1.66 07 2 Communicated 27 April 1966 by O. FROSTMAN and L. CARLESON

Holomorphic functions and Hausdorff dimension

By VICTOR L. SHAPIRO

1. Introduction

Let $D(z, r)$ represent the open disc with center z and radius r, and let $C(z, r)$ represent its boundary oriented in the usual counter-clockwise manner. We define the class A_α, $1 \leqslant \alpha \leqslant 2$, as follows:

$f(z)$ is in the class A_α if

(i) $f(z)$ is a continuous complex-valued function defined in $D(0, 1)$, and

(ii) there exist a constant K and a $\gamma > \alpha$ such that for $0 < \varrho < 1$ and $0 < r < 1 - \varrho$

$$\int_{D(0,\varrho)} \left| \int_{C(z,r)} f(\zeta)\, d\zeta \right|^2 dx\, dy \leqslant K r^{2+\gamma}.$$

We define the class B_α in the same manner as the class A_α except in (ii), we only require that $\gamma \geqslant \alpha$. It is clear that the class B_α is the natural widening of the class A_α.

We shall say that the relatively closed set $E \subset D(0, 1)$ [i.e. the complement of E in $D(0, 1)$ is open] is a removable set for the class A_α if the following fact holds: *If f is in A_α and f is holomorphic in $D(0, 1) \sim E$, then f is holomorphic in $D(0, 1)$.* E is a removable set for the class B_α is defined in a similar manner.

In this paper, we intend to establish the following result:

Theorem. *A necessary and sufficient condition that a relatively closed set E contained in $D(0, 1)$ be a removable set for the class A_α, $1 \leqslant \alpha \leqslant 2$, is that the Hausdorff dimension of E be $\leqslant \alpha$. Furthermore, the sufficiency condition is in a certain sense best possible, i.e., it is false for the class B_α.*

If $\alpha < 1$ and the Hausdorff dimension of $E \leqslant \alpha$, Besicovitch has shown that E is a removable set for the class of continuous functions in $D(0, 1)$. He has shown even more, namely that if E is a countable union of sets of finite length, then E is a removable set for this last named class of functions. For the details of his result see either [7, p. 197] or [1].

We next note that the sufficiency of the above theorem in the special case $\alpha = 2$ is essentially known already and is a corollary of [10; Theorem 1, p. 76].

2. Proof of the necessary condition

We first establish the necessary condition of the above theorem.

Since every set contained in $D(0, 1)$ is of Hausdorff dimension $\leqslant 2$, it follows that if E is a removable set for the class A_2 then E is of Hausdorff dimension $\leqslant 2$.

We can therefore suppose that $1 \leqslant \alpha < 2$ and that the Hausdorff dimension of $E = \beta$ where $\alpha < \beta \leqslant 2$. We shall establish the necessity of the above theorem by exhibiting a function f which is in A_α and which is holomorphic in $D(0, 1) \sim E$ but which is not holomorphic in $D(0, 1)$.

Since E is a relatively closed set contained in $D(0, 1)$ of Hausdorff dimension equal to β where $\alpha < \beta$, it follows from the definition of Hausdorff dimension, [6, p. 145], that there exists a closed set E_1 with $E_1 \subset E \subset D(0, 1)$ such that the Hausdorff dimension of E_1 is greater than α, i.e. the Hausdorff dimension of E_1 is β_1 where $\alpha < \beta_1 \leqslant \beta$. Frostman has shown [4, p. 90] that the capacity dimension and the Hausdorff dimension for closed sets are the same. Consequently, if we take $\gamma = (\beta_1 + \alpha)/2$, we have that $\alpha < \gamma < \beta_1 \leqslant \beta$ and furthermore that the γ-capacity of E_1 is positive, i.e. there exists a finite constant V and a probability measure μ (that is a non-negative Borel measure of total mass one) having its support contained in E_1 such that

$$\int_{E_1} |\zeta - z|^{-\gamma} d\mu(\zeta) \leqslant V \text{ for every } z. \tag{1}$$

We set

$$f(z) = \int_{E_1} (\zeta - z)^{-1} d\mu(\zeta) \tag{2}$$

and observe from (1) that $f(z)$ is well defined for every z.

