

Property (T) and rigidity for actions on Banach spaces

by

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1. Introduction and main results

1.a. Since its introduction by Kazhdan in [Ka], property (T) became a fundamental concept in mathematics with a wide range of applications to such areas as:

- The structure of infinite groups—finite generation and finite Abelianization of higher-rank lattices [Ka], obstruction to free or amalgamated splittings [Wa], [A], [M4], structure of normal subgroups [M2] etc.;
- Combinatorics—the first construction of expanders [M1] (see [Lu]);
- Operator algebras—factors of type II_1 whose fundamental group is countable [C] or even trivial [Po1]; rigidity theorems for the factors associated to the Kazhdan group [Po2];
- Ergodic theory—rigidity results related to orbit equivalence [Po3], [Hj]; the Banach–Ruziewicz problem [M3], [Su];
- Smooth dynamics—local rigidity [FM1], [FM2]; actions on the circle [N1] (and [PS], [Rz]).

It has also been an important tool in providing interesting (counter)examples: to Day’s “von Neumann conjecture” [Gr1, §5.6] and in the context of the Baum–Connes conjecture [HLS] (related to [Gr2]).

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Initially defined in terms of unitary representations, property (T) turned out to be equivalent to Serre’s property (FH) —a fixed-point property for affine isometric actions on Hilbert spaces that can be rephrased as cohomological vanishing. (The equivalence holds for σ -compact groups, in particular all locally compact second countable groups, and was proved by Delorme [D] and Guichardet [Gu]. As pointed out by de Cornulier [Cr], uncountable discrete groups that have Bergman’s cofinality property [Bn] have (FH) but fail (T) .) Some of the above applications use this latter characterization. Recently Shalom [Sh] described the *reduced* 1-cohomology with unitary coefficients for irreducible lattices in products of completely general locally compact groups. This led to a list of new rigidity results and added such lattices to the list of “naturally rigid” groups. For further details and more references on these topics, we suggest the monograph [HV] and the forthcoming [BHV].

1.b. Motivated by these broad themes: property (T) , property (FH) , lattices in semisimple groups and in general products, we study similar notions in the broader framework of Banach spaces rather than Hilbert spaces. Some of the results below apply to general *superreflexive* Banach spaces, whilst some are specific to the subclass of $L^p(\mu)$ -spaces with $1 < p < \infty$. (A Banach space is *superreflexive* if it admits an equivalent uniformly convex norm, see Proposition 2.3 below.)

One of the motivations to consider such questions came from the work of Fisher and Margulis [FM1], [FM2], in which an L^p -analogue of property (T) with $p \gg 2$ allowed them to weaken smoothness assumptions in their results.

The harder question of fixed-point results for *affine actions* on L^p for $p \gg 2$ (see Theorem B below) has applications e.g. for actions on the circle [N2], [BHV].

1.c. Let G be a topological group and B a Banach space. By a *linear isometric G -representation* on B , we shall mean a continuous homomorphism $\varrho: G \rightarrow O(B)$, where $O(B)$ denotes the (“orthogonal”) group of all invertible linear isometries $B \rightarrow B$ (see Lemma 2.4 for a clarification of the continuity assumption). We say that such a representation *almost has invariant vectors* if

$$\inf_{\|v\|=1} \text{diam}(\varrho(K)v) = 0 \quad \text{for all compact subsets } K \subseteq G. \quad (1.i)$$

Denote by $B^{\varrho(G)}$ the closed subspace of G -fixed vectors; the G -representation ϱ descends to a linear isometric G -representation ϱ' on $B' = B/B^{\varrho(G)}$ (see Remark 2.11 for more details in the case of superreflexive spaces). We shall use the following as a Banach space analogue of Kazhdan’s property (T) .

Definition 1.1. Let B be a Banach space. A topological group G is said to have property (T_B) if for any continuous linear isometric G -representation $\varrho: G \rightarrow O(B)$ the quotient G -representation $\varrho': G \rightarrow O(B/B^{\varrho(G)})$ does not almost have G -invariant vectors.

Note that if B is a Hilbert space, ϱ' is isomorphic to the restriction of ϱ to the orthogonal complement $(B^{\varrho(G)})^\perp$ of the subspace of $\varrho(G)$ -invariants. Thus for Hilbert spaces the above definition agrees with Kazhdan's property (T) .

Let μ be a σ -finite measure on a standard Borel space (X, \mathcal{B}) . We are most interested in the family $L^p(\mu)$, $1 < p < \infty$, of Banach spaces, which are close relatives of Hilbert spaces. They also possess a rich group of linear isometries $O(L^p(\mu))$.

THEOREM A. *Let G be a locally compact second countable group. If G has Kazhdan's property (T) then G has property (T_B) for Banach spaces B of the following types:*

- (i) $L^p(\mu)$ for any σ -finite measure μ and any $1 \leq p < \infty$;
- (ii) a closed subspace of $L^p(\mu)$ for any $1 < p < \infty$, $p \neq 4, 6, 8, \dots$;
- (iii) a quotient space of $L^p(\mu)$ for any $1 < p < \infty$, $p \neq \frac{4}{3}, \frac{6}{5}, \frac{8}{7}, \dots$.

If G has $(T_{L^p([0,1])})$ for some $1 < p < \infty$ then G has Kazhdan's property (T) .

1.d. Next we consider group actions by isometries on Banach spaces. By the Mazur–Ulam theorem, such actions are always affine with the linear part being isometric as well (working with *real* Banach spaces).

Definition 1.2. We say that G has property (F_B) if any continuous action of G on B by affine isometries has a G -fixed point.

When B is a Hilbert space this is precisely Serre's property (FH) . Delorme [D] and Guichardet [Gu] proved that properties (T) and (FH) are equivalent for σ -compact groups. Below we summarize the relations between properties (T) and (F_B) which hold for general groups.

THEOREM 1.3. *For a locally compact second countable group G we have*

- (1) (F_B) implies (T_B) for any Banach space B ;
- (2) (T) implies (F_B) for closed subspaces B of $L^p(\mu)$, where $1 < p \leq 2$; likewise for subspaces of L^1 and of the pseudo-normed spaces $L^p(\mu)$, $0 < p < 1$, except one obtains only bounded orbits instead of fixed points;⁽¹⁾
- (3) (T) also implies (F_B) for closed subspaces of $L^p(\mu)$ for $2 \leq p < 2 + \varepsilon$, where $\varepsilon = \varepsilon(G) > 0$ might depend on the Kazhdan group G .

⁽¹⁾ See Example 2.23 for an example without fixed point.

Remark 1.4. (1) is essentially due to Guichardet [Gu] as his proof of $(FH) \Rightarrow (T)$ applies to all Banach spaces. We give two proofs for (2) reducing the problem, in both, to one of the proofs of $(T) \Rightarrow (FH)$. We note that the particular case of $p=1$ in (2) is one of the results of [RS]. Statement (3) is due to Fisher and Margulis (unpublished). With their kind permission we have included their argument here (see §3.c).

The above results imply that any locally compact group G with Kazhdan's property (T) has property (T_{L^p}) for all $1 < p < \infty$, and has the fixed-point property (F_{L^p}) for $1 < p < 2 + \varepsilon(G)$. It turns out, however, that many Kazhdan groups (e.g. hyperbolic ones) do not have property (F_{L^p}) for large values of p .

Indeed, in his study of L^p -cohomology, Pansu [Pa] proved that $\mathrm{Sp}_{n,1}(\mathbf{R})$ and cocompact lattices in these groups have a non-trivial first L^p -cohomology $L^p H^1$ for all $p > 4n + 2$. This is equivalent to asserting that for $p > 4n + 2$ these groups admit fixed-point-free affine isometric actions on $L^p(G)$ with linear part being the regular representation. Hence these groups do not have property (F_{L^p}) for $p > 4n + 2$, whilst enjoying (T) .

More generally, $L^p H^1(\Gamma)$ and hence $H^1(\Gamma, \ell^p \Gamma)$ is non-zero for any non-elementary hyperbolic group when p is large enough. Indeed, Bourdon and Pajot identify this cohomology with a Besov space of functions on the boundary, which they prove to be non-trivial as soon as p is larger than the Hausdorff dimension of an Ahlfors-regular metric on the boundary, see [BP, Corollaire 6.2]. Again, this contradicts (F_{L^p}) for large p .

More recently, using Mineyev's homological bicomings [Mi], Yu [Y] gave a very short proof that any hyperbolic group Γ admits a *proper* action by affine isometries on $\ell^p(\Gamma \times \Gamma)$ if p is large enough. This is a strong negation of (F_{L^p}) for hyperbolic groups and all their infinite subgroups. The corresponding strengthening of the above mentioned [Pa], [BP] for rank-one Lie (or algebraic) groups G has been established by Cornuier–Tessera–Valette in [CTV]: For any $p > 1$ larger than the Hausdorff dimension of the boundary, there is a *proper* affine isometric action on $L^p(G)$ whose linear part is the regular representation. In particular, this holds for $\mathrm{Sp}_{n,1}(\mathbf{R})$ when $p > 4n + 2$.

1.e. Our next goal is now, by contrast, to establish (F_{L^p}) for certain groups. It is often remarked that property (T) for (simple) higher-rank Lie groups and their lattices is more robust than property (T) enjoyed by the rank-one groups $\mathrm{Sp}_{n,1}(\mathbf{R})$ and many other Gromov hyperbolic groups. In view of the preceding discussion of hyperbolic groups and $\mathrm{Sp}_{n,1}(\mathbf{R})$, the following result might be viewed as yet another evidence supporting this view.

THEOREM B. *Let $G = \prod_{i=1}^m \mathbf{G}_i(k_i)$, where k_i are local fields (of any characteristic), $\mathbf{G}_i(k_i)$ are k_i -points of Zariski connected simple k_i -algebraic groups \mathbf{G}_i . Assume that each simple factor $\mathbf{G}_i(k_i)$ has k_i -rank ≥ 2 .*

Then G and the lattices in G have property (F_B) for all $L^p(\mu)$ -related spaces B as in (i)–(iii) in Theorem A, assuming $1 < p < \infty$.

1.f. A broader class of spaces in which we propose to study properties (T_B) and (F_B) consists of *superreflexive* spaces, which can be defined as topological vector spaces isomorphic to uniformly convex Banach spaces.⁽²⁾ In this context we consider linear representations (resp. affine actions) which are *uniformly equicontinuous*; more concretely, for any given norm compatible with the topology, the class of all such linear representations (resp. affine actions) is that of *uniformly bounded* linear representations (resp. *uniformly Lipschitz* affine actions). It turns out that such representations (resp. actions) can always be viewed as isometric with respect to some equivalent norm that is *simultaneously* uniformly convex and uniformly smooth (Proposition 2.13).

Note that whether a given linear G -representation almost contains invariant vectors or not, in the sense of (1.i), does not depend on a particular norm among all mutually equivalent norms. Hence we can make the following definition.

Definition 1.5. Let B be a superreflexive topological vector space and G a locally compact second countable group. We say that G has property (\bar{T}_B) if for every uniformly equicontinuous linear representation ϱ of G on B the quotient G -representation on $B/B^{\varrho(G)}$ does not almost have invariant vectors.

Likewise, G has property (\bar{F}_B) if every uniformly equicontinuous affine G -action on B has a fixed point.

CONJECTURE 1.6. *Higher-rank groups $G = \prod_{i=1}^m \mathbf{G}_i(k_i)$ as in Theorem B and their lattices have property (\bar{F}_B) , and hence (\bar{T}_B) , for all superreflexive B .*

Remark 1.7. To support this conjecture, let us point out the following:

(1) Much of our proof of Theorem B is done in the broad context of uniformly equicontinuous affine actions on general superreflexive spaces, except for one argument— a version of the relative property (T_B) , whose proof is special to L^p -related spaces.

(2) Lafforgue proved [Lg] that the group $\mathrm{PGL}_3(\mathbf{Q}_p)$ has property (\bar{T}_B) for all superreflexive B (his result is actually stronger, in that he allows linear representations with *slowly growing*, rather *uniformly bounded* Lipschitz norms, see Theorem 3.2, Definition 0.2 and the discussion preceding it in [Lg]). Combined with our proof of Theorem B, it implies for example that $\mathrm{SL}_n(\mathbf{Q}_p)$, $n \geq 4$, has property (\bar{F}_B) .

(3) Shalom has proved (unpublished) that for Hilbert spaces \mathcal{H} higher-rank groups (and their lattices) have property $(\bar{F}_{\mathcal{H}})$, and hence $(\bar{T}_{\mathcal{H}})$, whilst rank-one groups have neither $(\bar{F}_{\mathcal{H}})$ nor $(\bar{T}_{\mathcal{H}})$.

⁽²⁾ For spaces that are only *strictly convex*, the fixed-point property always fails [BG], [HP].

1.g. One way to generalize the context of semisimple (non-simple) Lie/algebraic groups is simply to consider general products $G=G_1 \times \dots \times G_n$ of $n \geq 2$ arbitrary topological groups. In the absence of any assumption on the factors G_i , one can still establish splitting results for uniformly equicontinuous affine G -actions on superreflexive spaces.

THEOREM C. *Let $G=G_1 \times \dots \times G_n$ be a product of topological groups with a continuous action by uniformly equicontinuous affine maps on a superreflexive topological vector space B without G -fixed points. Assume that the associated linear G -representation ρ does not almost have non-zero invariant vectors.*

Then there is a G -invariant closed complemented affine subspace $\underline{B} \subseteq B$ and an affine equicontinuous G -equivariant isomorphism $\underline{B} \cong B_1 \oplus \dots \oplus B_n$, where each B_i is a superreflexive Banach space with an equicontinuous affine G -action factoring through $G \rightarrow G_i$.

Remarks 1.8. (1) If G has property (\bar{T}_B) then the assumption that ρ does not almost have invariant vectors is redundant.

(2) In the particular case where B is a Hilbert space and G is locally compact acting by affine isometries, a stronger result was established by Shalom in [Sh]: one assumes only that the *affine* G -action does not almost have fixed points. We replace Shalom's Hilbertian approach with an analogue of the geometric method used in the splitting theorem of [Mo2].

(3) This result can be reformulated in terms of the cohomology of the associated linear G -representation ρ on B as

$$H^1(G, B) \cong \bigoplus_{i=1}^n H^1(G_i, B^{\rho(\prod_{j \neq i} G_j)}).$$

It should be stressed that no such product formula holds in general. Not only does it fail for more general Banach spaces (Example 2.27), but even for Hilbert spaces one needs at least Shalom's assumption mentioned above. Compare the similar situation for the cohomological product formulas of [Sh] and [BMd].

1.h. When G is locally compact, we can as in the Lie case consider its lattices. One then calls a lattice $\Gamma < G$ *irreducible* if its projections to all G_i are dense. The above Theorem C can be used to establish a superrigidity result for irreducible lattices much in the way of [Sh]. (The general idea to use irreducibility in order to transfer results from $G_1 \times \dots \times G_n$ to Γ was also illustrated in [BMz], [BMd] and [MS]; it seems to originate from the work of Margulis and [BK]; lattices in products of completely general locally compact groups were first studied by Shalom [Sh].)

THEOREM D. *Let Γ be an irreducible uniform lattice in a locally compact σ -compact group $G=G_1 \times \dots \times G_n$. Let B be a superreflexive space with uniformly equicontinuous affine Γ -action. Assume that the associated linear Γ -representation does not almost have invariant vectors.*

Then there is a Γ -closed complemented affine subspace of B on which the Γ -action is a sum of actions extending continuously to G and factoring through $G \rightarrow G_i$.

Remark 1.9. More precisely, the conclusion means that there are superreflexive spaces E_i endowed each with a continuous uniformly equicontinuous affine G -action factoring through $G \rightarrow G_i$ and a Γ -equivariant affine continuous map $\bigoplus_{i=1}^n E_i \rightarrow B$. Equivalently, the cocycle $b: \Gamma \rightarrow B$ of the original Γ -action is cohomologous to a sum $b_1 + \dots + b_n$ of cocycles b_i ranging in a subspace $B_i \subseteq B$ on which the *linear* Γ -representation extends continuously to a G -representation factoring through G_i and such that b_i extends continuously to a cocycle $G \rightarrow G_i \rightarrow B_i$ (with respect to the corresponding linear G -representation). Moreover, $B_i \cong E_i$ as G -spaces.

