} v_b^-(i{+}1)^{\bar\varepsilon_{i+1}^-} v_w^-(i{+}1)),$$
$$f_{0i} = (v_w^+(i)v_b^+(i)^{\bar\varepsilon_i^+}\, v_w^-(i)^{\bar\varepsilon_i^-}\, v_b^-(i)v_w^-(i{+}1)v_b^-(i{+}1)^{\varepsilon_{i+1}^-} v_w^+(i{+}1)^{\varepsilon_{i+1}^+} v_b^+(i{+}1))$$

for $i{\in}[2,n{-}2]$, and

$$f_{10} = (\text{source } v_b^+(1)v_w^+(1)v_b^-(1)v_w^-(1) \text{ sink}),$$
$$f_{00} = (\text{sink } v_w^-(1)v_b^+(1) \text{ source}),$$
$$f_{11} = (v_b^-(1)v_w^+(1)v_b^+(2)v_w^+(2)^{\bar\varepsilon_2^+}\, v_b^-(2)^{\bar\varepsilon_2^-}\, v_w^-(2)),$$

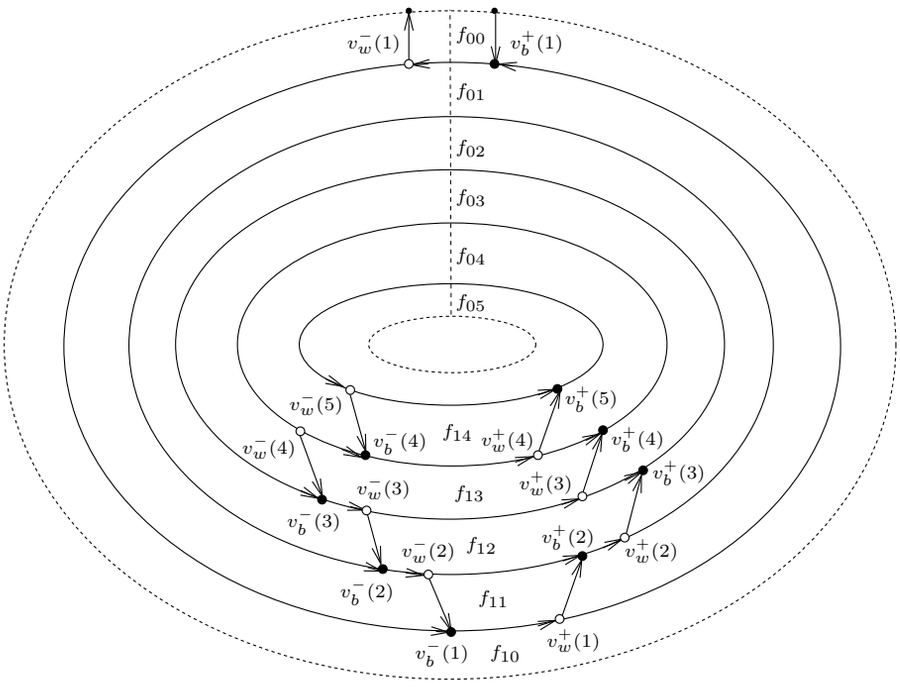


Figure 6. Vertices and faces of $N_{u,v}^\circ$ for Example 5.1.

$$
\begin{aligned}
f_{01} &= (v_w^+(1)v_b^+(1)v_w^-(1)v_b^-(1)v_w^-(2)v_b^-(2)^{\varepsilon_2^-}v_w^+(2)^{\varepsilon_2^+}v_b^+(2)),\\
f_{1\,n-1} &= (v_b^-(n-1)v_w^-(n-1)^{\varepsilon_{n-1}^-}v_b^+(n-1)^{\varepsilon_{n-1}^+}v_w^+(n-1)v_b^+(n)v_w^-(n)),\\
f_{0\,n-1} &= (v_w^+(n-1)v_b^+(n-1)^{\bar\varepsilon_{n-1}^+}v_w^-(n-1)^{\bar\varepsilon_{n-1}^-}v_b^-(n-1)v_w^-(n)v_b^+(n)),\\
f_{0n} &= (v_b^+(n)v_w^-(n)).
\end{aligned}
$$

Example 5.2. Vertices and faces of $N_{u,v}^\circ$ from Example 5.1 are shown in Figure 6. Consider the face f_{13}. As we have seen before in Example 3.1, $\varepsilon_3^+=\varepsilon_4^+=\varepsilon_4^-=0$ and $\varepsilon_3^-=1$, so the above description yields $f_{13}=(v_b^-(3)v_w^-(3)v_w^+(3)v_b^+(4)v_w^+(4)v_b^-(4)v_w^-(4))$.

In our description of $(N_{u,v}^\circ)^*$ below we use the following convention: whenever we say that there are $\alpha<0$ edges directed from vertex f to vertex f', it means that there are $|\alpha|$ edges directed from f' to f. It is easy to see that for any $i\in[0,n-1]$, the faces f_{0i} and f_{1i} have two common edges: $v_w^-(i+1)\to v_b^-(i)$ and $v_w^+(i)\to v_b^+(i+1)$. The starting points of each one of the edges are white, the endpoints are black, and in both cases face f_{1i} lies to the right of the edge. This means that in $(N_{u,v}^\circ)^*$ there are two edges directed from f_{0i} to f_{1i}. Similarly, the description above shows that $(N_{u,v}^\circ)^*$ has

(i) $1-\varepsilon_{i+1}$ edges directed from $f_{1\,i+1}$ to f_{1i},

(ii) $2-\varepsilon_{i+1}$ edges directed from f_{1i} to $f_{0\,i+1}$,

(iii) $1-\varepsilon_{i+1}$ edges directed from $f_{0\,i+1}$ to f_{0i},

(iv) ε_{i+1} edges directed from $f_{1\,i+1}$ to f_{0i}

for $i\in[1,n-2]$, one edge directed from f_{1n-1} to f_{0n} and one edge directed from f_{0n} to f_{0n-1}. Finally, for $|i-j|>1$ and $a,b\in\{0,1\}$, the vertices f_{ai} and f_{bj} in $(N^{\circ}_{u,v})^*$ are not connected by edges.

Next, we associate with every face f_{st} in $N^{\circ}_{u,v}$ a face weight y_{st}. We will see below that y_{st} are related to the parameters $c_i^{\pm}$ and d_i via a monomial transformation. At this point, however, let us examine the standard Poisson bracket on $\mathcal{F}_{N^{\circ}_{u,v}}$. As was explained in §2.4, this bracket is completely described by Proposition 2.2, which together with the above description of $(N^{\circ}_{u,v})^*$ implies the following Poisson relations for face weights:

$$\begin{aligned}
\{y_{0i},y_{1i}\}&=2y_{0i}y_{1i},\\
\{y_{0i},y_{0\,i+1}\}&=-(1-\varepsilon_{i+1})y_{0i}y_{0\,i+1},\\
\{y_{0i},y_{1\,i+1}\}&=-\varepsilon_{i+1}y_{0i}y_{1\,i+1},\\
\{y_{1i},y_{0\,i+1}\}&=(2-\varepsilon_{i+1})y_{1i}y_{0\,i+1},\\
\{y_{1i},y_{1\,i+1}\}&=-(1-\varepsilon_{i+1})y_{1i}y_{1\,i+1},
\end{aligned}\tag{5.1}$$

for $i\in[2,n-2]$, and

$$\begin{aligned}
\{y_{00},y_{01}\}&=-y_{00}y_{01},\\
\{y_{00},y_{10}\}&=y_{00}y_{10},\\
\{y_{10},y_{01}\}&=2y_{10}y_{01},\\
\{y_{10},y_{11}\}&=-y_{10}y_{11},\\
\{y_{0n},y_{0\,n-1}\}&=y_{0n}y_{0\,n-1},\\
\{y_{0n},y_{1\,n-1}\}&=-y_{0n}y_{1\,n-1},
\end{aligned}\tag{5.2}$$

and the rest of the brackets are zero.

Denote by $M(\lambda)$ the *boundary measurement* for the network $N^{\circ}_{u,v}$, and put

$$H_0=w_{\mathrm{in}}w_{\mathrm{out}}.$$

We have the following result.

PROPOSITION 5.1. *Let $X\in G^{u,v}$ be given by* (3.4), *then*

$$M(\lambda)=H_0((\lambda\mathbf{1}+X)^{-1}e_1,e_1)=-H_0m(-\lambda;X).$$

Proof. Clearly, $M(\lambda)$ is a power series in λ^{-1} with the coefficient of λ^{-k} equal to $(-1)^{k-1}$ times the sum of the weights of all paths from the source to the sink which cross the cut exactly k times. Moreover, the leading term of $M(\lambda)$ is $H_0\lambda^{-1}$. Let

$$M(\lambda)=\sum_{k=0}^{\infty}(-1)^k H_k\lambda^{-k-1}.$$

Since the weight of every path has a factor H_0, computing H_k/H_0 is equivalent to computing the boundary measurement between the first source and the first sink in the planar network obtained by concatenation of k copies of $N_{u,v}$. Therefore,

$$H_k/H_0=(X^k e_1,e_1)=h_k$$

and

$$M(\lambda)=H_0\sum_{k=0}^{\infty}(-1)^k h_k\lambda^{-k-1}=H_0\left(\lambda^{-1}\sum_{k=0}^{\infty}(-\lambda X)^k e_1,e_1\right)=H_0((\lambda\mathbf{1}+X)^{-1}e_1,e_1).\ \square$$

Let $\mathcal{R}_n$ denote the space of rational functions of the form Q/P, where P is a monic polynomial of degree n, Q is a polynomial of degree at most $n-1$, P and Q are coprime and $P(0)\neq 0$.

PROPOSITION 5.2. *The space of boundary measurements associated with the network $N^{\circ}_{u,v}$ is dense in $\mathcal{R}_n$.*

Proof. Let us first prove that any boundary measurement indeed belongs to $\mathcal{R}_n$. By Proposition 5.1, the roots of P are exactly the eigenvalues of $-X$, and hence the degree of P equals to n. Next, since

$$M(\lambda)=\sum_{k=0}^{\infty}(-1)^k H_k\lambda^{-k-1},$$

the value of $M(\lambda)$ at infinity equals zero, and hence the degree of Q is at most $n-1$.

By Proposition 5.1 and (4.1), the coprimality statement is equivalent to saying that X and its submatrix obtained by deleting the first row and column have no common eigenvalues. Suppose this is not true, and $\tilde{\lambda}$ is a common eigenvalue. Let $\widetilde{X}=X-\tilde{\lambda}\mathbf{1}$. Then $\det\widetilde{X}=\widetilde{X}^{[2,n]}_{[2,n]}=0$, and, by the Jacobi determinantal identity,

$$\widetilde{X}^{[2,n]}_{[1,n-1]}\widetilde{X}^{[1,n-1]}_{[2,n]}=\widetilde{X}^{[2,n-1]}_{[2,n-1]}\det\widetilde{X}+\widetilde{X}^{[2,n]}_{[2,n]}\widetilde{X}^{[1,n-1]}_{[1,n-1]}=0.$$

Thus either $\widetilde{X}^{[1,n-1]}_{[2,n]}$ or $\widetilde{X}^{[2,n]}_{[1,n-1]}$ is zero. Assume the latter is true (the other case can be treated similarly). Consider the classical adjoint $\widehat{X}$ of $\widetilde{X}$. Since $\widetilde{X}$ is degenerate, $\widehat{X}$ has

rank 1. As $\widehat{X}_{11}=\widehat{X}_{1n}=0$, either the first row or the first column of $\widehat{X}$ has all zero entries. Assume the latter is true. This means that every $(n-1)\times(n-1)$ minor based on the last $n-1$ rows of $\widetilde{X}$ equals zero, and so the $n\times(n-1)$ submatrix of $\widetilde{X}$ obtained by deleting the first column does not have full rank. Thus there is a non-zero vector w with $w_1=0$ such that $\widetilde{X}w=0$. Therefore, w is linearly independent with the eigenvector $(p_i^+(\lambda))_{i=1}^n$ of X constructed in Corollary 4.1 (ii), whose first component is equal to 1. We conclude that the dimension of the eigenspace of X corresponding to $\tilde{\lambda}$ is greater than 1. However, due to Lemma 4.3 and the invertibility of X, e_1 is a cyclic vector for X, which implies that all eigenspaces of X are 1-dimensional. This completes the proof of coprimality by contradiction. The case when the first row of $\widehat{X}$ is zero can be treated similarly.

To prove that $P(0)\neq 0$, we write $P(\lambda)=\lambda^n+p_{n-1}\lambda^{n-1}+\ldots+p_0$. Then the relation $Q(\lambda)=M(\lambda)P(\lambda)$ yields

$$\sum_{i=0}^{n}(-1)^i p_i H_{k+i}=0, \quad k\geqslant 0, \tag{5.3}$$

with $p_n=1$. The relations (5.3) for $k\in[0,n-1]$ provide a system of linear equations for $p_0,\ldots,p_{n-1}$. The determinant of this system equals $H_0^n\Delta_n^{(n-1)}$. It is well known (see, e.g. [19, Theorem 8.7.1]), that the coprimality of P and Q is equivalent to the non-vanishing of $\Delta_n^{(n-1)}$. So, $p_0=P(0)$ can be restored uniquely as $(-1)^n\Delta_n^{(n)}/\Delta_n^{(n-1)}$, which by Remark 4.3 is equal to $(-1)^n\det X$. It remains to recall that $\det X=d_1\ldots d_n\neq 0$.

The density statement follows easily from Theorem 4.1: given $M(\lambda)$, one builds Hankel determinants (4.3) and makes use of formulas (4.5) to restore X, provided H_0 and all the determinants in the denominator do not vanish. □

Remark 5.1. (i) Since $p_0\neq 0$, equations (5.3) extended to $k=-1,-2,\ldots$ can be used as a recursive definition of $H_{-1},H_{-2},\ldots$.

(ii) Since $m(-\lambda;X)=q(\lambda)/p(\lambda)$, where $p(\lambda)=\sum_{i=0}^n(-1)^i p_i\lambda^i$ is the characteristic polynomial of $-X$, the Cayley–Hamilton theorem implies that for any $k\in\mathbb{Z}$,

$$\sum_{i=0}^{n}(-1)^i p_i h_{k+i}=\left(\sum_{i=0}^{n}(-1)^i p_i X^{k+i}e_1,e_1\right)=(X^k p(-X)e_1,e_1)=0.$$

Therefore,

$$H_k=h_k H_0 \tag{5.4}$$

for any $k\in\mathbb{Z}$.

(iii) Write $Q(\lambda)=q_{n-1}\lambda^n+\ldots+q_0$. Similarly to (5.3) one gets

$$(-1)^{j+1}q_j=\sum_{i=j+1}^{n}(-1)^i p_i H_{i-j-1}, \quad j\in[0,n-1]. \tag{5.5}$$

The following proposition is a particular case of [22, Theorem 3.1].