We next show that

$$f(z) \text{ is a continuous function in the complex plane.} \tag{3}$$

To establish (3), fix z_0 and let $\varepsilon > 0$ be given. With $\sim G$ designating the complement of the set G, we observe that

$$\lim_{z \to z_0} \int_{\sim D(z_0, \varepsilon) \cap E_1} (\zeta - z)^{-1} d\mu(\zeta) = \int_{\sim D(z_0, \varepsilon) \cap E_1} (\zeta - z_0)^{-1} d\mu(\zeta).$$

Consequently, it follows from (2) that

$$\limsup_{z \to z_0} |f(z) - f(z_0)| \leqslant \int_{D(z_0, \varepsilon) \cap E_1} |\zeta - z_0|^{-1} d\mu(\zeta) + \limsup_{z \to z_0} \int_{D(z_0, \varepsilon) \cap E_1} |\zeta - z|^{-1} d\mu(\zeta). \tag{4}$$

Now if $|z - z_0| < \varepsilon$, then by (1)

$$\int_{D(z_0, \varepsilon)} |\zeta - z|^{-1} d\mu(\zeta) \leqslant (2\varepsilon)^{\gamma - 1} \int_{D(z_0, \varepsilon)} |\zeta - z|^{-\gamma} d\mu(\zeta) \leqslant V(2\varepsilon)^{\gamma - 1}. \tag{5}$$

Likewise from (1), the first integral on the right side of the inequality in (4) is majorized by $V\varepsilon^{\gamma-1}$. Consequently, we conclude from this last fact, (4), and (5) that

$$\limsup_{z\to z_0} |f(z)-f(z_0)| \leqslant V[\varepsilon^{\gamma-1}+(2\varepsilon)^{\gamma-1}]. \tag{6}$$

But γ is strictly greater than 1, and (3) therefore follows immediately from (6). It is clear from (2) and the fact that E_1 is a closed set that

$$f(z) \text{ is a holomorphic function in } \sim E_1. \tag{7}$$

To show that

$$f(z) \text{ is not a holomorphic function in } D(0, 1), \tag{8}$$

we choose r_1 with $0<r_1<1$ such that $E_1 \subset D(0, r_1)$, which can be done since E_1 is a closed set. Then E_1 also does not intersect the boundary of $D(0, r_1)$, and consequently it follows from Fubini's theorem and (2) that

$$\begin{aligned}\int_{C(0,r_1)} f(z)dz &= \int_{E_1} d\mu(\zeta)\int_{C(0,r_1)}(\zeta-z)^{-1}dz\\ &= -2\pi i\int_{E_1} d\mu(\zeta)\\ &= -2\pi i.\end{aligned}$$

This fact and Cauchy's theorem establish (8).

To complete the proof of the necessity, we need only show that for $0<\varrho<1$ and $0<r<1-\varrho$

$$\int_{D(0,\varrho)}\left|\int_{C(z,r)} f(\zeta)d\zeta\right|^2 dx\,dy \leqslant 4V\pi^3 r^{2+\gamma}, \tag{9}$$

where γ and V are defined in (1).

To establish (9), we first observe that it follows immediately from (1) and Fubini's theorem that

$$\mu[\bar{D}(z, r)\sim D(z, r)]=0 \text{ for every } z \text{ and every } r>0, \tag{10}$$

where $\bar{G}$ represents the closure of the set G.

Consequently, it follows from Fubini's theorem, (1), and (10) that

$$\begin{aligned}\int_{C(z,r)} f(s)\,ds &= \int_{E_1\cap\sim\bar{D}(z,r)} d\mu(\zeta)\int_{C(z,r)}(\zeta-s)^{-1}ds\\ &\quad+\int_{E_1\cap D(z,r)} d\mu(\zeta)\int_{C(z,r)}(\zeta-s)^{-1}ds\\ &= -2\pi i\mu[E_1\cap D(z,r)].\end{aligned}$$

We conclude that

$$\left|\int_{C(z,r)} f(\zeta)d\zeta\right| \leqslant 2\pi\mu[D(z,r)]. \tag{11}$$

Next, we observe from (1) that for $r>0$,

$$\mu[D(z,r)] \leqslant \int_{D(z,r)} |\zeta-z|^{\gamma}|\zeta-z|^{-\gamma}d\mu(\zeta)$$
$$\leqslant Vr^{\gamma}. \tag{12}$$

Designating the left side of the inequality in (9) by $I_{\varrho,r}$ and letting χ_G represent the indicator function of the set G, we consequently obtain from (11) and (12) that

$$I_{\varrho,r} \leqslant 4V\pi^2 r^{\gamma}\int_{D(0,\varrho)}\left|\int_{D(z,r)} d\mu(\zeta)\right| dx\,dy$$
$$\leqslant 4V\pi^2 r^{\gamma}\int_{D(0,1)} d\mu(\zeta)\left|\int_{D(0,\varrho)} \chi_{D(0,r)}(z-\zeta)dx\,dy\right|$$
$$\leqslant 4V\pi^3 r^{2+\gamma}.$$

(9) is therefore established, and the proof of the necessity is complete.

