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On the asymptotic distribution of eigenvalues

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Introduction

Let Ω be the union of a finite number of open, bounded and connected subsets of R^n, Δ the n-dimensional Laplace operator and ϱ a real-valued function defined in Ω. Consider the eigenvalue problems

$$\Delta f + \lambda \varrho f = 0$$

with f or its normal derivative vanishing at the boundary. It has been shown by Courant ([1] p. 321) that, when ϱ is a bounded Riemann integrable function and Ω satisfies a regularity condition, the asymptotic distribution of the eigenvalues is given by

$$N(\lambda) \sim (2\pi)^{-n} w_n \lambda^{n/2} \int_\Omega \varrho^{n/2}, \tag{1}$$

where $N(\lambda)$ stands for the number of eigenvalues smaller than λ, and w_n is the volume of the n-dimensional unit sphere. The object of this note is to show that (1) holds also when ϱ has a finite number of singular points y in Ω. More precisely, we assume that ϱ is $O(|x-y|^{-2\beta})$ in a neighbourhood of y, where $0<\beta<1$ when $n \geq 2$, and $0<\beta<1/2$ when $n=1$. The method adopted can also be used to treat cases when ϱ becomes singular on manifolds of dimension $< n$.

Preliminaries

We shall use the notations

$$(f,g)_O = \int_O f\bar{g}, \quad |f|_O^2 = (f,f)_O, \quad |f|_{O,p} = \left(\int_O |f|^p\right)^{1/p},$$

$$(\nabla f, \nabla g)_O = \int_O \nabla f \overline{\nabla g}, \quad |\nabla f|_O^2 = (\nabla f, \nabla f)_O,$$

where O is an open subset of R^n, the integrals are taken with ordinary Lebesgue measure and ∇f is the gradient of f, taken in the weak (distributional) sense. Whenever it is convenient we shall leave out the index O.

Define $$F_1(O) = (f;\ |\nabla f| + |f| < \infty)$$

and let $F_0(O)$ be the closure in $F_1(O)$ of all continuously differentiable functions with compact support in O. The elements of $F_0(O)$ vanish at the boundary of O at least in a weak sense. With the scalar product $(f, g) + (\nabla f, \nabla g)$ both $F_1(O)$ and $F_0(O)$ are Hilbert spaces.

An open set O is said to be permitted, if it consists of a finite number of bounded and connected subsets,

$$\text{the form } (f, g) \text{ is compact (i.e. completely continuous) in } F_1(O) \tag{2}$$

and
$$|f|_q \text{ is majorized by a constant times } |\nabla f| + |f|, \tag{3}$$

where $0 < q \leqslant 2n/(n-2)$ when $n > 2$, and $0 < q$ when $n = 1, 2$. These two properties hold if the boundary of O is sufficiently smooth (see [2] p. 471 and [3] respectively). In particular, they hold when O is the sum of a finite number of rectangles.

A function $\varrho \geqslant 0$ is said to be permitted in O if

$$\int_{O'} \varrho > 0 \text{ for every component } O' \text{ of } O \tag{4}$$

and
$$\int_O \varrho^{m/2} < \infty, \tag{5}$$

where $m = n$ when $n > 2$, and m is some number > 2 when $n = 1, 2$. Let us put

$$(\varrho f, g) = (\varrho f, g)_O = \int_O \varrho f \bar{g}.$$

Theorem 1. *If ϱ is a permitted function in a permitted set O, then*

$$\textit{the form } (\varrho f, g) \textit{ is compact in } F_1(O) \tag{6}$$

and
$$(\varrho f, f) + |\nabla f|^2 \sim |f|^2 + |\nabla f|^2 \textit{ in } F_1(O), \tag{7}$$

i.e. either side is majorized by a constant times the other for all f in $F_1(O)$.

Proof. By Hölder's inequality, (3) and (5),

$$(\varrho f, f) \leqslant |\varrho|_{m/2} |f|^2_{2m/(m-2)} \leqslant C |\varrho|_{m/2} (|f|^2 + |\nabla f|^2), \tag{8}$$

where C is a constant. Hence $(\varrho f, f)$ is bounded in $F_1(O)$. Put $\varrho_\lambda(x) = \min(\lambda, \varrho(x))$. It follows from (2) that $(\varrho_\lambda f, f)$ is compact in $F_1(O)$, and (8) applied to $\varrho - \varrho_\lambda$ shows that $((\varrho - \varrho_\lambda) f, f)$ tends to zero as $\lambda \to \infty$, uniformly on bounded sets in $F_1(O)$. From this (6) follows.