If one disregards a component of B where the linear Γ -representation ranges in a compact group of operators, this *sum of actions* is actually just a direct sum $\bigoplus B_i \subseteq B$ (see Remark 8.10).

Remark 1.10. A *uniform lattice* (in a locally compact group) is just a discrete cocompact subgroup; the theorem however also holds for certain non-uniform lattices, see §8 (Theorem 8.3). Similar arguments allow us to generalise slightly Shalom's *superrigidity for characters*, see Theorem 8.4.

Organization of the paper. In §2 we collect preliminary facts and lemmas on uniformly convex/smooth and superreflexive Banach spaces, linear representations and affine isometries on such spaces, special properties of L^p -spaces, and some general remarks and basic counterexamples. In §3 Theorem 1.3 is proved. Equivalence of properties (T) and (T_{L^p}) (Theorem A) is proved in §4. In §5 we discuss higher-rank groups and prove Theorem B. §6 studies minimal convex sets. §7 addresses product groups and proves the splitting theorem (Theorem C); it also proposes a proof of Theorem B that provides some evidence for Conjecture 1.6. In §8, we prove Theorem D. Appendix 9 describes Shalom's proof of a generalized Howe–Moore theorem.

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2. Preliminaries

This section contains basic definitions, background facts and some preliminary lemmas to be used in the proofs of our main results.

2.a. Banach Spaces

Let V be a Banach space; unless otherwise specified, we take the reals as scalar field. We denote by $S(V) = \{v \in V : \|v\| = 1\}$ its unit sphere. For $v \in B$ and $r > 0$ we denote by $B(v, r)$ and $\bar{B}(v, r)$ the open, respectively closed, ball of radius r around v .

A Banach space B is said to be *strictly convex* if its unit sphere does not contain straight segments, or equivalently if $\|(u+v)/2\| < 1$ whenever $u \neq v \in S(V)$. A Banach space V is called *uniformly convex* if the *convexity modulus* function

$$\delta(\varepsilon) = \inf \left\{ 1 - \frac{\|u+v\|}{2} : \|u\|, \|v\| \leq 1 \text{ and } \|u-v\| \geq \varepsilon \right\} \quad (2.i)$$

is positive, $\delta(\varepsilon) > 0$, whenever $\varepsilon > 0$.

We shall also use the notion of *uniform smoothness* of Banach spaces, which is easiest to define as the uniform convexity of the dual space V^* (see [BL, Appendix A]). Hence a Banach space V is *uniformly convex and uniformly smooth* (hereafter abbreviated *ucus*) if both V and its dual V^* are uniformly convex.

Facts 2.1. We refer to [BL] for the following facts.

(1) The function $\delta(\varepsilon)$ is non-decreasing and tends to 0 when ε tends to 0. If V is uniformly convex then $\delta(\varepsilon) \rightarrow 0 \Leftrightarrow \varepsilon \rightarrow 0$.

(2) Uniformly convex Banach spaces are reflexive. Hence the class of *ucus* Banach spaces is closed under taking duals. This class is also closed under the operations of taking closed subspaces and quotients.

(3) If V^* is strictly convex, in particular if V is uniformly smooth, then every $v \in S(V)$ has a unique supporting functional $v^* \in S(V^*)$, i.e. a unit functional with $\langle v, v^* \rangle = 1$.

(4) If V is *ucus* then the *duality map*

$$\begin{aligned} * : S(V) &\longrightarrow S(V^*), \\ x &\longmapsto x^*, \end{aligned}$$

is a *uniformly continuous* homeomorphism with a uniformly continuous inverse.

(5) To any non-empty bounded subset $E \subseteq V$ of a reflexive strictly convex Banach space V , one can associate a unique point $C(E) \in V$, the *circumcentre* of E (also known as the Chebyshev centre), defined as the unique $v \in V$ minimizing $\inf\{r > 0 : E \subseteq \bar{B}(v, r)\}$.

The existence of $x=C(E)$ in (5) follows from weak compactness of closed bounded convex sets (i.e. from reflexivity), whilst the uniqueness follows from uniform convexity. Note that somewhat contrary to the intuition, it was shown by Klee [Kl] that if $\dim(V)\geq 3$ and V is not a Hilbert space, then there exists a bounded subset $E\subseteq V$ for which $C(E)$ does not belong to the closed convex hull of E . The notion of circumcentre is also used in CAT(0) geometry. For CAT(0) spaces, the circumcentre $C(E)$ always lies in the closed convex hull of E .⁽³⁾

The following result can be found e.g. in [BL, §A.6 and §A.8].

THEOREM 2.2. *The following conditions on a topological vector space V are equivalent:*

- (1) V is isomorphic to a uniformly convex Banach space;
- (2) V is isomorphic to a uniformly smooth Banach space;
- (3) V is isomorphic to a ucus Banach space.

The space V is called *superreflexive* if these equivalent conditions hold. The class of superreflexive spaces is closed under taking duals, closed subspaces and quotients of topological vector spaces.

2.b. Linear representations

Let V be a topological vector space. We denote by $\text{GL}(V)$ the group of invertible linear transformations of V which are continuous together with their inverses.

Following the standard terminology [B1, §2, Definition 2, no. 1], a group G of transformations of V is *uniformly equicontinuous* (with respect to the uniform structure deduced from the topological vector space structure) if for any neighbourhood U of $0\in V$ there exists a neighbourhood W of 0 such that

$$x-y\in W \implies g(x)-g(y)\in U \text{ for all } g\in G. \quad (2.ii)$$

This definition will be applied to both linear groups and affine groups.

For a topological vector space V , we denote by $N(V)$ the (a priori possibly empty) set of norms on V defining the given topology. Elements of $N(V)$ will be called *compatible norms* and are pairwise equivalent.

The following key proposition is an equivariant version of Theorem 2.2. It enables us to reduce questions about uniformly equicontinuous linear representations on superreflexive spaces to isometric linear representations on ucus Banach spaces.

⁽³⁾ Note that Hilbert spaces are, in a sense, the most convex Banach spaces—they have the largest possible modulus of continuity $\delta(\varepsilon)$ among Banach spaces. On the other hand, Hilbert spaces have the smallest possible modulus of continuity among CAT(0) spaces. Thus, in a sense, CAT(0) spaces are more convex than (non-Hilbertian) Banach spaces.

PROPOSITION 2.3. (Invariant ucus norm) *For a superreflexive topological vector space V and a group of linear transformations G of V , the following conditions are equivalent:*

- (1) G is a uniformly equicontinuous group of linear transformations of V ;
- (2) G acts by uniformly bounded linear transformations with respect to any/all compatible norms on V ;
- (3) G acts by linear isometries with respect to some uniformly convex compatible norm on V ;
- (4) G acts by linear isometries with respect to some uniformly smooth compatible norm on V ;
- (5) G acts by linear isometries with respect to some uniformly convex and uniformly smooth compatible norm on V .

Proof. The main part of the proof is the implication [(3) and (4)] \Rightarrow (5); we begin by establishing this.

Let $N(V)$ denote the set of all compatible norms on V equipped with the metric

$$d(\|\cdot\|_1, \|\cdot\|_2) = \sup_{x \neq 0} \left| \log \frac{\|x\|_1}{\|x\|_2} \right|.$$

This is a complete metric space. Let $N(V)^G$ stand for the closed subspace of G -invariant norms in $N(V)$. Denoting by $\delta_{\|\cdot\|}$ the convexity modulus of $\|\cdot\| \in N(V)^G$, the subset $N_{\text{uc}}(V)^G$ of uniformly convex G -invariant norms on V is given by the countable intersection

$$N_{\text{uc}}(V)^G = \bigcap_{n=1}^{\infty} O_n, \quad \text{where } O_n = \{\|\cdot\| \in N(V)^G : \delta_{\|\cdot\|}(1/n) > 0\}.$$

Observe that the sets O_n are open. If $\|\cdot\|_0$ is some fixed G -invariant compatible uniformly convex norm (given in (3)) then any $\|\cdot\| \in N(V)^G$ can be viewed as a limit of uniformly convex norms $\|\cdot\| + \varepsilon \|\cdot\|_0$ as $\varepsilon \searrow 0$. Hence $N_{\text{uc}}(V)^G$ is a dense G_δ set in $N(V)^G$.

By duality between $N_{\text{uc}}(V^*)^G$ and the set $N_{\text{us}}(V)^G$ of uniformly smooth norms in $N(V)^G$, the latter is also a dense G_δ set in the Baire space $N(V)^G$. In particular $N_{\text{uc}}(V)^G \cap N_{\text{us}}(V)^G$ is not empty, as claimed.

Now we observe that (1) \Leftrightarrow (2) follows from the definitions and that (5) \Rightarrow [(3) and (4)] as well as [(3) or (4) or (5)] \Rightarrow (2) are trivial. Moreover, proving (2) \Rightarrow (3) will also yield (2) \Rightarrow (4) by duality, using the fact that the dual to a superreflexive space is superreflexive. Therefore it remains only to justify (2) \Rightarrow (3).

Let $\|\cdot\|$ be a compatible uniformly convex norm on V . The corresponding operator norms $\|g\| = \sup_{x \neq 0} \|gx\|/\|x\|$ are uniformly bounded by some $C < \infty$. Hence

$$\|x\|' = \sup_{g \in G} \|gx\|$$

defines a norm, equivalent to $\|\cdot\|$, and G -invariant. It is also uniformly convex. Indeed, if $\|x\|' = \|y\|' = 1$ and $\|(x+y)/2\|' > 1 - \alpha$, then for some $g \in G$

$$\left\| \frac{gx+gy}{2} \right\| > 1 - \alpha \quad \text{whilst} \quad \|gx\| \leq \|x\|' = 1 \quad \text{and} \quad \|gy\| \leq \|y\|' = 1.$$

Thus $\alpha \geq \delta_{\|\cdot\|}(\|gx - gy\|) \geq \delta_{\|\cdot\|}(\|x - y\|'/C)$. Hence the convexity moduli satisfy

$$\delta_{\|\cdot\|'}(\varepsilon) \geq \delta_{\|\cdot\|}(\varepsilon/C) > 0 \quad \text{for all } \varepsilon > 0. \quad \square$$

If G is a topological group, one should impose a continuity assumption on linear G -representations on V , that is on homomorphisms $\varrho: G \rightarrow \text{GL}(V)$. $\text{GL}(V)$ is naturally equipped with the operator norm (which is too strong for representation theory), and with the *weak* and the *strong* operator topologies. For uniformly equicontinuous representations the latter two topologies impose the same continuity assumption.

LEMMA 2.4. *Let G be a topological group, V a superreflexive topological vector space, and $\varrho: G \rightarrow \text{GL}(V)$ a homomorphism. Then the following are equivalent:*

- (1) ϱ is weakly continuous;
- (2) ϱ is strongly continuous;
- (3) the orbit maps $g \mapsto \varrho(g)u$ are continuous;
- (4) the action map $G \times V \rightarrow V$ is jointly continuous.

Since there is an invariant complete norm on V , this is a special case of a well-known fact holding for all Banach spaces, see [Mo1, §3.3.4] for references. We give an elementary proof in the present case.

Proof. Clearly it is enough to prove (1) \Rightarrow (4). Let $\|\cdot\|$ be a $\varrho(G)$ -invariant ucus norm on V . Assume $g_n \rightarrow e \in G$ and $u_n \rightarrow u \in S(V)$. Then

$$\begin{aligned} |\langle \varrho(g_n)u_n, u^* \rangle - 1| &\leq |\langle \varrho(g_n)u_n, u^* \rangle - \langle \varrho(g_n)u, u^* \rangle| + |\langle \varrho(g_n)u, u^* \rangle - 1| \\ &\leq \|u_n - u\| + |\langle \varrho(g_n)u, u^* \rangle - 1| \rightarrow 0. \end{aligned}$$

It follows that $\varrho(g_n)u_n \rightarrow u$, because

$$\left\langle \frac{\varrho(g_n)u + u}{2}, u^* \right\rangle \leq \left\| \frac{\varrho(g_n)u + u}{2} \right\| \leq 1 - \delta(\|\varrho(g_n)u - u\|)$$

and the left-hand side tends to 1. □

2.c. Invariant complements

One of the convenient properties of Hilbert spaces is the existence of a canonical complement M^\perp to any closed subspace M . Recall that a closed subspace X of a Banach space V is called *complemented* if there is another closed subspace $Y \subseteq V$ such that $V = X \oplus Y$ algebraically and topologically. This is equivalent to each of the following conditions:

- There is a continuous linear projection from V to X ;
- There is a closed subspace Y and a continuous linear projection $p: V \rightarrow Y$ with $\ker(p) = X$.

A classical theorem of Lindenstrauss and Tzafriri says that every infinite-dimensional Banach space which is not isomorphic to a Hilbert space admits a non-complemented closed subspace [LT]. However, for any uniformly equicontinuous linear representation ϱ of a group G on a superreflexive space V , the subspace of invariant vectors $V^{\varrho(G)}$ admits a canonical complement, described below.

In view of Proposition 2.3 we may assume that the representation is linear isometric with respect to a ucus norm on V , which allows us to use the duality map of the unit spheres $*: S(V) \rightarrow S(V^*)$.

Given any linear representation $\varrho: G \rightarrow \text{GL}(V)$ there is an associated dual (or contragredient) linear G -representation $\varrho^*: G \rightarrow \text{GL}(V^*)$ defined by

$$\langle x, \varrho^*(g)y \rangle = \langle \varrho(g^{-1})x, y \rangle, \quad g \in G, \quad x \in V, \quad y \in V^*.$$

If V is a Banach space and $\varrho: G \rightarrow \text{O}(V)$ is a linear isometric representation, then so is its dual $\varrho^*: G \rightarrow \text{O}(V^*)$, where V^* is equipped with the dual norm. Hence the dual to a uniformly equicontinuous representation on a superreflexive space is also of the same type.

Observation 2.5. If V is a ucus Banach space and $\varrho: G \rightarrow \text{O}(V)$, then the duality map $*: S(V) \rightarrow S(V^*)$ between the unit spheres intertwines the actions of $\varrho(G)$ and $\varrho^*(G)$. In particular it maps the set of $\varrho(G)$ -fixed unit vectors to the set of $\varrho^*(G)$ -fixed unit vectors.

PROPOSITION 2.6. *Let ϱ be a uniformly equicontinuous linear representation of G on a superreflexive space V , let $V^{\varrho(G)}$ denote the subspace of $\varrho(G)$ -fixed vectors in V , and let $V' = V'(\varrho)$ be the annihilator of $(V^*)^{\varrho^*(G)}$ in V . Then*

$$V = V^{\varrho(G)} \oplus V'(\varrho).$$

Furthermore, the decomposition is canonical in the following sense: if we denote by $p(\varrho)$ and $p'(\varrho)$ the associated projections, then for every morphism of uniformly equicontinuous

linear representations $\phi: (V_1, \varrho_1) \rightarrow (V_2, \varrho_2)$, the following diagrams are commutative:

$$\begin{array}{ccc} V_1 & \xrightarrow{\phi} & V_2 \\ p(\varrho_1) \downarrow & & \downarrow p(\varrho_2) \\ V_1 & \xrightarrow{\phi} & V_2 \end{array} \quad \begin{array}{ccc} V_1 & \xrightarrow{\phi} & V_2 \\ p'(\varrho_1) \downarrow & & \downarrow p'(\varrho_2) \\ V_1 & \xrightarrow{\phi} & V_2 \end{array} \quad (2.iii)$$

Remark 2.7. The conclusion fails if we drop the superreflexivity assumption, see Example 2.29.

