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Harnack estimates for quasi-linear degenerate parabolic differential equations

by

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Dedicated to the memory of Ennio De Giorgi

1. Main results

Let E be an open set in $\mathbb{R}^N$ and for $T>0$ let E_T denote the cylindrical domain $E\times(0,T]$. Consider quasi-linear, parabolic differential equations of the form

$$u_t-\operatorname{div}\mathbf{A}(x,t,u,Du)=B(x,t,u,Du)\quad\text{weakly in } E_T, \tag{1.1}$$

where the functions $\mathbf{A}\colon E_T\times\mathbb{R}^{N+1}\to\mathbb{R}^N$ and $B\colon E_T\times\mathbb{R}^{N+1}\to\mathbb{R}$ are only assumed to be measurable and subject to the structure conditions

$$\begin{cases} \mathbf{A}(x,t,u,Du)\cdot Du\geqslant C_0|Du|^p-C^p,\\ |\mathbf{A}(x,t,u,Du)|\leqslant C_1|Du|^{p-1}+C^{p-1}, \\ |B(x,t,u,Du)|\leqslant C|Du|^{p-1}+C^{p-1},\end{cases}\qquad \text{a.e. in } E_T, \tag{1.2}$$

where $p\geqslant 2$, C_0 and C_1 are given positive constants, and C is a given non-negative constant. A function

$$u\in C_{\mathrm{loc}}(0,T;L^2_{\mathrm{loc}}(E))\cap L^p_{\mathrm{loc}}(0,T;W^{1,p}_{\mathrm{loc}}(E)) \tag{1.3}$$

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is a local, weak solution to (1.1) if for every compact set $K \subset E$ and every sub-interval $[t_1, t_2] \subset (0, T]$ one has

$$\int_K u\varphi\, dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_K [-u\varphi_t + \mathbf{A}(x,t,u,Du)\cdot D\varphi]\, dx\, dt = \int_{t_1}^{t_2}\int_K B(x,t,u,Du)\varphi\, dx\, dt \tag{1.4}$$

for all bounded test functions

$$\varphi \in W^{1,2}_{\mathrm{loc}}(0,T;L^2(K)) \cap L^p_{\mathrm{loc}}(0,T;W^{1,p}_0(K)). \tag{1.5}$$

The parameters $\{N, p, C_0, C_1, C\}$ are the data, and we say that a generic constant $\gamma = \gamma(N, p, C_0, C_1, C)$ depends upon the data, if it can be quantitatively determined a priori only in terms of the indicated parameters.

For $\varrho > 0$ let K_ϱ be the cube centered at the origin on $\mathbb{R}^N$ with edge 2ϱ, and for $y \in \mathbb{R}^N$ let $K_\varrho(y)$ denote the homothetic cube centered at y. For $\theta > 0$ set also

$$Q^-_\varrho(\theta) = K_\varrho \times (-\theta\varrho^p, 0], \quad Q^+_\varrho(\theta) = K_\varrho \times (0, \theta\varrho^p],$$

and for $(y,s) \in \mathbb{R}^N \times \mathbb{R}$,

$$\begin{aligned}(y,s) + Q^-_\varrho(\theta) &= K_\varrho(y) \times (s - \theta\varrho^p, s],\\ (y,s) + Q^+_\varrho(\theta) &= K_\varrho(y) \times (s, s + \theta\varrho^p].\end{aligned}$$

Let u be a continuous, non-negative weak solution to (1.1)–(1.5), fix $(x_0, t_0) \in E_T$ such that $u(x_0, t_0) > 0$ and construct the cylinders

$$(x_0, t_0) + Q^\pm_{4\varrho}(\theta), \quad \text{where } \theta = \left(\frac{c}{u(x_0,t_0)}\right)^{p-2} \tag{1.6}$$

and c is a given positive constant. These cylinders are "intrinsic" to the solution, since their length is determined by the value of u at (x_0, t_0).

Theorem 1.1. (Intrinsic Harnack Inequality) *Let u be a continuous, non-negative, weak solution to* (1.1)–(1.5). *There exist positive constants c and γ depending only upon the data, such that for all intrinsic cylinders $(x_0, t_0) + Q^\pm_{4\varrho}(\theta)$ as in* (1.6), *contained in E_T, either $u(x_0, t_0) \leqslant \gamma C\varrho$, or*

$$u(x_0, t_0) \leqslant \gamma \inf_{K_\varrho(x_0)} u(x, t_0 + \theta\varrho^p), \quad \theta = \left(\frac{c}{u(x_0,t_0)}\right)^{p-2}. \tag{1.7}$$

Remark 1.1. The constants γ and c deteriorate as $p\to\infty$, in the sense that

$$\gamma(p), c(p) \to \infty \quad \text{as } p\to\infty.$$

However, they are "stable" as $p\to 2$, in the sense that there exist positive constants $\gamma(2)$ and $c(2)$, that can be determined a priori only in terms of the data, such that

$$\lim_{p\to 2}\gamma(p)=\gamma(2) \quad \text{and} \quad \lim_{p\to 2}c(p)=c(2).$$

Thus, by formally letting $p\to 2$ in (1.7), one recovers the classical Moser's Harnack inequality of [12].

The theorem has been stated for continuous solutions, to give meaning to $u(x_0,t_0)$. However, it continues to hold for non-negative weak solutions of (1.1)–(1.2) for almost all $(x_0,t_0)\in E_T$ and for corresponding cylinders $(x_0,t_0)+Q_\varrho(\theta)\subset E_T$. The intrinsic Harnack inequality, in turn, can be used to prove that local solutions of (1.1) are locally Hölder continuous within their domain of definition. This is the content of the next theorem.

THEOREM 1.2. (Harnack inequality and Hölder continuity) *Any locally bounded weak solution to* (1.1)–(1.2), *with no sign restriction*, *is locally Hölder continuous in* E_T. *A locally quantitative Hölder estimate is established in* §10.

The Hölder continuity of weak solutions of (1.1)–(1.2) was first established in [5]. The Harnack inequality (1.7) permits an independent proof. Summarizing, we have the following result.

COROLLARY 1.1. *Let* u *be a local*, *weak solution to* (1.1)–(1.5). *Then* u *is locally Hölder continuous in* E_T. *Moreover*, *if* u *is non-negative*, *it satisfies the intrinsic Harnack inequality in the form* (1.7).

The proof of these theorems is flexible enough to apply, by minor changes, to local weak solutions of equations of the porous medium type. These results are collected and stated in §11.

The singular case $1<p<2$ is still open and it will be the object of future investigations. Likewise, singular cases of quasi-linear versions of equations of the porous medium type remain to be investigated.

2. Novelty and significance

Equation (1.1) with the structure conditions (1.2) is a quasi-linear version of the degenerate, homogeneous equation

$$u_t-\sum_{i,j=1}^{N}(|Du|^{p-2}a_{ij}(x,t)u_{x_i})_{x_j}=0 \quad \text{weakly in } E_T, \tag{2.1}$$

where the coefficients a_{ij} are measurable and locally bounded in E_T and the matrix (a_{ij}) is almost everywhere positive definite in E_T. If $(a_{ij})=\mathbb{I}$, then (2.1) reduces to the degenerate, prototype parabolic p-Laplace equation

$$u_t - \operatorname{div} |Du|^{p-2} Du = 0 \quad \text{weakly in } E_T. \tag{2.2}$$

Both (2.1) and (2.2) satisfy the structure conditions (1.2) with $C=0$. Accordingly, non-negative, weak solutions of these equations satisfy the intrinsic Harnack inequality (1.7) with $C=0$.

2.1. The linear case $p=2$

The Harnack inequality for local, non-negative solutions of the heat equation ((1.7), with $p=2$ and $C=0$), was established independently by Hadamard [8] and Pini [15], by local representation of solutions in terms of heat potentials. In [12], Moser established the same Harnack inequality for weak solutions of (2.1) for $p=2$, by energy based, measure-theoretical arguments, and relying on a fine analysis of properties of parabolic BMO spaces. Moser's proof is non-linear in nature, and it can be extended to the quasi-linear versions (1.1)–(1.2) with $p=2$ ([17], [1]).

At almost the same time as Moser's paper [12], Ladyzhenskaya, Solonnikov and Uraltseva [9], established, by means of De Giorgi-type measure-theoretical arguments, that weak solutions of such quasi-linear equations (still for $p=2$), are locally bounded and locally Hölder continuous. It turns out that the Harnack inequality of Moser can be used to establish the Hölder continuity of solutions. On the other hand, it was observed in [4] that the Hölder continuity implies the Harnack inequality for non-negative solutions.

Thus a summary of the quasi-linear theory for the "linear" case $p=2$, is that Hölder continuity and Harnack inequality for non-negative solutions, present the same order of difficulties, and establishing either of them, requires independent measure-theoretical arguments.

2.2. The degenerate case $p>2$

Consider linear elliptic equations with bounded and measurable coefficients, of the form

$$\sum_{i,j=1}^{N} (a_{ij}(x) u_{x_i})_{x_j} = 0 \quad \text{weakly in } E \tag{2.3}$$

and their quasi-linear versions

$$\operatorname{div} \mathbf{A}(x,u,Du) = B(x,u,Du) \quad \text{weakly in } E, \tag{2.4}$$

where $\mathbf{A}$ and B satisfy the structure conditions (1.2). A seminal result of Moser [11] is that non-negative, local solutions of (2.3) satisfy the Harnack inequality. It was observed by Serrin [16] that the same Harnack estimate continues to hold for non-negative solutions of (2.4), for all $p>1$. On the other hand, De Giorgi [2] proved that solutions of (2.3) are locally Hölder continuous, and Ladyzhenskaya and Uraltseva [10] observed that indeed the same Hölder regularity continues to hold for solutions of (2.4), for all $p>1$. In either case, the extension from the "linear" case $p=2$ to the "non-linear" case $p\neq 2$ is possible by tracking down the topology of L^p versus the topology of L^2.