PROPOSITION 5.3. *The standard Poisson bracket on $\mathcal{F}_{N^\circ_{u,v}}$ induces a Poisson bracket on $\mathcal{R}_n$. This bracket is given by*

$$\{M(\lambda), M(\mu)\} = -(\lambda M(\lambda) - \mu M(\mu))\frac{M(\lambda)-M(\mu)}{\lambda-\mu}. \tag{5.6}$$

Remark 5.2. Using Proposition 5.1, one can deduce from (5.6) the Poisson brackets for the Weyl function $m(\lambda)=m(\lambda; X)$:

$$\{m(\lambda), m(\mu)\} = -(\lambda m(\lambda) - \mu m(\mu))\left(\frac{m(\lambda)-m(\mu)}{\lambda-\mu} + m(\lambda)m(\mu)\right). \tag{5.7}$$

(The derivation of (5.7) from (5.6) can be found in [13, Proposition 3].) Thus a combination of Theorem 2.1 and Propositions 5.1 and 5.3 provides a network-based proof of the fact that the standard Poisson–Lie structure on GL_n induces the Poisson bracket (5.7) on Weyl functions. This fact plays a useful role in the study of a multi-Hamiltonian structure of Toda flows.

5.2. To compute the face weights in terms of the factorization parameters $c_i^{\pm}$ and d_i, we introduce new notation which makes the formulas (4.5) more convenient. First of all, for any $l\in\mathbb{Z}$ and $i\in\mathbb{N}$, define, similarly to (4.3), Hankel determinants

$$\hat{\Delta}_i^{(l)} = \det(H_{\alpha+\beta+l-i-1})_{\alpha,\beta=1}^{i}; \tag{5.8}$$

we assume that $\hat{\Delta}_0^{(l)}=1$ for any $l\in\mathbb{Z}$. It follows from (5.4) that $\hat{\Delta}_i^{(l)}=H_0^i\Delta_i^{(l)}$.

Let us fix a pair of Coxeter elements (u, v) and set $\varepsilon=(\varepsilon_i)_{i=1}^n$,

$$x_{0i} = x_{0i}(\varepsilon) = \hat{\Delta}_i^{(\varkappa_i)}, \quad x_{1i} = x_{1i}(\varepsilon) = \hat{\Delta}_i^{(\varkappa_i+1)} \quad \text{and} \quad c_i = c_i^+ c_i^-. \tag{5.9}$$

Then formulas (4.5) become

$$d_i = \frac{x_{1i}x_{0\,i-1}}{x_{0i}x_{1\,i-1}} \quad \text{and} \quad c_i = \frac{x_{0\,i-1}x_{0\,i+1}}{x_{1i}^2}\left(\frac{x_{1\,i+1}}{x_{0\,i+1}}\right)^{\varepsilon_{i+1}}\left(\frac{x_{1\,i-1}}{x_{0\,i-1}}\right)^{2-\varepsilon_i}. \tag{5.10}$$

We now compute the face weights for $N^\circ_{u,v}$:

$$\begin{aligned} y_{0i} &= c_i^{-1} = x_{0\,i-1}^{1-\varepsilon_i} x_{0\,i+1}^{\varepsilon_{i+1}-1} x_{1i}^2 x_{1\,i-1}^{\varepsilon_i-2} x_{1\,i+1}^{-\varepsilon_{i+1}}, \\ y_{1i} &= \frac{c_i d_i}{d_{i+1}} = x_{1\,i-1}^{1-\varepsilon_i} x_{1\,i+1}^{\varepsilon_{i+1}-1} x_{0i}^{-2} x_{0\,i-1}^{\varepsilon_i} x_{0\,i+1}^{2-\varepsilon_{i+1}} \end{aligned} \tag{5.11}$$

for $i\in[1,n-1]$ and

$$y_{00}=\frac{1}{H_0}=x_{01}^{-1},\quad y_{10}=\frac{H_0^2}{H_1}=x_{01}^2x_{11}^{-1}\quad\text{and}\quad y_{0n}=d_n=x_{1n}x_{0\,n-1}x_{0n}^{-1}x_{1\,n-1}^{-1}. \tag{5.12}$$

We will rewrite (5.11) for $i=n-1$ in a slightly different way. Recall that $\varepsilon_n=0$. Due to (4.10),

$$\begin{aligned} x_{0n}^2x_{1n}^{-1}&=\hat\Delta_n^{(\varkappa_n-1)}=\hat\Delta_n^{(n-1)}(\det X)^{\varkappa_n-n},\\ x_{0n}&=\hat\Delta_n^{(\varkappa_n)}=\hat\Delta_n^{(n-1)}(\det X)^{\varkappa_n-n+1}. \end{aligned} \tag{5.13}$$

Thus, (5.11) yields

$$\begin{aligned} y_{0\,n-1}&=x_{0\,n-2}^{1-\varepsilon_{n-1}}x_{1\,n-1}^2x_{1\,n-2}^{\varepsilon_{n-1}-2}(\hat\Delta_n^{(n-1)})^{-1}\left(\frac{1}{\det X}\right)^{\varkappa_n-n+1},\\ y_{1\,n-1}&=x_{1\,n-2}^{1-\varepsilon_{n-1}}x_{0\,n-1}^{-2}x_{0\,n-2}^{\varepsilon_{n-1}}\hat\Delta_n^{(n-1)}\left(\frac{1}{\det X}\right)^{n-\varkappa_n}. \end{aligned}$$

Define

$$\mathbf{x}=\mathbf{x}(\varepsilon)=(x_i)_{i=1}^{2n}=\left(x_{01},x_{11},\dots,x_{0\,n-1},x_{1\,n-1},\hat\Delta_n^{(n-1)},\frac{\hat\Delta_n^{(n-2)}}{\hat\Delta_n^{(n-1)}}=\frac{1}{\det X}\right) \tag{5.14}$$

and $\mathbf{y}=\mathbf{y}(\varepsilon)=(y_i)_{i=1}^{2n}=(y_{00},y_{10},\dots,y_{0\,n-1},y_{1\,n-1})$. Then

$$y_i=\prod_{i=1}^{2n}x_j^{a_{ij}},$$

where $A=(a_{ij})_{i,j=1}^{2n}$ is an $n\times n$ block lower–triangular matrix with 2×2 blocks:

$$A=\begin{pmatrix} V_1 & 0 & 0 & \dots & 0\\ U & V_2 & 0 & \dots & 0\\ -V_2^T & U & V_3 & \dots & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & \dots & -V_{n-1}^T & U & V_n \end{pmatrix},$$

with

$$\begin{gathered} U=\begin{pmatrix} 0 & 2\\ -2 & 0\end{pmatrix},\quad V_1=\begin{pmatrix} -1 & 0\\ 2 & -1\end{pmatrix},\quad V_n=\begin{pmatrix} -1 & \varkappa_n-n+1\\ 1 & n-\varkappa_n\end{pmatrix},\\ V_i=\begin{pmatrix} \varepsilon_i-1 & -\varepsilon_i\\ 2-\varepsilon_i & \varepsilon_i-1\end{pmatrix},\quad i\in[2,n-1]. \end{gathered} \tag{5.15}$$

The matrix A is invertible, since $\det V_i=1$, $i\in[1,n-1]$, and $\det V_n=-1$.

Remark 5.3. Note that the expression for x_{2n} in terms of face weights is independent of ε:

$$x_{2n}=(y_{00}y_{10})^n(y_{01}y_{11})^{n-1}\dots(y_{0\,n-1}y_{1\,n-1}).$$

5.3. Given a pair of Coxeter elements (u, v), we want to define a cluster algebra with the compatible Poisson bracket given by (5.6). To this end, we use the strategy developed in [20]. The first step consists of finding a coordinate system on $\mathcal{R}_n$ such that written in terms of their logarithms, the Poisson bracket (5.6) becomes constant. Having in mind Proposition 2.1, we require this coordinate system to be given by a collection of regular functions on $\mathcal{R}_n$. Clearly, H_i, $i\geqslant 0$, are regular on $\mathcal{R}_n$, and hence so are $\hat{\Delta}_n^{(n-1)}$ and $\hat{\Delta}_n^{(n)}$. Besides, it was explained in the proof of Lemma 5.2 that $\hat{\Delta}_n^{(n-1)}$ and $\hat{\Delta}_n^{(n)}$ do not vanish on $\mathcal{R}_n$, and hence $(\hat{\Delta}_n^{(n-1)})^{-1}$ and $(\hat{\Delta}_n^{(n)})^{-1}$ are regular as well. Consequently, by Remark 5.1 (i), H_i are regular functions on $\mathcal{R}_n$ for $i<0$, and hence so are the Hankel determinants (5.8) for any $l\in\mathbb{Z}$ and $i\in\mathbb{N}$. Therefore, the components of $\mathbf{x}(\varepsilon)$ are regular functions on $\mathcal{R}_n$ and they are connected by an invertible monomial transformation to face weights $\mathbf{y}(\varepsilon)$ which satisfy the Poisson relations (5.1) and (5.2) of the required kind. Therefore we can use $\mathbf{x}(\varepsilon)$ as an initial cluster. Now, following [20], we have to compute the matrix which defines cluster transformations, based on the coefficient matrix of the bracket (5.6).

Define a $2n\times 2n$ matrix

$$B(\varepsilon)=-\begin{pmatrix} U & V_2 & 0 & \dots & 0\\ -V_2^T & U & V_3 & \dots & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & \dots & -V_{n-1}^T & U & V_n\\ 0 & \dots & 0 & -V_n^T & -\frac{1}{2}U \end{pmatrix} \tag{5.16}$$

with 2×2 block coefficients given by (5.15). Denote by $\widetilde{B}(\varepsilon)$ the $(2n-2)\times 2n$ submatrix of $B(\varepsilon)$ formed by the first $2n-2$ rows and consider the cluster algebra $\mathcal{A}_\varepsilon$ of rank $2n-2$ with the initial seed $\Sigma(\varepsilon)=(\mathbf{x}(\varepsilon), \widetilde{B}(\varepsilon))$, so that x_i, $i\in[1, 2n-2]$, are cluster variables and x_{2n-1} and x_{2n} are stable variables.

LEMMA 5.1. *The Poisson structure* (5.6) *is compatible with the cluster algebra* $\mathcal{A}_\varepsilon$.

Proof. Let us first revisit the standard Poisson structure on $\mathcal{F}_{N^\circ_{u,v}}$ described by (5.1) and (5.2). It is easy to see that in terms of the components of the vector $\mathbf{y}=\mathbf{y}(\varepsilon)$, this bracket can be written as $\{y_i, y_j\}=\omega_{ij}y_iy_j$, where the matrix $\Omega=\Omega(\varepsilon)=(\omega_{ij})_{i,j=1}^{2n}$ is given by

$$\Omega=\begin{pmatrix} \frac{1}{2}U & V_1 & 0 & \dots & 0\\ -V_1^T & U & V_2 & \dots & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & \dots & -V_{n-2}^T & U & V_{n-1}\\ 0 & \dots & 0 & -V_{n-1}^T & U \end{pmatrix}. \tag{5.17}$$

Therefore, the matrix of coefficients of the Poisson bracket (5.6) written in coordinates $\mathbf{x}(\varepsilon)$ is $\Omega^{\mathbf{x}}=A^{-1}\Omega(A^T)^{-1}$.

Note that Ω, defined by (5.17), is invertible. To see that, observe that the block-entries of Ω satisfy the relations $V_i^TUV_i=U$, $i\in[1,n-1]$, and $U^2=4\mathbf{1}_2$, which implies that Ω can be factored as

$$\Omega=\begin{pmatrix} \mathbf{1}_2 & 0 & \dots & 0 \\ \frac{1}{2}V_1^TU & \mathbf{1}_2 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \frac{1}{2}V_{n-1}^TU & \mathbf{1}_2 \end{pmatrix}\begin{pmatrix} \frac{1}{2}U & V_1 & 0 & \dots & 0 \\ 0 & \frac{1}{2}U & V_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{2}U & V_{n-1} \\ 0 & 0 & \dots & 0 & \frac{1}{2}U \end{pmatrix}. \tag{5.18}$$

Therefore, $\det\Omega=1$, and hence $\Omega^{\mathbf{x}}$ is invertible.

To find its inverse $A^T\Omega^{-1}A$, observe that if we define

$$J=\begin{pmatrix} 0 & \mathbf{1}_2 & 0 & \dots & 0 \\ 0 & 0 & \mathbf{1}_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \mathbf{1}_2 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

then $A=\Omega J^T+V_n\otimes E_{nn}$, with $E_{nn}=(\delta_{in}\delta_{jn})_{i,j=1}^n$. Thus we have

$$A^T\Omega^{-1}A=J\Omega^TJ^T-J(V_n\otimes E_{nn})+(V_n^T\otimes E_{nn})J^T+(V_n^T\otimes E_{nn})\Omega^{-1}(V_n\otimes E_{nn})=B(\varepsilon),$$

since, by (5.18), the lower-right 2×2 block of Ω^{-1} equals $-\frac{1}{2}U$ and $V_n^TUV_n=-U$.

So, $B(\varepsilon)$ is non-degenerate and skew-symmetric. Thus we can invoke Theorem 1.4 of [20]. According to equation (1.5) in the proof of this theorem, compatibility will follow from the condition $\widetilde{B}(\varepsilon)\Omega^{\mathbf{x}}=(D\ 0)$, where D is a $(2n-2)\times(2n-2)$ diagonal matrix. Since $B^{-1}(\varepsilon)=\Omega^{\mathbf{x}}$, this condition is obviously satisfied with $D=\mathbf{1}_{2n-2}$. □

Our goal is to prove the following result.

THEOREM 5.1. (i) *The cluster algebra $\mathcal{A}_\varepsilon$ does not depend on ε.*

(ii) *The localization of the complex form of $\mathcal{A}_\varepsilon$ with respect to the stable variables x_{2n-1} and x_{2n} is isomorphic to the ring of regular functions on $\mathcal{R}_n$.*

Proof. First, we will compute the cluster transformations (2.4) of the initial cluster $\mathbf{x}(\varepsilon)$ in the directions $(0i)$ and $(1i)$. The transformed variables are denoted $\bar{x}_{0i}$ and $\bar{x}_{1i}$,

respectively. By (5.16), for $i\in[1,n-2]$ the transformations in question are determined by the matrix

$$\begin{pmatrix} \varepsilon_i-1 & 2-\varepsilon_i & 0 & -2 & 1-\varepsilon_{i+1} & \varepsilon_{i+1} \\ -\varepsilon_i & \varepsilon_i-1 & 2 & 0 & \varepsilon_{i+1}-2 & 1-\varepsilon_{i+1} \end{pmatrix}.$$

Therefore, we have to consider the following cases.