At the same time (8) proves that, with a suitable constant C,

$$(\varrho f, f) + |\nabla f|^2 \leqslant C(|f|^2 + |\nabla f|^2) \quad \text{in } F_1(O).$$

To prove the reverse inequality, it suffices to show that there is no sequence $(f_j)_1^\infty$ such that

$$|\nabla f_j|^2 + |f_j|^2 = 1 \quad \text{and} \quad |\nabla f_j|^2 + (\varrho f_j, f_j) \to 0.$$

It is no restriction to assume that the sequence is weakly convergent to an element f in $F_1(O)$. Now

$$|f_j|^2 \to |f|^2 \quad \text{and} \quad (\varrho f_j, f_j) \to (\varrho f, f)$$

since the forms are compact. In particular $(\varrho f, f) = 0$, $|\nabla f_j| \to 0$ and $|f| = 1$. We also have

$$(\nabla f_j, \nabla f) + (f_j, f) \to |\nabla f|^2 + |f|^2$$

since the sequence is weakly convergent. Here

$$|(\nabla f_j, \nabla f)| \leqslant |\nabla f_j| \, |\nabla f|$$

tends to zero and (f_j, f) tends to $|f|^2$. Consequently

$$|f| = 1 \quad \text{and} \quad |\nabla f|^2 + (\varrho f, f) = 0.$$

But by (4), the last relation implies that $f = 0$, which is a contradiction. The proof is complete.

When ϱ is permitted in a permitted set O, we can use

$$((f, g)) = (\nabla f, \nabla g) + (\varrho f, g)$$

as a scalar product in $F(O) = F_0(O)$ or $F_1(O)$. Then, there is a compact, self-adjoint and linear transformation G defined in $F(O)$ such that

$$(\varrho f, g) = ((Gf, g))$$

for all f and g in $F(O)$.

From a theorem of Hilbert we have that

(*a*) $F(O)$ has an orthonormal basis, consisting of eigenfunctions of G;

(*b*) every eigenvalue μ of G is positive and every $\mu \neq 0$ has finite multiplicity; the eigenvalues are enumerable and 0 is the only possible limit point.

If $G\varphi = \mu\varphi$, it follows that

$$(\varrho\varphi, g) = \mu(\nabla\varphi, \nabla g) + \mu(\varrho\varphi, g) \quad \text{when } g \in F(O),$$

and from this by Green's formula that

$$\Delta\varphi + \lambda\varrho\varphi = 0 \tag{9}$$

with

$$\lambda = (1 - \mu)/\mu, \tag{10}$$

where Δ is the Laplace operator, taken in the weak sense. Further, if $F(O) = F_1(O)$, it follows that the normal derivative of φ vanishes at the boundary and, if $F(O) = F_0(O)$, that φ itself vanishes at the boundary. We shall in the following always interpret (9) in terms of the operator G, and λ and μ shall be connected by (10).

Our aim is to prove the asymptotic formula (1), using the well-known Weyl–Courant principle.

Weyl–Courant's principle

Let Ω be a permitted set and ϱ a permitted function in Ω, and let $(\Omega_j)_{j=0}^{\infty}$ be a division of Ω into permitted open subsets (their closures cover the closure of Ω). It is clear that ϱ is permitted in Ω_j unless (4) fails to hold in Ω_j. Let $(\Omega_j)_{j=0}^{s}$ be the sets for which this does not happen, and let the function $\sigma \geqslant 0$ satisfy (5) in Ω. Put

$$H = \sum_{j=0}^{s} \oplus F(\Omega_j),$$

where $F(\Omega_j) = F_0(\Omega_j)$ or $F_1(\Omega_j)$, and introduce the notations

$$(f, g) = \sum_{j=0}^{s} (f, g)_{\Omega_j}, \quad (\nabla f, \nabla g) = \sum_{j=0}^{s} (\nabla f, \nabla g)_{\Omega_j},$$

$$(\varrho f, g) = \sum_{j=0}^{s} (\varrho f, g)_{\Omega_j}, \quad (\sigma f, g) = \sum_{j=0}^{s} (\sigma f, g)_{\Omega_j}.$$