The parabolic theory is markedly different. Indeed, neither Moser's nor De Giorgi's ideas in the version of [9], nor Nash's approach [14] seem to apply when $p\neq 2$, even for the prototype case (2.2). Some progress was made in the mid 1980s, by the idea of *time-intrinsic* geometry, by which the time is scaled, roughly speaking by u^{p-2}. This permits one to establish that weak solutions of (1.1)–(1.2), for all $p>1$, are Hölder continuous in E_T [5, Chapters III and IV]. It was also observed that while the Harnack inequality in Moser's form is in general false for $p>2$, it might hold in this time-intrinsic geometry. Indeed, it was shown that (1.7), with $C=0$, holds for non-negative solutions of (2.2): the original results are in [3]; see [5, Chapter VI], for a complete account of the theory. The proof is based on the maximum principle and comparison functions constructed as variants of the Barenblatt similarity solutions

$$\Gamma_p(x,t)=\frac{1}{t^{N/\lambda}}\left(1-\gamma_p\left(\frac{|x|}{t^{1/\lambda}}\right)^{p/(p-1)}\right)_+^{(p-1)/(p-2)}, \quad t>0, \tag{2.5}$$

where

$$\gamma_p=\left(\frac{1}{\lambda}\right)^{1/(p-1)}\frac{p-2}{p}, \quad \lambda=N(p-2)+p. \tag{2.6}$$

As $p\to 2$, this tends pointwise to the fundamental solution of the heat equation. In this sense Γ_p is some sort of p-heat potential, and the approach can be regarded as paralleling that of Hadamard and Pini for the heat equation.

The issue of the Harnack inequality for non-negative solutions of equations of the type (1.1), with the full quasi-linear structure (1.2), while raised in [1], [17], [9] and [5], has since remained open.

The novelty of Theorem 1.1 is in producing a proof of the Harnack inequality (1.7) based only on measure-theoretical arguments. This bypasses any notion of maximum principle and potentials, and permits an extension to non-negative solutions of quasi-linear equations of the type of (1.1)–(1.2). Its significance is in paralleling Moser's measure-theoretical approach, in dispensing with Hadamard and Pini's potential representations.

It is worth noticing that the approach in this contribution substantially differs from the classical ideas of Moser [12], in that no properties of BMO spaces are used, nor covering arguments, nor cross-over estimates. Our arguments are only measure-theoretical in nature, and as such hold the promise of a wider applicability.

It is worth noticing that our method also differs from the one developed by Moser in [13], which makes no use of BMO spaces as well.

3. Main technical novelty: expansion of positivity

Let u be a non-negative, local solution of the heat equation in E_T. Let $(y,s)+Q_\varrho^-(1)$ with $p=2$ be a subset of E_T, and assume that

$$|\{x\in K_\varrho(y):u(x,s)<M\}|<\alpha|K_\varrho(y)|$$

for some $M>0$ and some $\alpha\in(0,1)$. Then there exists $\eta=\eta(\alpha)\in(0,1)$ such that

$$u\geqslant \eta M \quad \text{in } (y,s)+Q_{2\varrho}^+(1).$$

Thus, information on the measure of the "positivity set" of u at the time level s, over the cube $K_\varrho(y)$, translates into an expansion of the positivity set both in space (from $K_\varrho(y)$ to $K_{2\varrho}(y)$), and in time (from s to $s+4\varrho^2$). This fact continues to hold for quasi-linear versions of the heat equation and was established in [4]. A similar fact for $p>2$ is in general false, as one can verify from the Barenblatt solution (2.5)–(2.6). The main technical novelty of this investigation is that a similar fact continues to hold for the degenerate equations (1.1)–(1.2), in a time-intrinsic geometry.

LEMMA 3.1. *Let u be a non-negative, local, weak solution of* (1.1)–(1.2). *There exist positive constants γ and b, and $\eta\in(0,1)$, depending only upon the data and independent of (y,s), ϱ and M, such that if*

$$u(x,s)\geqslant M \quad \text{for all } x\in K_\varrho(y), \tag{3.1}$$

then either $M<\gamma C\varrho$, or

$$u(x,t)\geqslant \eta M \quad \text{for a.e. } x\in K_{2\varrho}(y) \tag{3.2}$$

for all

$$s+\frac{b}{(\eta M)^{p-2}}(2\varrho)^p\leqslant t\leqslant s+\frac{b}{(\eta M)^{p-2}}(4\varrho)^p. \tag{3.3}$$

Remark 3.1. The constants b and η are "stable" as $p\to2$, that is, there exist positive constants $b(2)$ and $\eta(2)$, such that $\lim_{p\to2} b(p)=b(2)$ and $\lim_{p\to2}\eta(p)=\eta(2)$.

4. Proof of Lemma 3.1—Preliminaries

4.1. Energy estimates

Let u be a local, weak solution to (1.1)–(1.2) in E_T; let k be any real number and consider the truncation of u given by

$$(u-k)_+ \equiv \max\{(u-k),0\}, \quad (u-k)_- \equiv \max\{-(u-k),0\}.$$

There exists a constant $\gamma=\gamma(\text{data})$ such that, for every cylinder $(y,s)+Q_\varrho^-(\theta)\subset E_T$, every $k\in\mathbb{R}$ and every piecewise smooth, non-negative function ζ vanishing on $\partial K_\varrho(y)$,

$$\begin{aligned} \operatorname*{ess\,sup}_{s-\theta\varrho^p<t<s} \int_{K_\varrho(y)} (u-k)_\pm^2 \zeta^p(x,t)\,dx - \int_{K_\varrho(y)} (u-k)_\pm^2 \zeta^p(x,s-\theta\varrho^p)\,dx \\ + C_0 \iint_{(y,s)+Q_\varrho^-(\theta)} |D(u-k)_\pm \zeta|^p\,dx\,d\tau \\ \leqslant \gamma \iint_{(y,s)+Q_\varrho^-(\theta)} [(u-k)_\pm^p |D\zeta|^p + (u-k)_\pm^2 |\zeta_t|]\,dx\,d\tau \\ + \gamma C^p \iint_{(y,s)+Q_\varrho^-(\theta)} [\chi_{\{(u-k)_\pm>0\}} + (u-k)_\pm^p]\zeta^p\,dx\,d\tau, \end{aligned} \tag{4.1}$$

where C_0 and C are the constants appearing in the structure conditions (1.2). Similar energy estimates hold for cylinders $(y,s)+Q_\varrho^+(\theta)\subset E_T$.

4.2. A De Giorgi-type lemma

Henceforth we will assume that u is non-negative, and for a fixed cylinder

$$(y,s)+Q_{2\varrho}^-(\theta)\subset E_T,$$

denote by $\mu_\pm$ and ω non-negative numbers such that

$$\mu_+ \geqslant \operatorname*{ess\,sup}_{(y,s)+Q_{2\varrho}^-(\theta)} u, \quad \mu_- \leqslant \operatorname*{ess\,inf}_{(y,s)+Q_{2\varrho}^-(\theta)} u \quad \text{and} \quad \omega \geqslant \mu_+ - \mu_-.$$

Denote by ξ and a fixed numbers in $(0,1)$.

LEMMA 4.1. *There exists a number ν depending upon the data and θ, ξ, ω and a, such that if*

$$|\{u\geqslant \mu_+ - \xi\omega\}\cap[(y,s)+Q_{2\varrho}^-(\theta)]| \leqslant \nu |Q_{2\varrho}^-(\theta)|, \tag*{$(4.2)_+$}$$

then either $\xi\omega < C\varrho$, or

$$u \leqslant \mu_+ - a\xi\omega \quad \textit{a.e. in } (y,s)+Q_\varrho^-(\theta). \tag*{$(4.3)_+$}$$

Likewise, *if*

$$|\{u\leqslant\mu_-+\xi\omega\}\cap[(y,s)+Q^-_{2\varrho}(\theta)]|\leqslant\nu|Q^-_{2\varrho}(\theta)|, \tag{4.2$_-$}$$

then either $\xi\omega<C\varrho$, *or*

$$u\geqslant\mu_-+a\xi\omega \quad \textit{a.e. in } (y,s)+Q^-_{\varrho}(\theta). \tag{4.3$_-$}$$

Proof. The statement is similar to Lemma 4.1 of [5, Chapter III]. We give a brief outline of the proof of (4.2)$_-$–(4.3)$_-$, to trace the precise dependence of ν on θ, a, ξ and ω. Assume that $(y,s)=(0,0)$ and for $n=0,1,2,...$, set

$$\varrho_n=\varrho+\frac{\varrho}{2^n},\quad K_n=K_{\varrho_n}\quad\text{and}\quad Q_n=K_n\times(-\theta\varrho_n^p,0].$$

Apply (4.1) over K_n and Q_n to $(u-k_n)_-$, for the levels

$$k_n=\mu_-+\xi_n\omega,\quad\text{where } \xi_n=a\xi+\frac{1-a}{2^n}\xi.$$

The cutoff function ζ is taken of the form $\zeta(x,t)=\zeta_1(x)\zeta_2(t)$, where

$$\zeta_1=\begin{cases}1, & \text{in } K_{n+1},\\ 0, & \text{in } \mathbb{R}^N\setminus K_n,\end{cases}\qquad |D\zeta_1|\leqslant\frac{1}{\varrho_n-\varrho_{n+1}}=\frac{2^{n+1}}{\varrho},$$

$$\zeta_2=\begin{cases}0, & \text{for } t<-\theta\varrho_n^p,\\ 1, & \text{for } t\geqslant-\theta\varrho_{n+1}^p,\end{cases}\qquad 0\leqslant\zeta_{2,t}\leqslant\frac{1}{\theta(\varrho_n^p-\varrho_{n+1}^p)}\leqslant\frac{2^{p(n+1)}}{\theta\varrho^p}.$$

The energy inequality (4.1), with these stipulations, yields that

$$\begin{aligned}\operatorname*{ess\,sup}_{-\theta\varrho_n^p<t<0}\int_{K_n}(u-k_n)_-^2\zeta^p(x,t)\,dx+\iint_{Q_n}|D(u-k_n)_-\zeta|^p\,dx\,d\tau\\
&\leqslant\gamma\frac{2^{np}}{\varrho^p}\biggl(\iint_{Q_n}(u-k_n)_-^p\,dx\,d\tau+\frac{1}{\theta}\iint_{Q_n}(u-k_n)_-^2\,dx\,d\tau\biggr)\\
&\quad+\gamma C\iint_{Q_n}(\chi_{\{u<k_n\}}+(u-k_n)_-^p)\,dx\,d\tau\\
&\leqslant\gamma\frac{2^{np}(\xi\omega)^p}{\varrho^p}\biggl(1+\frac{1}{\theta(\xi\omega)^{p-2}}+\Bigl(\frac{C\varrho}{\xi\omega}\Bigr)^p+(C\varrho)^p\biggr)|\{u<k_n\}\cap Q_n|\\
&\leqslant\gamma\frac{2^{np}(\xi\omega)^p}{\varrho^p}\biggl(1+\frac{1}{\theta(\xi\omega)^{p-2}}\biggr)|\{u<k_n\}\cap Q_n|,\end{aligned}$$

provided $\xi\omega\geqslant C\varrho$ and $\varrho<C^{-1}$, which we assume. Next, the first term on the left-hand side, is estimated below by

$$\int_{K_n}[(u-k_n)_-\zeta]^p\,dx\leqslant(\xi\omega)^{p-2}\int_{K_n}(u-k_n)_-^2\zeta^p\,dx.$$