Case 1. $\varepsilon_i=\varepsilon_{i+1}=0$. Then, by (3.11), $\varkappa_{i+1}=\varkappa_i+1$ and $\varkappa_{i-1}=\varkappa_i-1$, so $x_{0i}=\hat{\Delta}_i^{(\varkappa_i)}$ is transformed into

$$\bar{x}_{0i}=\frac{\hat{\Delta}_{i-1}^{(\varkappa_i-1)}(\hat{\Delta}_i^{(\varkappa_i+1)})^2+\hat{\Delta}_{i+1}^{(\varkappa_i+1)}(\hat{\Delta}_{i-1}^{(\varkappa_i)})^2}{\hat{\Delta}_i^{(\varkappa_i)}}.$$

Using (4.4) and (5.4), we rewrite the numerator as

$$\begin{aligned}\hat{\Delta}_{i-1}^{(\varkappa_i-1)}&(\hat{\Delta}_i^{(\varkappa_i)}\hat{\Delta}_i^{(\varkappa_i+2)}-\hat{\Delta}_{i-1}^{(\varkappa_i+1)}\hat{\Delta}_{i+1}^{(\varkappa_i+1)})+\hat{\Delta}_{i+1}^{(\varkappa_i+1)}(\hat{\Delta}_{i-1}^{(\varkappa_i)})^2\\ &=\hat{\Delta}_i^{(\varkappa_i)}\hat{\Delta}_{i-1}^{(\varkappa_i-1)}\hat{\Delta}_i^{(\varkappa_i+2)}+\hat{\Delta}_{i+1}^{(\varkappa_i+1)}((\hat{\Delta}_{i-1}^{(\varkappa_i)})^2-\hat{\Delta}_{i-1}^{(\varkappa_i-1)}\hat{\Delta}_{i-1}^{(\varkappa_i+1)})\\ &=\hat{\Delta}_i^{(\varkappa_i)}(\hat{\Delta}_{i-1}^{(\varkappa_i-1)}\hat{\Delta}_i^{(\varkappa_i+2)}-\hat{\Delta}_{i-2}^{(\varkappa_i)}\hat{\Delta}_{i+1}^{(\varkappa_i+1)}),\end{aligned}$$

and so

$$\bar{x}_{0i}=\hat{\Delta}_{i-1}^{(\varkappa_i-1)}\hat{\Delta}_i^{(\varkappa_i+2)}-\hat{\Delta}_{i-2}^{(\varkappa_i)}\hat{\Delta}_{i+1}^{(\varkappa_i+1)}. \tag{5.19}$$

Similarly, $x_{1i}=\hat{\Delta}_i^{(\varkappa_i+1)}$ is transformed into

$$\bar{x}_{1i}=\frac{\hat{\Delta}_{i-1}^{(\varkappa_i)}(\hat{\Delta}_{i+1}^{(\varkappa_i+1)})^2+\hat{\Delta}_{i+1}^{(\varkappa_i+2)}(\hat{\Delta}_i^{(\varkappa_i)})^2}{\hat{\Delta}_i^{(\varkappa_i+1)}},$$

which can be rewritten as

$$\bar{x}_{1i}=\hat{\Delta}_i^{(\varkappa_i-1)}\hat{\Delta}_{i+1}^{(\varkappa_i+2)}-\hat{\Delta}_{i-1}^{(\varkappa_i)}\hat{\Delta}_{i+2}^{(\varkappa_i+1)}. \tag{5.20}$$

Case 2. $\varepsilon_i=\varepsilon_{i+1}=2$. This case is similar to case 1. We have $\varkappa_{i+1}=\varkappa_i-1$ and $\varkappa_{i-1}=\varkappa_i+1$. Hence

$$\bar{x}_{0i}=\frac{\hat{\Delta}_{i-1}^{(\varkappa_i+1)}(\hat{\Delta}_{i+1}^{(\varkappa_i)})^2+\hat{\Delta}_{i+1}^{(\varkappa_i-1)}(\hat{\Delta}_i^{(\varkappa_i+1)})^2}{\hat{\Delta}_i^{(\varkappa_i)}}=\hat{\Delta}_i^{(\varkappa_i+2)}\hat{\Delta}_{i+1}^{(\varkappa_i-1)}-\hat{\Delta}_{i-1}^{(\varkappa_i+1)}\hat{\Delta}_{i+2}^{(\varkappa_i)} \tag{5.21}$$

and

$$\bar{x}_{1i}=\frac{\hat{\Delta}_{i+1}^{(\varkappa_i)}(\hat{\Delta}_{i-1}^{(\varkappa_i+1)})^2+\hat{\Delta}_{i-1}^{(\varkappa_i+2)}(\hat{\Delta}_i^{(\varkappa_i)})^2}{\hat{\Delta}_i^{(\varkappa_i+1)}}=\hat{\Delta}_{i-1}^{(\varkappa_i+2)}\hat{\Delta}_i^{(\varkappa_i-1)}-\hat{\Delta}_{i-2}^{(\varkappa_i+1)}\hat{\Delta}_{i+1}^{(\varkappa_i)}. \tag{5.22}$$

Case 3. $\varepsilon_i=0$, $\varepsilon_{i+1}=2$. We have $\varkappa_{i+1}=\varkappa_{i-1}=\varkappa_i-1$, and so x_{0i} is transformed into

$$\bar{x}_{0i}=\frac{(\hat{\Delta}_{i-1}^{(\varkappa_i)}\hat{\Delta}_{i+1}^{(\varkappa_i)})^2+\hat{\Delta}_{i-1}^{(\varkappa_i-1)}\hat{\Delta}_{i+1}^{(\varkappa_i-1)}(\hat{\Delta}_i^{(\varkappa_i+1)})^2}{\hat{\Delta}_i^{(\varkappa_i)}}.$$

The numerator of the above expression can be rewritten as

$$\begin{aligned}(\hat{\Delta}_{i-1}^{(\varkappa_i)}\hat{\Delta}_{i+1}^{(\varkappa_i)})^2&+(\hat{\Delta}_i^{(\varkappa_i)}\hat{\Delta}_i^{(\varkappa_i-2)}-(\hat{\Delta}_i^{(\varkappa_i-1)})^2)(\hat{\Delta}_i^{(\varkappa_i+1)})^2\\&=((\hat{\Delta}_{i-1}^{(\varkappa_i)}\hat{\Delta}_{i+1}^{(\varkappa_i)})^2-(\hat{\Delta}_i^{(\varkappa_i+1)}\hat{\Delta}_i^{(\varkappa_i-1)})^2)+\hat{\Delta}_i^{(\varkappa_i)}\hat{\Delta}_i^{(\varkappa_i-2)}(\hat{\Delta}_i^{(\varkappa_i+1)})^2\\&=\hat{\Delta}_i^{(\varkappa_i)}(\hat{\Delta}_i^{(\varkappa_i-2)}(\hat{\Delta}_i^{(\varkappa_i+1)})^2-\hat{\Delta}_i^{(\varkappa_i)}(\hat{\Delta}_{i-1}^{(\varkappa_i)}\hat{\Delta}_{i+1}^{(\varkappa_i)}+\hat{\Delta}_i^{(\varkappa_i+1)}\hat{\Delta}_i^{(\varkappa_i-1)})),\end{aligned}$$

and so

$$\bar{x}_{0i}=\hat{\Delta}_i^{(\varkappa_i-2)}(\hat{\Delta}_i^{(\varkappa_i+1)})^2-\hat{\Delta}_i^{(\varkappa_i)}(\hat{\Delta}_{i-1}^{(\varkappa_i)}\hat{\Delta}_{i+1}^{(\varkappa_i)}+\hat{\Delta}_i^{(\varkappa_i+1)}\hat{\Delta}_i^{(\varkappa_i-1)}).\tag{5.23}$$

On the other hand,

$$\bar{x}_{1i}=\frac{\hat{\Delta}_{i-1}^{(\varkappa_i)}\hat{\Delta}_{i+1}^{(\varkappa_i)}+(\hat{\Delta}_i^{(\varkappa_i)})^2}{\hat{\Delta}_i^{(\varkappa_i+1)}}=\hat{\Delta}_i^{(\varkappa_i-1)}.\tag{5.24}$$

Case 4. $\varepsilon_i=2$, $\varepsilon_{i+1}=0$. This case is similar to case 3. We have $\varkappa_{i+1}=\varkappa_{i-1}=\varkappa_i+1$. Hence

$$\bar{x}_{0i}=\frac{\hat{\Delta}_{i-1}^{(\varkappa_i+1)}\hat{\Delta}_{i+1}^{(\varkappa_i+1)}+(\hat{\Delta}_i^{(\varkappa_i+1)})^2}{\hat{\Delta}_i^{(\varkappa_i)}}=\hat{\Delta}_i^{(\varkappa_i+2)}\tag{5.25}$$

and

$$\begin{aligned}\bar{x}_{1i}&=\frac{(\hat{\Delta}_{i-1}^{(\varkappa_i+1)}\hat{\Delta}_{i+1}^{(\varkappa_i+1)})^2+\hat{\Delta}_{i-1}^{(\varkappa_i+2)}\hat{\Delta}_{i+1}^{(\varkappa_i+2)}(\hat{\Delta}_i^{(\varkappa_i)})^2}{\hat{\Delta}_i^{(\varkappa_i+1)}}\\&=\hat{\Delta}_i^{(\varkappa_i+3)}(\hat{\Delta}_i^{(\varkappa_i)})^2-\hat{\Delta}_i^{(\varkappa_i+1)}(\hat{\Delta}_{i-1}^{(\varkappa_i+1)}\hat{\Delta}_{i+1}^{(\varkappa_i+1)}+\hat{\Delta}_i^{(\varkappa_i)}\hat{\Delta}_i^{(\varkappa_i+2)}).\end{aligned}\tag{5.26}$$

Case 5. $\varepsilon_i=1$, $\varepsilon_{i+1}=2$. We have $\varkappa_{i+1}=\varkappa_i-1$ and $\varkappa_{i-1}=\varkappa_i$, so x_{0i} is transformed via (5.21) and x_{1i} via (5.24).

Case 6. $\varepsilon_i=2$, $\varepsilon_{i+1}=1$. We have $\varkappa_{i+1}=\varkappa_i$ and $\varkappa_{i-1}=\varkappa_i+1$, so x_{0i} is transformed via (5.25) and x_{1i} via (5.22).

Case 7. $\varepsilon_i=0$, $\varepsilon_{i+1}=1$. We have $\varkappa_{i+1}=\varkappa_i$ and $\varkappa_{i-1}=\varkappa_i-1$, so x_{0i} is transformed via (5.19) and x_{1i} via (5.24).

Case 8. $\varepsilon_i=1$, $\varepsilon_{i+1}=0$. We have $\varkappa_{i+1}=\varkappa_i+1$ and $\varkappa_{i-1}=\varkappa_i$, so x_{0i} is transformed via (5.25) and x_{1i} via (5.20).

Case 9. $\varepsilon_i{=}\varepsilon_{i+1}{=}1$. We have $\varkappa_{i+1}{=}\varkappa_{i-1}{=}\varkappa_i$, so x_{0i} is transformed via (5.25) and x_{1i} via (5.24).

Now, let $i{=}n{-}1$. In this situation transformations of the initial cluster are determined by the matrix

$$\begin{pmatrix} \varepsilon_{n-1}{-}1 & 2{-}\varepsilon_{n-1} & 0 & -2 & 1 & n{-}\varkappa_n{-}1 \\ -\varepsilon_{n-1} & \varepsilon_{n-1}{-}1 & 2 & 0 & -1 & \varkappa_n{-}n \end{pmatrix}.$$

Note that $\varkappa_n$ does not exceed $n{-}1$, so the last two elements in the first row are always non-negative, and the last two elements in the second row are always negative. By (5.13), they contribute to the corresponding relations $\hat{\Delta}_n^{(\varkappa_n)}$ and $\hat{\Delta}_n^{(\varkappa_n-1)}$, respectively. Therefore, we have to consider the following cases.

Case 10. $\varepsilon_{n-1}{=}0$. Then $\varkappa_{n-1}{=}\varkappa_n{-}1{=}\varkappa_{n-2}{+}1$, and $x_{0\,n-1}$ is transformed into

$$\bar{x}_{0\,n-1} = \frac{(\hat{\Delta}_{n-1}^{(\varkappa_n)})^2\hat{\Delta}_{n-2}^{(\varkappa_n-2)}+\hat{\Delta}_n^{(\varkappa_n)}(\hat{\Delta}_{n-2}^{(\varkappa_n-1)})^2}{\hat{\Delta}_{n-1}^{(\varkappa_n-1)}}.$$

Similarly to case 1, this gives (5.19) for $i{=}n{-}1$.

On the other hand,

$$\bar{x}_{1\,n-1} = \frac{(\hat{\Delta}_{n-1}^{(\varkappa_n-1)})^2+\hat{\Delta}_n^{(\varkappa_n-1)}\hat{\Delta}_{n-2}^{(\varkappa_n-1)}}{\hat{\Delta}_{n-1}^{(\varkappa_n)}},$$

which gives (5.24) for $i{=}n{-}1$. We thus see that the transformations in this case are exactly the same as in case 7.

Case 11. $\varepsilon_{n-1}{=}1$. Then $\varkappa_{n-1}{=}\varkappa_n{-}1{=}\varkappa_{n-2}$, and hence

$$\bar{x}_{0\,n-1} = \frac{(\hat{\Delta}_{n-1}^{(\varkappa_n)})^2+\hat{\Delta}_n^{(\varkappa_n)}\hat{\Delta}_{n-2}^{(\varkappa_n)}}{\hat{\Delta}_{n-1}^{(\varkappa_n-1)}},$$

which gives (5.25) for $i{=}n{-}1$.

Similarly,

$$\bar{x}_{1\,n-1} = \frac{(\hat{\Delta}_{n-1}^{(\varkappa_n-1)})^2+\hat{\Delta}_n^{(\varkappa_n-1)}\hat{\Delta}_{n-2}^{(\varkappa_n-1)}}{\hat{\Delta}_{n-1}^{(\varkappa_n)}},$$

which gives (5.24) for $i{=}n{-}1$. We thus see that the transformations in this case are exactly the same as in case 9.

Case 12. $\varepsilon_{n-1}{=}2$. Then $\varkappa_{n-1}{=}\varkappa_n{-}1{=}\varkappa_{n-2}{-}1$, and $x_{0\,n-1}$ transforms exactly as in the previous case.

ε	Direction	ε'
$\varepsilon_i{=}0$, $\varepsilon_{i+1}{=}2$	$(1,i)$	$\varepsilon'_i{=}1$, $\varepsilon'_{i+1}{=}1$
$\varepsilon_i{=}1$, $\varepsilon_{i+1}{=}0$	$(0,i)$	$\varepsilon'_i{=}0$, $\varepsilon'_{i+1}{=}1$
$\varepsilon_i{=}2$, $\varepsilon_{i+1}{=}0$	$(0,i)$	$\varepsilon'_i{=}1$, $\varepsilon'_{i+1}{=}1$
$\varepsilon_i{=}2$, $\varepsilon_{i+1}{=}1$	$(0,i)$	$\varepsilon'_i{=}1$, $\varepsilon'_{i+1}{=}2$
$\varepsilon_{n-1}{=}0$	$(0,n{-}1)$	$\varepsilon'_{n-1}{=}1$
$\varepsilon_{n-1}{=}1$	$(1,n{-}1)$	$\varepsilon'_{n-1}{=}2$

Table 1.