Proof. Choose a G -invariant uniformly convex and uniformly smooth norm on V , and the dual one on V^* (Proposition 2.3). For any unit vector $x \in V^{\varrho(G)}$ and arbitrary $y \in V'$,

$$1 = \langle x, x^* \rangle = \langle x - y, x^* \rangle \leq \|x - y\| \|x^*\| = \|x - y\|.$$

Thus $V^{\varrho(G)} \cap V' = \{0\}$ and $V^{\varrho(G)} \oplus V'$ is a closed subspace in V . It is also dense in V . Indeed if $\lambda \in V^*$ is a unit vector vanishing on V' it cannot vanish on $B^{\varrho(G)}$, because $\lambda \in (V^*)^{\varrho^*(G)}$ by the Hahn–Banach theorem, and hence $\lambda^* \in V^{\varrho(G)}$ and $\langle \lambda^*, \lambda \rangle = 1$. Thus $V^{\varrho(G)} \oplus V' = V$.

The last assertion follows from the fact that $\phi(V_1^{\varrho_1}) \subseteq V_2^{\varrho_2}$, and $\phi^*((V_2^*)^{\varrho_2}) \subseteq (V_1^*)^{\varrho_1}$ yields $\phi(V_2') \subseteq V_1'$. \square

COROLLARY 2.8. *The decomposition $V = V^{\varrho(G)} \oplus V'$ is preserved by the normalizer of $\varrho(G)$ in $\text{GL}(V)$.*

COROLLARY 2.9. *Let $G = G_1 \times G_2$ be any product of two groups and B be a superreflexive space with a uniformly equicontinuous linear G -representation ϱ . Then there is a canonical G -invariant decomposition*

$$V = V^{\varrho(G)} \oplus V_0 \oplus V_1 \oplus V_2$$

such that $V^{\varrho(G_i)} = V^{\varrho(G)} \oplus V_i$ for $i = 1, 2$.

PROPOSITION 2.10. *Let ϱ be a uniformly equicontinuous linear G -representation on a superreflexive space V . Then*

- (1) $V^{\varrho(G)}$ is isomorphic to V/V' as topological vector spaces;
- (2) V' is isomorphic to $V/V^{\varrho(G)}$ as G -representations;
- (3) $(V^{\varrho(G)})^*$ is isomorphic to $V^*/(V^*)'$ as topological vector spaces;
- (4) $(V')^*$ is isomorphic to $(V^*)'$ as G -representations;
- (5) V' almost has invariants if and only if $(V^*)'$ almost has invariants;

(6) if $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$ is an exact sequence of uniformly equicontinuous linear G -representations on superreflexive spaces, then V' almost has invariant vectors if and only if U' or W' does.

If V is equipped with a compatible uniformly convex and uniformly smooth G -invariant norm, then the natural isomorphisms in (1) and (3) are isometric.

Proof. Equip V with a G -invariant ucus norm (Proposition 2.3).

By the open mapping theorem the maps $p: V \rightarrow V^{\varrho(G)}$ and $p': V \rightarrow V'$ induce isomorphisms of topological vector spaces

$$(1) \tilde{p}: V/V' \rightarrow V^{\varrho(G)} \quad \text{and} \quad (2) \tilde{p}': V/V^{\varrho(G)} \rightarrow V'.$$

By Observation 2.5, $(V^{\varrho(G)})^*$ is $(V^*)^{\varrho^*(G)}$ and the latter is isomorphic to $V^*/(V^*)'$. This proves (3).

To see that (1) and (3) are isometric (with respect to the norms corresponding to any ucus G -invariant norm on V) we note that the isomorphisms above satisfy $\|(\tilde{p})^{-1}\| \leq 1$ and $\|(\tilde{p}')^{-1}\| \leq 1$, by the definition of the norm on a quotient space. Furthermore, for $v \in S(V^{\varrho(G)})$, we have $v^* \in S((V^*)^{\varrho^*(G)})$, hence

$$\|(\tilde{p})^{-1}(v)\|_{V/V'} = \inf\{\|v+v'\|_V : v' \in V'\} \geq \inf\{\langle v+v', v^* \rangle : v' \in V'\} = \langle v, v^* \rangle = 1.$$

Hence \tilde{p} is an isometry $V/V' \cong V^{\varrho(G)}$. Similarly, $(V^{\varrho(G)})^* = (V^*)^{\varrho^*(G)} \cong V^*/(V^*)'$.

In general Banach spaces the dual E^* of a subspace $E \subset F$ is isometric to the quotient F^*/E^\perp by the annihilator $E^\perp \subset F^*$ of E . Thus, with respect to a ucus norm on V and the above spaces, $(V')^*$ is isometric to $V^*/(V^*)^{\varrho^*(G)}$ as Banach spaces, while the latter is isomorphic to $(V^*)'$ as a topological vector space, by (2). Whence (4).

(5) Assume that there exist $x_n \in S(V')$ with $\text{diam}(\varrho(K) \cdot x_n) \rightarrow 0$. The uniformly continuous map $*$: $S(V) \rightarrow S(V^*)$ takes vectors $x_n \in S(V')$ to vectors $x_n^* \in S(V^*)$ with

$$\text{diam}(\varrho^*(K) \cdot x_n^*) \rightarrow 0.$$

Since the x_n^* 's are uniformly separated from $(V^*)^{\varrho^*(G)}$, their normalized projection y_n^* to $(V^*)'$ still satisfy $\text{diam}(\varrho^*(K) \cdot y_n^*) \rightarrow 0$.

(6) As U' maps into V' , if U' almost has invariants, then so does V' . If W' almost has invariants, then so does $(W^*)'$, hence $(V^*)'$, hence V' . On the other hand, assume that V' almost has invariant unit vectors v_n . Assume for simplicity that $U=U'$, $V=V'$ and $W=W'$. Note that W is isomorphic to V/U , and denote by $\pi: V \rightarrow W$ the projection. Then either $\pi(v_n)$ converges to $0 \in W$, then there exist u_n such that $v_n - u_n$ converges to $0 \in V$, and the normalized sequence $\{u_n/\|u_n\|\}_{n=1}^\infty$ is almost invariant in U , or there exists a subsequence v_{n_k} with $\inf_k \|\pi(v_{n_k})\| > 0$, and then the normalized sequence $\{\pi(v_{n_k})/\|\pi(v_{n_k})\|\}_{n=1}^\infty$ is almost invariant in W . \square

Remark 2.11. For a ucus Banach space V , Definition 1.1 of property (T_V) can be rephrased as follows: For any representation $\varrho: G \rightarrow \mathcal{O}(V)$, the restriction $\varrho': G \rightarrow \mathcal{O}(V')$ of ϱ to the invariant subspace V' complement to $V^{\varrho(G)}$ does not almost have invariant vectors, i.e. for some compact $K \subseteq G$ and $\varepsilon > 0$,

$$\text{for all } v \in S(V') \text{ there exists } g \in K \text{ such that } \|\varrho(g)v - v\| \geq \varepsilon.$$

Hence item (4) gives the following result.

COROLLARY 2.12. *Let V be a ucus Banach space, and G be a locally compact group. Then G has property (T_V) if and only if it has (T_{V^*}) .*

2.d. Affine actions

The affine group $\text{Aff}(V)$ of a real affine space V (a vector space who forgot its origin) consists of invertible maps satisfying

$$T(tx + (1-t)y) = tT(x) + (1-t)T(y), \quad t \in \mathbf{R}, \quad x, y \in V.$$

The group $\text{Aff}(V)$ is a semi-direct product $\text{Aff}(V) = \text{GL}(V) \ltimes V$, i.e. an invertible affine map T has the form $T(x) = Lx + b$, where $L \in \text{GL}(V)$ is linear invertible.

An affine action of a group G on V , i.e. a homomorphism $G \rightarrow \text{Aff}(V)$, has the form

$$g \cdot x = \varrho(g)x + c(g),$$

where $\varrho: G \rightarrow \text{GL}(V)$ is a linear G -representation (we call it the *linear part* of the action) and $c: G \rightarrow V$ is a ϱ -cocycle, namely an element of the Abelian group

$$Z^1(\varrho) = \{c: G \rightarrow V : c(gh) = \varrho(g)c(h) + c(g) \text{ for all } g, h \in G\}. \quad (2.\text{iv})$$

The group $Z^1(\varrho)$ of ϱ -cocycles contains the subgroup of ϱ -coboundaries

$$B^1(\varrho) = \{c(g) = v - \varrho(g)v : v \in V\}. \quad (2.\text{v})$$

$Z^1(\varrho)$ describes all affine G -actions on V with linear part ϱ , and $B^1(\varrho)$ corresponds to those affine actions which have a G -fixed point (namely v in (2.v)). This description involves the choice of reference point—the origin—in the space. Two cocycles differing by a coboundary can be thought of defining the same affine action viewed from different reference points. The first cohomology of G with ϱ -coefficients is the Abelian group

$$H^1(G, \varrho) = Z^1(\varrho) / B^1(\varrho).$$

It describes different types of actions in the above sense. $H^1(G, \varrho) = 0$ if and only if any affine G -action on V with linear part ϱ has a fixed point.

For a Banach space V , denote by $\text{Isom}(V)$ the group of isometries of V as a metric space. It is a classical theorem of Mazur–Ulam that any surjective isometry T of a (real) Banach space V is necessarily affine $T(x) = Lx + c$ with linear part $L \in \text{O}(V)$ being isometric. (This theorem is elementary when V is strictly convex; compare Lemma 6.1.) Hence $\text{Isom}(V) = \text{O}(V) \ltimes V$.

Now suppose that V is a superreflexive topological vector space. Recall that a group G of affine self maps is uniformly equicontinuous if it satisfies (2.ii). This condition is equivalent to uniform equicontinuity of the linear part $\varrho: G \rightarrow \text{GL}(V)$.

PROPOSITION 2.13. *For a superreflexive topological vector space V and a group G of transformations of V , the following conditions are equivalent:*

- (1) G is a uniformly equicontinuous group of affine transformations of V ;
- (2) G acts by uniformly Lipschitz affine transformations with respect to any/all compatible norms on V ;
- (3) G acts by affine isometries with respect to some compatible norm on V ;
- (4) G acts by affine isometries with respect to some uniformly convex and uniformly smooth compatible norm on V .

Proof. Apply Proposition 2.3 to the linear part of the affine action, using Mazur–Ulam to deduce in (3) that the action is affine. \square

If G is a topological group acting by affine transformations on a topological vector space V , continuity of the action

$$G \times V \rightarrow V, \quad g \cdot x = \varrho(g)x + c(g),$$

is equivalent to continuity of the linear part $G \times V \rightarrow V$ and the continuity of the cocycle $c: G \rightarrow V$. Indeed $c(g) = g \cdot 0$, and $\varrho(g)x = g \cdot x - c(g)$.

Hence, in the context of topological groups, affine actions should be assumed continuous, and $Z^1(G, \varrho)$ will include only continuous cocycles $c: G \rightarrow V$ (we assume that the linear part ϱ is continuous as well). If G is a locally compact σ -compact group, then $Z^1(\varrho)$ has a natural structure of a Fréchet space with respect to the family of semi-norms

$$\|c\|_K = \sup_{g \in K} \|c(g)\|_V,$$

where $K \subseteq G$ runs over a countable family of compact subsets which cover G and $\|\cdot\|_V$ is a norm inducing the topology of V . Moreover, if G is compactly generated (e.g. if G has property (T)), say by K_0 , then $\|c\|_{K_0}$ is a norm on $Z^1(\varrho)$ (note that any cocycle $c \in Z^1(\varrho)$

is completely determined by its values on a generating set), and $Z^1(\varrho)$ is a Banach space with respect to this norm. We remark that in general $B^1(\varrho)$ is not closed in $Z^1(\varrho)$ (this is the idea behind the $(F_V) \Rightarrow (T_V)$ argument of Guichardet—see §3).

LEMMA 2.14. *For a uniform equicontinuous affine action of a group G on a super-reflexive space V , the following are equivalent:*

- (1) *there exists a bounded G -orbit;*
- (2) *all G -orbits are bounded;*
- (3) *G fixes a point in V ;*
- (4) *G preserves a (Borel regular) probability measure on V .*

Note that the notion of a subset $E \subseteq V$ being bounded, means that for any open neighbourhood U of $0 \in V$ there is some $t \in \mathbf{R}$ so that $E \subseteq tU$. This notion agrees with the notion of being bounded with respect to any compatible norm on V .

Proof. Introduce a G -invariant uniformly convex norm on V (Proposition 2.13). The only non-trivial implications are $(4) \Rightarrow (1) \Rightarrow (3)$. For the first, let μ be a G -invariant probability on V . Since V is a countable union of closed bounded sets, there is a closed bounded set $A \subseteq V$ with $\mu(A) > \frac{1}{2}$. For all $g \in G$ we have $\mu(gA) > \frac{1}{2}$ hence $gA \cap A \neq \emptyset$. It follows that the G -orbit of every point of A is bounded.

The latter implication follows by considering the circumcentre (compare §2.a) of the given bounded G -orbit. \square

PROPOSITION 2.15. *Let V be a ucus Banach space. Then*

- (1) *any finite (or compact) group has properties (T_V) and (F_V) ;*
- (2) *properties (T_V) and (F_V) pass to quotient groups;*
- (3) *if $G = G_1 \times \dots \times G_n$ is a finite product of topological groups then G has property (T_V) (resp. (F_V)) if and only if all G_i have this property.*

Proof. (1) and (2) are straightforward, (3) follows from Corollary 2.9. \square

2.e. Special properties of $L^p(\mu)$ -spaces

In this subsection we collect some special properties of the Banach spaces $L^p(\mu)$ which will be used in the proofs.

By an $L^p(\mu)$, or $L^p(X, \mu)$ space we mean the usual space of equivalence classes (modulo null sets) of measurable p -integrable functions $f: X \rightarrow \mathbf{R}$, where μ is a positive σ -finite measure defined on a standard Borel space (X, \mathcal{B}) . If $1 < p < \infty$ then $L^p(\mu)$ is ucus, whilst $L^1(\mu)$ and $L^\infty(\mu)$ are not (they are not even strictly convex). For $1 \leq p < \infty$ the dual to $L^p(\mu)$ is $L^q(\mu)$ where $1 < q \leq \infty$ is determined by $q = p/(p-1)$.

The space $L^p([0, 1], \text{Lebesgue measure})$ is usually denoted by L^p . Any $L^p(\mu)$ -space with *non-atomic* finite or σ -finite measure μ is isometrically isomorphic to L^p . Indeed let $\varphi \in L^1(\mu)$ be a strictly positive measurable function with integral 1, and let μ_1 be given by $d\mu_1 = \varphi d\mu$. Then

$$\begin{aligned} L^p(\mu) &\longrightarrow L^p(\mu_1), \\ f &\longmapsto f \cdot \varphi^{-1/p}, \end{aligned}$$

is a surjective isometry. Since any non-atomic standard probability space is isomorphic to $[0, 1]$ as a measure space, $L^p(\mu_1) \cong L^p$. If μ is purely atomic then a similar argument gives an isomorphism of $L^p(\mu)$ with a finite- or infinite-dimensional ℓ^p space. A general $L^p(\mu)$ space is therefore isometrically isomorphic to a direct sum of L^p and ℓ^p components.

More generally, for another Banach space B , one defines the spaces $L^p(\mu, B)$ of B -valued function classes by means of the Bochner integral. We refer the reader to [DU] for details; we recall here that the dual of $L^p(\mu, B)$ is $L^q(\mu, B^*)$ through the natural pairing for all $1 \leq p < \infty$, but only when B has the Radon–Nikodým property—this includes all ucus spaces (see again [DU]). These spaces will be used in §8.b in order to *induce* isometric (linear or affine) actions.

Banach [Ba] and Lamperti [Li] (see also [FJ, Theorem 3.25]) classified the linear isometries of $L^p(\mu)$ as follows.

THEOREM 2.16. (Banach, Lamperti) *For any $1 < p < \infty$, where $p \neq 2$, any linear isometry U of $L^p(X, \mathcal{B}, \mu)$ has the form*

$$Uf(x) = f(T(x))h(x) \left(\frac{dT_*\mu}{d\mu}(x) \right)^{1/p},$$

where T is a measurable, measure-class preserving map of (X, μ) , and h is a measurable function with $|h(x)| = 1$ almost everywhere.