3. Proof of the best possible condition

The proof of the best possible condition of the above theorem in the case $\alpha=2$ is particularly simple. We take a function $g(x)$ which is in class C^1 on the real line, which vanishes outside the closed interval $[-\frac{1}{2},\frac{1}{2}]$, and which takes the value one in $[-\frac{1}{4},\frac{1}{4}]$. We take E to be the intersection of the open unit disc with the strip $-\frac{1}{2}\leqslant x\leqslant\frac{1}{2}$ and define $f(z)$ in the complex plane by $f(z)=-ig(x)$. Then E is of Hausdorff dimension 2 and is relatively closed with respect to $D(0,1)$. Furthermore, $f(z)$ is continuous in the complex plane, holomorphic in $D(0,1)\sim E$, but not holomorphic in $D(0,1)$. To complete the proof of the best possibility in the special case $\alpha=2$, we need only show that f is in class B_2.

In order to do this set $K=\sup_{-\infty<x<\infty}|dg(x)/dx|$. Then by Green's theorem, for $r>0$ and $\zeta=\xi+i\eta$,

$$\left|\int_{C(z,r)} f(\zeta)\,d\zeta\right| = \left|\int_{D(z,r)} dg(\xi)/d\xi\,d\xi\,d\eta\right| \leqslant K\pi r^2.$$

Therefore for $0<\varrho<1$ and $0<r<1-\varrho$

$$\int_{D(0,\varrho)}\left|\int_{C(z,r)} f(\zeta)\,d\zeta\right|^2 dx\,dy \leqslant K^2\pi^3 r^4,$$

and we conclude that $f(z)$ is in B_2.

To handle the situation when $1\leqslant\alpha<2$, we proceed in a similar manner, though the situation now is slightly more complicated.

For $1<\alpha<2$, we set $q=2^{1/(\alpha-1)}$, and on the interval $[-\frac{1}{2}, \frac{1}{2}]$, we construct a symmetric Cantor set, $Q_{\alpha-1}$, corresponding to q^{-1} (i.e., at the first stage, we take out the open interval $(-\frac{1}{2}+q^{-1}, \frac{1}{2}-q^{-1})$. We proceed in this manner so that at the nth stage we have taken out 2^{n-1} open intervals, leaving 2^n closed intervals each of length q^{-n}.) As is well-known, [5], the Hausdorff dimension of $Q_{\alpha-1}$ is $\alpha-1$.

For Q_0, that is for $\alpha=1$, we take any perfect set of Hausdorff dimension zero constructed on the interval $[-\frac{1}{2}, \frac{1}{2}]$ which contains the points $\frac{1}{2}$ and $-\frac{1}{2}$.

For $1\leqslant\alpha<2$, we shall designate by $g_\alpha(x)$ the Lebesgue-Cantor function constructed on $[-\frac{1}{2}, \frac{1}{2}]$ corresponding to Q_α which is defined on the rest of the real line by setting $g_\alpha(x)=0$ for $x<-\frac{1}{2}$ and $g_\alpha(x)=1$ for $x>\frac{1}{2}$. Then as is well-known [5, p. 173], $g_\alpha(x)$ is in Lip$(\alpha-1)$ on the real line (Lip 0 being interpreted here as continuous), that is there is a constant K_α such that

$$|g_\alpha(x_1)-g_\alpha(x_2)|\leqslant K_\alpha|x_1-x_2|^{\alpha-1} \tag{13}$$

for every x_1 and x_2.