As a scalar product in the Hilbert space H we use

$$((f, g)) = (\nabla f, \nabla g) + (\varrho f, g) + (\sigma f, g).$$

It is clear that $(\varrho f, f) \leqslant ((f, f))$ is compact in H, and hence

$$(\varrho f, g) = ((Gf, g)), \quad (f, Gf, g \in H)$$

defines a compact, self-adjoint and linear transformation G from H to H such that $1 > G > 0$. An eigenfunction φ of G with the eigenvalue

$$\mu = (1 + \lambda)^{-1}$$

satisfies

$$\Delta \varphi + (\lambda \varrho - \sigma)\, \varphi = 0$$

in every Ω_j, and its normal derivative vanishes at the boundary if $F(\Omega_j) = F_1(\Omega_j)$. Otherwise $\varphi \in F_0(\Omega_j)$ vanishes itself at the boundary. If $f = 0$ except in one Ω_j,

Gf has the same property. Hence G is the direct sum of its restrictions G_j to Ω_j, $0 \leqslant j \leqslant s$. By the spectral theorem, $F(\Omega_j)$ has an orthonormal basis consisting of eigenfunctions of G_j, $0 \leqslant j \leqslant s$, and since these are also eigenfunctions of G and constitute an orthonormal basis of H, we have, provided every eigenvalue is counted with its multiplicity.

Theorem 2. *The eigenvalues of G are the union of the eigenvalues of the G_j.*

Now let $(\varphi_i)_1^\infty$ be a complete orthonormal set of eigenfunctions of G and $(\mu_i)_1^\infty$ the corresponding eigenvalues, ordered so that $\mu_1 \geqslant \mu_2 \geqslant \dots$. Then, if

$$\mu_j = (1+\lambda_j)^{-1},$$

we have $\lambda_1 \leqslant \lambda_2 \leqslant \dots$. Let

$$N(\lambda) = N(\lambda, \varrho, \sigma, H) = \sum_{\lambda_j < \lambda} 1$$

be the number of eigenvalues below λ. We have

Theorem 3. *$N(\lambda, \varrho, \sigma, H)$ is a non-decreasing function of ϱ and H and a non-increasing function of σ.*

Proof. It suffices to prove that $\lambda_j(\varrho, \sigma, H)$ has the reverse properties. The minimum–maximum principle gives

$$\mu_j = \mu_j(\varrho, \sigma, H) = \inf_L \sup_{f \in L} (\varrho f, f)/((f, f)),$$

where L runs through all subspaces of H of codimension $< j$. Since

$$\mu_j = (1+\lambda_j)^{-1},$$

we get
$$\lambda_j = \lambda_j(\varrho, \sigma, H) = \sup_L \inf_{f \in L} (|\nabla f|^2 + (\sigma f, f))/(\varrho f, f).$$

Hence, it is clear that λ_j is a non-increasing function of ϱ and a non-decreasing function of σ. Next, let $H' \supset H$ be of the same type as H. Since cod $L < j$ in H, there is a subspace $M' \subset H'$ of dimension $< j$ such that $f \in H$ and $f \perp M'$ implies $f \in L$. Hence,

$$\lambda_j(\varrho, \sigma, H) = \sup_{M'} \alpha(M'),$$

where $\alpha(M') = \inf (|\nabla f|^2 + (\sigma f, f))/(\varrho f, f)$ when $f \in H$ and $f \perp M'$.

Replacing H by H', we get a new function

$$\alpha'(M') \leqslant \alpha(M').$$

Since
$$\sup_{M'} \alpha'(M') = \lambda_j(\varrho, \sigma, H'),$$

$$\lambda_j(\varrho, \sigma, H) \geqslant \lambda_j(\varrho, \sigma, H').$$

This completes the proof.

We conclude this section by proving two lemmas, which will be used later.

Lemma 1. *If $\varrho \geqslant 1$ is a permitted function in a permitted set O, then*

$$N(\lambda, \varrho, 0, F_1(O)) \leqslant N((1+v)\lambda + v, \varrho, \sigma, F_1(O)),$$

where

$$v = C\,|\sigma|_{m/2}$$

with C depending only on m and O.