Let $\mu = \mu_a + \mu_c$ be the decomposition of μ into its atomic and continuous parts ($\mu_a = \mu|_A$, where $A \subseteq X$ is the (at most countable) set of atoms of μ). Then

$$L^p(\mu) = L^p(\mu_c) \oplus L^p(\mu_a) \cong L^p \oplus \ell^p(A), \quad \text{or just } L^p(\mu) \cong \ell^p(A),$$

the latter case occurs if $\mu = \mu_a$ is a purely atomic measure. Note that it follows from the Banach–Lamperti theorem that this decomposition is preserved by any linear isometry of $L^p(\mu)$. As $\ell^p(A)$ has a much smaller group of linear (or affine) isometries than L^p , we could restrict our attention only to the latter. However we shall not make use of this “simplification”.

Another useful tool in the study of L^p -spaces is the Mazur map.

THEOREM 2.17. ([BL, Theorem 9.1]) *Let μ be a σ -finite measure. For any $1 \leq p, q < \infty$ the Mazur map $M_{p,q}: L^p(\mu) \rightarrow L^q(\mu)$ defined by*

$$M_{p,q}(f) = \text{sign}(f)|f|^{p/q}$$

is a (non-linear) map which induces a uniformly continuous homeomorphism between the unit spheres $M_{p,q}: S(L^p(\mu)) \rightarrow S(L^q(\mu))$.

(Note that if $p, q \neq 1$ and $p^{-1} + q^{-1} = 1$ then the restriction of $M_{p,q}$ to the unit spheres is just the duality map $*$: $S(L^p(\mu)) \rightarrow S(L^p(\mu)^*)$.)

In the proofs of Theorems A and B, the results for subspaces and quotients are deduced from the $L^p(\mu)$ case using the following theorem of Hardin about extension of isometries defined on subspaces of $L^p(\mu)$. The formulation we give here is not quite identical to the original, but it easily follows from it and from its proof (see [Ha, Theorem 4.2] or [FJ, Theorem 3.3.14]).

THEOREM 2.18. (Hardin) *Let (X, \mathcal{B}, μ) be a measure space. For every closed subspace $F \subseteq L^p(X, \mu)$, there is a canonical extension $F \subseteq \widetilde{F} \subseteq L^p(\mu)$ which is isometric to $L^p(X', \mu')$ for some other measure space (X', μ') . Furthermore, if $1 < p \notin 2\mathbf{Z}$, then every linear isometry $U: F \rightarrow L^p(Y, \nu)$ extends uniquely to a surjective linear isometry*

$$\widetilde{U}: \widetilde{F} \longrightarrow \widetilde{UF} \subseteq L^p(Y, \nu).$$

Remark 2.19. If $\mathcal{B}' \subseteq \mathcal{B}$ is the minimal sub σ -algebra with respect to which all the functions in F are measurable, then $\widetilde{F} = L^p(X, \mathcal{B}', \mu)$.

A straightforward consequence is the following result.

COROLLARY 2.20. *Let $1 < p \notin 2\mathbf{Z}$, and let $F \subseteq L^p(X, \mu)$ be a closed subspace. Let ϱ be a linear isometric representation of the group G on F . Then there is some linear isometric G -representation ϱ' of G on some other space $L^p(X', \mu')$, and a linear G -equivariant isometric embedding $F \hookrightarrow L^p(X', \mu')$.*

Another important fact about $L^p(\mu)$ -spaces, this time for $p \in (0, 2]$, is that $B = L^p(\mu)$ has an embedding $j: B \rightarrow \mathcal{H}$ into the unit sphere of a Hilbert space so that

$$\langle j(x), j(y) \rangle = \|x - y\|^p.$$

Having such an embedding is equivalent (via the classical result of Schoenberg, see [BHV]) to the following result.

PROPOSITION 2.21. For $0 < p < 2$ and any $s > 0$ the function $f \mapsto e^{-s\|f\|^p}$ is positive definite on $L^p(\mu)$, i.e. for any finite collection $f_i \in L^p(\mu)$ and any $\lambda_i \in \mathbf{C}$,

$$\sum_{i,j} e^{-s\|f_i - f_j\|^p} \lambda_i \bar{\lambda}_j \geq 0.$$

In fact, more is known: it was shown by Bretagnolle, Dacunha-Castelle and Krivine [BDK] (cf. [WW, Theorem 5.1]) that, for $1 \leq p \leq 2$, a Banach space X is isometric to a closed subspace of $L^p(\mu)$ if and only if $e^{-s\|\cdot\|^p}$ is a positive definite function on X for any $s > 0$.

2.f. Some easy counterexamples and remarks

Example 2.22. $((T_B) \not\Rightarrow (F_B))$ Let B be a Banach space with $O(B) \cong \mathbf{Z}/2\mathbf{Z}$, i.e. a space where the only linear isometries are the identity and the antipodal map $x \mapsto -x$. A trivial example of such a space is the line $B = \mathbf{R}$, but it is not hard to construct such spaces of arbitrary dimensions even within the class of ucus Banach spaces (by considering e.g. sufficiently asymmetric convex sets in Hilbert space and choosing the corresponding norm). Clearly for such a space any group has property (T_B) . However the groups \mathbf{Z} or \mathbf{R} or any group G with sufficiently large Abelianization $G/[G, G]$ would fail to have property (F_B) for it would admit an isometric action by translations: $n \cdot x := x + nx_0$, where $0 \neq x_0 \in B$ is arbitrary. However groups with trivial Abelianization would also have property (F_B) on such an asymmetric Banach space B .

Example 2.23. $((T) \not\Rightarrow (F_B))$ Suppose G is locally compact and non-compact (e.g. $G = \mathrm{SL}_3(\mathbf{R})$ or $G = \mathrm{SL}_3(\mathbf{Z})$). Fix a Haar measure dg on G and let

$$B = L_0^1(G) = \{f \in L^1(G) : \int f dg = 0\}$$

be the codimension-1 subspace of functions with 0 mean. Then B is isometric to the affine subspace $\{f \in L^1(G) : \int f dg = 1\}$ on which G acts isometrically by translations without fixed points. Hence G does not have property (F_B) . This Banach space is not ucus. Notice that in this example all orbits are bounded regardless of G .

Remark 2.24. Haagerup and Przybyszewska [HP] showed that any locally compact group G admits a proper isometric action on the strictly convex space $\bigoplus_{n=1}^{\infty} L^{2n}(G)$.

Example 2.25. $((T) \not\Rightarrow (T_B))$ Let G be as in Example 2.23. Consider the space $B = C_0(G)$ of continuous real-valued functions on G which tend to 0 at ∞ with the sup (L^∞) -norm. The action of G on B by translations is a linear isometric action. A function $f \in B$ which decays very slowly forms an ‘‘almost invariant vector’’. On the other hand there are no non-zero invariant vectors. Hence G does not have property (T_B) .

Remark 2.26. Since any separable Banach space is a quotient of ℓ^1 , Example 2.25 shows that case (iii) of Theorem A cannot be extended to $p=1$.

Example 2.27. (Remarks 1.8 (3)) Let $G=G_1 \times G_2$ be any product of non-compact locally compact groups (e.g. $G=\mathbf{Z} \times \mathbf{Z}$). Let $B=L_0^1(G)$ as in Example 2.23. Then $H^1(G, B) \neq 0$, but there are no non-zero G_i -fixed vectors in the associated linear representation. Thus the product formula of Remarks 1.8 (3) cannot hold for B .

Let us make some remarks about Kazhdan's property (T) and property (T_B) as in Definition 1.1 and Remark 2.11. Given a unitary representation (ϱ, \mathcal{H}) of a locally compact group G , a compact subset $K \subseteq G$ and $\varepsilon > 0$, one says that a vector $0 \neq v \in \mathcal{H}$ is (K, ε) -almost invariant if

$$\sup_{g \in K} \|\varrho(g)v - v\| < \varepsilon \|v\|.$$

A locally compact group G has Kazhdan's property (T) if and only if it satisfies the following equivalent conditions:

(1) For any unitary G -representation (ϱ, \mathcal{H}) there exist a compact $K \subseteq G$ and $\varepsilon > 0$ so that the G -representation ϱ' on $(\mathcal{H}^{\varrho(G)})^\perp \cong \mathcal{H}/\mathcal{H}^{\varrho(G)}$ has no (K, ε) -almost invariant vectors;

(2) There exist a compact $K \subseteq G$ and $\varepsilon > 0$ so that all non-trivial irreducible unitary G -representations (ϱ, \mathcal{H}) have no (K, ε) -almost invariant vectors;

(3) There exist a compact $K \subseteq G$ and $\varepsilon > 0$ so that for all unitary G -representations (ϱ, \mathcal{H}) the G -representation ϱ' on $(\mathcal{H}^{\varrho(G)})^\perp \cong \mathcal{H}/\mathcal{H}^{\varrho(G)}$ has no (K, ε) -almost invariant vectors.

In the above, (3) clearly implies both (1) and (2). In showing (1) \Rightarrow (3) one uses the fact that the category of Hilbert spaces and unitary representations is closed under ℓ^2 -sums and L^2 -integration. The fact that any unitary representation decomposes as an L^2 -integral of irreducible ones gives (2) \Rightarrow (3).

Remark 2.28. Definition 1.1 (Remark 2.11) of property (T_B) is modeled on (1) above. There does not seem to be any reasonable theory of irreducible representations (and decomposition into irreducibles) for Banach spaces other than Hilbert ones. Hence form (2) of property (T) does not seem to have a Banach space generalization. As for (3), for any given $1 < p < \infty$ the class of $L^p(\mu)$ -spaces is closed under taking ℓ^p -sums (and L^p -integrals) and hence for groups with property (T_{L^p}) an analogue of (3) holds, namely there exist $K \subseteq G$ and $\varepsilon > 0$ which are good for all $\varrho: G \rightarrow \mathcal{O}(L^p)$. Also, if a group G has property (T_B) for all ucus Banach spaces B (conjecturally all higher-rank groups and their lattices) then for every ucus Banach space B there is a pair (K, ε) which is good for all linear isometric representations $G \rightarrow \mathcal{O}(B)$. This uses the fact that $L^2(\mu, B)$ is ucus if B is (see Lemma 8.6 below).

Finally, we justify Remark 2.7.

Example 2.29. Let G be a discrete group and consider the Banach space $B = \ell^\infty(G)$ with the (linear isometric) regular G -representation ϱ . Then one shows that the space $B^{\varrho(G)}$ (which consists of the constant functions) admits a G -invariant complement (if and) *only if G is amenable*. Indeed, the Riesz space (or Banach lattice) structure of B allows one to take the “absolute value” of any linear functional on B ; renormalizing the absolute value of any non-zero invariant functional would yield an invariant mean on G . Alternatively, one can argue similarly on the Banach space of continuous functions on any compact topological G -space.

We point out that nevertheless the space B' is well-defined for any topological vector G -space B ; in the case at hand, we have $B' = B$ which shows why it cannot be a complement for $B^{\varrho(G)} \neq 0$.

3. Proof of Theorem 1.3

3.a. Guichardet: $(F_B) \Rightarrow (T_B)$

Proof. Assume that G does not have property (T_E) , where E is a Banach space, and let $\varrho: G \rightarrow O(E)$ be a representation such that $E/E^{\varrho(G)}$ admits almost invariant vectors. In order to show that $H^1(G, \varrho) \neq \{0\}$ it suffices to prove that $B^1(G, \varrho) \subseteq Z^1(G, \varrho)$ is not closed.

As was mentioned in §2, the space of ϱ -cocycles $Z^1(G, \varrho)$ is always a Fréchet space (and even a Banach space if G is compactly generated). Note that $B^1(G, \varrho)$ is the image of the bounded linear map

$$\tau: E \longrightarrow Z^1(G, \varrho), \quad (\tau(v))(g) = v - \varrho'(g)v.$$

If $\tau(E)$ were closed, and hence a Fréchet space, the open mapping theorem would imply that $\tau^{-1}: B^1(G, \varrho) \rightarrow E/E^{\varrho(G)}$ is a bounded map. That would mean that for some $M < \infty$ and a compact $K \subseteq G$,

$$\|v\| \leq M \|\tau(v)\|_K = M \sup_{g \in K} \|\varrho(g)v - v\|, \quad v \in E/E^{\varrho(G)},$$

contrary to the assumption that ϱ almost contains invariant vectors. □

3.b. (T) \Rightarrow (F L^p), $0 < p \leq 2$

Proof. Let G be a locally compact group with Kazhdan's property (T) acting by affine isometries on a closed subspace $B \subseteq L^p(\mu)$ with $0 < p \leq 2$. Using Proposition 2.21 and a slight modification of a Delorme–Guichardet argument for (T) \Rightarrow (FH) we shall prove that such an action has bounded orbits. For $1 < p \leq 2$ uniform convexity of $B \subseteq L^p(\mu)$ yields a G -fixed point using Lemma 2.14.

Proposition 2.21 allows us to define a family, indexed by $s > 0$, of Hilbert spaces \mathcal{H}_s , embeddings $U_s: B \rightarrow \mathcal{S}(\mathcal{H}_s)$ and unitary representations $\pi_s: G \rightarrow \mathcal{O}(\mathcal{H}_s)$ with the following properties:

- (1) the image $U_s(B)$ spans a dense subspace of \mathcal{H}_s ;
- (2) $\langle U_s(x), U_s(y) \rangle = e^{-s\|x-y\|^p}$ for all $x, y \in B$;
- (3) $U_s(gx) = \pi_s(g)U_s(x)$ for all $x \in B$ and $g \in G$.

Indeed, one constructs \mathcal{H}_s as the completion of the pre-Hilbert space whose vectors are finite linear combinations $\sum_i a_i x_i$ of points $x_i \in B$, and the inner product is given by

$$\left\langle \sum_i a_i x_i, \sum_j b_j y_j \right\rangle = \sum_{i,j} a_i \bar{b}_j e^{-s\|x_i - y_j\|^p}.$$

The representation π_s can be constructed (and is uniquely determined) by property (3).

Since G is assumed to have Kazhdan's property (T), for some compact subset $K \subseteq G$ and $\varepsilon > 0$, any unitary G -representation with (K, ε) -almost invariant vectors has a non-trivial invariant vector.

Let $x_0 \in B$ be fixed. The isometric G -action is continuous, so Kx_0 is a compact and hence bounded subset of B , hence

$$R_0 = \sup_{g \in K} \|gx_0 - x_0\| < \infty.$$

For the unit vectors $u_s = U_s(x_0) \in \mathcal{S}(\mathcal{H}_s)$ we have

$$\min_{g \in K} |\langle \pi_s(g)u_s, u_s \rangle| \geq e^{-sR_0^p} \rightarrow 1, \quad \text{as } s \rightarrow 0.$$

In particular, for a sufficiently small $s > 0$, $\max_{g \in K} \|\pi_s(g)u_s - u_s\| < \varepsilon$. Let us fix such an s , and rely on property (T) to deduce that π_s has an invariant vector $v \in \mathcal{S}(\mathcal{H}_s)$.

We claim that G must have bounded orbits for its affine isometric action on B . Indeed, otherwise there would exist a sequence $g_n \in G$ so that

$$\|g_n x - y\| \rightarrow \infty \quad \text{and hence} \quad \langle \pi_s(g_n)U_s(x), U_s(y) \rangle \rightarrow 0$$

for all $x, y \in B$. This implies that $\langle \pi_s(g_n)w, u \rangle \rightarrow 0$ for any $w, u \in \text{span}(U_s(B))$ and, since $\text{span}(U_s(B))$ is dense in \mathcal{H}_s , for any $w, u \in \mathcal{H}_s$. Taking $w = u = v$, we get a contradiction. Therefore the affine isometric G -action on B has bounded orbits, and hence fixes a point in case $1 < p \leq 2$. \square

3.c. Fisher–Margulis: $(T) \Rightarrow (F_{L^p}), p < 2 + \varepsilon(G)$

Let G have Kazhdan’s property (T) . Fix a compact generating subset K of G .