On the other hand,

$$\bar{x}_{0n-1} = \frac{(\hat{\Delta}_{n-1}^{(\varkappa_n-1)})^2\hat{\Delta}_{n-2}^{(\varkappa_n+1)} + \hat{\Delta}_{n}^{(\varkappa_n-1)}(\hat{\Delta}_{n-2}^{(\varkappa_n)})^2}{\hat{\Delta}_{n-1}^{(\varkappa_n)}}.$$

Similarly to case 2, this gives (5.22) for $i{=}n{-}1$. We thus see that the transformations in this case are exactly the same as in case 6.

Let (u', v') be an arbitrary pair of Coxeter elements, ε' be the corresponding n-tuple built by (3.7) and (3.10).

LEMMA 5.2. *For any Coxeter elements u' and v', the seed $\Sigma(\varepsilon'){=}(\mathbf{x}(\varepsilon'), \widetilde{B}(\varepsilon'))$ belongs to $\mathcal{A}_\varepsilon$.*

Proof. First we will show that, in certain cases, mutations of the seed $\Sigma(\varepsilon)$ transform it into a seed equivalent to $\Sigma(\varepsilon')$ for an appropriately chosen ε'. These situations are listed in Table 1. In this table, only the entries at which ε and ε' differ are specified. In the first four rows, i is assumed to be less than $n{-}1$. The second column describes the direction of the seed mutation: under the mutation in direction (s,i), the cluster variable x_{si} is being transformed. It should also be understood that each mutation is followed by the permutation of variables with indices $(0,i)$ and $(1,i)$ in the new cluster, which results in permuting columns and rows $2i{-}1$ and $2i$ in the matrix obtained via the corresponding matrix mutation. In particular, if $\mathbf{x}(\varepsilon')$ is obtained from $\mathbf{x}(\varepsilon)$ via the cluster transformation in direction (s,i), then $\mathbf{x}(\varepsilon)$ is obtained from $\mathbf{x}(\varepsilon')$ via the cluster transformation in direction $(1{-}s,i)$.

We will only provide justification for rows one and five of the table. The remaining cases can be treated similarly. If $\varepsilon_i{=}0$ and $\varepsilon_{i+1}{=}2$, let ε' be defined by $\varepsilon'_i{=}1$, $\varepsilon'_{i+1}{=}1$ and $\varepsilon'_j{=}\varepsilon_j$ for $j{\neq}i, i{+}1$. Then it is easy to check that the matrix mutation in the direction $(1,i)$ followed by the permutation of rows and columns $2i{-}1$ and $2i$ transforms $B(\varepsilon)$ into

$B(\varepsilon')$. Note also that when ε is replaced by ε', the corresponding sequence $\varkappa=(\varkappa_i)_{i=1}^n$ transforms into a sequence $\varkappa'$ that differs from $\varkappa$ only in the component $\varkappa'_i=\varkappa_i-1$. This means that $\mathbf{x}'=\mathbf{x}(\varepsilon')$ differs from $\mathbf{x}(\varepsilon)$ only in the components $x_{0i}(\varepsilon')=\hat\Delta_i^{(\varkappa_i-1)}=\bar x_{1i}(\varepsilon)$ (cf. (5.23) in case 3) and $x_{1i}(\varepsilon')=\hat\Delta_i^{(\varkappa_i)}=x_{0i}(\varepsilon)$. Thus, we see that the seed mutation in direction $(1,i)$ of the initial seed of $\mathcal{A}_\varepsilon$ transforms it into a seed equivalent to the initial seed of $\mathcal{A}_{\varepsilon'}$.

Now consider the case $\varepsilon_{n-1}=0$ and $\varepsilon'_{n-1}=1$. Then $\varkappa_{n-1}=\varkappa_n-1$, $\varkappa'_{n-1}=\varkappa_{n-1}-1$ and $\varkappa'_n=\varkappa_n-1$. The fact that the matrix mutation in direction $(1,n-1)$ followed by the permutation of rows and columns $2n-1$ and $2n$ transforms $B(\varepsilon)$ into $B(\varepsilon')$ becomes easy to check once we recall that, by (3.11), $n-1-\varkappa_n$ is always non-negative. As was shown in case 11 above, $\bar x_{1\,n-1}(\varepsilon)=\hat\Delta_{n-1}^{(\varkappa_n-2)}=\hat\Delta_{n-1}^{(\varkappa'_{n-1})}=x_{0\,n-1}(\varepsilon')$. Also $\bar x_{0\,n-1}(\varepsilon)=\hat\Delta_{n-1}^{(\varkappa_{n-1})}=\hat\Delta_{n-1}^{(\varkappa'_{n-1}+1)}=x_{1\,n-1}(\varepsilon')$, which completes the check.

To complete the proof of the lemma, it suffices to show that, for any ε', the seed $\Sigma(\varepsilon')$ is a seed in $\mathcal{A}_{\varepsilon^{(0)}}$ for $\varepsilon^{(0)}=(2,0,...,0)$. This can be done by induction on $\sum_{i=2}^{n-1}\varepsilon'_i$. Indeed, if $\varepsilon'_{n-1}\ne0$, then $\Sigma(\varepsilon')$ can be obtained via a single mutation from $\Sigma(\varepsilon)$, where ε differs from ε' only in the $(n-1)$-st component: $\varepsilon_{n-1}=\varepsilon'_{n-1}-1$ (see the last two rows of Table 1). Otherwise, if $i\in[2,n-2]$ is the largest index such that $\varepsilon'_i\ne0$, then, using the table again, we see that $\Sigma(\varepsilon')$ can be obtained via a sequence of mutations from $\Sigma(\varepsilon)$, where $\varepsilon=(\varepsilon'_1,...,\varepsilon'_{i-1},\varepsilon'_i-1,0,...,0)$. The intermediate transformations of the n-tuple ε in this case are $(\varepsilon'_1,...,\varepsilon'_{i-1},\varepsilon'_i-1,0,...,0,1,0)$, $(\varepsilon'_1,...,\varepsilon'_{i-1},\varepsilon'_i-1,0,...,1,0,0)$, $...,(\varepsilon'_1,...,\varepsilon'_{i-1},\varepsilon'_i-1,1,0,...,0,0)$. □

The first statement of Theorem 5.1 follows immediately. We can now drop the dependence on ε in the cluster algebra $\mathcal{A}_\varepsilon$ and denote it simply by $\mathcal{A}$.

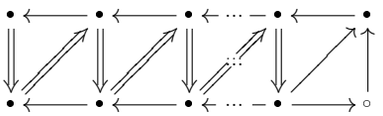
LEMMA 5.3. *For any $j\in[1,n-1]$ and $k\in\mathbb{Z}$, the function*

$$x(j,k)=\hat\Delta_j^{(k)}x_{2n}^{\max\{0,k+j+1-2n\}} \tag{5.27}$$

is a cluster variable in $\mathcal{A}$.

Proof. Consider the cluster $\mathbf{x}=\mathbf{x}(\varepsilon)$ corresponding to $\varepsilon=(2,0,...,0)$. In this case $x_{2j-1}=\hat\Delta_j^{(j-1)}$, $j\in[1,n]$, and $x_{2j}=\hat\Delta_j^{(j)}$, $j\in[1,n-1]$. The matrix $B(\varepsilon)$ can be conveniently represented by a planar graph Γ, whose vertices are represented by nodes of a $2\times n$ rectangular grid. Vertices in the top row (listed left to right) correspond to cluster variables $x_1,x_3,...,x_{2n-1}$, and vertices in the bottom row correspond to cluster variables $x_2,x_4,...,x_{2n}$. We will label the jth vertex in the sth row by (s,j), $s=0,1$, $j\in[2,n]$ ($s=0$ corresponds to the top row, $s=1$ to the bottom row). In accordance with (5.16), Γ has edges $(s,j)\to(s,j-1)$ for $s=0,1$ and any $j\in[2,n-1]$, edges $(0,n)\to(0,n-1)$, $(1,n-1)\to$

$(1,n)$, $(1,n-1)\to(0,n)$, $(1,n)\to(0,n)$ and double edges $(0,j)\to(1,j)$ for $j\in[1,n-1]$ and $(1,j)\to(0,j+1)$ for $j\in[1,n-2]$:



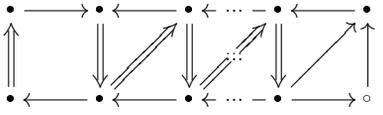
(5.28)

We marked the vertex corresponding to the stable variable x_{2n} differently, as it plays a special role in what follows. In particular, we will occasionally perturb a two-row structure of transformations of the graph Γ by "moving around" the white vertex. Note also that, in view of the definition (5.27), the cluster variable associated with the vertex (s,j) of Γ is $x(j,s+j-1)$ for $j\in[1,n-1]$.

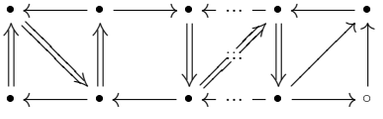
Denote by T_p the cluster transformation in direction p. Let us consider the result of the composition

$$T=T_{2n-2}\circ T_{2n-4}\circ\ldots\circ T_4\circ T_2\circ T_{2n-3}\circ\ldots\circ T_3\circ T_1.$$

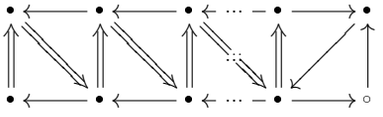
An application of T_1 transforms $x_1=H_0$ into $\tilde{x}_1=(H_1^2+\hat{\Delta}_2^{(1)})/H_0=H_2$ and the graph Γ into



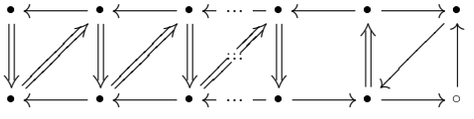
Next, an application of T_3 transforms x_3 into $\tilde{x}_3=\hat{\Delta}_2^{(3)}$ (here we use (4.4) with $i=l=2$) and the graph Γ into



Continuing in the same fashion and using in the jth step relation (4.4) with $i=l=j$, we conclude that an application of $T_{2n-3}\circ\ldots\circ T_3\circ T_1$ to the initial cluster transforms Γ into



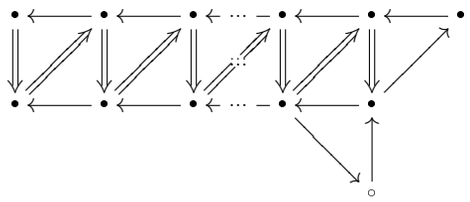
with the variable x_{2i-1} replaced by $\tilde{x}_{2i-1}=\hat{\Delta}_i^{(i+1)}$ for all $i\in[1,n-1]$. Similarly, the subsequent application of $T_{2n-4}\circ\ldots\circ T_4\circ T_2$ transforms Γ into



and replaces x_{2i} by $\tilde{x}_{2i}=\hat{\Delta}_i^{(i+2)}$ for all $i\in[1,n-2]$. Finally, T_{2n-2} transforms $x_{2n-2}=\hat{\Delta}_{n-1}^{(n-1)}$ into

$$\begin{aligned}\tilde{x}_{2n-2}&=(\hat{\Delta}_{n-1}^{(n-1)})^{-1}(x_{2n}(\hat{\Delta}_{n-1}^{(n)})^2+\hat{\Delta}_{n-2}^{(n)}\hat{\Delta}_n^{(n-1)})\\&=(\hat{\Delta}_{n-1}^{(n-1)})^{-1}x_{2n}((\hat{\Delta}_{n-1}^{(n)})^2+\hat{\Delta}_{n-2}^{(n)}\hat{\Delta}_n^{(n)})=x_{2n}\hat{\Delta}_{n-1}^{(n+1)},\end{aligned}$$

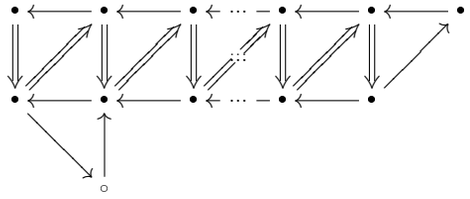
where we used (4.10). The corresponding transformation of the graph Γ is



(5.29)

To summarize, T results in the transformation $x(j,s+j-1)\mapsto x(j,s+j+1)$ for $s=0,1$, $j\in[1,n-1]$, and in replacing the initial graph Γ (see (5.28)) with $T(\Gamma)$ (see (5.29)). Observe also that the subgraphs of Γ and $T(\Gamma)$ spanned by black vertices coincide.

Arguing in exactly the same fashion, we deduce that for $r=1,\dots,n-2$, an application of T^r results in a cluster $T^r(\mathbf{x})$ with the corresponding graph $T^r(\Gamma)$ such that (i) the subgraphs of Γ and $T^r(\Gamma)$ spanned by black vertices coincide and (ii) the white vertex is connected by simple edges to the vertices $(1,n-r)$ and $(1,n-r-1)$, so as to form a cyclically oriented triangle. In particular, the graph associated with $T^{n-2}(\mathbf{x})$ is

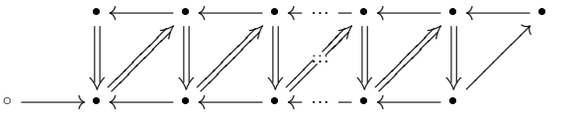


Furthermore, the cluster variable in $T^r(\mathbf{x})$ associated with the vertex (s,j), $s=0,1$, $j\in[1,n-1]$, in $T^r(\Gamma)$ is $x(j,s+j-1+2r)$. This claim relies on repeated applications of the relations

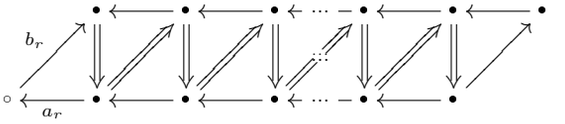
$$\begin{aligned}x(1,k-1)x(1,k+1)&=x_{2n}^{\delta_{k+2-2n,0}}x(1,k)^2+x(2,k),\\x(j,k-1)x(j,k+1)&=x_{2n}^{\delta_{k+j+1-2n,0}}x(j,k)^2+x(j-1,k)x(j+1,k),\quad j\in[2,n-2],\\x(n-1,k-1)x(n-1,k+1)&=x_{2n}^{\delta_{k-n,0}}x(n-1,k)^2+x_{2n}^{\max\{0,k+1-n\}}x(n-2,k-1)\hat{\Delta}_n^{(n-1)},\end{aligned}$$

which, in turn, follow easily from (5.27), (4.4) and (4.10).

The same pattern of transformations for cluster variables remains valid also for $r \geqslant n-1$. However, the graph associated with $T^{n-1}(\mathbf{x})$ has the form



and the graph associated with $T^r(\mathbf{x})$ for $r > n-1$ has the form



where multiplicities a_r and b_r are given by $a_r = 2(r-n)+1$ and $b_r = 2(r-n)+2$. Thus we have shown that $x(j, l+j-1)$ is a cluster variable in $\mathcal{A}$ for any $j \in [1, n-1]$, and $l \geqslant 0$.