Next, we take the set F_α in the complex plane to be $F_\alpha=\{x+iy;\ x \text{ in } Q_\alpha\}$ and define E_α as $E_\alpha=F_\alpha\cap D(0, 1)$. Now, as is well-known, the Hausdorff dimension of E_α is equal to α, [5]. We define $f_\alpha(z)=-ig_\alpha(x)$ and observe that $f_\alpha(z)$ is continuous in the complex plane, holomorphic in $D(0, 1)\sim E_\alpha$, but not holomorphic in $D(0, 1)$. Consequently to establish the best possibility of the theorem for $1\leqslant\alpha<2$, it only remains to show that $f_\alpha(z)$ is in B_α. We shall accomplish this by showing that for $0<\varrho<1$ and $0<r<1-\varrho$,

$$\int_{D(0,\varrho)}\left|\int_{C(z,r)} f_\alpha(\zeta)d\zeta\right|^2 dx\,dy\leqslant 2^{\alpha+1}K_\alpha\pi^2r^{\alpha+2}, \tag{14}$$

where K_α is the constant in (13).

To establish (14), we first observe from the evenness of $g(x+r\cos\theta)$ as a function of θ that

$$\int_{C(z,r)} f_\alpha(\zeta)\,d\zeta=2r\int_0^\pi g_\alpha(x+r\cos\theta)\cos\theta\,d\theta$$

$$=2r\int_0^{\pi/2}[g_\alpha(x+r\cos\theta)-g_\alpha(x-r\cos\theta)]\cos\theta\,d\theta.$$

Since $g_\alpha(x)$ is a bounded non-decreasing function of x, we conclude that for every z and for $r>0$

$$\left|\int_{C(z,r)} f_\alpha(\zeta)\,d\zeta\right|\leqslant\pi r[g_\alpha(x+r)-g_\alpha(x-r)]. \tag{15}$$

From the definition of $g_\alpha(x)$, it follows that there exists a probability measure μ_α having its support on Q_α such that for every x and every $r>0$

$$g_\alpha(x+r)-g_\alpha(x-r)=\int_{-\infty}^{\infty}\chi_{[-r,r]}(t-x)\,d\mu_\alpha(t), \tag{16}$$

where $\chi_{[-r,r]}$ is the indicator function for the interval $[-r, r]$.

Designating the expression on the left side of the inequality in (14) by $I_{\varrho,r}$, we then obtain from (13), (15), and (16) that

$$I_{\varrho,r} \leqslant 2^{\alpha-1}K_\alpha \pi^2 r^{2+\alpha-1}\int_{-\varrho}^{\varrho} dy \int_{-\varrho}^{\varrho} dx \int_{-\infty}^{\infty} \chi_{[-r,r]}(t-x)\, d\mu_\alpha(t)$$

$$\leqslant 2^{\alpha}K_\alpha \pi^2 r^{2+\alpha-1}\int_{-\infty}^{\infty} d\mu_\alpha(t) \int_{-\varrho}^{\varrho} \chi_{[-r,r]}(t-x)\, dx$$

$$\leqslant 2^{\alpha+1}K_\alpha \pi^2 r^{2+\alpha}.$$

(14) is consequently established, and the proof of the best possibility is complete.

4. Proof of the sufficient condition

To establish the sufficient condition of the theorem, we need only show by Morera's theorem that

$$\int_{\partial\tau} f(\zeta)\, d\zeta = 0 \quad \text{for every} \quad \tau \subset D(0,1), \tag{17}$$

where τ designates a two simplex, i.e., closed triangle, and $\partial\tau$ is oriented in the usual counter-clockwise manner.

For $\alpha=2$, (17) follows easily from the definition of A_2 and [10; Theorem 1, p. 76]. We shall therefore suppose in the sequel that $1\leqslant\alpha<2$.

Suppose then that τ_0 is a fixed two simplex and that $\tau_0 \subset D(0, r_1)$ where $0<r_1<1$. To prove the sufficient condition of the theorem, we need only show that

$$\int_{\partial\tau_0} f(\zeta)\, d\zeta = 0. \tag{18}$$

To this end, we choose r_2, r_3 and r_4 such that

$$0<r_1<r_2<r_3<r_4<1 \quad \text{where} \quad \tau_0 \subset D(0, r_1), \tag{19}$$

and select a real-valued function $\lambda(z)$ which is in class $C^{(\infty)}$ and takes the value one in $D(0, r_1)$ and the value zero outside of $D(0, r_2)$. Using the facts that $f(z)$ and $\lambda(z)$ are bounded in $\bar{D}(0, r_4)$ and there exists a constant K_1 such that $|\lambda(z+\zeta)-\lambda(z)| \leqslant K_1|\zeta|$ for every z and ζ, we obtain that for z in $D(0, r_3)$ and $0<r<r_4-r_3$,