LEMMA 3.1. *There exist a constant $C < \infty$ and $\varepsilon > 0$ such that for any G -action by affine isometries on a closed subspace $B \subseteq L^p(\mu)$ with $p \in (2 - \varepsilon, 2 + \varepsilon)$ and any $x \in B$ there exists a point $y \in B$ with*

$$\|x - y\| \leq C \operatorname{diam}(K \cdot x), \quad \operatorname{diam}(K \cdot y) < \frac{\operatorname{diam}(K \cdot x)}{2}.$$

Proof. By contradiction there exists a sequence of subspaces $B_n \subseteq L^{p_n}$ with $p_n \rightarrow 2$, affine isometric G -actions on B_n and points $x_n \in B_n$ so that, after a rescaling to achieve $\operatorname{diam}(K \cdot x_n) = 1$, we have

$$\operatorname{diam}(K \cdot y) \geq \frac{1}{2} \quad \text{for all } y \in B(x_n, n) \tag{3.i}$$

Passing to an ultraproduct of the spaces B_n with the marked points x_n and the corresponding G -actions, one obtains an isometric (hence also affine) G -action on a Hilbert space \mathcal{H} , because the limit of L^p -parallelograms as $p \rightarrow 2$ is the parallelogram identity, which characterizes Hilbert spaces. (The action is well-defined because K generates G and we ensured $\operatorname{diam}(K \cdot x_n) = 1$.) If G is a topological group, one needs to ensure continuity of the limit action by selecting uniformly K -equicontinuous sets of vectors (as in [Sh, Lemma 6.3]; compare also [CCS]). Due to (3.i), this G -action has no fixed points, contradicting property (FH) , and hence (T) , of G . \square

Proof of (F_B) for $B \subseteq L^p(\mu)$, $2 \leq p < 2 + \varepsilon(G)$. Now consider an arbitrary affine isometric G -action on a closed subspace $B \subseteq L^p$ with $|p - 2| < \varepsilon$, where $\varepsilon = \varepsilon(G) > 0$ is as in the lemma. Define a sequence $x_n \in B$ inductively, starting from an arbitrary x_0 . Given x_n , let $R_n = \operatorname{diam}(K \cdot x_n)$. Then, applying the lemma, there exists x_{n+1} within the ball $B(x_n, CR_n)$ so that

$$\operatorname{diam}(K \cdot x_{n+1}) < \frac{R_n}{2}.$$

We get $R_n < R_0/2^n$ and

$$\sum_{n=1}^{\infty} \|x_{n+1} - x_n\| < \infty.$$

The limit of the Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ is a G -fixed point. \square

Question 3.2. For a given group G with property (T) , what can be said about the following invariant?

$$p(G) := \inf\{p : G \text{ fails to have } (F_B) \text{ for some closed subspace } B \subseteq L^p\}.$$

For instance, Pansu’s aforementioned result [Pa] shows that for $G = \operatorname{Sp}_{n,1}(\mathbf{R})$ one has

$$p(G) \leq 4n + 2.$$

4. Proof of Theorem A

We start with the first assertion of the theorem: $(T) \Rightarrow (T_B)$ for B being an L^p -related space as in (i), (ii) or (iii) in the theorem. We first reduce to the case (i), where $B = L^p(\mu)$ with $1 \leq p < \infty$. Then, using Corollary 2.20 of Hardin's extension theorem, $(T_{L^p(\mu)})$ implies (T_B) for subspaces $B \subseteq L^p(\mu)$, where $p \neq 4, 6, \dots$ as in (ii), and the duality argument (Corollary 2.12) gives the result for quotients of $L^q(\mu)$ with $q \neq \frac{4}{3}, \frac{6}{5}, \dots$ as in (iii). Hence it suffices to prove $(T) \Rightarrow (T_{L^p(\mu)})$ for $1 \leq p < \infty$. We give two proofs for this implication.

Let us note that our restriction on p and q when taking subspaces/quotients comes from our use of Hardin's theorem.

Question 4.1. Does property (T) imply property (T_B) for any closed subspace and any quotient B of $L^p(\mu)$ for any $1 < p < \infty$?

4.a. Property (T) implies $(T_{L^p(\mu)})$

Proof. Assuming that a locally compact group G fails to have property $(T_{L^p(\mu)})$ for some $1 \leq p < \infty$, we are going to show that G does not have (T). We may and will assume $p \neq 2$; write $B = L^p(\mu)$ and $\mathcal{H} = L^2(\mu)$. Using Remark 2.11, there is a representation

$$\varrho: G \longrightarrow \mathrm{O}(B)$$

so that for the canonical complement B' of $B^{e(G)}$ the restriction $\varrho': G \rightarrow \mathrm{O}(B')$ almost has invariant vectors, i.e. there exist unit vectors $v_n \in \mathrm{S}(B')$ so that

$$f_n(g) = \|\varrho(g)v_n - v_n\|$$

converges to 0 uniformly on compact subsets of G .

We shall obtain a related unitary, or orthogonal, representation $\pi: G \rightarrow \mathrm{O}(\mathcal{H})$ using the following lemma.

LEMMA 4.2. *For $p \neq 2$, the conjugation $U \mapsto M_{p,2} \circ U \circ M_{2,p}$ by the non-linear Mazur map sends $\mathrm{O}(B)$ to $\mathrm{O}(\mathcal{H})$.*

Proof. This follows from Banach–Lamperti's description of $\mathrm{O}(B)$ (Theorem 2.16) by calculation. \square

Let us then define $\pi: G \rightarrow \mathrm{O}(\mathcal{H})$ by $\pi(g) = M_{p,2} \circ \varrho(g) \circ M_{2,p}$. Note that $M_{p,2}$ maps $B^{e(G)}$ onto $\mathcal{H}^{\pi(G)}$.

As $\mathrm{S}(B')$ is uniformly separated (in fact is at distance 1) from $B^{e(G)}$, the uniform continuity of the Mazur map (Theorem 2.17) implies that $u_n = M_{p,2}(v_n)$ is a sequence in

$S(\mathcal{H})$ such that $\text{dist}(u_n, \mathcal{H}^{\pi(G)}) \geq \delta > 0$ and $\varphi_n(g) = \|\pi(g)u_n - u_n\| \rightarrow 0$ uniformly on compact subsets of G . Let w_n denote the projections of u_n to $\mathcal{H} = (\mathcal{H}^{\pi(G)})^\perp$. Then

$$\|w_n\| \geq \delta > 0 \quad \text{and} \quad \|\pi(g)w_n - w_n\| \leq \varphi_n(g) \rightarrow 0$$

uniformly on compact sets. Thus the restriction π' of π to \mathcal{H}' does not have G -invariant vectors, but almost does. Hence G does not have Kazhdan's property (T). \square

Remark 4.3. In fact, the above proof has established the following more specific statement. Let G act measurably on a σ -finite measure space. Denote by ϱ_p the associated linear isometric representation on L^p , namely the quasi-regular representation twisted by the p th root of the Radon–Nikodým derivative. Then, the existence of almost invariant vectors in $L^p/(L^p)^{\varrho_p(G)}$ is independent of $1 \leq p < \infty$.

4.b. Property (T) implies $(T_{L^p(\mu)})$

Proof. For $1 < p \leq 2$ we have $(T) \Rightarrow (F_{L^p(\mu)}) \Rightarrow (T_{L^p(\mu)})$ by Theorem 1.3 (1) and (2). Using duality (Corollary 2.12), this implication extends to $L^p(\mu)$ with $2 < p < \infty$. \square

4.c. Property (T_{L^p}) implies (T)

Proof. Assume that G is not Kazhdan, i.e. G admits a unitary representation π almost containing (but not actually containing) non-trivial invariant vectors. Connes and Weiss [CW] showed how to find such a representation of the form $L_0^2(\mu)$. More precisely, they construct a measure-preserving, *ergodic* G -action on a probability space (X, μ) which admits a sequence $\{E_n\}_{n=1}^\infty$ of *asymptotically invariant* measurable subsets, namely

$$\text{for all } g \in G, \quad \mu(gE_n \triangle E_n) \rightarrow 0 \quad \text{whilst} \quad \mu(E_n) = \frac{1}{2}. \quad (4.i)$$

Consider the unitary G -representation π' on $L_0^2(\mu)$ —the space of *zero mean* square integrable functions, which is the orthogonal complement of the constants. Then π' does not have non-trivial invariant vectors because of ergodicity; but it almost does, namely $f_n = 2 \cdot \mathbf{1}_{E_n} - 1$.

For a given $1 \leq p < \infty$, consider the linear isometric G -representation ϱ on $B = L^p(\mu)$, $\varrho(g)f(x) = f(g^{-1}x)$. Then $B^{\varrho(G)} = \mathbf{R}\mathbf{1}$ —the constants, and its canonical complement is

$$B' = L_0^p(\mu) = \{f \in L^p(\mu) : \int f d\mu = 0\}.$$

The above sequence $\{f_n\}_{n=1}^\infty$ lies in $L_0^p(\mu)$, consists of unit vectors and still satisfies

$$\|\varrho(g)f_n - f_n\|_p \rightarrow 0.$$

Hence failing to have Kazhdan's property (T) a group G does not have $(T_{L^p(\mu)})$ either.

In the original paper [CW], Connes and Weiss considered discrete groups. In a similar context the case of locally compact groups was also considered by Glasner and Weiss (see [GW, §3] and references therein). One way to treat the non-discrete case, is the following: start from a unitary representation π of a given locally compact secondly countable G which has almost invariant vectors but no invariant ones, and apply the original Connes–Weiss Gaussian construction to the restriction $\pi|_\Gamma$ of π to some dense countable subgroup $\Gamma \subseteq G$. This gives an ergodic measure-preserving Γ -action on a probability space (X, μ) with an asymptotically invariant sequence $\{E_n\}_{n=1}^\infty$ on X . The fact that the representation $\pi|_\Gamma$ came from G is manifested by the fact that it is continuous in the topology on Γ induced from G . It can be shown to imply that the Γ -representation on $L_0^2(X, \mu)$ is also continuous, hence extends to G , and thus the Γ -action on (X, μ) extends to a measurable G -action. This construction gives a uniform convergence in (4.i) on compact subsets of G . \square

5. Fixed-point property for higher-rank groups

5.a. The goal of this section is to prove Theorem B; we start with some preliminaries for the *linear* part.

The first ingredient needed for the proof is an analogue of Howe–Moore's theorem on vanishing of matrix coefficients, or rather its corollary analogous to Moore's ergodicity theorem, extended to the framework of uniformly equicontinuous representations on superreflexive Banach spaces. The ucus Banach space version of Howe–Moore is due to Shalom (unpublished). With his kind permission we have included the argument in Appendix 9. Here we shall use the following corollary, which we formulate for the case of simple groups.

COROLLARY 5.1. (Banach space analogue of Moore's theorem) *Let k be a local field and let $G = \mathbf{G}(k)$ be the k -points of a Zariski connected isotropic simple k -algebraic group \mathbf{G} . Let G^+ be the image of the simply connected form $\tilde{G} = \tilde{\mathbf{G}}(k)$ in G under the cover map. Let $H \subseteq G^+$ be a closed non-compact subgroup.*

Then for any superreflexive space B and any continuous uniformly equicontinuous linear G -representation $\varrho: G^+ \rightarrow \text{GL}(B)$, $B^{\varrho(H)} = B^{\varrho(G^+)}$ and the canonical complements with respect to both $\varrho(G^+)$ and $\varrho(H)$ coincide, and can be denoted just by B' .

Proof. By Proposition 2.3, we may assume that B is a ucus Banach space and ϱ is a linear isometric representation $\varrho: G \rightarrow \mathrm{O}(B)$. Now the statement follows readily from Theorem 9.1. \square

5.b. The second ingredient is the *strong relative property (T)*. It will be used to prove Claim 5.5 below which is the only part that is specific to L^p -like spaces. The rest of the argument applies to all affine isometric actions on ucus Banach spaces, or all uniformly equicontinuous affine actions on a superreflexive space.

Definition 5.2. Let $H \ltimes U$ be a semi-direct product of locally compact groups. We shall say that it has

- the *strong relative property (T)* if for any unitary representation π of $H \ltimes U$ for which H almost has non-trivial invariant vectors, U has invariant vectors;
- the *strong relative property (T_B)*, where B is a Banach space, if for any linear isometric representation $\varrho: H \ltimes U \rightarrow \mathrm{O}(B)$ the linear isometric H -representation

$$\varrho': H \longrightarrow \mathrm{O}(B/B^{e(U)})$$

does not almost have non-trivial invariant vectors.

Remarks 5.3. (1) The first definition is a variant of the “relative property (T)”. The latter usually refers to a pair of groups $G_0 \subseteq G$ and requires that any unitary G -representation with G -almost invariant vectors, has non-trivial G_0 -invariant vectors. The strong relative property (T) for $H \ltimes U$ implies, but is not equivalent to, the relative property (T) for $(H \ltimes U, U)$. In fact $\mathrm{SL}_2(\mathbf{R}) \ltimes \mathbf{R}^2$ has the strong relative property (T), and thus the relative property (T) as well, whilst its lattice $\mathrm{SL}_2(\mathbf{Z}) \ltimes \mathbf{Z}^2$ does not have the strong relative property (T) even though the pair $(\mathrm{SL}_2(\mathbf{Z}) \ltimes \mathbf{Z}^2, \mathbf{Z}^2)$ has the relative property (T). (For the latter, cf. Burger’s appendix in [HV]. For the former, consider the representation on $\ell^2(\mathbf{Z}^2)$ induced by the affine action on \mathbf{Z}^2 .)

(2) If B is a ucus Banach space, then the canonical splitting with respect to $\varrho(U)$, namely $B = B^{e(U)} \oplus B'$ is preserved by $\varrho(H)$ which normalizes $\varrho(U)$ (Corollary 2.8). Hence, as in Remark 2.11, for a ucus space B the strong relative property (T_B) requires that the *restriction* of $\varrho(H)$ to B' does not almost have invariant vectors. The strong relative property (T_ℋ) for a Hilbert space \mathcal{H} is equivalent to the strong relative property (T).

LEMMA 5.4. *A semi-direct product $H \ltimes U$ with the strong relative property (T) has the strong relative property (T_B) for all L^p -related Banach spaces B of types (i), (ii) and (iii) as in Theorem A.*

Proof. This is analogous to the proof of $(T) \Rightarrow (T_B)$ given in §4.a. First observe that the extension Theorem 2.20 and a duality argument (based on Proposition 2.10) reduce the statement to the case (i) of $B=L^p(\mu)$.

Thus we assume that $B=L^p(\mu)$ with $p \neq 2$, and $\varrho: H \times U \rightarrow O(B)$ is a linear isometric representation. Let $B=B^{\varrho(U)} \oplus B'$ be the canonical splitting with respect to U . It is preserved by $\varrho(H)$ because H normalizes U . Now let $\pi=M_{p,2} \circ \varrho \circ M_{2,p}$ be the conjugate of ϱ by the Mazur map. Then π is an orthogonal representation $\pi: H \times U \rightarrow O(\mathcal{H})$, where $\mathcal{H}=L^2(\mu)$ (Lemma 4.2).

If $H \times U$ fails to have the strong relative property (T_B) , then there exist $x_n \in S(B')$ so that $\|\varrho(h)x_n - x_n\| \rightarrow 0$ uniformly on compact subsets of H . Uniform continuity of $M_{p,2}$ and the fact that $\text{dist}(S(B'), S(B^{\varrho(U)}))=1$, imply that for $v_n=M_{p,2}(x_n)$,

$$\text{dist}(v_n, \mathcal{H}^{\varrho(U)}) \geq \delta > 0 \quad \text{and} \quad \|\pi(h)v_n - v_n\| \rightarrow 0$$

uniformly on compact subsets of H . Taking projections of v_n to \mathcal{H}' , we show that in this case $H \times U$ does not have the strong relative property (T) . \square

5.c. Proof of Theorem B

We first show that we can assume that G is (the k -points of a) connected and simply connected algebraic group. Assuming that Theorem B is known for \tilde{G}_0 and lattices therein, we will prove it for G and its lattices. For any affine isometric action of G on B there is an associated action of \tilde{G}_0 , inflated via the covering map $\tilde{G}_0 \rightarrow G$. \tilde{G}_0 has a fixed point by assumption, hence G has a compact orbit, as the cokernel of the covering map is compact [M5, Theorem I.2.3.1 (b)]. It follows that G has a fixed point as well. A similar argument applies to lattices: For a given lattice Γ in G its inverse image $\tilde{\Gamma}$ by the covering map is a lattice in \tilde{G}_0 , and its projection is of finite index in Γ . Every affine isometric action of Γ gives rise to an affine isometric action of $\tilde{\Gamma}$, which, by assumption, has a fixed point. It follows that Γ has a finite orbit, and therefore fixes a point.