Therefore,

$$\begin{aligned}\operatorname*{ess\,sup}_{-\theta\varrho_n^p<t<0}\frac{1}{(\xi\omega)^{p-2}}\int_{K_n}[(u-k_n)_-\zeta]^p(x,t)\,dx+\iint_{Q_n}|D(u-k_n)_-\zeta|^p\,dx\,d\tau \\ \leqslant\gamma\frac{2^{np}(\xi\omega)^p}{\varrho^p}\bigg(1+\frac{1}{\theta(\xi\omega)^{p-2}}\bigg)|A_n|,\end{aligned}\tag{4.4}$$

where we have set

$$A_n=\{u<k_n\}\cap Q_n.$$

Combining this with the embedding of Proposition 3.1 of [5, Chapter I], gives that

$$\begin{aligned}\bigg(\frac{1-a}{2^n}\bigg)^p(\xi\omega)^p|A_{n+1}| &\leqslant\iint_{Q_{n+1}}(u-k_n)_-^p\,dx\,d\tau \\ &\leqslant\iint_{Q_n}[(u-k_n)_-\zeta]^p\,dx\,d\tau \\ &\leqslant\bigg(\iint_{Q_n}[(u-k_n)_-\zeta]^{p\cdot(N+p)/N}\,dx\,d\tau\bigg)^{N/(N+p)}|A_n|^{p/(N+p)} \\ &\leqslant\gamma\bigg(\operatorname*{ess\,sup}_{-\theta\varrho_n^p<t<0}\int_{K_n(t)}[(u-k_n)_-\zeta]^p\,dx\bigg)^{(p/N)\cdot N/(N+p)} \\ &\qquad\times\bigg(\iint_{Q_n}|D(u-k_n)_-\zeta|^p\,dx\,d\tau\bigg)^{N/(N+p)}|A_n|^{p/(N+p)} \\ &\leqslant\gamma\frac{2^{np}(\xi\omega)^p}{\varrho^p}\bigg(1+\frac{1}{\theta(\xi\omega)^{p-2}}\bigg)(\xi\omega)^{(p-2)\cdot p/(N+p)}|A_n|^{1+p/(p+N)}.\end{aligned}$$

To render the estimate dimensionless, set $Y_n=|A_n|/|Q_n|$. Then

$$Y_{n+1}\leqslant\frac{\gamma4^{np}}{(1-a)^p}\frac{1+\theta(\xi\omega)^{p-2}}{[\theta(\xi\omega)^{p-2}]^{N/(N+p)}}Y_n^{1+p/(N+p)}.\tag{4.5}$$

By Lemma 4.1 of [5, Chapter I], $Y_n\to0$ as $n\to\infty$, provided

$$Y_0\leqslant\bigg(\frac{1-a}{\gamma(\text{data})}\bigg)^{N+p}\frac{[\theta(\xi\omega)^{p-2}]^{N/p}}{[1+\theta(\xi\omega)^{p-2}]^{(p+N)/p}}=\nu.\tag{4.6}$$

Thus, this choice of ν yields $Y_\infty=0$, which is equivalent to $(4.3)_-$. Similar arguments for the corresponding statement $(4.2)_+$–$(4.3)_+$ yield the same expression in (4.6) with the proper interpretation of Y_0. □

4.3. A variant of Lemma 4.1

Assume now that some information is available on the "initial data" relative to the cylinder $(y,s)+Q^+_{2\varrho}(\theta)$, say for example

$$u(x,s) \geqslant \xi M \quad \text{for a.e. } x \in K_{2\varrho}(y) \tag{4.7}$$

for some $M>0$ and $\xi\in(0,1]$. Then, writing the energy inequalities (4.1) for $(u-k)_-$, for $k\leqslant\xi M$, over the cylinder $(y,s)+Q^+_{2\varrho}(\theta)$, the integral extended over $K_{2\varrho}$ at the time level $t=s$, vanishes in view of (4.7). Moreover, by taking cutoff functions $\zeta(x,t)=\zeta_1(x)$ independent of t, also the integral involving ζ_t, on the right-hand side of (4.1) vanishes. We may now repeat the same arguments as in the previous proof for $(u-\xi_n M)_-$, over the cylinders $\widetilde{Q}_n$, where

$$\xi_n = a\xi + \frac{1-a}{2^n}\xi, \quad \widetilde{Q}_n = K_n \times (0, \theta(2\varrho)^p].$$

This leads to an analog of (4.4) without the factor $1+1/\theta(\xi\omega)^{p-2}$ on the right-hand side, with Q_n replaced by $\widetilde{Q}_n$, and with A_n replaced by

$$\tilde{A}_n = \{u < \xi_n M\} \cap \widetilde{Q}_n,$$

provided $\xi M > C\varrho$. Proceeding as before gives an analog of (4.5) in the form

$$\widetilde{Y}_{n+1} \leqslant \frac{\gamma 4^{np}}{(1-a)^p}[\theta(\xi M)^{p-2}]^{p/(N+p)} \widetilde{Y}_n^{1+p/(N+p)},$$

where $\widetilde{Y}_n = |\tilde{A}_n|/|\widetilde{Q}_n|$. This, in turn, implies that $\widetilde{Y}_n \to 0$ as $n\to\infty$, provided

$$\widetilde{Y}_0 \leqslant \frac{\delta}{\theta(\xi M)^{p-2}} \tag{4.8}$$

for a constant $\delta\in(0,1)$ depending only upon the data and a, and independent of ξ, M, ϱ and θ. We summarize this in the following result.

LEMMA 4.2. *Let M and ξ be positive numbers such that both* (4.7) *and* (4.8) *hold. Then either $\xi\omega < C\varrho$, or*

$$u \geqslant a\xi M \quad \textit{a.e. in } K_\varrho(y) \times (s, s+\theta(2\varrho)^p]. \tag{4.9}$$

5. Proof of Lemma 3.1—Continued

5.1. Changing the time variables

By taking $\theta=\delta(\xi M)^{2-p}$, condition (4.8) is always satisfied and yields

$$u\biggl(x, s+\frac{\delta\varrho^p}{(\xi M)^{p-2}}\biggr)\geqslant a\xi M \quad \text{for a.e. } x\in K_\varrho(y).$$

Next, observe that if (4.7) holds for some $\xi\in(0,1)$, it continues to hold for all $\xi_\tau\leqslant\xi$, and the conclusion of Lemma 4.2 continues to hold with ξ replaced by ξ_τ, provided in (4.8) we choose $\theta=\delta(\xi_\tau M)^{2-p}$. For $\tau>0$ let

$$\xi_\tau=\frac{\xi}{f(\tau)}, \quad \text{where } f(\tau)=e^{\tau/(p-2)}, \tag{5.1}$$

and let θ be chosen accordingly. Then for all $\tau\geqslant 0$,

$$u\biggl(x, s+\biggl[\frac{f(\tau)}{\xi M}\biggr]^{p-2}\delta\varrho^p\biggr)\geqslant a\frac{\xi M}{f(\tau)} \quad \text{for a.e. } x\in K_\varrho(y).$$

Set

$$w(x,\tau)\overset{\text{def}}{=}\frac{f(\tau)}{\xi M}(\delta\varrho^p)^{1/(p-2)}\,u\biggl(x, s+\biggl[\frac{f(\tau)}{\xi M}\biggr]^{p-2}\delta\varrho^p\biggr). \tag{5.2}$$

Corollary 5.1. *Let* (4.7) *hold. Then for a.e.* $x\in K_\varrho(y)$ *and all* $\tau\geqslant 0$,

$$w(x,\tau)\geqslant a(\delta\varrho^p)^{1/(p-2)}\overset{\text{def}}{=}k_0. \tag{5.3}$$

5.2. Relating w to the evolution equation

Since $u\geqslant 0$, by formal calculations, we get that

$$\begin{aligned} w_\tau &= \biggl(\frac{f(\tau)}{\xi M}(\delta\varrho^p)^{1/(p-2)}\biggr)^{p-1}u_t+\frac{1}{p-2}\frac{f(\tau)}{\xi M}(\delta\varrho^p)^{1/(p-2)}\,u \\ &\geqslant \biggl(\frac{f(\tau)}{\xi M}(\delta\varrho^p)^{1/(p-2)}\biggr)^{p-1}[\operatorname{div}\mathbf{A}(x,t,u,Du)+B(x,t,u,Du)] \\ &= \operatorname{div}\tilde{\mathbf{A}}(x,\tau,w,Dw)+\widetilde{B}(x,\tau,w,Dw), \end{aligned} \tag{5.4}$$

where $\tilde{\mathbf{A}}: E_T\times\mathbb{R}^{N+1}\to\mathbb{R}^N$ and $\widetilde{B}: E_T\times\mathbb{R}^{N+1}\to\mathbb{R}$ satisfy the structure conditions

$$\begin{cases} \tilde{\mathbf{A}}(x,\tau,w,Dw)\cdot Dw\geqslant C_0|Dw|^p-\widetilde{C}^p, \\ |\tilde{\mathbf{A}}(x,\tau,w,Dw)|\leqslant C_1|Dw|^{p-1}+\widetilde{C}^{p-1}, \qquad \text{a.e. in } E_T, \\ |\widetilde{B}(x,\tau,w,Dw)|\leqslant C|Dw|^{p-1}+\widetilde{C}^{p-1}, \end{cases} \tag{5.5}$$

where C_0, C_1 and C are the constants appearing in the structure condition (1.2), and

$$\widetilde{C}(\tau) = C\frac{f(\tau)}{\xi M}(\delta\varrho^p)^{1/(p-2)}. \tag{5.6}$$

The formal differential inequality (5.4) can be made rigorous by starting from the weak formulation (1.4), by operating the corresponding change of variables from t into τ, and by taking test functions $\varphi\geqslant 0$. We will be using (5.4) in space-time domains contained in $K_{8\varrho}(y)\times\mathbb{R}^+$, where $y\in E$ is a point for which (4.7) holds. In what follows we assume that y coincides with the origin and write energy estimates for $(w-k)_-$, of the type of (4.1), over cylinders $Q^+_{8\varrho}(\theta)\subset E_T$. Precisely

$$\begin{aligned} \operatorname*{ess\,sup}_{0<\tau<\theta(8\varrho)^p} \int_{K_{8\varrho}} (w-k)_-^2\zeta^p(x,\tau)\,dx + \iint_{Q^+_{8\varrho}(\theta)} |D(w-k)_-\zeta|^p\,dx\,d\tau \\ \leqslant \gamma \iint_{Q^+_{8\varrho}(\theta)} [(w-k)_-^p|D\zeta|^p + (w-k)_-^2|\zeta_t|]\,dx\,d\tau \\ +\gamma\widetilde{C}^p(\theta)\iint_{Q^+_{8\varrho}(\theta)} [\chi_{\{(w-k)_->0\}} + (w-k)_-^p]\zeta^p\,dx\,d\tau \end{aligned} \tag{5.7}$$

for a non-negative, piecewise smooth cutoff function that vanishes on the parabolic boundary of $Q^+_{8\varrho}(\theta)$.