Proof. By Hölder's inequality, (3) and (5),

$$(\sigma f, f) \leqslant C\,|\sigma|_{m/2}(|\nabla f|^2 + |f|^2).$$

Hence, since $\varrho \geqslant 1$ in O,

$$(|\nabla f^2| + (\sigma f, f))/(\varrho f, f) \leqslant (1+v)\ |\nabla f|^2/(\varrho f, f) + v.$$

Consequently, with L running through all subspaces of $F_1(O)$ of codimension $< j$,

$$\lambda_j(\varrho, \sigma, F_1(O)) = \sup_L \inf_{f \in L} (|\nabla f|^2 + (\sigma f, f))/(\varrho f, f)$$
$$\leqslant \sup_L \inf_{f \in L} ((1+v)\,|\nabla f|^2/(\varrho f, f) + v) = (1+v)\,\lambda_j(\varrho, 0, F_1(O)) + v.$$

Thus, $\lambda_j(\varrho, 0, F_1(O)) < \lambda$ implies $\lambda_j(\varrho, \sigma, F_1(O)) < (1+v)\lambda + v$ and the lemma follows.

Lemma 2. *If $G\varphi = \mu\varphi$ and $\mu = (1+\lambda)^{-1}$, then the support of φ cannot be contained in the set where $\lambda\varrho - \sigma < 0$.*

Proof. $G\varphi = \mu\varphi$ gives $(\varrho\varphi, \varphi) = \mu((\varphi, \varphi))$, i.e.

$$|\nabla\varphi|^2 = ((\lambda\varrho - \sigma)\,\varphi, \varphi),$$

and hence we have the lemma.

The asymptotic formula

The case, where Ω is a n-dimensional rectangle, ϱ a constant and $H = F_0(\Omega)$ or $F_1(\Omega)$, is classical. Since $(\nabla f, \nabla g)$ is invariant under translations and orthogonal transformations, we can assume that

$$\Omega = (x; 0 < x_k < a_k, 1 \leqslant k \leqslant n).$$

Then the eigenfunctions are

$$\prod_{j=1}^{n} \sin \pi l_j x_j a_j^{-1}, \quad \text{if} \quad H = F_0(\Omega),$$

and $$\prod_{j=1}^{n} \cos \pi l_j x_j a_j^{-1}, \quad \text{if} \quad H = F_1(\Omega),$$

where $l_j = 1, 2, 3, \ldots$ in the first and $0, 1, 2, 3, \ldots$ in the second case. Thus, the eigenvalues are

$$\pi^2 \varrho^{-1} \sum_{j=1}^{n} (l_j/a_j)^2$$

which gives $$N(\lambda, \varrho, 0, H) = (2\pi)^{-n} w_n \int_\Omega (\lambda\varrho)^{n/2} + O(\lambda^{(n-1)/2}),$$

where w_n is the volume of the n-dimensional unit sphere, $(w_1 = 2)$. This estimate will be used later. We shall also need

Lemma 3. *If $a_i \leqslant a$ and $a_l \leqslant b$ when $l \neq i$, then*

$$N(\lambda, \varrho, 0, F_1(\Omega)) \leqslant 2^{n-1}(1 + b^{n-1}(\lambda\varrho)^{(n-1)/2})\,(1 + a(\lambda\varrho)^{\frac{1}{2}}).$$

Proof. The number of non-negative integral solutions of

$$\sum_{j=1}^{n} (l_j/a_j)^2 < \pi^{-2} \varrho\lambda$$

is majorized by $$\prod_{j=1}^{n} (1 + \pi^{-1} a_j (\varrho\lambda)^{\frac{1}{2}})$$

so that the lemma follows.

Now, to simplify the notations, write

$$\bar{N}(\varrho, \sigma, H) = \limsup \lambda^{-n/2} N(\lambda, \varrho, \sigma, H), \quad (\lambda \to \infty),$$

$$\underline{N}(\varrho, \sigma, H) = \liminf \lambda^{-n/2} N(\lambda, \varrho, \sigma, H), \quad (\lambda \to \infty)$$

and $N(\varrho, \sigma, H) = \bar{N} = \underline{N}$ when the limits are equal. Also, put

$$M(\varrho, \Omega) = (2\pi)^{-n} w_n \int_\Omega \varrho^{n/2}.$$

When $$\Omega = (\Omega_j)_{j=0}^{p}$$

is a sum of rectangles, and $\underline{\varrho}$ and $\bar{\varrho}$ are constants in these rectangles such that $\underline{\varrho} \leqslant \varrho$ and $0 < \bar{\varrho} \geqslant \varrho$, Theorem 2 and Theorem 3 give

$$\Sigma' N(\lambda, \underline{\varrho}, 0, F_0(\Omega_j)) \leqslant N(\lambda, \varrho, 0, F_0(\Omega)) \leqslant N(\lambda, \varrho, 0, F_1(\Omega)) \leqslant \Sigma N(\lambda, \bar{\varrho}, 0, F_1(\Omega_j)),$$

where Σ' denotes that we only sum over such j that $\varrho > 0$ in Ω_j. Hence,

$$M(\underline{\varrho}, \Omega) \leqslant \underline{N}(\varrho, 0, F_0(\Omega)) \leqslant \bar{N}(\varrho, 0, F_1(\Omega)) \leqslant M(\bar{\varrho}, \Omega).$$