Hereafter we will assume that G is (the k -points of a) connected and simply connected group. In that case G decomposes into a direct product of simply connected almost simple groups $G=\prod_i G_i$ [M5, Proposition I.1.4.10].

In view of (the independent) §8.a and §8.b, more specifically Proposition 8.8 (2) and the discussion following Definition 8.2, property (F_B) for $G=\prod_i G_i$ is inherited by its lattices. Thus it suffices to consider the ambient group $G=\prod_i G_i$ only. By Proposition 2.15 (3) the statement reduces to that about almost-simple factors G_i .

So we are left proving the theorem for $G=\mathbf{G}(k)$, a higher-rank connected, simply-connected, almost-simple algebraic group. Using Proposition 2.13, we assume that B

is a reflexive Banach space and we consider a G -action on B by affine isometries, with $\varrho: G \rightarrow \mathcal{O}(B)$ denoting the linear part of the action. Let $B = B^{\varrho(G)} \oplus B'$ be the canonical decomposition and $\varrho': G \rightarrow \mathcal{O}(B')$ denote the corresponding subrepresentation.

Claim 5.5. (For L^p -like spaces) G contains a direct product $A \times H$ so that

- (1) the restriction $\varrho'|_H: H \rightarrow \mathcal{O}(B')$ does not almost contain invariant vectors;
- (2) A is a 1-dimensional split torus, and in particular it is not compact.

Proof. Any higher-rank almost-simple group $G = \mathbf{G}(k)$ is known to contain a subgroup whose simply-connected cover is isomorphic to either $G_0 = \mathrm{SL}_3(k)$ or $G_0 = \mathrm{Sp}_4(k)$ [M5, Theorem I.1.6.2]. In the first case, $G_0 = \mathrm{SL}_3(k)$ contains the semi-direct product $H_0 \times U_0 = \mathrm{SL}_2(k) \ltimes k^2$ embedded in $\mathrm{SL}_3(k)$ as

$$\left\{ \begin{pmatrix} a & b & x \\ c & d & y \\ 0 & 0 & 1 \end{pmatrix} : ad - bc = 1 \right\},$$

where $U_0 \cong k^2$ is the subgroup given by $a = d = 1$ and $b = c = 0$. It is normalized by the copy H_0 of $\mathrm{SL}_2(k)$ embedded in the upper left corner. Let $A_0 \subseteq \mathrm{SL}_3(k)$ be the subgroup $\mathrm{diag}[\lambda, \lambda, \lambda^{-2}]$, $\lambda \in k^*$, which centralizes H_0 in G_0 , and let A and $H \times U$ denote the corresponding subgroups in G .

The semi-direct product $\mathrm{SL}_2(k) \ltimes k^2$ is known to have the strong relative property (T) . Hence it also has the strong relative property (T_B) for L^p -related spaces B (Lemma 5.4). By Corollary 5.1, we have $B^{\varrho(G)} = B^{\varrho(U)}$ and we have denoted by B' the common canonical complement. Then (1) follows from the strong relative property (T_B) for $H \times U$, while (2) is clear from the construction.

In the second case, G contains a copy of $G_0 = \mathrm{Sp}_4(k)$, which is usually defined as a subgroup of $\mathrm{SL}_4(k)$ by

$$\mathrm{Sp}_4(k) = \{g \in \mathrm{SL}_4(k) : {}^t g J g = J\}, \quad \text{where } J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

The semi-direct product $H_0 \times U_0$ embedded in $\mathrm{SL}_4(k)$ is

$$\left\{ \begin{pmatrix} A & B \\ 0 & {}^t A^{-1} \end{pmatrix} : A \in \mathrm{SL}_2(k) \text{ and } {}^t B = A^{-1} B ({}^t A) \right\},$$

with H_0 denoting the image $A \mapsto \mathrm{diag}[A, {}^t A^{-1}]$ of $\mathrm{SL}_2(k)$, and U_0 the normal Abelian subgroup

$$\left\{ \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} : {}^t B = B \right\}.$$

The semi-direct product $H_0 \times U_0$ actually lies in $\mathrm{Sp}_4(k)$, it is isomorphic to $\mathrm{SL}_2(k) \times S^2(k)$, where $S^2(k)$ is the space of symmetric bilinear forms on k^2 with the natural $\mathrm{SL}_2(k)$ action. This semi-direct product is also known to have the strong relative property (T), and therefore the strong relative property (T_B) . H_0 is centralized by

$$A_0 = \{\mathrm{diag}[\lambda, \lambda, \lambda^{-1}, \lambda^{-1}] : \lambda \in k^*\}.$$

As in the $G_0 = \mathrm{SL}_3(k)$ case, we conclude that the corresponding product $A \times H \subseteq G$ satisfies (1) and (2). The claim is proved. \square

We now turn to the affine isometric G -action defined by a ϱ -cocycle $c \in Z^1(\varrho)$. We shall prove that $c \in B^1(\varrho)$, i.e. that G has a global fixed point. Write $c(g) = c_0(g) + c'(g)$ with $c_0(g) \in B^{\varrho(G)}$ and $c'(g) \in B'$, where $B = B^{\varrho(G)} \oplus B'$ is the canonical splitting. Then $c_0: G \rightarrow B$ is a homomorphism into the (additive) Abelian group. As G has compact Abelianization, $c_0(g) \equiv 0$, which means that the affine G -action preserves each affine subspace $p + B'$. Hence both the affine G -action and the representation can be restricted to B' .

Claim 5.5 provides an input for the following general lemma.

LEMMA 5.6. *Let a direct product of topological groups $A \times H$ act by affine isometries on a Banach space B . Suppose that the associated linear isometric representation ϱ restricted to H does not almost have invariant vectors. Then the affine action of A has bounded orbits in B . In particular, if B is uniformly convex, then A has a fixed point in B .*

Remark 5.7. In the uniformly convex case, this follows of course from the stronger splitting theorem (Theorem C); compare also with Theorem 7.1 below for the weaker assumption that the *product* does not almost have invariant vectors.

Proof. Let $\varrho: A \times H \rightarrow \mathrm{O}(B)$ and $c \in Z^1(\varrho)$ denote the associated linear isometric representation and the translation cocycle. The commutation relation between any $h \in H$ and $a \in A$ gives

$$c(h) + \varrho(h)c(a) = c(ha) = c(ah) = c(a) + \varrho(a)c(h),$$

which can be rewritten as

$$(I - \varrho(h))c(a) = (I - \varrho(a))c(h).$$

By the assumption on $\varrho(H)$, there exist a compact subset $K \subseteq H$ and $\varepsilon > 0$ so that $\max_{h \in K} \|\varrho(h)v - v\| \geq \varepsilon \|v\|$ for all $v \in B$. Let $R = \max_{h \in K} \|c(h)\| < \infty$. Then for $a \in A$,

$$\varepsilon \|c(a)\| \leq \max_{h \in K} \|(I - \varrho(h))c(a)\| \leq 2R.$$

Hence $\sup_{a \in A} \|c(a)\| \leq 2R/\varepsilon$, i.e. the A -orbit of 0 is bounded. If B is uniformly convex then the circumcentre of this orbit is an A -fixed point as in Lemma 2.14. \square

We restrict the G -action to B' , since G has no additive characters. It follows from Claim 5.5 and Lemma 5.6 that for some (1-dimensional) split torus A , there is an A -fixed point. This point is unique; indeed, if x and y are A -fixed, then $x-y \in B'$ is an A -invariant vector for the linear representation ϱ . Since A is non-compact, the ucus analogue of Moore's ergodicity (Corollary 5.1) implies that $B^{\varrho(A)} = B^{\varrho(G)}$. Hence $x-y=0$, as claimed. Being unique, it is also fixed by any element commuting with A .

Let now $S \supseteq A$ be a maximal split torus. Recall (Cartan decomposition) that there is a compact subgroup $M < G$ such that $G = MSM$. (We refer to [M5, Theorem I.2.2.1], recalling that we placed ourselves in the case of simply connected algebraic groups; in fact, there is even a "positive semigroup" $S^+ \subseteq S$ such that $G = MS^+M$, but we shall not need this.)

At this point, we observe that if any group G of isometries of any metric space can be written as a product of finitely many subgroups with bounded orbits, then G itself has bounded orbits. Since $S \supseteq A$ is commutative, it fixes the unique A -fixed point. In particular, S has bounded orbits. Since M is compact, it has bounded orbits. In conclusion, it follows that $G = MSM$ has bounded orbits and hence a fixed point (Lemma 2.14), concluding the proof of Theorem B.

6. Minimal sets

Let B be a strictly convex reflexive Banach space and G a group acting on B by affine isometries. Consider the ordered category \mathcal{C} of non-empty closed convex G -invariant subsets of B endowed with G -equivariant isometric maps and inclusion order. The goal of this section is to study minimal elements of \mathcal{C} (regardless of whether they exist). In §7 we shall prove their existence, under conditions (see Corollary 7.5).

The Mazur–Ulam theorem states that a surjective isometry between (real) Banach spaces is affine. It is not known (and probably not true under no further assumptions) whether the analogue of the Mazur–Ulam theorem holds in the general context of convex subsets of Banach spaces. However, for subsets of strictly convex spaces it is obviously true.

LEMMA 6.1. *Let $C \subseteq B$ be a convex subset. Then every isometric map $C \rightarrow B$ is affine.*

Proof. It is enough to show that for all $x, y \in C$ and for every $0 < t < 1$ the point $p = tx + (1-t)y$ is determined metrically. This is true since, by strict convexity,

$$\bar{B}(x, (1-t)\|x-y\|) \cap \bar{B}(y, t\|x-y\|) = \{p\}. \quad \square$$

In particular the morphisms of \mathcal{C} are affine. Another useful geometric property of closed convex sets in B is the existence of a nearest point projection.

LEMMA 6.2. *Let C be a non-empty closed convex subset of B . Then for every $x \in B$ there exists a unique point $\pi_C(x) \in C$ such that $\|x - \pi_C(x)\| = d(x, C)$.*

Proof. The uniqueness follows from strict convexity. By the Hahn–Banach theorem, C is weakly closed since it is closed and convex; therefore, by reflexivity and the Banach–Alaoglu theorem, we have a nested family $C \cap \bar{B}(x, d)$ of weakly compact sets as $d \searrow d(x, C)$; its intersection yields existence. \square

The map $\pi_C: B \rightarrow C$ is called the *nearest point projection* on C . We remark that it is not continuous in general. It is continuous for uniformly convex Banach spaces and non-expanding for Hilbert spaces. Still, the distance between a point and its projection is always a 1-Lipschitz function.

LEMMA 6.3. *Let C be a non-empty closed convex subset of B . Then the function*

$$x \mapsto \|\pi_C(x) - x\|$$

from B to \mathbf{R} is 1-Lipschitz.

Proof. For any $x, y \in B$,

$$\|\pi_C(x) - x\| \leq \|\pi_C(y) - x\| \leq \|\pi_C(y) - y\| + \|y - x\|. \quad \square$$

LEMMA 6.4. *If $C \in \mathcal{C}$ is a minimal element, then any convex G -invariant continuous (or lower semi-continuous) function $\varphi: C \rightarrow \mathbf{R}$ is constant.*

Proof. If φ were to assume two distinct values $s < t$, then $\varphi^{-1}((-\infty, s])$ would be a strictly smaller element of \mathcal{C} . \square

LEMMA 6.5. *Let $C, C' \in \mathcal{C}$ with C minimal. Then the nearest point projection*

$$\pi = \pi_{C'}|_C: C \rightarrow C'$$

is affine.

Proof. For every $x, y \in C$ and $t \in [0, 1]$, the definition of π implies that

$$\begin{aligned} \|\pi(tx + (1-t)y) - (tx + (1-t)y)\| &\leq \|(t\pi(x) + (1-t)\pi(y)) - (tx + (1-t)y)\| \\ &\leq t\|\pi(x) - x\| + (1-t)\|\pi(y) - y\|. \end{aligned} \quad (6.i)$$

It follows that the function $C \rightarrow \mathbf{R}$, $x \mapsto \|\pi(x) - x\|$, is convex. Clearly it is G -invariant, and by Lemma 6.3 it is continuous, hence Lemma 6.4 implies that $\|\pi(x) - x\|$

is constant on C . This constant must be $d(C, C')$; as both the right-hand side and the left-hand side in (6.i) equal $d(C, C')$, it follows that

$$\|(t\pi(x) + (1-t)\pi(y)) - (tx + (1-t)y)\| = \|\pi(tx + (1-t)y) - (tx + (1-t)y)\|.$$

Therefore, by the uniqueness part of Lemma 6.2, we must have

$$t\pi(x) + (1-t)\pi(y) = \pi(tx + (1-t)y). \quad \square$$

LEMMA 6.6. *If $C \in \mathcal{C}$ is minimal and $T: C \rightarrow B$ is a G -equivariant affine map, then there exists a $\varrho(G)$ -invariant vector $b \in B$ such that $T(c) = c + b$ for all $c \in C$.*

Proof. The map $C \rightarrow \mathbf{R}$, $x \mapsto \|Tx - x\|$, is G -invariant, continuous and convex, hence by Lemma 6.4 it has a constant value $d \geq 0$. Since B is strictly convex and C is convex, the affine map $\sigma(x) = Tx - x$ from C to the sphere of radius d in B must be constant. Its value $b = \sigma(C)$ is the desired ($\varrho(G)$ -invariant) translation vector. \square

COROLLARY 6.7. *The map $\pi_C: C \rightarrow C'$ from Lemma 6.5 is in fact a translation.*

COROLLARY 6.8. *If $C, C' \in \mathcal{C}$ are minimal, then they are equivariantly isometric. Moreover, any equivariant isometry $C \rightarrow C'$ is a translation by a $\varrho(G)$ -invariant vector.*

Proof. By Corollary 6.7, $\pi_{C'}|_C: C \rightarrow C'$ is an isometry; it is G -equivariant and hence onto, by the minimality of C' . The second claim follows from Lemmas 6.1 and 6.6. \square

7. Actions of product groups and splitting

7.a. The main goal of this section is to prove Theorem C. By Proposition 2.13, we may assume the affine action to be isometric with respect to a ucus norm on a Banach space B . The main step is the following theorem.

THEOREM 7.1. *Let $G = G_1 \times G_2$ be a product of topological groups with a continuous action by affine isometries on a uniformly convex Banach space B without G -fixed point. Assume that the associated linear G -representation ϱ does not almost have non-zero invariant vectors. Then there exists a non-zero $\varrho(G_i)$ -invariant vector for some $i \in \{1, 2\}$.*

The proof of Theorem 7.1 uses minimal sets (in analogy to [Mo2]); notice that we are in the setting of §6, since uniformly convex spaces are reflexive and strictly convex [BL, Appendix A]. More precisely, we show the following result.

PROPOSITION 7.2. *Let G and B be as above. Then there exists a minimal non-empty closed convex G_1 -invariant subset in B . In fact, any non-empty closed convex G_1 -invariant subset contains such a minimal subset.*

Proof of Theorem 7.1. Proposition 7.2 provides a minimal non-empty closed convex G_1 -invariant set $C \subseteq B$. If there is no non-zero $\varrho(G_1)$ -invariant vector, Lemma 6.6 (applied to G_1) shows that G_2 fixes every point of C . Since G_1 preserves C and G has no fixed point, C cannot consist of a single point. Picking two distinct points $x, y \in C$ yields the non-zero $\varrho(G_2)$ -invariant vector $x - y$. \square

Recall that uniform convexity is characterized by the positivity δ defined in §2.a. Moreover, δ is a positive, non-decreasing function which tends to zero at zero. Defining

$$\delta^{-1}(t) = \sup\{\varepsilon : \delta(\varepsilon) \leq t\},$$

δ^{-1} is easily seen to share the same properties. Furthermore, for every $\varepsilon > 0$, $\delta^{-1} \circ \delta(\varepsilon) \geq \varepsilon$.