6. Proof of Lemma 3.1—Concluded

6.1. Expanding the positivity of w

The bound from below of Corollary 5.1, valid for all $\tau>0$, will be expanded in the space variables over the cube $K_{2\varrho}$ for "times" τ sufficiently large. For this, set

$$\mathcal{Q}_{4\varrho}(\theta) = K_{4\varrho}\times((4\varrho)^p\theta, (8\varrho)^p\theta].$$

PROPOSITION 6.1. *Let* (4.7) *hold and let* k_0 *be defined by* (5.3). *Then for every* $\nu>0$, *there exist* $\sigma\in(0,1)$ *depending only upon the data*, $\gamma=\gamma(\sigma)$ *depending only upon* σ *and the data, and* $\theta=\theta(k_0,\sigma)$ *depending only upon* k_0, σ *and the data, such that either* $\xi M<\gamma(\sigma)C\varrho$, *or*

$$|\{w<\sigma k_0\}\cap\mathcal{Q}_{4\varrho}(\theta)| \leqslant \nu|\mathcal{Q}_{4\varrho}(\theta)|. \tag{6.1}$$

Proof. In (5.7) take ζ that equals 1 on $\mathcal{Q}_{4\varrho}(\theta)$, and such that $|D\zeta|\leqslant(4\varrho)^{-1}$ and $|\zeta_t|\leqslant[\theta(4\varrho)^p]^{-1}$. Take also levels

$$k_j = \frac{1}{2^j}k_0 \quad \text{for } j=0,1,...,j_*, \quad \text{where } j_*\in\mathbb{N} \text{ is to be chosen.}$$

Discarding the first term on the left-hand side gives

$$\iint_{\mathcal{Q}_{4\varrho}(\theta)} |D(w-k_j)_-|^p\,dx\,d\tau \leqslant \frac{\gamma k_j^p}{(4\varrho)^p}|\mathcal{Q}_{4\varrho}(\theta)|(1+\theta^{-1}k_j^{2-p}+\widetilde{C}^p(4\varrho)^p k_j^{-p}).$$

Choose

$$\theta = k_{j_*}^{2-p} = \left(\frac{2^{j_*}}{k_0}\right)^{p-2}.$$

From the definition (5.6) of $\widetilde{C}$ and the definition (5.3) of k_0, we estimate

$$\widetilde{C}^p(4\varrho)^p k_j^{-p} \leqslant \gamma(j_*;\text{data})\left(\frac{\varrho C}{\xi M}\right)^p.$$

Therefore, if $\xi M > \gamma(j_*)C\varrho$, the last term is majorized by an absolute constant depending only upon the data, and the previous inequality becomes

$$\iint_{\mathcal{Q}_{4\varrho}(\theta)} |D(w-k_j)_-|^p\,dx\,d\tau \leqslant \frac{\gamma k_j^p}{(4\varrho)^p}\big|\mathcal{Q}_{4\varrho}(\theta)\big| \tag{6.2}$$

for a constant γ depending only upon the data and independent of j_*. Set

$$A_j(\tau)=\{w(\cdot,\tau)<k_j\}\cap K_{4\varrho}, \quad A_j=\{w<k_j\}\cap\mathcal{Q}_{4\varrho}(\theta).$$

Therefore

$$|A_j| = \int_{\theta(4\varrho)^p}^{\theta(8\varrho)^p} |A_j(\tau)|\,d\tau.$$

By the measure-theoretical Lemma 2.2 of [5, Chapter I],

$$(k_j-k_{j+1})|A_{j+1}(\tau)| \leqslant \frac{\gamma\varrho^{N+1}}{|K_{4\varrho}\setminus A_j(\tau)|}\int_{k_{j+1}<w(\cdot,\tau)<k_j}|Dw|\,dx$$

for all $\tau\in(\theta(4\varrho)^p,\theta(8\varrho)^p]$. For all such τ, by Corollary 5.1, one has

$$|K_{4\varrho}\setminus A_j(\tau)| \geqslant |K_\varrho|.$$

Therefore

$$\frac{1}{2}k_j|A_{j+1}(\tau)| \leqslant \gamma\varrho\int_{k_{j+1}<w(\cdot,\tau)<k_j}|Dw|\,dx.$$

Integrate this in $d\tau$ over $(\theta(4\varrho)^p,\theta(8\varrho)^p)$ and majorize the resulting integral on the right-hand side by Hölder's inequality and by means of (6.2), to obtain that

$$\begin{aligned}\frac{1}{2}k_j|A_{j+1}| &\leqslant \gamma\varrho\left(\iint_{A_j\setminus A_{j+1}}|Dw|^p\,dx\,d\tau\right)^{1/p}|A_j\setminus A_{j+1}|^{(p-1)/p}\\ &\leqslant \gamma\varrho\left(\iint_{\mathcal{Q}_{4\varrho}(\theta)}|D(w-k_j)_-|^p\,dx\,d\tau\right)^{1/p}|A_j\setminus A_{j+1}|^{(p-1)/p}\\ &\leqslant \gamma k_j|\mathcal{Q}_{4\varrho}(\theta)|^{1/p}|A_j\setminus A_{j+1}|^{(p-1)/p}.\end{aligned}$$

From this, by taking the $p/(p-1)$-power of both sides, we get the recursive inequalities

$$|A_{j+1}|^{p/(p-1)} \leqslant \gamma |\mathcal{Q}_{4\varrho}(\theta)|^{1/(p-1)} |A_j \setminus A_{j+1}|.$$

Now add these for $j=0,1,...,j_*-1$, and majorize the sum on the right-hand side by the corresponding telescopic series. This gives

$$(j_*-1)|A_{j_*}|^{p/(p-1)} \leqslant \gamma |\mathcal{Q}_{4\varrho}(\theta)|^{p/(p-1)}.$$

Rewrite this as

$$|A_{j_*}| \leqslant \left(\frac{\gamma}{j_*}\right)^{(p-1)/p} |\mathcal{Q}_{4\varrho}(\theta)|. \tag{6.3}$$

This proves the proposition for the choices

$$\sigma = \frac{1}{2^{j_*}} \quad \text{and} \quad \nu = \left(\frac{\gamma}{j_*}\right)^{(p-1)/p}. \tag{6.4}$$

□

PROPOSITION 6.2. *Assume that* (4.7) *holds. There exist* $\sigma \in (0,1)$ *and* $\gamma(\sigma)>1$, *that can be determined a priori only in terms of the data*, *such that either* $\xi M<\gamma(\sigma)C\varrho$, *or*

$$w(\cdot,\tau) \geqslant \frac{1}{2}\sigma k_0 \quad \textit{a.e. in } K_{2\varrho} \times \left(\frac{(6\varrho)^p}{(\sigma k_0)^{p-2}}, \frac{(8\varrho)^p}{(\sigma k_0)^{p-2}} \right]. \tag{6.5}$$

Proof. Apply $(4.2)_-$–$(4.3)_-$ of Lemma 4.1 to w over the cylinder

$$\mathcal{Q}_{4\varrho}(\theta) = (0,\tau_*) + Q^-_{4\varrho}(\theta) \quad \text{for } \tau_* = \theta(8\varrho)^p.$$

The parameter $\xi\omega$ is replaced by σk_0 and $\mu_- \geqslant 0$ is neglected. Taking into account (4.6), and choosing $a=\frac{1}{2}$ gives

$$w(x,\tau) \geqslant \tfrac{1}{2}\sigma k_0 \quad \text{for a.e. } (x,\tau) \in (0,\tau_*) + Q^-_{2\varrho}(\theta),$$

provided $\xi M > \gamma(\sigma)C\varrho$ and

$$\frac{|\{w<\sigma k_0\} \cap \mathcal{Q}_{4\varrho}(\theta)|}{|\mathcal{Q}_{4\varrho}(\theta)|} \leqslant \gamma^{-(N+p)} \frac{[\theta(\sigma k_0)^{p-2}]^{N/p}}{[1+\theta(\sigma k_0)^{p-2}]^{(p+N)/p}} = \delta_*,$$

where δ_* depends only upon the data. Choosing now $\nu = \delta_*$ from (6.4) determines σ and therefore θ quantitatively. □

6.2. Expanding the positivity of u

Return to the definitions (5.1)–(5.3) of $f(\cdot)$, w and k_0. As τ ranges over the interval in (6.5), $f(\tau)$ ranges over

$$b_1 \stackrel{\text{def}}{=} \exp\left(\frac{2^{p-2}6^p}{(p-2)\sigma^{p-2}\delta}\right) \leqslant f(\tau) \leqslant \exp\left(\frac{2^{p-2}8^p}{(p-2)\sigma^{p-2}\delta}\right) \stackrel{\text{def}}{=} b_2,$$

where b_1 and b_2 are constants that can be determined a priori only in terms of the data and are independent of ϱ, M and u. Translating Proposition 6.2 in terms of u and t gives

$$u(x,t) \geqslant \frac{\sigma\xi M}{4b_2} \stackrel{\text{def}}{=} \eta M \quad \text{for a.e. } x\in K_{2\varrho}(y)$$

for all times

$$s+\left(\frac{\bar{b}}{\eta M}\right)^{p-2}(2\varrho)^p \leqslant t \leqslant s+\left(\frac{\bar{b}}{\eta M}\right)^{p-2}(4\varrho)^p$$

for a proper $\bar{b}$ depending only upon the data. Lemma 3.1 then follows with $b=\bar{b}^{p-2}$.