$$\left|\int_{C(z,r)} \lambda(\zeta) f(\zeta)\, d\zeta\right| \leqslant \left|\int_{C(0,r]} [\lambda(z+\zeta)-\lambda(z)] f(z+\zeta)\, d\zeta\right| + |\lambda(z)|\left|\int_{C(z,r)} f(\zeta)\, d\zeta\right|$$

$$\leqslant K_2 r^2 + K_2\left|\int_{C(z,r)} f(\zeta)\, d\zeta\right|,$$

where K_2 is a constant. Consequently, it follows from the definition of the class A_α and from Minkowski's inequality that there exists a constant K_3 and there exists a constant γ with $\alpha<\gamma<2$ such that

$$\int_{D(0,r_3)}\left|\int_{C(z,r)}\lambda(\zeta)f(\zeta)\right|^2 dx\,dy \leqslant K_3 r^{2+\gamma}$$

$$\text{for } 0<r<r_4-r_3. \tag{20}$$

(We recall that we are dealing with $1\leqslant\alpha<2$ and with no loss in generality, we can suppose that $\gamma<2$.)

Next, we introduce the two dimensional torus $T_2=\{(x,y);\ -\pi<x\leqslant\pi$ and $-\pi<y\leqslant\pi\}$ and define

$$\begin{aligned} f_1(z) &= \lambda(z)f(z) \text{ for } z \text{ in } D(0,1),\\ &=0 \qquad\quad\ \text{for } z \text{ in } T_2\sim D(0,1). \end{aligned} \tag{21}$$

We then extend f_1 by periodicity to the whole complex plane, i.e.

$$f_1[x+2m\pi+i(y+2n\pi)] = f_1(x+iy)$$

for m and n integers, and observe that $f_1(z)$ is a continuous function on the complex plane and furthermore from (19), (20), and (21) that there is a constant K_3 such that

$$\int_{T_2}\left|\int_{C(z,r)} f_1(\zeta)\,d\zeta\right|^2 dx\,dy \leqslant K_3 r^{2+\gamma}$$

$$\text{for } 0<r<\min\,[r_3-r_2,\, r_4-r_3]. \tag{22}$$

We next set

$$\left.\begin{aligned} f_1(z) &= u_1(x,y)+iv_1(x,y),\\ f(z) &= u(x,y)+iv(x,y),\\ \lambda(z) &= \psi(x,y) \end{aligned}\right\} \tag{23}$$

and observe that $u_1(x,y)$ and $v_1(x,y)$ are periodic continuous functions, $u_1(x,y)=v_1(x,y)=0$ in $T_2\sim\bar{D}(0,r_2)$, and $u_1(x,y)=u(x,y)$ and $v_1(x,y)=v(x,y)$ in $\bar{D}(0,r_1)$.

We first of all infer from these facts that (18) will be established if we show

$$\left.\begin{aligned} \int_{\partial\tau_0} u_1(x,y)\,dx - v_1(x,y)\,dy &= 0,\\ \int_{\partial\tau_0} u_1(x,y)\,dy + v_1(x,y)\,dx &= 0. \end{aligned}\right\} \tag{24}$$

Next we set

$$\left.\begin{aligned} g &= -[u\partial\psi/\partial y + v\partial\psi/\partial x] && \text{for } (x,y) \text{ in } D(0,1),\\ h &= u\partial\psi/\partial x - v\partial\psi/\partial y && \text{for } (x,y) \text{ in } D(0,1),\\ g &= h = 0 && \text{for } (x,y) \text{ in } T_2\sim D(0,1), \end{aligned}\right\} \tag{25}$$

and define g and h throughout the rest of the plane by periodicity of period 2π in each variable. We observe that g and h are continuous functions in the plane and that

$$g = h = 0 \text{ in } \bar{D}(0, r_1) \text{ and in } T_2 \sim D(0, r_2). \tag{26}$$

We furthermore observe from the fact that f is holomorphic in $D(0, 1) \sim E$ that

$$\left.\begin{aligned} &(\pi r^2)^{-1} \int_{C(x,y,r)} u_1(\xi, \eta)\, d\xi - v_1(\xi, \eta)\, d\eta \to g(x, y), \\ &(\pi r^2)^{-1} \int_{C(x,y,r)} u_1(\xi, \eta)\, d\eta + v_1(\xi, \eta)\, d\xi \to h(x, y), \\ &\qquad \text{as } r \to 0 \text{ for } (x, y) \text{ in } T_2 \sim [\bar{D}(0, r_2) \cap E], \end{aligned}\right\} \tag{27}$$

where we are now writing $C(x+iy, r)$ as $C(x, y, r)$.