Proof of Proposition 7.2. Let $C_0 \subseteq B$ be any non-empty closed convex G_1 -invariant subset; we will show that C_0 contains a minimal subset (if no initial C_0 was prescribed, one may choose $C_0 = B$).

Pick any $p \in C_0$ and let $C_1 \subseteq C_0$ be the closed convex hull of the G_1 -orbit of p . By Hausdorff's maximal principle, we can choose a maximal chain \mathcal{D} of non-empty closed convex G_1 -invariant subsets of C_1 . If $b_C := \pi_C(0)$ is bounded as C ranges over \mathcal{D} , then for some $R > 0$ we have a nested family of non-empty sets $\bar{B}(0, R) \cap C$ which are weakly compact by reflexivity, the Hahn–Banach theorem and the Banach–Alaoglu theorem. In particular the intersection $\bigcap \mathcal{D}$ is non-empty, thus providing a minimal set for G_1 . Therefore, we may from now on assume, by contradiction, that the (non-decreasing) net $R_C := \|b_C\|$ is unbounded over $C \in \mathcal{D}$. Let $\mathcal{D}' \subseteq \mathcal{D}$ be the cofinal segment defined by $R_C > 0$. We will obtain a contradiction by showing that for every compact $K \subseteq G$, $\text{diam}(\varrho(K)\hat{b}_C)$ tends to zero along $C \in \mathcal{D}'$, where $\hat{b}_C = b_C/R_C$.

Indeed, choose $K_i \subseteq G_i$ compact with $K \subseteq K_1 \times K_2$ and let $L = \max_{g \in K_1 \times K_2} \|g \cdot 0\|$. The choice of b_C implies that $g \cdot b_C \neq 0$ and $R_C \leq \|(b_C + g \cdot b_C)/2\|$ for all $g \in G$. Therefore, setting $x = b_C/\|g \cdot b_C\|$ and $y = g \cdot b_C/\|g \cdot b_C\|$, the convexity modulus $\delta_{C,g} := \delta(\|x - y\|)$ gives

$$\begin{aligned} R_C &\leq \left\| \frac{b_C + g \cdot b_C}{2} \right\| \leq \left\| \frac{x + y}{2} \right\| \|g \cdot b_C\| \leq (1 - \delta_{C,g})(\|g \cdot b_C - g \cdot 0\| + \|g \cdot 0\|) \\ &\leq (1 - \delta_{C,g})(R_C + L) \leq R_C \left(1 + \frac{L}{R_C} - \delta_{C,g} \right) \end{aligned}$$

for all $g \in K_1$. Therefore $\delta_C := \sup_{g \in K_1} \delta_{C,g} \leq L/R_C \rightarrow 0$ along $C \in \mathcal{D}'$, and hence

$$\sup_{g \in K_1} \frac{\|g \cdot b_C - b_C\|}{\|g \cdot b_C\|} \leq \delta^{-1}(\delta_C) \rightarrow 0.$$

Using $\|g \cdot b_C\| \leq \|g \cdot b_C - g \cdot 0\| + L \leq R_C + L$, it follows that

$$\sup_{g \in K_1} \frac{\|g \cdot b_C - b_C\|}{R_C} \rightarrow 0 \quad \text{along } C \in \mathcal{D}'. \quad (7.i)$$

On the other hand, for every $g \in G_2$, the function $z \mapsto \|g \cdot z - z\|$ is continuous, convex and G_1 -invariant; therefore, it is bounded by $\|g \cdot p - p\|$ on C_1 . Setting $L' = \max_{g \in K_2} \|g \cdot p - p\|$, it follows now that for all $k = (g_1, g_2) \in K$ we have

$$\begin{aligned} R_C \|\varrho(k) \cdot \hat{b}_C - \hat{b}_C\| &= \|k \cdot b_C - b_C - k \cdot 0\| \leq \|g_1 \cdot b_C - b_C\| + \|g_2 g_1 \cdot b_C - g_1 \cdot b_C\| + L \\ &\leq \|g_1 \cdot b_C - b_C\| + L' + L. \end{aligned}$$

Thus, in view of (7.i), $\text{diam}(\varrho(K) \hat{b}_C)$ goes to zero, as claimed. \square

Proof of Theorem C. We adopt the notation and the assumptions of the theorem; let ϱ be the linear part of the action. Assume first that $n=2$. Since we have in particular $B^{e(G)}=0$, Corollary 2.9 yields a canonical splitting $B = B^{e(G_1)} \oplus B^{e(G_2)} \oplus B_0$ invariant under $\varrho(G)$. Decomposing the cocycle $G \rightarrow B$ along this splitting shows that up to affine isometry we may assume that the affine G -space B splits likewise as affine product of affine spaces with corresponding linear parts. However, Theorem 7.1 shows that the resulting affine G -action on B_0 must have a fixed point, since $B_0^{e(G_i)}=0$. Therefore, we obtain a G -invariant affine subspace G -isometric to $B^{e(G_1)} \oplus B^{e(G_2)}$ in B , as claimed.

In order to obtain the general case $n \geq 2$, we only need to observe that Corollary 2.9 applied to the product $G_1 \times \prod_{i \geq 2} G_i$ allows us to apply induction on n . \square

Remark 7.3. The above proof characterizes the subspaces $B_i \subseteq B$ appearing in the statement of Theorem C as follows: upon possibly replacing the B_i 's with the corresponding linear subspaces (which corresponds to replacing the cocycles with cohomologous cocycles), we have $B_i = B^{e(G_i)}$ for $G_i' = \prod_{j \neq i} G_j$.

7.b. A more geometric approach to Theorem B and a step towards

Conjecture 1.6

Before continuing towards the superrigidity theorem, let us explain a more geometric, and seemingly more general, approach to prove $(T_B) \Rightarrow (F_B)$, which is based on minimal sets. First we shall formulate a very general statement in the vein of Conjecture 1.6.

THEOREM 7.4. *Let B be a ucus Banach space and G be a topological group with property (T_B) and compact Abelianization. Then for any continuous affine isometric action of G on B there is a minimal non-empty closed convex subset $C \subseteq B$. Moreover, $\text{Aut}_G(C)$ is trivial, $C \subseteq B'$ and C is unique, up to translations by a $\varrho(G)$ -invariant vector.*

The proof of Theorem 7.4 relies on the following consequence of our discussion of minimal sets.

COROLLARY 7.5. *Let G be a topological group with a continuous action by affine isometries on a uniformly convex Banach space B . Assume that the associated linear representation does not almost have non-zero invariant vectors. Then there exists a unique minimal non-empty closed convex G -invariant subset $C_0 \subseteq B$. Moreover, there are no non-trivial G -equivariant isometries of C_0 .*

Remark 7.6. In view of the additional statement of Proposition 7.2, the set C_0 is contained in every non-empty closed convex G -invariant subset. Thus it is indeed the (non-empty) intersection of all those subsets.

Proof. For the existence of C_0 , we may apply Proposition 7.2 if $G = G_1 \times 1$ has no fixed point, or otherwise take such a fixed point for C_0 . Both uniqueness and the additional statement follow now from Corollary 6.8. \square

Proof of Theorem 7.4. Since G has compact Abelianization, the $\varrho(G)$ -invariant subspace B' is in fact G -invariant as an affine space, as the projection of the cocycle to $B^{\varrho(G)}$ must be a homomorphism. It follows that every minimal non-empty closed convex G -invariant set is contained in some coset of B' . The existence and uniqueness of such a subset C inside B' follows from Corollary 7.5. The fact that any two such sets are different by a $\varrho(G)$ -invariant vector is a consequence of Corollary 6.8. \square

Let us now describe an alternative proof for Theorem B. Let B be an L^p -related Banach space as in Theorem B. We reduce to the case where G is (the k -points of a) connected, simply-connected and almost-simple as in §5. Now G either contains a copy of $\mathrm{SL}_3(k)$ or a copy of $\mathrm{Sp}(4, k)$ which, in each case, contains a semidirect product $H \ltimes U$ with the strong relative property (T_B) (see Lemma 5.4 and the proof of Claim 5.5). We decompose $B = B^{\varrho(U)} \oplus B'$ according to that U -action; note that, by the Howe–Moore theorem, $B^{\varrho(U)} = B^{\varrho(G)}$. Then B' is invariant under the affine action of G , and H does not almost have invariant vectors in B' . Hence, by Corollary 7.5 there is a unique minimal non-empty closed convex H -invariant subset $C \subseteq B'$ and it has no non-trivial automorphisms which commute with the H -action. Since, by Claim 5.5, the centralizer of H is non-compact, it follows by the Howe–Moore theorem that C is reduced to a point. Now one finishes the proof as in §5 using a Cartan decomposition.

8. Induction and superrigidity

Let $\Gamma \subset G = G_1 \times \dots \times G_n$ be a lattice in a product of $n \geq 2$ locally compact groups. Under an irreducibility assumption, the splitting theorem (Theorem C) implies a superrigidity result for uniformly equicontinuous affine Γ -actions on superreflexive spaces B . As before, such an action can be viewed as an affine isometric Γ -action on a ucus Banach space B .

It therefore suffices to apply the splitting theorem to the *induced* G -action on an *induced space* $L^p(G/\Gamma, B)$ (compare [Sh] for the Hilbertian case).

The goal of this section is to address the various (mostly technical) issues that arise when carrying out this programme. We begin by preparing for a statement (Theorem 8.3 below) that will then imply a more general form of Theorem D.

8.a. Let G be a locally compact group and $\Gamma \subset G$ a lattice. The induction procedure will work smoothly if Γ is uniform (i.e. cocompact); in order to treat some non-uniform cases, one introduces the following definition.

Definition 8.1. ([M5, §III.1.8]) The lattice Γ is *weakly cocompact* if the G -representation $L_0^2(G/\Gamma)$, i.e. the canonical complement of the trivial representation in $L^2(G/\Gamma)$, does not almost have non-zero invariant vectors.

One verifies that any cocompact lattice is weakly cocompact. If G has property (T), then all its lattices are weakly cocompact. This also holds if G is any (topologically) connected semisimple Lie group ([Bk], compare also [M5, Remark III.1.12]). By Remark 4.3, this definition does not depend on considering $L^2(G/\Gamma)$ rather than $L^p(G/\Gamma)$ for some other $1 \leq p < \infty$.

Definition 8.2. (See [Sh, §1.II]) Let $p > 0$. The lattice Γ is *p-integrable* if either (i) it is uniform, or (ii) it is finitely generated and for some (or equivalently any) finite generating set $S \subseteq \Gamma$, there is a Borel fundamental domain $\mathcal{D} \subseteq G$ (with null boundary) such that

$$\int_{\mathcal{D}} \|\chi(g^{-1}h)\|_S^p dh < \infty \quad \text{for all } g \in G,$$

where $\|\cdot\|_S$ is the word-length associated to S and $\chi: G \rightarrow \Gamma$ is defined by

$$\chi^{-1}(e) = \mathcal{D} \quad \text{and} \quad \chi(g\gamma^{-1}) = \gamma\chi(g).$$

This formulation is a bit awkward so as to include all uniform lattices, since (ii) would otherwise fail when G is not compactly generated. Condition (ii) holds (with any $p \geq 1$) for all irreducible lattices in higher-rank semisimple Lie/algebraic groups, see [Sh, §2]; it holds likewise for Rémy's Kac–Moody lattices [Ry].

Finally, given a product structure $G = G_1 \times \dots \times G_n$, we say that a lattice $\Gamma \subset G$ is *irreducible* if its projection to each G_i is dense.

THEOREM 8.3. *Let Γ be an irreducible lattice in a locally compact σ -compact group $G = G_1 \times \dots \times G_n$. Assume that Γ is weakly cocompact and p -integrable for some $p > 1$. Let B be a ucus Banach space with a Γ -action by affine isometries.*

If the associated linear Γ -representation does not almost have invariant vectors, then there is a Γ -closed complemented affine subspace of B on which the Γ -action is a sum of actions extending continuously to G and factoring through $G \rightarrow G_i$. (Compare Remark 1.9.)

Theorem 8.3 indeed implies Theorem D in the wider generality of weakly cocompact p -integrable lattices, since Proposition 2.13 allows us to assume that the topological vector space of Theorem D is in fact a ucus Banach space with a Γ -action by affine isometries.

A (simpler) application of the same techniques implies the following result.

THEOREM 8.4. *Let Γ be an irreducible lattice in a locally compact σ -compact group $G = G_1 \times \dots \times G_n$. Assume that Γ is weakly cocompact and p -integrable for some $p > 1$.*

Then any homomorphism $\Gamma \rightarrow \mathbf{R}$ extends continuously to G .

This result was established by Shalom in the case of cocompact lattices [Sh, Theorem 0.8] (actually, his proof holds in the setting of square-integrable lattices). It is therefore not surprising that our results imply the generalisation stated in Theorem 8.4 above (see the end of this section).

8.b. Induction

Throughout this section, G is a locally compact second countable group and $\Gamma \subset G$ is a lattice. In particular, the Haar measure induces a standard Lebesgue space structure on G/Γ .

Remark 8.5. Even though Theorem 8.3 and Theorem D were stated in the more general setting of σ -compact groups, it is indeed enough to treat the second countable case: one can reduce to the latter by a structural result of Kakutani–Kodaira [KK] (the details of the straightforward reduction are expounded at length in [Mo2]).

Let B be any Banach space and $1 < p < \infty$. We consider the Banach space $E = L^p(G/\Gamma, B)$, as in §2.e.

LEMMA 8.6. *If B is uniformly convex or ucus, then so is E .*

Proof. This follows from a result of Figiel and Pisier; see Theorem 1.e.9, point (i) in [LT2, Vol. II]. \square

Suppose now that B is endowed with a linear isometric Γ -representation ϱ . Then E can be canonically isometrically identified with the space of those Bochner-measurable Γ -equivariant function classes $f: G \rightarrow B$ such that $\|f\|_B: G/\Gamma \rightarrow \mathbf{R}$ is p -integrable (the latter

condition is symbolized by the notation $L^{[p]}$):

$$E \cong L^{[p]}(G, B)^{\varrho(\Gamma)}. \quad (8.i)$$

Here, we choose to interpret Γ -equivariance as $f(g\gamma) = \varrho(\gamma)^{-1}f(g)$. The isomorphism (8.i) can be e.g. realized by restricting equivariant maps to any Borel fundamental domain $\mathcal{D} \subseteq G$ for Γ , since $\mathcal{D} \cong G/\Gamma$ as Lebesgue spaces. This identification allows us to endow E with a continuous linear isometric G -representation by left multiplication. This G -representation is called the *induced representation*. If we choose a fundamental domain $\mathcal{D} \subseteq G$ and consider the corresponding map χ as in Definition 8.2, then this G -representation reads as follows for $f \in E = L^p(G/\Gamma, B)$:

$$(hf)(g\Gamma) = \varrho(\chi(g)^{-1}\chi(h^{-1}g))f(h^{-1}g\chi(h^{-1}g)\Gamma) \quad (8.ii)$$

(a good indication that the model (8.i) is more natural!).