To recover $x(j, 1-j-l)$ for $j \in [1, n-1]$, $l \geqslant 0$, we act in a similar way, starting with the cluster corresponding to $\varepsilon = (2, 2, ..., 2, 0)$ and repeatedly applying a composition of cluster transformations

$$T_{2n-3} \circ ... \circ T_3 \circ T_1 \circ T_{2n-2} \circ ... \circ T_4 \circ T_2.$$

Thus, $x(j, k)$ is a cluster variable in $\mathcal{A}$ for $k \in \mathbb{Z} \setminus [2-j, j-2]$. To complete the proof, it suffices to notice that by (3.11), the range of possible values for $\varkappa_j$ is $[1-j, j-1]$, and thus for any $k \in [2-j, j-2]$ there exists a cluster $\mathbf{x}(\varepsilon)$ given by (5.14) such that $x(j, k)$ is one of its variables. Therefore, this variable belongs to $\mathcal{A}$ by Lemma 5.2. □

Since $x(1, k) = H_k$ for $k \leqslant 2n-2$, the following statement is readily apparent from Lemma 5.3.

COROLLARY 5.1. *For any $k \leqslant 2n-2$, H_k is a cluster variable in $\mathcal{A}$.*

To prove the second statement of Theorem 5.1, we would like to apply Proposition 2.1 in the situation when $V = \mathcal{R}_n$, $\mathcal{A}$ is the cluster algebra discussed above and $\mathbf{x}$ is given by (5.14). Clearly $\mathcal{R}_n$ is Zariski open in $\mathbb{C}^{2n}$, as the complement of the union of hypersurfaces $\hat{\Delta}_n^{(n-1)} = 0$ and $p_0 = 0$. Condition (ii) is satisfied by construction, and (iii) follows from cases 1–12 discussed above. It remains to check condition (i).

Observe that the ring of regular functions on $\mathcal{R}_n$ is generated by $2(n+1)$ functions $p_0, ..., p_{n-1}$, $q_0, ..., q_{n-1}$, p_0^{-1}, $(\hat{\Delta}_n^{(n-1)})^{-1}$. Clearly, the last two generators belong to $\mathcal{A}^*$. Recall that the coefficients p_i satisfy the relations (5.3). These relations for $k \in [-2, n-3]$ provide a system of linear equations, and by Corollary 5.1, the coefficients of this system belong to $\mathcal{A}^*$. The determinant of the system is $\hat{\Delta}_n^{(n-3)} = \hat{\Delta}_n^{(n-1)} / p_0^2$, so it does not vanish on $\mathcal{R}_n$. Therefore, the coefficients $p_0, ..., p_{n-1}$ belong to $\mathcal{A}^*$. Finally, the coefficients $q_0, ..., q_{n-1}$ belong to $\mathcal{A}^*$ due to the relations (5.5) that involve $p_0, ..., p_{n-1}$ and H_k for $k < n$. □

5.4. The cluster algebra $\mathcal{A}$ built above is tightly connected to the cluster algebra studied in [10] and [25]. We denote the latter $\mathcal{A}_2$, since to get it from $\mathcal{A}$ one has to fix the values of both stable variables x_{2n-1} and x_{2n} at 1. Another cluster algebra, intermediate between $\mathcal{A}$ and $\mathcal{A}_2$, is obtained by fixing the value of x_{2n} at 1; it is denoted $\mathcal{A}_1$.

The exchange matrix of $\mathcal{A}_2$ is obtained from $\widetilde{B}(\varepsilon)$ by deleting the last two rows. If we take $\varepsilon=(2,1,...,1,0)$ and rearrange the cluster variables as $x_{01},...,x_{0\,n-1},x_{11},...,x_{1\,n-1}$, the exchange matrix will be given by

$$\begin{pmatrix} 0 & -C \\ C & 0 \end{pmatrix},$$

where C is the Cartan matrix for A_{n-1}. This gives precisely the initial cluster considered in [10] and [25]. Other clusters related to Q-systems (in what follows we call them *Q-clusters*) are obtained from the initial one using the exchange relation, which is identical to (4.4). It is shown in [10, Lemma 1.3] that Q-clusters correspond bijectively to *Motzkin paths*, that is, integer sequences $\{m_1,...,m_{n-1}\}$ such that $|m_j-m_{j+1}|\leqslant 1$. It follows immediately from (3.11) that $\{\varkappa_1,...,\varkappa_{n-1}\}$ is a Motzkin path starting at 0. It is easy to check that this gives a bijection between Q-clusters corresponding to Motzkin paths starting at 0 and clusters $\mathbf{x}(\varepsilon)$ studied above: the former are truncations $\mathbf{x}_2(\varepsilon)$ obtained from $\mathbf{x}(\varepsilon)$ by deleting the stable coordinates. Any other Motzkin path is a translate of a Motzkin path starting at 0. The corresponding Q-clusters are described by the following statement.

LEMMA 5.4. *Let* $\mathbf{x}_1(\varepsilon)=((\hat\Delta_i^{(\varkappa_i)},\hat\Delta_i^{(\varkappa_i+1)})_{i=1}^{n-1},\hat\Delta_n^{(n-1)})$ *be a cluster in* $\mathcal{A}_1$ *obtained by the truncation of* $\mathbf{x}(\varepsilon)$, *and* $r\in\mathbb{Z}$.

(i) *The r-shift* $\mathbf{x}_1^r(\varepsilon)=((\hat\Delta_i^{(\varkappa_i+r)},\hat\Delta_i^{(\varkappa_i+r+1)})_{i=1}^{n-1},\hat\Delta_n^{(n-1+r)})$ *is a cluster in* $\mathcal{A}_1$, *and its exchange matrix coincides with that of* $\mathbf{x}_1(\varepsilon)$.

(ii) *Let* $\mathbf{x}_2(\varepsilon)=(\hat\Delta_i^{(\varkappa_i)},\hat\Delta_i^{(\varkappa_i+1)})_{i=1}^{n-1}$ *be the further truncation of* $\mathbf{x}_1(\varepsilon)$, *then its r-shift* $\mathbf{x}_2^r(\varepsilon)=(\hat\Delta_i^{(\varkappa_i+r)},\hat\Delta_i^{(\varkappa_i+r+1)})_{i=1}^{n-1}$ *is a Q-cluster corresponding to the Motzkin path* $\{\varkappa_1+r,...,\varkappa_{n-1}+r\}$ *and its exchange matrix coincides with that of* $\mathbf{x}_2(\varepsilon)$.

Proof. (i) For $r=1$ and $\varepsilon=(2,0,...,0)$ the proof consists of applying $T_{2n-3}\circ...\circ T_1$ to the graph Γ in (5.28) with the white vertex deleted (see the proof of Lemma 5.3, and take into account that $x_{2n}=1$ implies via (4.10) that $\hat\Delta_n^{(n-1+r)}=\hat\Delta_n^{(n-1)}$ for any $r\in\mathbb{Z}$). To extend this results to any other value ε' it suffices to use the cluster transformation taking $\mathbf{x}(\varepsilon)$ to $\mathbf{x}(\varepsilon')$. The case $r>1$ follows by induction. The case $r=-1$ is treated similarly to the case $r=1$ with $T_{2n-3}\circ...\circ T_1$ replaced by $T_2\circ...\circ T_{2n-2}$, and the case $r<-1$ follows by backward induction.

Statement (ii) follows immediately from (i). □

Consequently, all cluster variables in all Q-clusters (R_{α,m_α} in the notation of [10]) form a subset of $\{x_2(j,k)\colon j\in[1,n-1]$ and $k\in\mathbb{Z}\}$, where $x_2(j,k)$ are obtained from $x(j,k)$ defined in (5.27) by setting both stable variables to 1. The correspondence is given by $R_{\alpha,m_\alpha}\leftrightarrow x_2(\alpha,m_\alpha)$.

We conclude this section with a proposition that, in light of the above fact, implies the central positivity result in [10] (Theorem 9.15).

PROPOSITION 5.4. (i) *For any ε and any $j\in[1,n-1]$ and $k\in\mathbb{Z}$, $x(j,k)$ is a Laurent polynomial in $\mathbf{x}(\varepsilon)$ with non-negative integer coefficients.*

(ii) *For any ε and any $j\in[1,n-1]$ and $k,r\in\mathbb{Z}$, $x_1(j,k)=x(j,k)|_{x_{2n}=1}$ is a Laurent polynomial in $\mathbf{x}_1^r(\varepsilon)$ with non-negative integer coefficients.*

Proof. (i) Define the parameters c_i and d_i by (5.10). Pick a pair (u,v) of Coxeter elements corresponding to ε and consider the element $X\in G^{u,v}$ defined by (3.4) with factorization parameters d_i, $c_i^-=c_i$ and $c_i^+=1$. Then $H_i=H_0h_i(X)$. This means that for any $j\in[1,n-1]$ and $k\in\mathbb{Z}$,

$$x(j,k)=H_0^j x_{2n}^{\max\{0,k+j+1-2n\}}\Delta_j^{(k)}(X),$$

where by $\Delta_j^{(k)}(X)$ we mean the determinant defined in (4.3) built from $h_i(X)$. By Corollary 4.2, $\Delta_j^{(k)}$ is the product of a Laurent monomial in the variables from $\mathbf{x}(\varepsilon)$ with coefficient 1 and the minor $(X^{k-\varkappa_j})_{[1,j]}^{[1,j]}$. If $k-\varkappa_j\geqslant 0$, then, by Lindström's lemma, this minor is equal to the sum of products of path weights over all collections of non-intersecting paths leading from the j lowest sources to the j lowest sinks in the network obtained by concatenating $k-\varkappa_j$ copies of the network $N_{u,v}$. Thus $(X^{k-\varkappa_j})_{[1,j]}^{[1,j]}$ is a polynomial in the factorization parameters c_i and d_i with non-negative integer coefficients, and the claim follows, since cluster variables and factorization parameters are connected by a monomial transformation with no coefficients. On the other hand, if $k-\varkappa_j<0$, then, by a well-known determinantal identity,

$$(X^{k-\varkappa_j})_{[1,j]}^{[1,j]}=(\det X)^{k-\varkappa_j}(X^{\varkappa_j-k})_{[j+1,n]}^{[j+1,n]}=(d_1\dots d_n)^{k-\varkappa_j}(X^{\varkappa_j-k})_{[j+1,n]}^{[j+1,n]},$$

and the previous argument applies.

(ii) By Lemma 5.4(i), $\mathbf{x}_1^r(\varepsilon)$ is indeed a cluster in $\mathcal{A}_1$, and its exchange matrix corresponds to $\widetilde{B}(\varepsilon)$. Define the parameters $c_i^{(r)}$ and $d_i^{(r)}$ by (5.10) with every Hankel determinant $\hat{\Delta}_i^{(l)}$ replaced by $\hat{\Delta}_i^{(l+r)}$. Pick a pair (u,v) of Coxeter elements corresponding to ε and consider the element $X\in G^{u,v}$ defined by (3.4) with factorization parameters $d_i=d_i^{(r)}$, $c_i^-=c_i^{(r)}$ and $c_i^+=1$. Then $H_{i+r}=H_rh_i(X)$ for $i\in[0,\dots,2n-1]$. Recursion (5.3) together with Remark 5.1 imply that, in fact, $H_{i+r}=H_rh_i(X)$ for all $i\in\mathbb{Z}$. Therefore, $x(j,k)=H_r^j x_{2n}^{\max\{0,k+j+1-2n\}}\Delta_j^{(k-r)}(X)$, and the rest of the proof is identical to (i). □

Remark 5.4. (i) In fact, we can refine Proposition 5.4 (i) and prove Laurent positivity of $x(j,k)$ with respect to shifted clusters as well. However, this proof needs additional tools in cluster algebra theory, and will be published elsewhere.

(ii) If the factorization parameters in (3.4) are positive, then the matrix X is totally non-negative, and so are the matrices X^k, for $k{=}1,2,...$, and JX^kJ with $J{=}\operatorname{diag}((-1)^i)_{i=1}^n$, for $k{=}-1,-2,...$. This indicates, in particular, a connection between Q-systems and totally non-negative matrices and their network interpretation. This connection is explored in [9, §7].

(iii) The quantization of the cluster algebra considered in this subsection is the subject of the forthcoming paper [4].

6. Coxeter–Toda lattices

6.1. The goal of this section is to establish a connection between the cluster algebra $\mathcal{A}$ defined above and transformations of Coxeter–Toda flows. First, consider the Toda hierarchy defined by (1.1). Equations on X induce an evolution of the corresponding Weyl function $m(\lambda;X)$, which can be most conveniently described in terms of its Laurent coefficients h_i. The following proposition is well known in the case of the usual (tridiagonal) Toda flows.

PROPOSITION 6.1. *If $X{=}X(t)$ satisfies the Lax equation* (1.1)*, then the coefficients $h_i(X){=}(X^ie_1,e_1)$ of the Laurent expansion of the Weyl function $m(\lambda;X)$ evolve according to the equations*

$$\frac{d}{dt}h_i(X)=h_{i+k}(X)-h_k(X)h_i(X).$$

Proof. If X satisfies (1.1), then so does X^i. By rewriting X^k as

$$\pi_+(X^k)+\pi_-(X^k)+\pi_0(X^k),$$

we get

$$\begin{aligned}\frac{d}{dt}h_i(X)&=\big(\big[X^i,-\tfrac12(\pi_+(X^k)-\pi_-(X^k))\big]e_1,e_1\big)\\&=\tfrac12(X^i(X^k-2\pi_+(X^k)-\pi_0(X^k))e_1,e_1)-\tfrac12((-X^k+2\pi_-(X^k)+\pi_0(X^k))X^ie_1,e_1)\\&=(X^{i+k}e_1,e_1)-(\pi_0(X^k)e_1,e_1)(X^ie_1,e_1)\\&=h_{i+k}(X)-h_k(X)h_i(X).\end{aligned}$$

□

Now, let (u,v) be a pair of Coxeter elements. Coxeter–Toda flows on $G^{u,v}/\mathbf{H}$ are induced by the restriction of the Toda hierarchy to $G^{u,v}$. To get a more detailed description of Coxeter–Toda flows, we choose the parameters $c_i{=}c_i^+c_i^-$ and d_i corresponding to

the factorization (3.4) of a generic element in $G^{u,v}$ as coordinates on the open dense set in $G^{u,v}/\mathbf{H}$. Indeed, c_i and d_i are invariant under conjugation by diagonal matrices (cf. Remark 4.2) and are clearly independent as functions on $G^{u,v}/\mathbf{H}$.

LEMMA 6.1. *The standard Poisson–Lie structure on* GL_n *induces the following Poisson brackets for the variables* c_i *and* d_i:

$$\{c_i, c_{i+1}\} = (\varepsilon_{i+1} - 1)c_i c_{i+1}, \quad \{d_i, d_j\} = 0, \quad \{c_i, d_i\} = -c_i d_i, \quad \{c_i, d_{i+1}\} = c_i d_{i+1} \tag{6.1}$$

and the rest of the brackets are zero.

Proof. In view of Theorem 2.1, it is sufficient to compute Poisson brackets for c_i and d_i induced by the Poisson brackets (5.1) and (5.2) for face weights of the network $N^{\circ}_{u,v}$. The first equation is an easy consequence of the equality $y_{0i} = c_i^{-1}$, $i \in [1, n-1]$, (cf. (5.11)) and the Poisson relations for y_{0i} described in (5.1) and (5.2).