If ϱ is bounded and Jordan measurable, the first and the last term can be made arbitrarily close by choosing a fine subdivision of Ω, and hence

$$N(\varrho, 0, F) = M(\varrho, \Omega), \tag{11}$$

where $$F = F_0(\Omega) \text{ or } F_1(\Omega).$$

This is the asymptotic formula in this very regular case. We shall see that the same formula holds also when ϱ has moderate singularities, more precisely, if

(*a*) Ω is a finite sum of rectangles,

(*b*) $\varrho \geq 0, \int_\Omega \varrho > 0$ and ϱ is bounded except for a finite number of singular points y in Ω, where

$$\varrho(x) = O(|x-y|^{-2\beta}) \tag{12}$$

with $0<\beta<1$ when $n \geq 2$, and $0<\beta<\frac{1}{2}$ when $n=1$,

and

(*c*) ϱ is Jordan measurable.

For convenience, the norm $|x|$ is defined as $\max_j |x_j|$. We have made the first assumption, since we are interested only in the singularities of the function ϱ and not in the complications that arise from the boundary. For generalization to more general regions we refer to [1]. The third condition implies that $\varrho^{n/2}$ is Riemann integrable.

Now, let Ω_0 be such a sum of rectangular neighbourhoods of the y that $\Omega - \Omega_0$ is also a sum of rectangles. Then we have by Theorem 2 and Theorem 3

$$N(\lambda, \varrho, 0, F_0(\Omega-\Omega_0)) \leq N(\lambda, \varrho, 0, F_0(\Omega)) \leq N(\lambda, \varrho, 0, F_1(\Omega)) \leq N(\lambda, \varrho, 0, F_1(\Omega_0)) + N(\lambda, \varrho, 0, F_1(\Omega-\Omega_0))$$

and hence

$$M(\varrho, \Omega-\Omega_0) \leq \underline{N}(\varrho, 0, F_0(\Omega)) \leq \bar{N}(\varrho, 0, F_1(\Omega) \leq \bar{N}(\varrho, 0, F_1(\Omega_0)) + M(\varrho, \Omega-\Omega_0).$$

Here we see that in order that

$$N(\varrho, 0, F(\Omega)) = M(\varrho, \Omega)$$

with $F=F_0$ or F_1, it is sufficient to prove that

$$\bar{N}(\varrho, 0, F_1(\Omega_0)) \to 0$$

as the diameter of Ω_0 tends to zero. Hence, if

$$D_t \text{ stands for the cube } |x|<t$$

we have, putting

$$\xi(x) = |x|^{-2\beta},$$

that it suffices to show that

$$\bar{N}(\xi, 0, F_1(D_r)) \to 0 \quad \text{when } r \to 0. \tag{13}$$

Consider
$$N(\lambda) = N(\lambda, \xi, 0, F_1(D)),$$

where $D = D_r$. When $\xi \geqslant 1$ in D and $\sigma \geqslant 0$ satisfies (5), we have according to Lemma 1
$$N(\lambda, \xi, 0, F_1(D)) \leqslant N((1+v)\lambda + v, \xi, \sigma, F_1(D)),$$

where
$$v = C\,|\sigma|_{m/2}$$

with C depending on m and D and hence on r.

We put
$$\sigma(x) = |x|^{-2(\beta+\varepsilon)}$$

$$(\beta < \beta + \varepsilon < 1 \text{ when } n \geqslant 2, \quad \beta < \beta + \varepsilon < \tfrac{1}{2} \text{ when } n = 1)$$

inside a cube D' with its centre at the origin and
$$\sigma(x) = 0$$

outside, and choose D' so small that $v \leqslant 1$. Then
$$N(\lambda, \xi, 0, F_1(D)) \leqslant N(\lambda', \xi, \sigma, F_1(D)),$$

where $\lambda' = 2\lambda + 1$. Consider $D_s \subset D$. By Theorem 2 and Theorem 3
$$N(\lambda', \xi, \sigma, F_1(D)) \leqslant N(\lambda', \xi, 0, F_1(D - D_s)) + N(\lambda', \xi, \sigma, F_1(D_s)). \tag{14}$$

Now determine $s = s(\lambda)$ so that $D_s \subset D'$ and
$$\lambda' \xi - \sigma < 0 \text{ in } D_s.$$

This is possible, if e.g.
$$\lambda' = 2\lambda + 1 = s^{-2\varepsilon} \tag{15}$$

and λ is sufficiently large.