7. Stabilizing η in Lemma 3.1, as $p\to 2$

The proof shows that the constants b and η in (3.2)–(3.3) depend on p as

$$\bar{b} \approx \exp\left(\gamma_b \frac{h^{p-2}}{p-2}\right) \quad \text{and} \quad \eta \approx \exp\left(-\gamma_\eta \frac{k^{p-2}}{p-2}\right)$$

for constants $\gamma_b, \gamma_\eta, h, k>1$ depending only upon the data and independent of p. Thus the ratio $(\bar{b}/\eta)^{p-2}$ that determines the "waiting time" needed to preserve positivity, deteriorates as $p\to\infty$. However it is "stable" as $p\to 2$ and (3.3) remains meaningful for p near 2. On the other hand, $\eta(p)\to 0$, as $p\to 2$, and (3.2) becomes vacuous. The next lemma realizes a stable dependence of $\eta(p)$ for p near 2.

Lemma 7.1. *Let u be a non-negative, local, weak solution of* (1.1)–(1.2) *in E_T. There exist constants $\gamma_*>1$, $b_*, \eta_*\in(0,1)$ and $p_*>2$, depending only upon the data and independent of (y,s), ϱ, M and p, such that if*

$$u(x,s) \geqslant M \quad \textit{for all } x\in K_\varrho(y) \tag{7.1}$$

and $2<p\leqslant p_*$, then either $M<C\gamma_*\varrho$, or

$$u(x,t) \geqslant \eta_* M \quad \textit{for all } x\in K_{2\varrho}(y) \tag{7.2}$$

for all

$$s+\frac{b_*}{M^{p-2}}(4\varrho)^p \leqslant t \leqslant s+\frac{b_*}{M^{p-2}}(8\varrho)^p. \tag{7.3}$$

Remark 7.1. The constants γ_*, b_* and η_* are "stable" as $p\to 2$, that is there exist positive constants $b(2)$, $\eta(2)$ and $\gamma(2)$ such that

$$\lim_{p\to 2} b_*(p)=b(2), \quad \lim_{p\to 2}\eta_*(p)=\eta(2) \quad \text{and} \quad \lim_{p\to 2}\gamma_*(p)=\gamma(2).$$

In particular, the same conclusion continues to hold for the "linear case" $p=2$.

7.1. Proof of Lemma 7.1

Assume that (y,s) is the origin of $\mathbb{R}^{N+1}$. The assumption (7.1) implies that

$$|\{u(\cdot,0)<M\}\cap K_{8\varrho}|<(1-8^{-N})|K_{8\varrho}|. \tag{7.4}$$

PROPOSITION 7.1. *There exist numbers $b_*,\xi_*\in(0,1)$ depending only upon the data, and independent of u, M, ϱ and p, such that either $M\leqslant C\varrho$, or*

$$|\{u(\cdot,t)<\xi_*M\}\cap K_{8\varrho}|<(1-32^{-N})|K_{8\varrho}|$$

for all $0<t<b_*M^{2-p}(8\varrho)^p$.

Proof. Write the energy inequality (4.1) for $(u-M)_-$ over $Q^+_{8\varrho}(\theta)$ for $\theta=b_*M^{2-p}$, where b_* is to be chosen. The cutoff function ζ is taken independent of t, equals 1 on $K_{\sigma_*8\varrho}$, for some $\sigma_*\in(0,1)$ to be chosen, vanishes on the boundary of $K_{8\varrho}$ and

$$|D\zeta|\leqslant\frac{1}{8\varrho(1-\sigma_*)}.$$

These choices in (4.1) give that

$$\int_{K_{\sigma_*8\varrho}}(u-M)^2_-(x,t)\,dx\leqslant\int_{K_{8\varrho}}(u-M)^2_-(x,0)\,dx+\frac{\gamma M^p}{(1-\sigma_*)^p\varrho^p}|Q^+_{8\varrho}(\theta)|$$

for all $0<t<b_*M^{2-p}(8\varrho)^p$, provided $M>C\varrho$. Estimate from below

$$\begin{aligned}\int_{K_{\sigma_*8\varrho}}(u-M)^2_-(x,t)\,dx&\geqslant\int_{K_{\sigma_*8\varrho}\cap\{u(\cdot,t)<\xi_*M\}}(u-M)^2_-(x,t)\,dx\\&\geqslant(1-\xi_*)^2M^2|\{u(\cdot,t)<\xi_*M\}\cap K_{\sigma_*8\varrho}|.\end{aligned}$$

Next, by using (7.1), estimate from above

$$\int_{K_{8\varrho}}(u-M)^2_-(x,0)\,dx\leqslant M^2(1-8^{-N})|K_{8\varrho}|.$$

By the definition of $Q^+_{8\varrho}(\theta)$, with $\theta=b_*M^{2-p}$, the last term is majorized by

$$\frac{\gamma b_* M^2}{(1-\sigma_*)^p}|K_{8\varrho}|.$$

Combining these estimates yields

$$|\{u(\cdot,t)<\xi_*M\}\cap K_{\sigma_*8\varrho}|\leqslant\left[\frac{1-8^{-N}}{(1-\xi_*)^2}+\frac{\gamma b_*}{(1-\sigma_*)^p(1-\xi_*)^2}\right]|K_{8\varrho}|.$$

Finally,

$$\begin{aligned}|\{u(\cdot,t)<\xi_*M\}\cap K_{8\varrho}|&\leqslant|\{u(\cdot,t)<\xi_*M\}\cap K_{\sigma_*8\varrho}|+|K_{8\varrho}\setminus K_{\sigma_*8\varrho}|\\&\leqslant\left[\frac{1-8^{-N}}{(1-\xi_*)^2}+\frac{\gamma b_*}{(1-\sigma_*)^p(1-\xi_*)^2}+(1-\sigma_*^N)\right]|K_{8\varrho}|\end{aligned}$$

for all $0<t<b_*M^{2-p}(8\varrho)^p$. Choose ξ_* so small that

$$\frac{1-8^{-N}}{(1-\xi_*)^2}\leqslant 1-16^{-N}.$$

Then, ξ_* being fixed, choose σ_* and b_* so small that the term in square brackets on the right-hand side is majorized by $1-32^{-N}$. □

To proceed, set $t_*=\theta(8\varrho)^p$ and consider the cylinder with "vertex" at $(0,t_*)$:

$$Q^*_{8\varrho}(\theta)=(0,t_*)+Q^-_{8\varrho}(\theta),\quad\text{where }\theta=b_*M^{2-p}.$$

PROPOSITION 7.2. *For every $\nu_*\in(0,1)$ there exist constants $p_*>2$, $\eta_*\in(0,1)$ and $\gamma_*>1$, depending only upon the data and independent of u, M and ϱ, such that for all $2<p<p_*$, either $M\leqslant C\gamma_*\varrho$, or*

$$|\{u<2\eta_*M\}\cap Q^*_{4\varrho}(\theta)|\leqslant\nu_*|Q^*_{4\varrho}(\theta)|.$$

Proof. Write down the energy inequalities in (4.1), for $(u-k_j)_-$, over the cylinder $Q^*_{8\varrho}(\theta)$ for a cutoff function ζ that equals 1 on $Q^*_{4\varrho}(\theta)$, and is such that $|D\zeta|\leqslant(4\varrho)^{-1}$ and $|\zeta_t|\leqslant[\theta(4\varrho)^p]^{-1}$. The levels k_j are taken as

$$k_j=\frac{\xi_*M}{2^j}\quad\text{for } j=0,1,...,j_*,\text{ where } j_*\in\mathbb{N}\text{ is to be chosen.}$$

Discarding the first term on the left-hand side gives

$$\begin{aligned}\iint_{Q^*_{4\varrho}(\theta)}|D(u-k_j)_-|^p\,dx\,d\tau&\leqslant\frac{\gamma k_j^p}{(2\varrho)^p}(1+k_j^{2-p}M^{p-2}b_*^{-1})|Q^*_{4\varrho}(\theta)|\\&\leqslant\frac{\gamma k_j^p}{(2\varrho)^p}\left(1+\left(\frac{2^{j_*}}{\xi_*}\right)^{p-2}\frac{1}{b_*}\right)|Q^*_{4\varrho}(\theta)|,\end{aligned}$$

provided $M>C2^{j_*}\varrho$. Such a j_* will be chosen shortly depending only upon the data and independent of u, M, ϱ and p. Assuming momentarily that such a choice has been made, choose p_* such that $2<p_*<2+j_*^{-1}$ and let $2<p\leqslant p_*$. This yields the energy estimates

$$\iint_{Q^*_{4\varrho}(\theta)} |D(u-k_j)_-|^p\,dx\,d\tau \leqslant \frac{\gamma k_j^p}{(2\varrho)^p}|Q^*_{4\varrho}(\theta)| \tag{7.5}$$

for a constant γ depending only upon the data and independent of u, M, ϱ and p, provided $M>C\gamma_*\varrho$ for $\gamma_*=2^{j_*}$. The energy estimate (7.5), derived for $2<p\leqslant p_*$, is formally analogous to the energy estimates (6.2), valid for all $p>2$. They only differ in the meaning of the parameter θ that determines the time-length of the cylinders $\mathcal{Q}_{4\varrho}(\theta)$ and $Q^*_{4\varrho}(\theta)$, respectively. In the former, θ was taken "large" of the order of $k_{j_*}^{2-p}$ so that $\theta^{-1}k_{j_*}^{2-p}\approx 1$. This is precisely the effect of the intrinsic geometry. In the latter, since $p\approx 2$, it suffices to take $\theta\approx M^{2-p}$, since $2^{j_*(p-2)}\approx 1$ for p sufficiently close to 2. The proof of Proposition 7.2 can now be concluded as in Proposition 6.1. Precisely, setting

$$A_j=\{u<k_j\}\cap Q^*_{4\varrho}(\theta) \quad \text{for } \theta=b_*M^{2-p}$$

and proceeding as in that context, we arrive at the analog of (6.3):

$$|A_{j_*}|\leqslant\left(\frac{\gamma}{j_*}\right)^{(p-1)/p}|Q^*_{4\varrho}(\theta)|$$

for a constant γ depending only upon the data and independent of u, M, ϱ and p. This proves the proposition for the choices

$$2\eta_*=\frac{\xi_*}{2^{j_*}} \quad \text{and} \quad \nu_*=\left(\frac{\gamma}{j_*}\right)^{(p-1)/p}. \qquad \square$$