By (5.11) and (5.12), $y_{0i}y_{1i} = d_i/d_{i+1}$ for $i \in [0, n-1]$ (here $d_0 = 1$). Therefore,

$$\left\{\log \frac{d_i}{d_{i+1}}, \log \frac{d_j}{d_{j+1}}\right\} = \{\log y_{0i}y_{1i}, \log y_{0j}y_{1j}\}, \quad i, j \in [0, n-1],$$

which equals the sum of the entries of the 2×2 block of Ω in rows $2i+1, 2i+2$ and columns $2j+1, 2j+2$. By (5.17), each such block is proportional either to U, or to V_k, or to V_k^T, $k \in [1, n-1]$, given by (5.15). It is easy to see that the sum of the entries for each of these matrices equals zero, and hence $\{d_i/d_{i+1}, d_j/d_{j+1}\} = 0$ for all $i, j \in [0, n-1]$. In particular, this holds for $i = 0$, which can be rewritten as $\{d_1, d_j/d_{j+1}\} = 0$ for all $j \in [0, n-1]$. Taking into account that $d_j = d_1(d_2/d_1) \dots (d_j/d_{j-1})$, we get the second formula in (6.1).

Similarly,

$$\left\{\log c_i, \log \frac{d_{j+1}}{d_j}\right\} = \left\{\log \frac{1}{c_i}, \log \frac{d_j}{d_{j+1}}\right\} = \{\log y_{0i}, \log y_{0j}y_{1j}\},$$

for $i \in [1, n-1]$ and $j \in [0, n-1]$, which equals the sum of the two upper entries of the 2×2 block of Ω in rows $2i+1, 2i+2$ and columns $2j+1, 2j+2$. By (5.17), if such a block is non-trivial, it is equal either to U, or to V_k, or to $-V_k^T$, $k \in [1, n-1]$, given by (5.15). Since the sum of the two upper entries equals 2 for U and -1 in the other two cases, we conclude that $\{\log c_i, \log d_{j+1}/d_j\} = 2\delta_{i,j} - \delta_{i,j+1} - \delta_{i,j-1}$ for $i \in [1, n-1]$ and $j \in [0, n-1]$. In particular, for $j = 0$ one gets $\{\log c_i, \log d_1\} = -\delta_{i1}$ for $i \in [1, n-1]$. Rewriting d_j via d_1 and d_{i+1}/d_i as before, one gets $\{\log c_i, \log d_j\} = -\delta_{i,j} + \delta_{i,j-1}$, $i \in [1, n-1]$, $j \in [1, n]$, which is equivalent to the last two equations in (6.1). □

Remark 6.1. We could have also computed the brackets (6.1) by specializing the general formulas obtained in [26] for Poisson brackets for factorization parameters of an arbitrary double Bruhat cell in a standard semisimple Poisson–Lie group.

Due to their invariance under conjugation by elements of $\mathbf{H}$, Hamiltonians

$$F_k(X) = \frac{1}{k} \operatorname{tr} X^k$$

of the Toda flows, when restricted to a Coxeter double Bruhat cell $G^{u,v}$, can be expressed as functions of c_i and d_i, which, in turn, serve as Hamiltonians for Coxeter–Toda flows on $G^{u,v}/\mathbf{H}$. The easiest way to write down F_k as a function of c_i and d_i explicitly is to observe that $\operatorname{tr} X^k$ is equal to the sum of the weights of all paths that start and end at the same level in the planar network obtained by concatenation of k copies of $N_{u,v}$. In the case $k{=}1$, we only need to use $N_{u,v}$ itself, which leads to the following formula for F_1: define I^- and I^+ by (3.2) and write $I^- \cup I^+ = \{1{=}i_1{<}...{<}i_m{=}n\}$, then

$$F_1 = F_1(c,d) = d_1 + \sum_{l=1}^{k-1} \sum_{j=i_l+1}^{i_{l+1}} (d_j + c_{j-1}d_{j-1} + ... + c_{j-1} \dots c_{i_l} d_{i_l}). \tag{6.2}$$

One can use (6.2) and (6.1) to write equations for the first Coxeter–Toda flow generated by F_1 on $G^{u,v}/\mathbf{H}$ as a system of evolution equations for c_i and d_i.

Example 6.1. (i) For our running Example 3.1, $I^- \cup I^+ = \{1,3,4,5\}$, so (6.2) becomes

$$F_1 = d_1 + d_2 + c_1 d_1 + d_3 + c_2 d_2 + c_2 c_1 d_1 + d_4 + c_3 d_3 + d_5 + c_4 d_4.$$

(ii) Let $v = s_{n-1} \dots s_1$. Then $I^- \cup I^+ = [1,n]$ and formula (6.2) reads

$$F_1(c,d) = d_1 + d_2 + c_1 d_1 + ... + d_n + c_{n-1} d_{n-1}.$$

If, in addition, $u = v^{-1}$, then $\varepsilon = (2,0,...,0)$ and F_1 and (6.1) generate the Hamiltonian equations

$$\begin{aligned} \frac{d}{dt} d_i &= \{d_i, F_1\} = \{d_i, c_i d_i + c_{i-1} d_{i-1}\} = d_i(c_i d_i - c_{i-1} d_{i-1}), \\ \frac{d}{dt} c_i &= \{c_i, F_1\} = \{c_i, d_i + d_{i+1} + c_{i-1} d_{i-1} + c_i d_i + c_{i+1} d_{i+1}\} \\ &= c_i(d_{i+1} - d_i + c_{i-1} d_{i-1} - c_i d_i). \end{aligned}$$

Then a change of variables $r_{2i-1} = d_i$, $i \in [1,n]$, and $r_{2i} = c_i d_i$, $i \in [1,n-1]$, results in the equations of the *open Volterra lattice*:

$$\frac{d}{dt} r_i = r_i(r_{i+1} - r_{i-1}), \quad i \in [1, 2n-1]; \quad r_0 = r_{2n} = 0.$$

Another change of variables, $a_i = c_i d_i^2$ and $b_i = d_i + c_{i-1} d_{i-1}$, leads to the equations of motion of the Toda lattice presented in the introduction. Note that a_i and b_i are the

subdiagonal and the diagonal matrix entries, respectively, in a lower Hessenberg representative of an element in $G^{u,v}/\mathbf{H}$ defined by the parameters c_i and d_i.

(iii) If $u=v=s_{n-1}\dots s_1$, then $\varepsilon=\{2,1,\dots,1,0\}$, and the Hamiltonian equations generated by F_1 and (6.1) produce the system

$$\frac{d}{dt}d_i = d_i(c_id_i - c_{i-1}d_{i-1}) \quad \text{and} \quad \frac{d}{dt}c_i = c_i(d_{i+1} - d_i + c_{i+1}d_{i+1} - c_id_i).$$

After the change of variables $\tilde{c}_i=c_id_i$, this system turns into the *relativistic Toda lattice*

$$\frac{d}{dt}d_i = d_i(\tilde{c}_i - \tilde{c}_{i-1}) \quad \text{and} \quad \frac{d}{dt}\tilde{c}_i = \tilde{c}_i(d_{i+1} - d_i + \tilde{c}_{i+1} - \tilde{c}_{i-1}).$$

Proposition 6.1 combined with Theorem 4.1 suggests a method to solve Coxeter–Toda lattices explicitly, following the strategy originally applied in [28] to the usual Toda lattice. In order to find a solution with initial conditions $c_i(0)$ and $d_i(0)$ to the Coxeter–Toda equation on $G^{u,v}/\mathbf{H}$ generated by the Hamiltonian F_k, we first define

$$m^0(\lambda) = m(\lambda; X(0)) = \sum_{i=0}^{\infty} \frac{h_i^0}{\lambda^{i+1}}$$

to be the Weyl function of any representative $X(0)\in G^{u,v}$ of the element in $G^{u,v}/\mathbf{H}$ with coordinates $c_i(0)$ and $d_i(0)$. Let $M(\lambda;t)=\sum_{i=0}^{\infty} H_i(t)\lambda^{-i-1}$ be the solution to a linear system on $\mathcal{R}_n$ described in terms of the Laurent coefficients $H_i(t)$ by

$$\frac{d}{dt}H_i(t) = H_{i+k}(t), \quad i=0,1,\dots,$$

with initial conditions $H_i(0)=h_i^0$. For $i<0$, define $H_i(t)$ via (5.3), where $(-1)^{n-i}p_i$ are the coefficients of the characteristic polynomial of $X(0)$.

PROPOSITION 6.2. *The solution with initial conditions $c_i(0)$ and $d_i(0)$ to the k-th Coxeter–Toda equation on $G^{u,v}/\mathbf{H}$ is given by the formulas* (4.5) *with*

$$h_i = h_i(t) = \frac{H_i(t)}{H_0(t)}, \quad i\in\mathbb{Z}.$$

Proof. An easy calculation shows that $h_i=h_i(t)=H_i(t)/H_0(t)$, $i\geqslant 0$, give the solution to the system presented in Proposition 6.1 with initial conditions $h_i(0)=h_i^0$. Thus the function $m(\lambda,t)=\sum_{i=0}^{\infty} h_i(t)\lambda^{-i-1}$ evolves in the way prescribed by the kth Toda flow and therefore coincides with $m(\lambda;X(t))$, where $X(t)$ is the solution of (1.1) with the initial condition $X(0)$. Since the coefficients of the characteristic polynomial are preserved by Toda flows, Remark 5.1 (ii) implies that for $i<0$ we also have $h_i(t)=h_i(X(t))$. Finally, since the Moser map is invertible on $G^{u,v}/\mathbf{H}$, we see that the system in Proposition 6.1 is, in fact, equivalent to the kth Toda flow on $G^{u,v}/\mathbf{H}$, which completes the proof. □

We see that for any pair of Coxeter elements (u,v), the Coxeter–Toda flows are equivalent to the same evolution of Weyl functions. We want to exploit this fact to construct, for any two pairs (u,v) and (u',v') of Coxeter elements, a transformation between $G^{u,v}/\mathbf{H}$ and $G^{u',v'}/\mathbf{H}$ which is Poisson and maps the kth Coxeter–Toda flow into the kth Coxeter–Toda flow. We call such a transformation a *generalized Bäcklund–Darboux transformation*. The term "Bäcklund transformation" has been used broadly over the years for any transformation that maps solutions of one non-linear equation into solutions of another. To justify the use of Darboux's name, we recall that traditionally a Bäcklund–Darboux transformation consists of interchanging factors in some natural factorization of the Lax operator associated with a given integrable system. In the case of Coxeter–Toda flows, the same number and type of elementary factors appear in the Lax matrix associated with any Coxeter double Bruhat cell. Hence we use the term "generalized Bäcklund–Darboux transformation", even though in our case the rearrangement of factors is accompanied by a transformation of the factorization parameters.

Let (u,v) and (u',v') be two pairs of Coxeter elements and let $\varepsilon=(\varepsilon_i)_{i=1}^n$, $\varepsilon'=(\varepsilon'_i)_{i=1}^n$ be the corresponding n-tuples defined by (3.7) and (3.10). We construct a map

$$\sigma_{u,v}^{u',v'}\colon G^{u,v}/\mathbf{H}\longrightarrow G^{u',v'}/\mathbf{H}$$

using the following procedure. Consider the cluster algebra $\mathcal{A}$ defined in §5. Fix a seed $\Sigma(\varepsilon)=(\mathbf{x}(\varepsilon),\widetilde{B}(\varepsilon))$ in $\mathcal{A}$, where $\mathbf{x}(\varepsilon)$ is given by (5.14) and $B(\varepsilon)$ by (5.16). Let $T_\varepsilon^{\varepsilon'}$ be the sequence of cluster transformations defined in the proof of Lemma 5.2 which transforms $\Sigma(\varepsilon)$ into the seed $\Sigma(\varepsilon')$. Next, for an element in $G^{u,v}/\mathbf{H}$ with coordinates c_i and d_i, consider its representative $X\in G^{u,v}$, the corresponding Weyl function $m(\lambda;X)$ and the sequence of moments $h_i(X)$, $i\in\mathbb{Z}$. Apply the transformation $\tau_{u,v}$ by assigning values to the cluster variables in the cluster $\mathbf{x}(\varepsilon)$ according to the formulas (5.8), (5.9) and (5.14), with H_i replaced by $h_i(X)$. Then apply the transformation $T_\varepsilon^{\varepsilon'}$ to $\mathbf{x}(\varepsilon)$ to obtain the cluster $\mathbf{x}(\varepsilon')$. Finally, apply the transformation $\varrho_{u',v'}$ by using equations (5.10) with ε replaced by ε' and with the components of $\mathbf{x}(\varepsilon)$ replaced by those of $\mathbf{x}(\varepsilon')$ to compute the parameters c'_i and d'_i serving as coordinates of an element in $G^{u',v'}/\mathbf{H}$. This concludes the construction of $\sigma_{u,v}^{u',v'}$.

THEOREM 6.1. *The map $\sigma_{u,v}^{u',v'}\colon G^{u,v}/\mathbf{H}\to G^{u',v'}/\mathbf{H}$ is a birational transformation which preserves the Weyl function, maps Coxeter–Toda flows on $G^{u,v}/\mathbf{H}$ into matching Coxeter–Toda flows on $G^{u',v'}/\mathbf{H}$ and is Poisson with respect to Poisson structures on $G^{u,v}/\mathbf{H}$ and $G^{u',v'}/\mathbf{H}$ induced by the standard Poisson–Lie bracket on* GL_n.

Proof. The moments $h_j(X)$ are polynomial functions of c_i and d_i for $i\geqslant 0$, and rational functions of c_i and d_i for $i<0$. The values we assign to the cluster variables in

$\mathbf{x}(\varepsilon)$ are thus rational functions of c_i and d_i. This, combined with the rationality of $T_\varepsilon^{\varepsilon'}$ and the equations (5.10), shows that the map $\sigma_{u,v}^{u',v'}$ is rational. It is easy to see that its inverse is $\sigma_{u',v'}^{u,v}$ which implies birationality. The claim that $\sigma_{u,v}^{u',v'}$ preserves the Weyl function is simply a restatement of Lemma 5.2, which implies that if the clusters $\mathbf{x}(\varepsilon)$ and $\mathbf{x}(\varepsilon')$ are obtained from a function $M(\lambda)\in\mathcal{R}_n$ according to (5.8), (5.9) and (5.14), then $T_\varepsilon^{\varepsilon'}$ transforms $\mathbf{x}(\varepsilon)$ into $\mathbf{x}(\varepsilon')$. The rest of the statement of the theorem is a consequence of the invariance of the Weyl function, since Poisson structures on $G^{u,v}/\mathbf{H}$ and $G^{u',v'}/\mathbf{H}$ induce the same Poisson bracket on $\mathcal{R}_n$ compatible with $\mathcal{A}$ and, by Proposition 6.1, Coxeter–Toda flows generated by the Hamiltonians F_k on $G^{u,v}/\mathbf{H}$ and $G^{u',v'}/\mathbf{H}$ induce the same evolution of the Weyl function. □

To illustrate Theorem 6.1, in Table 2 below we list elementary generalized Bäcklund–Darboux transformations corresponding to cluster transformations from a fixed cluster $\mathbf{x}(\varepsilon)$ into an adjacent cluster $\mathbf{x}(\varepsilon')$. The table can be viewed in parallel with Table 1. Expressions for the transformed variables c_j' and d_j' are obtained by combining the formulas for cluster transformations with the equations (5.10). The variables which are not listed are left unchanged.