Then, by Lemma 2, the second term on the right side of (14) vanishes, and we get
$$N(\lambda) \leqslant N(\lambda', \xi, 0, F_1(D - D_s)).$$

Now it is easy to estimate the right side.

We choose p numbers such that
$$r = r_0 > r_1 > \dots r_p = s.$$

A more precise choise will be made later. Put $D_j = D_{r_j}$. Then by Theorem 2 and Theorem 3,
$$N(\lambda) \leqslant \sum_{j=0}^{p-1} N(\lambda', \xi, 0, F_1(D_j - D_{j+1})). \tag{16}$$

Now $D_j - D_{j+1}$ is obviously a sum of a fixed number of rectangular regions of diameter $\leqslant r_j$, having one side equal to $(r_j - r_{j+1})$. Further, $\xi \leqslant r_{j+1}^{-2\beta}$ in $D_j - D_{j+1}$. Hence, by (15), (16), Lemma 3 and Theorem 3, there is a constant C such that

$$N(\lambda) \leqslant C \sum_{j=0}^{p-1} (1 + r_j^{n-1} r_{j+1}^{-\beta(n-1)} \lambda^{(n-1)/2}) (1 + r_{j+1}^{-\beta}(r_j - r_{j+1}) \lambda^{\frac{1}{2}})$$

and hence

$$\lambda^{-n/2} N(\lambda) \leqslant C \sum_{j=0}^{p-1} ((\lambda)^{-(n-1)/2} + r_j^{n-1} r_{j+1}^{-\beta(n-1)}) ((\lambda)^{-\frac{1}{2}} + r_{j+1}^{-\beta}(r_j - r_{j+1})).$$

By virtue of (15) we may write

$$\lambda^{-n/2} N(\lambda) \leqslant C \sum_{j=0}^{p-1} (s^{(n-1)\varepsilon} + r_j^{n-1} r_{j+1}^{-\beta(n-1)}) (s^{\varepsilon} + r_{j+1}^{-\beta}(r_j - r_{j+1}))$$

provided we increase the constant.

We now choose the numbers r_j so that

$$2 \leqslant r_j / r_{j+1} \leqslant 4 \quad \text{for all } j. \tag{17}$$

It is easy to see that this is always possible if $s < r/2$. Then we have

$$s = r_p \leqslant 2^{-p} r$$

and hence

$$s^{\varepsilon} p \to 0 \text{ as } s \to 0. \tag{18}$$

Further, (17) gives, with still another C,

$$\lambda^{-n/2} N(\lambda) \leqslant C \sum_{j=0}^{p-1} (s^{(n-1)\varepsilon} + r_j^{(n-1)(1-\beta)}) (s^{\varepsilon} + r_j^{-\beta}(r_j - r_{j+1})).$$

Since $1 - \beta > 0$, we obtain the following majorant for the right side

$$C(s^{(n-1)\varepsilon} + r^{(n-1)(1-\beta)}) \left(s^{\varepsilon} p + \int_0^r t^{-\beta} \, dt \right),$$

and hence by (18)

$$\limsup \lambda^{-n/2} N(\lambda) \leqslant C r^{(n-1)(1-\beta)} \int_0^r t^{-\beta} \, dt = O(r^{n(1-\beta)}),$$

which tends to zero with r and the proof is finished.

Remark. Using the Weyl–Courant principle and the fact that ξ is homogeneous of order -2β, it is easy to see that

$$\lambda_j(\xi, 0, F_1(D_r)) = r^{2(\beta-1)} \lambda_j(\xi, 0, F_1(D_1)).$$

This gives

$$N(\lambda, \xi, 0, F_1(D_r)) = N(r^{2(1-\beta)}\lambda, \xi, 0, F_1(D_1)).$$

Hence (13) is a consequence of

$$\bar{N}(\xi, O, F_1(D_1)) < \infty,$$

and this follows if we put $r=1$ in the proof above.

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