7.2. Proof of Lemma 7.1—Concluded

It suffices to show that $\nu_*\in(0,1)$ can be chosen a priori, depending only upon the data and independent of u, M, ϱ and p, such that

$$u(x,t)>\eta_*M \quad \text{for all } (x,t)\in Q^*_{2\varrho}(\theta), \text{ with } \theta=b_*M^{2-p}. \tag{7.6}$$

This follows from $(4.2)_-$–$(4.3)_-$ of Lemma 4.1, with $\mu_-=0$, $\xi=2\eta_*$, $a=\frac{1}{2}$, $\omega=M$ and ϱ replaced by 2ϱ. Set

$$Y_0=\frac{|\{u<2\eta_*M\}\cap Q^*_{4\varrho}(\theta)|}{|Q^*_{4\varrho}(\theta)|}=\frac{|A_{j_*}|}{|Q^*_{4\varrho}(\theta)|}.$$

Then, by virtue of Lemma 4.1, and (4.6), the conclusion (7.6) holds true if

$$Y_0\leqslant\frac{1}{\gamma(\text{data})}\frac{(b_*\eta_*^{p-2})^{N/p}}{(1+b_*\eta_*^{p-2})^{(N+p)/p}}=\nu_*.$$

8. Proof of Theorem 1.1

Fix $(x_0,t_0)\subset E_T$, assume that $u(x_0,t_0)>0$, and construct the cylinders

$$(x_0,t_0)+Q^{\pm}_{4\varrho}(\theta)\subset E_T$$

as in (1.6), where the constant $c\geqslant 1$ is to be determined. The change of variables

$$x\longmapsto \frac{x-x_0}{\varrho},\quad t\longmapsto u(x_0,t_0)^{p-2}\frac{t-t_0}{\varrho^p},$$

maps these cylinders into $Q^{\pm}$, where

$$Q^+=K_4\times(0,4^pc^{p-2}]\quad\text{and}\quad Q^-=K_4\times(-4^pc^{p-2},0].$$

Denoting again by (x,t) the transformed variables, the rescaled function

$$v(x,t)=\frac{1}{u(x_0,t_0)}u\biggl(x_0+\varrho x,t_0+\frac{t\varrho^p}{u(x_0,t_0)^{p-2}}\biggr)$$

is a bounded, non-negative, weak solution of

$$\begin{cases} v_t-\operatorname{div}\bar{\mathbf{A}}(x,t,v,Dv)=\overline{B}(x,t,v,Dv) & \text{weakly in } Q=Q^+\cup Q^-,\\ v(0,0)=1, \end{cases}\tag{8.1}$$

where $\bar{\mathbf{A}}$ and $\overline{B}$ satisfy the structure conditions

$$\begin{cases} \bar{\mathbf{A}}(x,t,v,Dv)\cdot Dv\geqslant C_0|Dv|^p-\overline{C}^p,\\ |\bar{\mathbf{A}}(x,t,v,Dv)|\leqslant C_1|Dv|^{p-1}+\overline{C}^{p-1}, \\ |\overline{B}(x,t,v,Dv)|\leqslant C\varrho|Dv|^{p-1}+\varrho\overline{C}^{p-1}, \end{cases}\qquad \text{where } \overline{C}=\frac{C\varrho}{u(x_0,t_0)}\tag{8.2}$$

and C_0, C_1 and C are as in (1.2). Theorem 1.1 is a consequence of the following result.

Proposition 8.1. *There exist constants $\gamma_0\in(0,1)$ and $\gamma_1,\gamma_2>1$ that can be quantitatively determined a priori only in terms of the data, and independent of $u(x_0,t_0)$, such that either $u(x_0,t_0)\leqslant\gamma_2C\varrho$, or*

$$v(x,\gamma_1)\geqslant\gamma_0\quad\textit{for all } x\in K_1.$$

Proof. For $\tau\in[0,1)$, introduce the family of nested cylinders $\{Q_\tau\}_\tau$ with the same "vertex" $(0,0)$, and the families of non-negative numbers $\{m_\tau\}_\tau$ and $\{n_\tau\}_\tau$, defined by

$$Q_\tau=Q_\tau^-(1)=K_\tau\times(-\tau^p,0],\quad m_\tau=\sup_{Q_\tau}v\quad\text{and}\quad n_\tau=(1-\tau)^{-\beta},$$

where $\beta>1$ is to be chosen. Let τ_0 be the largest root of the equation $m_\tau=n_\tau$. Such a largest root exists since $m_0=n_0=1$, $n_\tau\to\infty$ as $\tau\to1$ and m_τ remains bounded. By the continuity of v, there exists $(\bar{x},\bar{t}\,)\in\overline{Q}_{\tau_0}$ such that

$$v(\bar{x},\bar{t}\,)=n_{\tau_0}=(1-\tau_0)^{-\beta}. \tag{8.3}$$

Moreover, $(\bar{x},\bar{t}\,)+Q_{(1-\tau_0)/2}\subset Q_{(1+\tau_0)/2}\subset Q_1$. Therefore, by the definition of m_τ and n_τ,

$$\sup_{(\bar{x},\bar{t})+Q_{(1-\tau_0)/2}} v\leqslant \sup_{Q_{(1+\tau_0)/2}} v\leqslant 2^\beta(1-\tau_0)^{-\beta}.$$

The parameter τ_0 is only known qualitatively, and β has to be chosen. The arguments below have the role of eliminating the qualitative knowledge of τ_0 by a quantitative choice of β. □

8.1. Local largeness of v near $(\bar{x},\bar{t}\,)$

The largeness of v at $(\bar{x},\bar{t}\,)$ as expressed by (8.3), propagates to a full space-time neighborhood nearby $(\bar{x},\bar{t}\,)$. To render this quantitative, set

$$M_0=2^\beta(1-\tau_0)^{-\beta},\quad R_0=\frac{1-\tau_0}{2}\quad\text{and}\quad \theta_0=M_0^{2-p},$$

and consider the cylinder $(\bar{x},\bar{t}\,)+Q^-_{R_0}(\theta_0)$. Set also

$$\xi=1-\frac{1}{2^{\beta+1}}\quad\text{and}\quad a=\frac{1-\dfrac{3}{2}\dfrac{1}{2^{\beta+1}}}{1-\dfrac{1}{2^{\beta+1}}}.$$

Proposition 8.2. *Either* $\overline{C}\geqslant1$, *or*

$$|\{v>2^{-(1+\beta)}M_0\}\cap[(\bar{x},\bar{t}\,)+Q^-_{R_0}(\theta_0)]|>\nu|Q^-_{R_0}(\theta_0)|, \tag{8.4}$$

where

$$\nu=\left(\frac{1-a}{\gamma(\text{data})}\right)^{N+p}\frac{\xi^{N(p-2)/p}}{(1+\xi^{p-2})^{(p+N)/p}}.$$

Proof. Assume that $\overline{C}<1$. If (8.4) is violated, apply Lemma 4.1 in the form $(4.2)_+$–$(4.3)_+$, with the choices $\mu_+=\omega=M_0$, $\theta=\theta_0=M_0^{2-p}$ and $\varrho=R_0$, to conclude that

$$v(\bar{x},\bar{t}\,)\leqslant M_0(1-a\xi)=\tfrac{3}{4}(1-\tau_0)^{-\beta},$$

contradicting (8.3). The condition for this to occur is in (4.6), with the proper meaning of the symbols, and it coincides with (8.4) being violated. □

Remark 8.1. The indicated expressions of ξ, a and ν imply that ν is bounded below by a quantitative positive constant $\nu(\text{data})$, independent of τ_0, and "stable" as $p\to 2$. We continue to denote such a constant by ν.

PROPOSITION 8.3. *Assume that* (8.4) *holds. Then for every* $\lambda\in(0,1)$ *and every* $\nu_0\in(0,1)$, *there exist* $(y,s)\in(\bar{x},\bar{t})+Q^-_{R_0}(\theta_0)$, *a number* $\eta_0\in(0,1)$ *and a cylinder*

$$(y,s)+Q^-_{2\eta_0 R_0}(\theta_0)\subset(\bar{x},\bar{t})+Q^-_{R_0}(\theta_0)$$

such that either $\overline{C}\geqslant 1$, *or*

$$|\{v<\lambda 2^{-(\beta+1)}M_0\}\cap[(y,s)+Q^-_{\eta_0 R_0}(\theta_0)]|\leqslant\nu_0|Q^-_{\eta_0 R_0}(\theta_0)|. \tag{8.5}$$

The number η_0 *depends only upon* ν_0 *and the data, and is independent of* τ_0, ϱ, M, u *and* p. *In particular, it is "stable" as* $p\to 2$.

We assume Proposition 8.3 for the moment and proceed to prove Theorem 1.1.