Elementary generalized Bäcklund–Darboux transformations can be conveniently interpreted in terms of equivalent transformations of perfect networks introduced in [29]. The three types of equivalent transformations are shown in Figure 7. Instead of trying to describe the general case, we will provide an example.

Example 6.2. Consider the network from Example 5.1. Recall that $\varepsilon=(2,2,1,0,0)$ and set $i=2$. So, $\varepsilon_2=2$ and $\varepsilon_3=1$, which corresponds to the fourth row of Table 2. The corresponding transformation consists of the following steps:

(i) Type-2 transformation with $x=v_b^+(3)$, $y=v_b^-(3)$ and $w_1=w_4=1$, $w_2=c_2^+$, $w_3=c_3^-$.

(ii) Type-3 transformation with $x=v_b^+(3)$, $y=v_w^-(3)$, $x'=v_w^+(2)$, $y'=v_w^-(2)$ and $w_1=c_2^+$, $w_2=c_2^-$, $w_3=d_3$, $w_4=1$.

(iii) Type-1 transformation with $x=v_w^+(2)$, $y=v_w^-(2)$ and $w_1=c_2^+/(1+c_2)$, $w_2=1$, $w_3=d_2$, $w_4=c_1^-$.

(iv) The gauge group action at $v_b^+(3)$ which takes the triple of weights

$$\left(d_3, \frac{c_2^+}{1+c_2}, \frac{1}{1+c_2}\right) \quad \text{to} \quad \left(1, \frac{d_3c_2^+}{1+c_2}, \frac{d_3}{1+c_2}\right).$$

(v) The gauge group action at $v_w^-(2)$ which takes the triple of weights

$$(1+c_2, 1, c_1^-) \quad \text{to} \quad (1, 1+c_2, c_1^-(1+c_2)).$$

(vi) The gauge group action at $v_w^+(2)$ which takes the triple of weights

$$\left(1+c_2, \frac{d_3c_2^+}{1+c_2}, d_2\right) \quad \text{to} \quad \left(d_2(1+c_2), \frac{d_3c_2^+}{d_2(1+c_2)}, 1\right).$$

ε	ε'	Transformation	Inverse
$\varepsilon_i=0$ $\varepsilon_{i+1}=2$	$\varepsilon'_i=1$ $\varepsilon'_{i+1}=1$	$c'_i=\dfrac{c_id_i}{d_{i+1}}$ $d'_i=\dfrac{d_id_{i+1}}{d_{i+1}+c_id_i}$ $d'_{i+1}=d_{i+1}+c_id_i$	$c_i=\dfrac{c'_id'_{i+1}}{d'_i(1+c'_i)^2}$ $d_i=d'_i(1+c'_i)$ $d_{i+1}=\dfrac{d'_{i+1}}{1+c'_i}$
$\varepsilon_i=1$ $\varepsilon_{i+1}=0$	$\varepsilon'_i=0$ $\varepsilon'_{i+1}=1$	$c'_i=\dfrac{c_id_{i+1}}{d_i(1+c_i)^2}$ $c'_{i+1}=c_{i+1}(1+c_i)$ $d'_i=d_i(1+c_i)$ $d'_{i+1}=\dfrac{d_{i+1}}{1+c_i}$	$c_i=\dfrac{c'_id'_i}{d'_{i+1}}$ $c_{i+1}=\dfrac{c'_{i+1}d'_{i+1}}{d'_{i+1}+c'_id'_i}$ $d_{i+1}=d'_{i+1}+c'_id'_i$ $d_i=\dfrac{d'_id'_{i+1}}{d'_{i+1}+c'_id'_i}$
$\varepsilon_i=2$ $\varepsilon_{i+1}=0$	$\varepsilon'_i=1$ $\varepsilon'_{i+1}=1$	$c'_i=\dfrac{c_id_{i+1}}{d_i(1+c_i)^2}$ $c'_{i+1}=c_{i+1}(1+c_i)$ $d'_{i+1}=\dfrac{d_{i+1}}{1+c_i}$ $d'_i=d_i(1+c_i)$	$c_i=\dfrac{c'_id'_i}{d'_{i+1}}$ $c_{i+1}=\dfrac{c'_{i+1}d'_{i+1}}{d'_{i+1}+c'_id'_i}$ $d_{i+1}=d'_{i+1}+d'_ic'_i$ $d_i=\dfrac{d'_id'_{i+1}}{d'_{i+1}+c'_id'_i}$
$\varepsilon_i=2$ $\varepsilon_{i+1}=1$	$\varepsilon'_i=1$ $\varepsilon'_{i+1}=2$	$c'_i=\dfrac{c_id_{i+1}}{d_i(1+c_i)^2}$ $c'_{i-1}=c_{i-1}(1+c_i)$ $d'_i=d_i(1+c_i)$ $d'_{i+1}=\dfrac{d_{i+1}}{1+c_i}$	$c_i=\dfrac{c'_id'_i}{d'_{i+1}}$ $c_{i-1}=\dfrac{c'_{i-1}d'_{i+1}}{d'_{i+1}+c'_id'_i}$ $d_{i+1}=d'_{i+1}+c'_id'_i$ $d_i=\dfrac{d'_id'_{i+1}}{d'_{i+1}+c'_id'_i}$
$\varepsilon_{n-1}=0$	$\varepsilon'_{n-1}=1$	$c'_{n-1}=\dfrac{c_{n-1}d_{n-1}}{d_n}$ $d'_n=d_n+c_{n-1}d_{n-1}$ $d'_{n-1}=\dfrac{d_nd_{n-1}}{d_n+c_{n-1}d_{n-1}}$	$c_{n-1}=\dfrac{c'_{n-1}d'_n}{d'_i(1+c'_{n-1})^2}$ $d_{n-1}=d'_{n-1}(1+c'_{n-1})$ $d_n=\dfrac{d'_n}{1+c'_{n-1}}$
$\varepsilon_{n-1}=1$	$\varepsilon'_{n-1}=2$	$c'_{n-1}=\dfrac{c_{n-1}d_{n-1}}{d_n}$ $c'_{n-2}=\dfrac{c_{n-2}d_n}{d_n+c_{n-1}d_{n-1}}$ $d'_n=d_n+c_{n-1}d_{n-1}$ $d'_{n-1}=\dfrac{d_nd_{n-1}}{d_n+c_{n-1}d_{n-1}}$	$c_{n-1}=\dfrac{c'_{n-1}d'_n}{d'_{n-1}(1+c'_{n-1})^2}$ $c_{n-2}=c'_{n-2}(1+c'_{n-1})$ $d_{n-1}=d'_{n-1}(1+c'_{n-1})$ $d_n=\dfrac{d'_n}{1+c'_{n-1}}$

Table 2.

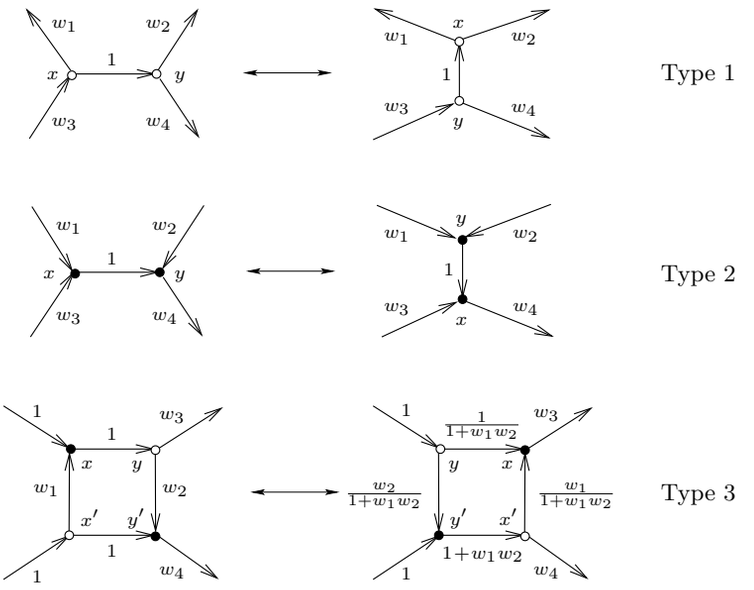


Figure 7. Equivalent transformations of perfect networks.

Thus, at the end we have

$$(c_2^-)' = \frac{c_2^-}{1+c_2}, \quad (c_2^+)' = \frac{d_3 c_2^+}{d_2(1+c_2)}, \quad \text{and hence} \quad c_2' = \frac{d_3 c_2}{d_2(1+c_2)}.$$

Besides,

$$(c_1^-)' = c_1^-(1+c_2), \quad (c_1^+)' = c_1^+, \quad \text{and hence} \quad c_1' = c_1(1+c_2).$$

Finally,

$$d_2' = d_2(1+c_2) \quad \text{and} \quad d_3' = \frac{d_3}{1+c_2}.$$

All these expressions coincide with those given in the fourth row of Table 2.

The transformations of the relevant part of the network during the first two steps are shown in Figure 8. The transformations of the relevant part of the network during the remaining four steps are shown in Figure 9.

We can make the transformations $\sigma_{u,v}^{u',v'}$ more explicit by using Corollary 4.2. Below we write $A_{[i]}$ for the determinant of the leading principal $i \times i$ submatrix of a matrix A. Pick an element $X = X(c,d) \in G^{u,v}$ defined by (3.4) with factorization parameters d_i, $c_i^- = c_i$ and $c_i^+ = 1$.

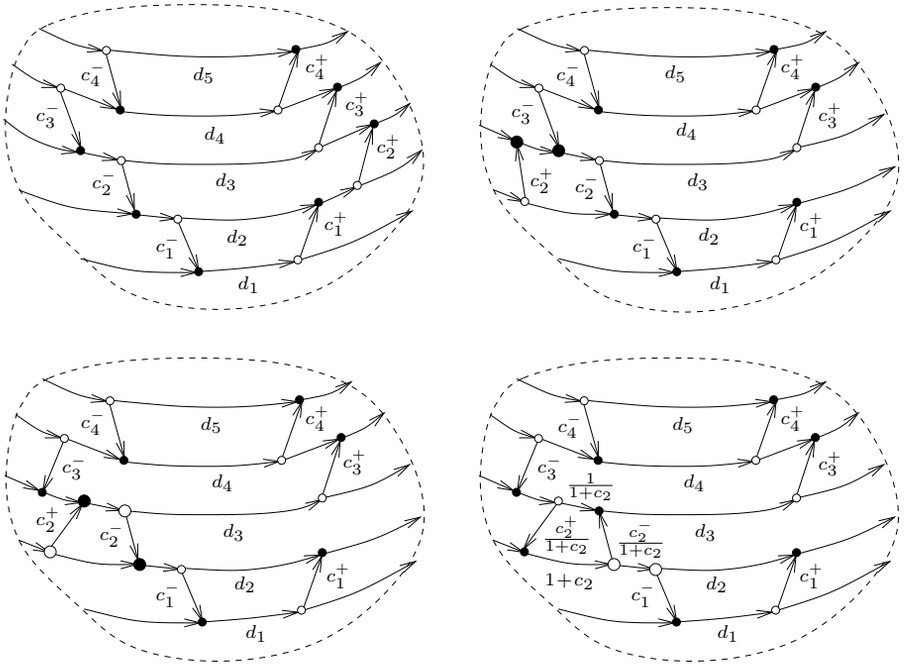


Figure 8. Elementary generalized Bäcklund–Darboux transformation: steps (i) and (ii).

PROPOSITION 6.3. *The map $\sigma_{u,v}^{u',v'}$ transforms the coordinates c_i and d_i on $G^{u,v}/\mathbf{H}$ into the coordinates c_i' and d_i' on $G^{u',v'}/\mathbf{H}$ given by the formulas*

$$d_i' = \frac{(X^{\delta\varkappa_i+1})_{[i]}(X^{\delta\varkappa_{i-1}})_{[i-1]}}{(X^{\delta\varkappa_i})_{[i]}(X^{\delta\varkappa_{i-1}+1})_{[i-1]}},$$

$$c_i' = c_i d_i^2 \frac{(X_{[i-1]})^{\varepsilon_i}}{(X_{[i+1]})^{\varepsilon_{i+1}}} \frac{(X^{\delta\varkappa_{i-1}})_{[i-1]}(X^{\delta\varkappa_{i+1}})_{[i+1]}}{(X^{\delta\varkappa_i+1})_{[i]}^2} \times \left(\frac{(X^{\delta\varkappa_{i+1}+1})_{[i+1]}}{(X^{\delta\varkappa_{i+1}})_{[i+1]}}\right)^{\varepsilon_{i+1}} \left(\frac{(X^{\delta\varkappa_{i-1}+1})_{[i-1]}}{(X^{\delta\varkappa_{i-1}})_{[i-1]}}\right)^{2-\varepsilon_i},$$

where ε and $\varkappa$ (resp. ε' and $\varkappa'$) are the n-tuples (3.10), (3.11) *associated with (u,v) (resp. (u',v')), and $\delta\varkappa_j = \varkappa_j' - \varkappa_j$ for $j \in [1,n]$.*

Proof. The claim follows from formulas (4.5), (4.16) and (4.17), and an easy computation that shows that

$$\frac{\Gamma_{i-1}\Gamma_{i+1}}{\Gamma_i^2} = c_i d_i^2 \frac{(d_1 \dots d_{i-1})^{\varepsilon_i}}{(d_1 \dots d_{i+1})^{\varepsilon_{i+1}}} = c_i d_i^2 \frac{(X_{[i-1]})^{\varepsilon_i}}{(X_{[i+1]})^{\varepsilon_{i+1}}}. \qquad \square$$

Example 6.3. Let $v = u^{-1} = u' = v' = s_{n-1} \dots s_1$. Then $G^{u,v}/\mathbf{H}$ is the set of Jacobi matrices (1.2), which serves as the phase space for the finite non-periodic Toda lattice,

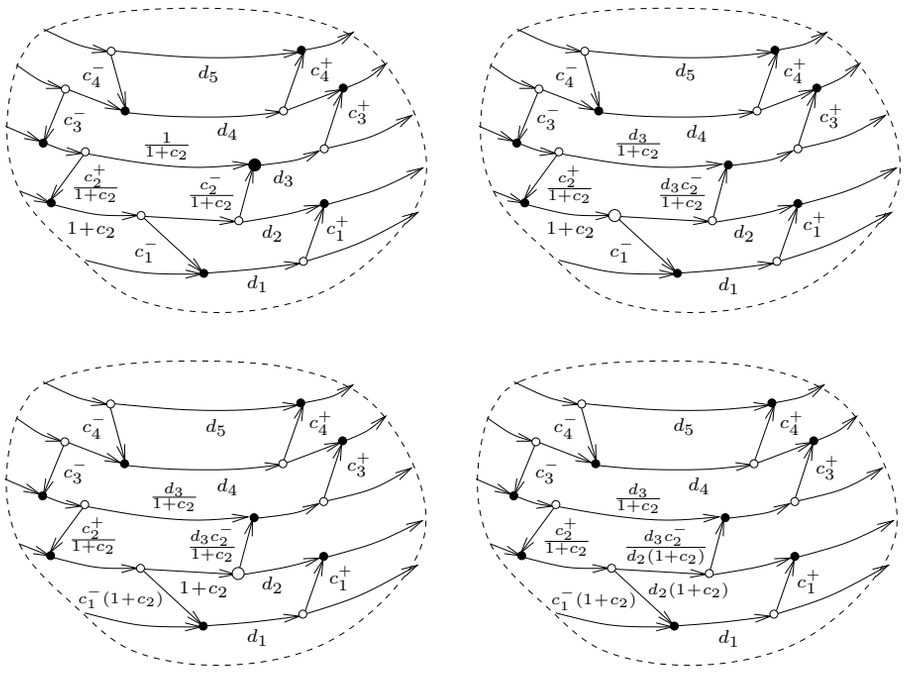


Figure 9. Elementary generalized Bäcklund–Darboux transformation: steps (iii)–(vi).

and $G^{u',v'}/\mathbf{H}$ can be viewed as a phase space for the relativistic Toda lattice. Combining Theorem 6.1 with Example 6.1, we obtain the following corollary of Proposition 6.3.