LEMMA 8.7. *Assume Γ to be weakly cocompact in G . If the linear Γ -representation does not almost have invariant vectors, then the induced linear G -representation does not either.*

Proof. The proof given by Margulis in the unitary case [M5, Proposition III.1.11] holds without changes (recalling that we can apply weak cocompactness in the L^p setting by Remark 4.3). \square

Suppose now that B is endowed with an isometric Γ -action—not necessarily linear anymore. We want to endow E with a continuous affine isometric G -action by identifying E with a space of Γ -equivariant function classes $G \rightarrow B$ as before, except that equivariance is now understood with respect to the affine Γ -action. Formally, there is nothing to change to the special case of linear action considered above; the action is defined by left G -translation of equivariant maps, so that, via the natural identification, we get for $f \in E = L^p(G/\Gamma, B)$ the action

$$(hf)(g\Gamma) = \chi(g)^{-1}\chi(h^{-1}g)f(h^{-1}g\chi(h^{-1}g)\Gamma), \quad (8.iii)$$

in complete analogy with (8.ii). However, the L^p integrability property might be lost. The condition (ii) of Definition 8.2 is a straightforward sufficient condition to retain integrability; cocompactness of Γ is also enough, because it ensures that one can choose \mathcal{D} in such a way that for any compact $C \subseteq G$ the set $\{\eta \in \Gamma : \mathcal{D}\eta \cap C \neq \emptyset\}$ is finite [B2, §VII.2, Example 12]. Compare [Sh, §2] (and [Mo2, Appendix B]).

In conclusion, we may always consider the continuous *induced (affine) isometric G -action* on E when Γ is p -integrable.

By construction, the linear part of the induced affine action coincides with the induced linear G -representation on E considered earlier. If we denote by $b: \Gamma \rightarrow B$ the cocycle of the original affine Γ -action, then comparing (8.ii) with (8.iii) shows that the cocycle $\tilde{b}: G \rightarrow E$ of the induced affine action is given by

$$\tilde{b}(h)(g\Gamma) = b(\chi(g)^{-1}\chi(h^{-1}g)). \quad (8.iv)$$

Moreover, the correspondence $b \mapsto \tilde{b}$ induces a (topological) isomorphism

$$H^1(\Gamma, B) \longrightarrow H^1(G, E).$$

At this point, we record the following result.

PROPOSITION 8.8. *Keep the notation of this section.*

- (1) *If Γ has property (F_B) , then so does G .*
- (2) *If G has property (F_E) and Γ is p -integrable, then Γ has property (F_B) .*

Proof. For (1), consider any continuous affine isometric G -action on B ; then there is a Γ -fixed point $b \in B$. The corresponding orbit map $G \rightarrow B$ descends to a continuous map $G/\Gamma \rightarrow B$. The image of the normalized invariant measure on G/Γ in B being preserved by G , it follows from Lemma 2.14 that there is a G -fixed point.

For (2), consider an affine isometric Γ -action on B and endow E with the induced affine action as in the discussion above. Then there is a G -fixed point $f \in E$. It follows from the description of E as space of equivariant maps that f is essentially constant and that its essential value is a Γ -fixed point of B . \square

8.c. Superrigidity

In order to prove Theorem 8.3, we now analyse the interplay between the induction constructions and the setting of irreducible lattices

$$\Gamma \subset G = G_1 \times \dots \times G_n,$$

as in the beginning of this section. We will roughly imitate the arguments given by Shalom when he deduces Corollary 4.2 in [Sh].

Keep all the above notation and write $G'_i = \prod_{j \neq i} G_j$. First we observe that the irreducibility of Γ implies that for each i it is a well-posed definition to consider the maximal (possibly zero) linear subspace $B_i \subseteq B$ on which the linear Γ -representation ϱ extends to a continuous G -representation $\varrho_i: G \rightarrow G_i \rightarrow \mathcal{O}(B_i)$ factoring through G_i ; moreover B_i is automatically closed by maximality.

The induced space E is ucus by Lemma 8.6. The isometric (affine) G -action on E has no fixed point, by the very same argument given to prove Proposition 8.8(2). On the other hand, the linear part does not have almost invariant vectors, by Lemma 8.7. Thus Theorem C applies: there is a G -invariant closed complemented affine subspace $\underline{E} \subseteq E$ and an affine isometric G -equivariant isomorphism $\underline{E} \cong E_1 \oplus \dots \oplus E_n$, where each E_i is a ucus space with an affine isometric G -action factoring through $G \rightarrow G_i$. In view of Remark 7.3, there is no loss of generality in assuming that E_i is the space of G'_i -fixed points under the induced linear representation. One verifies readily the following lemma.

LEMMA 8.9. *The map $B_i \rightarrow E \cong L^{[p]}(G, B)^{e(\Gamma)}$, which with $v \in B_i$ associates the function $G \rightarrow B$ defined by $g \mapsto \rho_i(g^{-1})v$, yields an isometric isomorphism of (linear) G -spaces $B_i \cong E_i$.*

Indeed, since the image of Γ in G_i is dense, the Fubini–Lebesgue theorem implies that any map $f: G \rightarrow B$ in E that is G'_i -invariant in the linear representation on E is an orbit map, as in the lemma.

At this point we observe that if the subspaces B_i had trivial intersection, we would indeed have found a subspace $\bigoplus_i B_i \cong \bigoplus_i E_i$ of B on which the affine Γ -action extends continuously to G as requested. In general, we have a Γ -equivariant affine map

$$\bigoplus_i E_i \longrightarrow \sum_i B_i \subseteq B$$

induced by the maps of Lemma 8.9. Alternatively, we can think of this map as follows. The cocycle induced as in (8.iv) decomposes as a sum of cocycles

$$\tilde{b} = \bigoplus_i \tilde{b}_i: G \longrightarrow E, \quad \tilde{b}_i: G \longrightarrow G_i \longrightarrow E_i,$$

and in turn, by Lemma 8.9, each \tilde{b}_i is the cocycle induced under the correspondence (8.iv) from a cocycle $b_i: \Gamma \rightarrow B_i$; the affine Γ -action on $\sum_i B_i$ is determined by the cocycle $\sum_i b_i$. This completes the proof of Theorem 8.3.

Remark 8.10. As mentioned in Remark 1.9, the obstruction to extending the affine Γ -action on some subspace of B is confined within a compact group. Indeed, the only reason we might end up with a *sum* of actions extending to G through various G_i , rather than with a direct sum (which then extends globally to G), is the possibility that $B_i \cap B_j \neq 0$ for some $i \neq j$. But then the linear representation of Γ on $B_i \cap B_j$ extends continuously to G in two different ways, *both* through G_i and through G_j . This may indeed happen but forces the image of Γ in $O(B_i \cap B_j)$ to be compact; see examples and discussion in [Mo2].

Let us only mention the most basic example: $\Gamma \subset G = G_1 \times G_2$ with $G_i = \mathbf{Z} \rtimes \{\pm 1\}$ and $\Gamma = \mathbf{Z}^2 \rtimes \{\pm 1\}$. Then Γ acts affinely isometrically without fixed point on $B = \mathbf{R}$ (by $(n, m; \varepsilon) \cdot x = \varepsilon x + n + m$) and the associated linear representation does not almost have invariant vectors. However, it is easy to check that this action does not extend to G . Instead, it is a sum of actions extending to G_i with sum map $\mathbf{R} \oplus \mathbf{R} \rightarrow B = \mathbf{R}$. Here $B_1 = B_2 = B$.

Proof of Theorem 8.4. Recall that the space of homomorphisms $\Gamma \rightarrow \mathbf{R}$ is precisely the space of affine isometric Γ -actions on \mathbf{R} with the trivial representation as linear part. By Remark 4.3, the G -representation on $L_0^p(G/\Gamma)$ does not almost have invariant vectors. Therefore, using p -integrable induction, one deduces Theorem 8.4 from Theorem C very exactly as Shalom deduced [Sh, Theorem 0.8] from [Sh, Theorem 3.1]. \square

Appendix 9. The Howe–Moore theorem on Banach spaces

In this appendix we sketch the proof of a version of the well known Howe–Moore theorem on vanishing of matrix coefficients for unitary representations, extended to the framework of ucus Banach spaces. This generalization is due to Shalom (unpublished) and we state it here with a sketch of the proof for the reader’s convenience.

THEOREM 9.1. *Let I be a finite set, $k_i, i \in I$, be local fields, \mathbf{G}_i be connected semisimple simply-connected k_i -groups, $G_i = \mathbf{G}_i(k_i)$ be the locally compact group of k_i -points, and*

$$G = \prod_{i \in I} G_i.$$

Let B be a ucus Banach space and $\varrho: G \rightarrow \mathcal{O}(B)$ be a continuous isometric linear representation, such that $B^{\varrho(G_i)} = \{0\}$ for each $i \in I$. Then all matrix coefficients

$$c_{x,\lambda}(g) = \langle \varrho(g)x, \lambda \rangle, \quad x \in B, \lambda \in B^*,$$

vanish at infinity, i.e. $c_{x,\lambda} \in C_0(G)$.

Notice that we can (and will) assume that the \mathbf{G}_i have no k_i -anisotropic factors, since the group of k_i -points of such factors are compact.

Proof. By contradiction, assume that for some $g_n \rightarrow \infty$ in G , $v \in S(B)$ and $\lambda \in S(B^*)$ one has

$$\inf |\langle \varrho(g_n)x, \lambda \rangle| = \varepsilon > 0.$$

We shall prove that at least one simple factor G_i of G has a non-trivial $\varrho(G_i)$ invariant vector.

Let $G=KAK$ be a Cartan decomposition of G (here $K=\prod_i K_i$ and $A=\prod_i A_i$, where $G_i=K_iA_iK_i$ is the Cartan decomposition for G_i). We first show that without loss of generality one may assume that $g_n \in A$.

LEMMA 9.2. (KAK reduction) *There exists a sequence $a_n \rightarrow \infty$ in the Cartan subgroup $A \subseteq G$ and non-zero vectors $y, z \in B$ so that*

$$\varrho(a_n)y \xrightarrow{w} z \neq 0,$$

where \xrightarrow{w} denotes the weak convergence.

Proof. Write $g_n = k_n a_n k'_n$, where $k_n, k'_n \in K$ and $a_n \in A$. Then $a_n \rightarrow \infty$ since $g_n \rightarrow \infty$. Upon passing to a subsequence, $k'_n \rightarrow k' \in K$ and $k_n \rightarrow k \in K$. Let

$$y_n = \varrho(k'_n)x, \quad y = \varrho(k')x, \quad \mu_n = \varrho^*(k_n^{-1})\lambda \quad \text{and} \quad \mu = \varrho^*(k^{-1})\lambda,$$

where ϱ^* is the dual (contragradient) G -representation on B^* . Using the weak-compactness of the unit ball of B , we may also assume that

$$\varrho(a_n)y \xrightarrow{w} z.$$

We shall show that $\langle z, \mu \rangle = \lim_{n \rightarrow \infty} \langle \varrho(g_n)x, \lambda \rangle$ which is bounded away from zero, hence implying that $z \neq 0$.

Recall that in a uniformly convex Banach space B the weak and the strong topologies agree on the unit sphere $S(B)$: indeed if $y_n \xrightarrow{w} y$ are unit vectors, then

$$1 - \delta(\|y_n - y\|) \geq \frac{\|y_n + y\|}{2} \geq \left\langle \frac{y_n + y}{2}, y^* \right\rangle \rightarrow 1.$$

Hence $\delta(\|y_n - y\|) \rightarrow 0$ and $\|y_n - y\| \rightarrow 0$. For the same reason, we also have $\|\mu_n - \mu\| \rightarrow 0$ in $S(B^*)$. For an arbitrary $\xi \in B^*$,

$$|\langle \varrho(a_n)y_n, \xi \rangle - \langle \varrho(a_n)y, \xi \rangle| \leq \|y_n - y\| \|\xi\| \rightarrow 0.$$

Hence $\varrho(a_n)y_n \xrightarrow{w} z$. In general, if $z_n \xrightarrow{w} z$ in B and $\mu_n \xrightarrow{w} \mu$ in B^* then $\langle z_n, \mu_n \rangle \rightarrow \langle z, \mu \rangle$, because weakly convergent sequences are bounded in norm and

$$|\langle z_n, \mu_n \rangle - \langle z, \mu \rangle| \leq |\langle z_n, \mu_n - \mu \rangle| + |\langle z_n - z, \mu \rangle| \leq \left(\sup_n \|z_n\| \right) \|\mu_n - \mu\|^* + |\langle z_n - z, \mu \rangle| \rightarrow 0.$$

Therefore,

$$\langle \varrho(g_n)x, \lambda \rangle = \langle \varrho(a_n k'_n)x, \varrho^*(k_n^{-1})\lambda \rangle = \langle \varrho(a_n)y_n, \mu_n \rangle \rightarrow \langle z, \mu \rangle,$$

implying that $|\langle z, \mu \rangle| \geq \varepsilon$, which in particular means that $z \neq 0$. □

LEMMA 9.3. (Generalized Mautner lemma) *Suppose that $\{a_n\}_{n=1}^\infty$ and h in G satisfy $a_n^{-1}ha_n \rightarrow 1_G$ in G . If $y, z \in B$ are such that $\varrho(a_n)y \xrightarrow{w} z$ then $\varrho(h)z = z$. In particular, if $\varrho(a_n)z = z$ then $\varrho(h)z = z$.*

Proof. (Strong) continuity of ϱ gives

$$\|\varrho(ha_n)y - \varrho(a_n)y\| = \|\varrho(a_n^{-1}ha_n)y - y\| \rightarrow 0.$$

At the same time $\varrho(a_n)y \xrightarrow{w} z$ and $\varrho(ha_n)y \xrightarrow{w} \varrho(h)z$. Hence $\varrho(h)z = z$. \square

We can now prove Theorem 9.1 in the case of $G = \mathrm{SL}_2(k)$, where k is a local field. Assuming that $\varrho: \mathrm{SL}_2(k) \rightarrow \mathrm{O}(B)$ has some matrix coefficient not vanishing at infinity, we get, by Lemma 9.2, a sequence $a_n \rightarrow \infty$ in A , and *non-zero* vectors $y, z \in B$ with

$$\varrho(a_n)y \xrightarrow{w} z.$$

Let H be the unipotent (horocyclic) subgroup defined by $H = \{h \in G : a_n^{-1}ha_n \rightarrow e\}$. It is normalized by a_n , and, by Lemma 9.3, z is a (non-trivial) $\varrho(H)$ -invariant vector. We may assume that $\|z\| = 1$. The matrix coefficient $f(g) = \langle \varrho(g)z, z^* \rangle$ is a continuous function on G , which is bi- H -invariant:

$$f(gh) = \langle \varrho(g)\varrho(h)z, z^* \rangle = \langle \varrho(g)z, z^* \rangle = f(g), \quad (9.i)$$

$$f(hg) = \langle \varrho(g)z, \varrho^*(h^{-1})z^* \rangle = \langle \varrho(g)z, z^* \rangle = f(g) \quad (9.ii)$$

for all $g \in G$ and $h \in H$. The proof can now be completed as in the original unitary Howe–Moore theorem. By (9.i), f can be viewed as a continuous function f_0 on the punctured plane $G/H = k^2 \setminus \{(0, 0)\}$, and, by (9.ii), f_0 is constant on each horizontal line $\ell_s = \{(t, s) : t \in k\}$, $s \neq 0$, where we identify H with the upper triangular unipotent subgroup, by choosing an appropriate basis for k^2 . By continuity, f_0 is a constant on $\{(t, 0) : t \neq 0\}$. Since $f_0(0, 1) = f(e) = 1$, this constant is 1.

This implies that z is $\varrho(A)$ -invariant, because $\langle \varrho(a)z, z^* \rangle = f(a) = f(e) = 1$ whilst z^* attains its norm only on z .

Thus z is fixed by the upper triangular group $AH \subseteq G$ and f descends to a continuous function f_1 on the projective line $\mathbf{P}(k^2) = G/AH$. The H -action on $\mathbf{P}(k^2)$ has a dense orbit. Thus f_1 is constant 1, and so is f :

$$\langle \varrho(g)z, z^* \rangle = f(g) = f(e) = 1, \quad g \in G.$$

Thus the unit vector z is $\varrho(G)$ -invariant, completing the proof in the case of $G = \mathrm{SL}_2(k)$.

The proof of the unitary Howe–Moore theorem for a semisimple Lie group $G = \prod_i G_i$ (cf. Zimmer [Z] and Margulis [M5]) relies only on the reduction to the Cartan subgroup (Lemma 9.2), the structure of such groups, the case of $\mathrm{SL}_2(k)$ and on the Mautner lemma. Thus the “unitary” argument can be applied almost *verbatim* to the present setup of ucus Banach spaces. \square

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