COROLLARY 8.1. *Assume that* (8.4) *holds. There exist* $(y,s)\in(\bar{x},\bar{t})+Q^-_{R_0}(\theta_0)$ *and a number* $\eta_0\in(0,1)$ *such that either* $\overline{C}\geqslant 1$, *or*

$$v(x,s)\geqslant\tfrac{1}{8}(1-\tau_0)^{-\beta}\quad\textit{for all } x\in K_r(y),\quad\textit{where } r=\eta_0R_0=\tfrac{1}{2}\eta_0(1-\tau_0).$$

Proof. In Proposition 8.3, choose $\lambda=\frac{1}{2}$ and let ν_0 be determined by (4.6) of Lemma 4.1, with the choices $\mu_-=0$, $\omega=M_0$, $\xi=2^{-(\beta+2)}$, $a=\frac{1}{2}$ and $\varrho=\eta_0R_0$. Then Proposition 8.3 identifies a cylinder $(y,s)+Q^-_{2\eta_0R_0}(\theta_0)$ for which (8.5) holds. The conclusion then follows from Lemma 4.1. □

8.2. Proof of Theorem 1.1, assuming Proposition 8.3

Apply Lemma 3.1 to the weak solution v of (8.1) with the structure conditions (8.2), for the choices $M=\frac{1}{8}(1-\tau_0)^{-\beta}$ and $\varrho=r$. Then either

$$(1-\tau_0)^{-\beta}u(x_0,t_0)\leqslant 2\gamma C\varrho,$$

or

$$v(x,t)\geqslant\eta M\quad\text{for all } x\in K_{2r}(y)$$

for all t in the range

$$t_0\stackrel{\text{def}}{=}s+\frac{b}{(\eta M)^{p-2}}(2r)^p\leqslant t\leqslant s+\frac{b}{(\eta M)^{p-2}}(4r)^p\stackrel{\text{def}}{=}t_1.$$

By iteration, for all $n=2,3,...$, either

$$\eta^n(1-\tau_0)^{-\beta}u(x_0,t_0)\leqslant 2\gamma C\varrho,$$

or

$$v(x,t)\geqslant \eta^n M \quad \text{for all } x\in K_{2^n r}(y)$$

for all t in the range

$$t_{n-1}+\frac{b}{(\eta^n M)^{p-2}}(2^{n+1}r)^p\leqslant t\leqslant t_{n-1}+\frac{b}{(\eta^n M)^{p-2}}(2^{n+2}r)^p=t_n.$$

Without loss of generality we may assume that $\eta_0(1-\tau_0)$ is a negative, integral power of 2. Then choosing n so that $2^n r=2$, the cube $K_2(y)$ covers the cube K_1 centered at $x=0$, and

$$v(x,t)\geqslant \eta^n M \quad \text{for all } x\in K_1 \text{ and all } t_{n-1}<t<t_n.$$

For the indicated choice of n,

$$\eta^n M=\tfrac{1}{8}\eta^n(1-\tau_0)^{-\beta}=2^{-3\beta-1}(2^\beta\eta)^n\eta_0^\beta=\gamma_0$$

for the choices of β so that $2^\beta\eta=1$ and $\gamma_0=2^{-3\beta-1}\eta_0^\beta$. On the other hand, since $\bar t$, and hence s, ranges over $(-1,0)$, the range of t includes the time level

$$t_*=4^p b\gamma_0^{2-p}\stackrel{\text{def}}{=}c.$$

9. Proof of Proposition 8.3

Write down the energy estimate (4.1) for $(v-k)_+$ for $k=\frac{1}{2}(1-\tau_0)^{-\beta}$, over the pair of coaxial cylinders with the same "vertex"

$$(\bar x,\bar t\,)+Q^-_{R_0/2}(\theta)\subset(\bar x,\bar t\,)+Q^-_{R_0}(\theta)\subset Q_{(1+\tau_0)/2}.$$

The non-negative, piecewise smooth cutoff ζ is taken to be equal to 1 on the smallest of these cylinders, to vanish on the parabolic boundary of the largest, and such that

$$0\leqslant\zeta_t\leqslant\frac{4^p}{\theta R_0^p}\quad\text{and}\quad|D\zeta|\leqslant\frac{4^p}{R_0^p},\quad\text{where }\theta=M_0^{2-p}.$$

Recalling that v solves (8.1) with the structure conditions (8.2) gives

$$\iint_{(\bar x,\bar t\,)+Q^-_{R_0/2}(\theta)}|D(v-k)_+|^p\,dx\,d\tau\leqslant\gamma\frac{k^p}{R_0^p}|Q^-_{R_0}(\theta)|,$$

provided $\overline{C}\leqslant 1$. Introduce the change of variables

$$x \longmapsto \frac{2(x-\bar{x})}{R_0}, \quad t \longmapsto \frac{2^p(t-\bar{t})}{\theta R_0^p}, \quad w=\frac{v}{k}.$$

This maps $(\bar{x},\bar{t})+Q^-_{R_0/2}(\theta)$ into $Q_1=K_1\times(-1,0]$, and the previous energy estimate takes the form

$$\iint_{Q_1} |Dw|^p \, dx \, d\tau \leqslant \gamma \quad \text{and} \quad |\{w>1\}\cap Q_1|>\nu$$

for a constant γ depending only upon the data.

LEMMA 9.1. *There exists a time level* $\bar{s}\in\left(-1,-\frac{1}{4}\nu\right]$ *such that*

$$\int_{K_1} |Dw(\cdot,\bar{s})|^p \, dx \leqslant \frac{2\gamma}{\nu} \quad \textit{and} \quad |\{w(\cdot,\bar{s})>1\}\cap K_1| \geqslant \tfrac{1}{2}\nu. \tag{9.1}$$

Proof. Introduce the two subsets of $(-1,0]$,

$$T_1=\left\{t\in(-1,0]: \int_{K_1} |Dw(\cdot,t)|^p \, dx > \frac{4\gamma}{\nu}\right\},$$
$$T_2=\left\{t\in(-1,0]: |\{w(\cdot,t)>1\}\cap K_1| \geqslant \tfrac{1}{2}\nu\right\}.$$

From the definition of T_1,

$$\frac{4\gamma}{\nu}|T_1| < \iint_{Q_1} |Dw|^p \, dx \, dt \leqslant \gamma.$$

Therefore $|T_1|<\frac{1}{4}\nu$. From the definition of T_2,

$$\nu < |\{w>1\}\cap Q_1| = \int_{-1}^{0} |\{w(\cdot,t)>1\}\cap K_1| \, dt$$
$$= \int_{T_2} |\{w(\cdot,t)>1\}\cap K_1| \, dt + \int_{(-1,0]-T_2} |\{w(\cdot,t)>1\}\cap K_1| \, dt \leqslant |T_2|+\tfrac{1}{2}\nu.$$

Therefore $|T_2|>\frac{1}{2}\nu$. □

By the results of [7], (9.1) implies that for every fixed $\bar{\lambda},\bar{\nu}\in(0,1)$, there exist at least one point $\bar{y}\in K_1$ and a constant $\bar{\varepsilon}\in(0,1)$, that can be determined a priori only in terms of γ and ν, such that

$$K_{\bar{\varepsilon}}(\bar{y})\subset K_1 \quad \text{and} \quad |\{w(\cdot,\bar{s})>\bar{\lambda}\}\cap K_{\bar{\varepsilon}}| > (1-\bar{\nu})|K_{\bar{\varepsilon}}|.$$

Returning to the original coordinates, and the original function v, there exists $\bar{y}\in K_{R_0}$ such that $K_{\bar{\varepsilon}R_0}(\bar{y})\subset K_{R_0}(\bar{x})$, and

$$|\{v(\cdot,\bar{s})<\bar{\lambda}2^{-(\beta+1)}M_0\}\cap K_{\bar{\varepsilon}R_0}(\bar{y})| < \bar{\nu}|K_{\bar{\varepsilon}R_0}|. \tag{9.2}$$

9.1. Proof of Proposition 8.3—Concluded

The estimate in (9.2), established for some time level $\bar{s}$, can be extended to a cylinder by suitably modifying the various constants. Set $s=\bar{s}+\bar{\theta}(\bar{\varepsilon}R_0)^p$, and write down the energy estimates (4.1) over the pair of cylinders

$$(\bar{y},s)+Q^-_{\bar{\varepsilon}R_0/2}(\bar{\theta})\subset(\bar{y},s)+Q^-_{\bar{\varepsilon}R_0}(\bar{\theta})\quad\text{for } \bar{\theta}=\bar{\nu}^pM_0^{2-p},$$

where $\bar{\nu}$ is the number appearing in (9.2), and we may assume that $s<t$ without loss of generality. The estimate is written for $(v-\bar{\lambda}k)_-$, where $\bar{\lambda}$ is the number appearing in (9.2), and

$$k=\tfrac{1}{2}(1-\tau_0)^{-\beta}=2^{-(\beta+1)}M_0.$$

The cutoff function is taken to be independent of t, equal to 1 on the smaller cylinder, vanishing on the lateral boundary of the larger cylinder and such that $|D\zeta|\leqslant 4(\bar{\varepsilon}R_0)^{-1}$. Recalling that v solves (8.1) with the structure conditions (8.2), and neglecting the term involving Dv, gives that

$$\int_{K_{\bar{\varepsilon}R_0/2}(\bar{y})}(v-\bar{\lambda}k)^2_-(x,t)\,dx\leqslant\int_{K_{\bar{\varepsilon}R_0}(\bar{y})}(v-\bar{\lambda}k)^2_-(x,\bar{s})\,dx+\frac{\gamma k^p}{(\bar{\varepsilon}R_0)^p}|Q^-_{\bar{\varepsilon}R_0}(\bar{\theta})|$$

for all $-\bar{\theta}(\bar{\varepsilon}R_0)^p<t<s$. The constant γ depends only upon the data and is independent of k, $\bar{\varepsilon}$ and M_0, provided $\overline{C}\leqslant 1$. Having fixed $\lambda\in(0,1)$, set $\bar{\lambda}=\frac{1}{2}(1+\lambda)$ and estimate the left-hand side from below, by extending the integration over the smaller sets

$$\{v(\cdot,t)<\lambda k\}.$$

Thus

$$\begin{aligned}\int_{K_{\bar{\varepsilon}R_0/2}(\bar{y})}(v-\bar{\lambda}k)^2_-(x,t)\,dx&\geqslant\int_{K_{\bar{\varepsilon}R_0/2}(\bar{y})\cap\{v(\cdot,t)<\lambda k\}}(v-\bar{\lambda}k)^2_-(x,t)\,dx\\&>\tfrac{1}{4}(1-\lambda)^2k^2|\{v(\cdot,t)<\lambda k\}\cap K_{\bar{\varepsilon}R_0/2}(\bar{y})|\end{aligned}$$

for all $-\bar{\theta}(\bar{\varepsilon}R_0)^p<t<s$. The right-hand side is estimated above by using (9.2) and the expression of $\bar{\theta}$. An upper bound is given by $\gamma k^2\bar{\nu}|K_{\bar{\varepsilon}R_0/2}|$ for a constant γ depending only upon the data and independent of $\bar{\nu}$ and k. Combining these estimates, we get that

$$|\{v(\cdot,t)<\lambda k\}\cap K_{\bar{\varepsilon}R_0/2}(\bar{y})|\leqslant\gamma\bar{\nu}|K_{\bar{\varepsilon}R_0/2}| \tag{9.3}$$

for all $-\bar{\theta}(\bar{\varepsilon}R_0)^p<t<s$. Having fixed $\nu_0\in(0,1)$, choose $\bar{\nu}\leqslant\nu_0$. By choosing a smaller $\bar{\nu}$ if necessary, we may assume that $\bar{\nu}^{-1}$ is an integer. Then, partition the cube $K_{\bar{\varepsilon}R_0/2}(\bar{y})$, up to a set of measure zero, into $\bar{\nu}^{-N}$ pairwise disjoint cubes congruent to $K_{\bar{\nu}\bar{\varepsilon}R_0/2}$, and let y_j, for $j=1,...,\bar{\nu}^{-N}$, be the center of such cubes. The collection of cylinders

$$(y_j,s)+Q^-_{\eta_0R_0}(\theta)\quad\text{for } j=1,...,\bar{\nu}^{-N},\quad\text{where } \eta_0=\tfrac{1}{2}\bar{\nu}\bar{\varepsilon},$$

is a partition, up to a set of measure zero, of the cylinder $(\bar{y},s)+Q^-_{\bar{\varepsilon}R_0/2}(\bar{\theta})$, into $\bar{\nu}^{-N}$ sub-cylinders each congruent to $Q^-_{\eta_0R_0}(\theta)$. By virtue of (9.3), for $\gamma\bar{\nu}=\nu_0$, (8.5) holds true for at least one of these sub-cylinders.