COROLLARY 6.1. *If the entries a_i and b_i of the Jacobi matrix L evolve according to the equations of the Toda lattice, then the functions*

$$d_i' = \frac{(L^{2-i})_{[i]}(L^{2-i})_{[i-1]}}{(L^{1-i})_{[i]}(L^{3-i})_{[i-1]}} \quad \textit{and} \quad \tilde{c}_i' = a_i \frac{(L^{-i})_{[i+1]}(L^{3-i})_{[i-1]}}{(L^{1-i})_{[i]}(L^{2-i})_{[i]}}$$

solve the relativistic Toda lattice.

Proof. First, observe that the element X featured in Proposition 6.3 is a tridiagonal matrix whose non-zero off-diagonal entries are $X_{i\,i+1}=d_i$ and $X_{i+1\,i}=c_id_i$. The matrix L associated with the same parameters c_i and d_i is related to X via $L=DXD^{-1}$, where $D=\operatorname{diag}(1, d_1, d_1d_2, ..., d_1 \dots d_{n-1})$. This means that $(L^k)_{[i]}=(X^k)_{[i]}$ for any i and k, and $a_i=L_{i+1\,i}=c_id_i^2$. Furthermore, $\varepsilon_i=0$ for $i\in[2, ..., n]$, $\varkappa_i=i-1$ and $\varkappa_i'=0$ for $i\in[1, ..., n]$. The claim then follows from Example 6.1 (iii) and the formulas of Proposition 6.3. □

6.2. It is natural to ask if the classical Darboux transformation

$$X = X_-X_0X_+ \longmapsto D(X) = X_0X_+X_-$$

can also be interpreted in terms of the cluster algebra $\mathcal{A}$. The transformation D constitutes a step in the *LU-algorithm* for computing eigenvalues of a matrix X. A connection of the LU-algorithm (as well as similar numerical algorithms, such as QR and Cholesky algorithms) to integrable systems of Toda type is well documented, see, e.g. [8] and [34]. For an arbitrary semisimple Lie group, a restriction of such a transformation to a Coxeter double Bruhat cell of type $G^{u,u}$ was studied, under the name of *factorization dynamics*, in [23]. We collect some relevant simple facts about the transformation $\mathcal{D}$.

PROPOSITION 6.4. *Let* $X \in \mathbf{N}_- \mathbf{B}_+$. *Then*

(i) *for any* $i \in \mathbb{Z}$, $h_i(D(X)) = h_{i+1}(X)/h_1(X)$;

(ii) *for any* $u, v \in S_n$, *if* $X \in G^{u,v}$ *then* $D(X) \in G^{u,v}$;

(iii) D *descends to a rational Poisson map* $\mathcal{D}\colon G^{u,v}/\mathbf{H} \to G^{u,v}/\mathbf{H}$ *which coincides with the time-one map of the Hamiltonian flow generated by the Hamiltonian*

$$F(X) = \tfrac{1}{2} \operatorname{tr} \log^2 X.$$

Proof. (i) For $i \geqslant 0$, we have

$$h_{i+1}(X) = (X_-(X_0X_+X_-)^i X_0X_+e_1, e_1) = d_1(D(X)^i e_1, e_1) = h_1(X)h_i(D(X)).$$

The case $i < 0$ can be treated similarly.

(ii) It suffices to observe that if $Y_1 \in \mathbf{N}_-$ and $Y_2 \in \mathbf{B}_+$ then both statements $Y_1Y_2 \in G^{u,v}$ and $Y_2Y_1 \in G^{u,v}$ are equivalent to $Y_1 \in \mathbf{B}_+ v \mathbf{B}_+$ and $Y_2 \in \mathbf{B}_- u \mathbf{B}_-$.

(iii) Claim (ii) implies that D descends to a rational map from $G^{u,v}/\mathbf{H}$ to $G^{u,v}/\mathbf{H}$. The rest of the claim is an immediate corollary of general results in [23, §7.1]. □

For a pair of Coxeter elements (u, v), Proposition 6.4 (i) allows us to completely describe the action of $\mathcal{D}$ on $G^{u,v}/\mathbf{H}$ in terms of a simple map on $\mathcal{R}_n$. Namely, define $\eta\colon \mathcal{R}_n \to \mathcal{R}_n$ by $\eta(M(\lambda)) = \lambda M(\lambda) - H_0$. Equivalently, η can be described by

$$\eta\Big(\sum_{i=0}^{\infty} H_i \lambda^{-i-1}\Big) = \sum_{i=0}^{\infty} H_{i+1} \lambda^{-i-1}.$$

Then Proposition 6.4 (i) implies that on $G^{u,v}/\mathbf{H}$,

$$\mathcal{D} = \varrho_{u,v} \circ \mathbf{x}_{u,v} \circ \eta \circ m_{u,v},$$

where the maps $\varrho_{u,v}$, $\mathbf{x}_{u,v}$ and $m_{u,v}$ were defined in the introduction.

Remark 6.2. As we have seen in §5.4, the shift $H_i \mapsto H_{i+1}$ plays an important role in the study of Q-systems in [10].

To tie together the cluster algebra $\mathcal{A}$ and the Darboux transformation $\mathcal{D}$, we have to descend to the cluster algebra $\mathcal{A}_1$ introduced in §5.4. We will only need to fix the stable variable x_{2n} to be equal to 1. In view of (5.14), this means that we are dealing with double Bruhat cells in SL_n rather than in GL_n. In order to emphasize a similarity between the classical Darboux transformation $\mathcal{D}$ and the generalized Bäcklund–Darboux transformation $\sigma_{u,v}^{u',v'}$, we express the former similarly to (1.5).

PROPOSITION 6.5. *$\mathcal{D}=\varrho_{u,v}\circ T_{\mathcal{D}}\circ\tau_{u,v}$, where $T_{\mathcal{D}}$ is a sequence of cluster transformations in $\mathcal{A}$.*

Proof. Note that in the graphical representation of the matrix $B(\varepsilon)$ that we employed in the proof of Lemma 5.3, passing to the cluster algebra $\mathcal{A}'$ amounts to erasing the white vertex and all corresponding edges in the graph Γ. Consider the cluster corresponding to $\varepsilon=(2,0,...,0)$. By Lemma 5.4 (i), the shift $H_i\mapsto H_{i+1}$ is achieved by an application of $T_{2n-3}\circ...\circ T_1$. This means that for $v=u^{-1}=s_{n-1}\dots s_1$ we can choose $T_{2n-3}\circ...\circ T_1$ for $T_{\mathcal{D}}$. Then, for an arbitrary pair of Coxeter elements (u,v), $T_{\mathcal{D}}$ can be defined as

$$T_{\mathcal{D}}=T_{w^{-1},w}^{u,v}\circ(T_{2n-3}\circ...\circ T_1)\circ T_{u,v}^{w^{-1},w}$$

with $w=s_{n-1}\dots s_1$. □

Acknowledgments

We wish to express gratitude to A. Berenstein, P. Di Francesco, R. Kedem and A. Zelevinsky for useful comments. A. V. would like to thank the University of Michigan, where he spent a sabbatical term in Spring 2009 and where this paper was finished. He is grateful to Sergey Fomin for warm hospitality and stimulating working conditions. M. S. expresses his gratitude to the Stockholm University and the Royal Institute of Technology, where he worked on this manuscript in the Fall 2008 during his sabbatical leave. M. G. was supported in part by NSF Grant DMS #0801204. M. S. was supported in part by NSF Grants DMS #0800671 and PHY #0555346. A. V. was supported in part by ISF Grant #1032/08.

References

[1] AKHIEZER, N. I., *The Classical Moment Problem and Some Related Questions in Analysis*. Hafner, New York, 1965.

[2] BERENSTEIN, A., FOMIN, S. & ZELEVINSKY, A., Parametrizations of canonical bases and totally positive matrices. *Adv. Math.*, 122 (1996), 49–149.

[3] — Cluster algebras. III. Upper bounds and double Bruhat cells. *Duke Math. J.*, 126 (2005), 1–52.

[4] Berenstein, A. & Kazhdan, D., Quantum Hankel algebras, clusters, and canonical bases. In preparation.

[5] Brockett, R. W. & Faybusovich, L. E., Toda flows, inverse spectral transform and realization theory. *Systems Control Lett.*, 16 (1991), 79–88.

[6] Cantero, M. J., Moral, L. & Velázquez, L., Five-diagonal matrices and zeros of orthogonal polynomials on the unit circle. *Linear Algebra Appl.*, 362 (2003), 29–56.

[7] Deift, P., Li, L. C., Nanda, T. & Tomei, C., The Toda flow on a generic orbit is integrable. *Comm. Pure Appl. Math.*, 39 (1986), 183–232.

[8] Deift, P., Li, L. C. & Tomei, C., Matrix factorizations and integrable systems. *Comm. Pure Appl. Math.*, 42 (1989), 443–521.

[9] Di Francesco, P. & Kedem, R., Q-system cluster algebras, paths and total positivity. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 6 (2010), Paper 014, 36.

[10] — Q-systems, heaps, paths and cluster positivity. *Comm. Math. Phys.*, 293 (2010), 727–802.

[11] Fallat, S. M., Bidiagonal factorizations of totally nonnegative matrices. *Amer. Math. Monthly*, 108 (2001), 697–712.

[12] Faybusovich, L. & Gekhtman, M., Elementary Toda orbits and integrable lattices. *J. Math. Phys.*, 41 (2000), 2905–2921.

[13] — Poisson brackets on rational functions and multi-Hamiltonian structure for integrable lattices. *Phys. Lett. A*, 272 (2000), 236–244.

[14] — Inverse moment problem for elementary co-adjoint orbits. *Inverse Problems*, 17 (2001), 1295–1306.

[15] Fomin, S. & Zelevinsky, A., Double Bruhat cells and total positivity. *J. Amer. Math. Soc.*, 12 (1999), 335–380.

[16] — Total positivity: tests and parametrizations. *Math. Intelligencer*, 22 (2000), 23–33.

[17] — Cluster algebras. I. Foundations. *J. Amer. Math. Soc.*, 15 (2002), 497–529.

[18] — Cluster algebras. II. Finite type classification. *Invent. Math.*, 154 (2003), 63–121.

[19] Fuhrmann, P. A., *A Polynomial Approach to Linear Algebra*. Universitext. Springer, New York, 1996.

[20] Gekhtman, M., Shapiro, M. & Vainshtein, A., Cluster algebras and Poisson geometry. *Mosc. Math. J.*, 3 (2003), 899–934, 1199.

[21] — Poisson geometry of directed networks in a disk. *Selecta Math.*, 15 (2009), 61–103.

[22] — Poisson geometry of directed networks in an annulus. Preprint, 2009. `arXiv:0901.0020 [math.QA]`.

[23] Hoffmann, T., Kellendonk, J., Kutz, N. & Reshetikhin, N., Factorization dynamics and Coxeter–Toda lattices. *Comm. Math. Phys.*, 212 (2000), 297–321.

[24] Karlin, S. & McGregor, J., Coincidence probabilities. *Pacific J. Math.*, 9 (1959), 1141–1164.

[25] Kedem, R., Q-systems as cluster algebras. *J. Phys. A*, 41 (2008), 194011, 14 pp.

[26] Kogan, M. & Zelevinsky, A., On symplectic leaves and integrable systems in standard complex semisimple Poisson-Lie groups. *Int. Math. Res. Not.*, 32 (2002), 1685–1702.

[27] Lindström, B., On the vector representations of induced matroids. *Bull. London Math. Soc.*, 5 (1973), 85–90.

[28] Moser, J., Finitely many mass points on the line under the influence of an exponential potential–an integrable system, in *Dynamical Systems, Theory and Applications* (Seattle, WA, 1974), Lecture Notes in Physics, 38, pp. 467–497. Springer, Berlin–Heidelberg, 1975.

[29] Postnikov, A., Total positivity, Grassmannians, and networks. Preprint, 2006. arXiv:math/0609764 [math.CO].
[30] Reshetikhin, N., Integrability of characteristic Hamiltonian systems on simple Lie groups with standard Poisson Lie structure. *Comm. Math. Phys.*, 242 (2003), 1–29.
[31] Reyman, A. & Semenov-Tian-Shansky, M., Group-theoretical methods in the theory of finite-dimensional integrable systems, in *Encyclopaedia of Mathematical Sciences*, 16, pp. 116–225. Springer, Berlin–Heidelberg, 1994.
[32] Simon, B., *Orthogonal Polynomials on the Unit Circle.* Part 1. American Mathematical Society Colloquium Publications, 54. Amer. Math. Soc., Providence, RI, 2005.
[33] Stieltjes, T. J., Recherches sur les fractions continues, in *Œuvres complètes/Collected papers.* Vol. II, pp. 402–566. Noordhoff, Groningen, 1918.
[34] Watkins, D. S., Isospectral flows. *SIAM Rev.*, 26 (1984), 379–391.
[35] Yakimov, M., Symplectic leaves of complex reductive Poisson–Lie groups. *Duke Math. J.*, 112 (2002), 453–509.
[36] Yang, S. W. & Zelevinsky, A., Cluster algebras of finite type via Coxeter elements and principal minors. *Transform. Groups*, 13 (2008), 855–895.
[37] Zelevinsky, A., Connected components of real double Bruhat cells. *Int. Math. Res. Not.*, 21 (2000), 1131–1154.

Michael Gekhtman
Department of Mathematics
University of Notre Dame
Notre Dame, IN 46556
U.S.A.
mgekhtma@nd.edu

Michael Shapiro
Department of Mathematics
Michigan State University
East Lansing, MI 48823
U.S.A.
mshapiro@math.msu.edu

Alek Vainshtein
Department of Mathematics and
Department of Computer Science
University of Haifa
Haifa, Mount Carmel 31905
Israel
alek@cs.haifa.ac.il

Received June 26, 2009
Received in revised form July 12, 2010