10. The intrinsic Harnack inequality implies Hölder continuity

Local weak solutions u of (1.1), with no sign restrictions, are locally Hölder continuous. Such a local behavior was established in [5, Chapter III], along with locally quantitative Hölder estimates.

The intrinsic Harnack inequality of Theorem 1.1 can be used to establish locally quantitative Hölder estimates for local, weak solutions u of (1.1), thereby providing an alternative proof to [5].

Fix a point in E_T, which, up to a translation, we take to be the origin of $\mathbb{R}^{N+1}$, and for $\varrho_0>0$ consider the cylinder $Q^{p-2}=K_{\varrho_0}\times(-\varrho_0^2,0]$, with "vertex" at $(0,0)$, and set

$$M_0=\sup_{Q^{p-2}} u, \quad m_0=\inf_{Q^{p-2}} u \quad \text{and} \quad \omega_0=\operatorname*{osc}_{Q^{p-2}} u=M_0-m_0.$$

With ω_0 at hand, construct now the cylinder of intrinsic geometry

$$Q_0=K_{\varrho_0}\times(-\theta_0\varrho_0^p,0], \quad \text{where} \quad \theta_0=\left(\frac{c}{\omega_0}\right)^{p-2}$$

and c is a constant to be determined later in terms only of the data and independent of u and ϱ_0. If $\omega_0>c\varrho_0$, then $Q_0\subset Q^{p-2}$.

Proposition 10.1. *Either $\omega_0\leqslant c\varrho_0$, or there exist numbers $\gamma>1$, δ and $\varepsilon\in(0,1)$, that can be quantitatively determined only in terms of the data and independent of u and ϱ_0, such that setting*

$$\omega_n=\delta\omega_{n-1}, \quad \theta_n=\left(\frac{c}{\omega_n}\right)^{p-2}, \quad \varrho_n=\varepsilon\varrho_{n-1} \quad \textit{and} \quad Q_n=Q^-_{\varrho_n}(\theta_n),$$

for $n\in\mathbb{N}$, it holds that $Q_{n+1}\subset Q_n$, and either

$$\operatorname*{osc}_{Q_n} u\leqslant\frac{4\gamma}{\varepsilon}C\varrho_n \quad \textit{or} \quad \operatorname*{osc}_{Q_n} u\leqslant\omega_n.$$

Proof. We exhibit constants c, δ and ε depending only upon the data, such that if the statement holds for n, it continues to hold for $n+1$. Thus assume that Q_n has been constructed and that the statement holds up to n. Set

$$M_n=\sup_{Q_n} u, \quad m_n=\inf_{Q_n} u \quad \text{and} \quad P_0=\left(0,-\tfrac{1}{2}\theta_n\varrho_n^p\right).$$

The point P_0 is roughly speaking the "mid-point" of Q_n. The two functions M_n-u and $u-m_n$ are non-negative weak solutions of (1.1) in Q_n. Either of these satisfies the intrinsic Harnack inequality with respect to P_0, if its "intrinsic waiting time",

$$\left(\frac{c}{M_n-u(P_0)}\right)^{p-2}\varrho_n^p \quad \text{or} \quad \left(\frac{c}{u(P_0)-m_n}\right)^{p-2}\varrho_n^p,$$

is of the order of $\theta_n \varrho_n^p$. At least one of the two inequalities

$$M_n - u(P_0) > \tfrac{1}{4}\omega_n \quad \text{and} \quad u(P_0) - m_n > \tfrac{1}{4}\omega_n$$

must hold. Assuming that the first holds true, apply Theorem 1.1. By possibly modifying the constant c appearing in (1.7) that determines the "waiting time", either

$$\gamma C \varrho_n \geqslant M_n - u(P_0) > \tfrac{1}{4}\omega_n, \tag{10.1}$$

or

$$\inf_{Q^-_{\varrho_n/4}(\theta_n)} (M_n - u) \geqslant \frac{1}{\gamma}(M_n - u(P_0)) > \frac{1}{4\gamma}\omega_n. \tag{10.2}$$

Choosing

$$\delta = 1 - \frac{1}{4\gamma} \quad \text{and} \quad \varepsilon = \frac{1}{4}\delta^{(p-2)/p},$$

one verifies that $Q_{n+1} \subset Q^-_{\varrho_n/4}(\theta_n) \subset Q_n$. Then, if (10.1) occurs,

$$\operatorname*{osc}_{Q_{n+1}} u \leqslant \widetilde{\gamma} C \varrho_{n+1} \quad \text{for } \widetilde{\gamma} = \frac{4\gamma}{\varepsilon}.$$

If (10.2) occurs, then

$$M_n \geqslant \sup_{Q_{n+1}} u + \frac{1}{4\gamma}\omega_n.$$

From this, subtracting $\inf_{Q_{n+1}} u$ from both sides yields that

$$\omega_n \geqslant \operatorname*{osc}_{Q_{n+1}} u + \frac{1}{4\gamma}\omega_n.$$

Thus

$$\operatorname*{osc}_{Q_{n+1}} u \leqslant \delta \omega_n = \omega_{n+1}.$$

□

11. Further results: Equations of the porous media type

Consider quasi-linear, degenerate, parabolic differential equations of the form

$$u_t - \operatorname{div} \mathbf{A}(x,t,u,Du) = B(x,t,u,Du) \quad \text{weakly in } E_T, \tag{11.1}$$

where the functions $\mathbf{A}: E_T \times \mathbb{R}^{N+1} \to \mathbb{R}^N$ and $B: E_T \times \mathbb{R}^{N+1} \to \mathbb{R}$ are only assumed to be measurable and subject to the structure conditions

$$\begin{cases} \mathbf{A}(x,t,u,Du)\cdot Du \geqslant C_0 |u|^{m-1}|Du|^2 - C^2, \\ |\mathbf{A}(x,t,u,Du)| \leqslant C_1 |u|^{m-1}|Du| + C, \\ |B(x,t,u,Du)| \leqslant C|u|^{m-1}|Du| + C, \end{cases} \quad \text{a.e. in } E_T, \tag{11.2}$$

where $m \geqslant 1$, C_0 and C_1 are given positive constants, and C is a given non-negative constant. A function

$$u \in C_{\mathrm{loc}}(0,T;L^2_{\mathrm{loc}}(E)) \quad \text{such that } |u|^{(m+1)/2} \in L^2_{\mathrm{loc}}(0,T;W^{1,2}_{\mathrm{loc}}(E)) \tag{11.3}$$

is a local, weak solution to (11.1) if for every compact set $K \subset E$ and every sub-interval $[t_1,t_2] \subset (0,T]$ the integral equality (1.4) holds for all φ as in (1.5) for $p=2$. For $(x_0,t_0) \in E_T$, assume that $u(x_0,t_0)>0$, and consider cylinders of the type

$$(x_0,t_0)+Q^{\pm}_{\varrho}(\theta) \quad \text{for } p=2 \text{ and } \theta = \left(\frac{c}{u(x_0,t_0)}\right)^{m-1}. \tag{11.4}$$

Local, weak solutions to (11.1)–(11.2) are locally bounded and locally Hölder continuous in E_T [6]. Therefore, they have pointwise values in E_T and the boxes in (11.4) are well defined. These cylinders are intrinsic to the solution, since their length is determined by the value of u at (x_0,t_0).

THEOREM 11.1. (Intrinsic Harnack inequality) *Let u be a continuous, non-negative, weak solution to* (11.1)–(11.2). *There exist positive constants c and γ depending only upon the data, such that for all intrinsic cylinders $(x_0,t_0)+Q^{\pm}_{4\varrho}(\theta)$ as in* (11.4), *contained in E_T, either $u(x_0,t_0) \leqslant \gamma C \varrho$ or*

$$u(x_0,t_0) \leqslant \gamma \inf_{K_\varrho(x_0)} u(x,t_0+\theta\varrho^2), \quad \theta = \left(\frac{c}{u(x_0,t_0)}\right)^{m-1}. \tag{11.5}$$

The theorem has been stated for continuous solutions, to give meaning to $u(x_0,t_0)$. However, it continues to hold for non-negative, weak solutions of (11.1)–(11.2) for almost all $(x_0,t_0) \in E_T$ and for corresponding cylinders $(x_0,t_0)+Q_\varrho(\theta) \subset E_T$. The intrinsic Harnack inequality, in turn, can be used to prove that local solutions of (11.1) are locally Hölder continuous.

THEOREM 11.2. (Harnack inequality and Hölder continuity) *Any locally bounded weak solution to* (11.1)–(11.2), *with no sign restriction, is locally Hölder continuous in E_T.*

A locally quantitative Hölder estimate can be established as a minor variant of the arguments of §10. The only difference is that $M_n - u$ and $u - m_n$ are not solutions of (11.1). However, the proof in §§3–9, only uses De Giorgi-type lemmas (such as Lemma 4.1) for the truncations $(u-k)_\pm$, and the corresponding Harnack estimates can be restated as local bounds for $M_n - u$ and $u - m_n$.

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