We continue along these lines and observe that (22) can be interpreted in the following manner:

$$\left.\begin{aligned} &\text{there exist } \gamma \text{ with } \alpha < \gamma < 2, \text{ a constant } K_3, \\ &\text{and } r_0 \text{ with } 0 < r_0 < 1 \text{ such that for } 0 < r < r_0, \\ &\qquad \int_{T_2} \left| \int_{C(x,y,r)} u_1\, d\xi - v_1\, d\eta \right|^2 dx\, dy \leqslant K_3 r^{2+\gamma} \\ &\text{and} \\ &\qquad \int_{T_2} \left| \int_{C(x,y,r)} u_1\, d\eta + v_1\, d\xi \right|^2 dx\, dy \leqslant K_3 r^{2+\gamma} \end{aligned}\right\} \tag{28}$$

Using (27), (28), and the theory of double trigonometric series, we shall establish (24) and consequently the theorem.

To this end, we introduce the notation $X = (x, y)$, $M = (m, n)$, and $(M, X) = mx + ny$, and write the Fourier series of u_1 and v_1 on T_2, designated by $S[u_1]$ and $S[v_1]$ respectively, as

$$S[u_1] = \sum_M u_1^{\wedge}(M)\, e^{i(M, X)} \text{ and } S[v_1] = \sum_M v_1^{\wedge}(M)\, e^{i(M, X)}, \tag{29}$$

where M represents an integral lattice point.

Now, with $|M| = (M, M)^{\frac{1}{2}}$,

$$\int_{C(0,r)} e^{i(M,X)} dx = -in \int_{D(0,r)} e^{i(M,X)} dx\, dy$$

$$= -(2\pi i)\, n J_1(|M| r)\, r\, |M|^{-1}$$

and

$$\int_{C(0,r)} e^{i(M,X)} dy = (2\pi i)\, m J_1(|M| r)\, r\, |M|^{-1},$$

where J_1 is the Bessel function of the first kind and order 1.

Consequently, it follows from the Riesz-Fischer theorem, the fact that u_1 and v_1 are continuous functions and from (28) that

$$\left.\begin{array}{l}\text{there exist } \gamma \text{ with } \alpha<\gamma<2, \text{ a constant } K_4,\\ \text{and } r_0 \text{ with } 0<r_0<1 \text{ such that for } 0<r<r_0\\ \qquad \sum_M |u_1^\wedge(M)n+v_1^\wedge(M)m|^2|J_1(|M|r)|^2|M|^{-2}\leqslant K_4 r^\gamma,\\ \text{and}\\ \qquad \sum_M |u_1^\wedge(M)m-v_1^\wedge(M)n|^2|J_1(|M|r)|^2|M|^{-2}\leqslant K_4 r^\gamma.\end{array}\right\} \tag{30}$$

As is well-known, $J_1(t)t^{-1}$ is a continuous function on the interval $(0, \infty)$ and $\lim_{t\to 0} J_1(t)t^{-1}=2^{-1}$. Therefore, there exists a $t_0>0$ such that for $0<t<t_0$, $|J_1(t)|t^{-1}>\frac{1}{4}$. Consequently from (30) we obtain that for $0<r<r_0$,

$$\left.\begin{array}{l}\sum_{|M|\leqslant t_0 r^{-1}} |u_1^\wedge(M)n+v_1^\wedge(M)m|^2 r^2\leqslant 4^2K_4 r^\gamma,\\ \text{and}\\ \sum_{|M|\leqslant t_0 r^{-1}} |u_1^\wedge(M)m-v_1^\wedge(M)n|^2 r^2\leqslant 4^2K_4 r^\gamma.\end{array}\right\} \tag{31}$$

Next, let β be such that $\alpha<\beta<\gamma<2$. Then we conclude from (31) that there exists a constant K_β such that for $0<r<r_0$

$$\left.\begin{array}{l}\sum_{t_0(2r)^{-1}\leqslant|M|\leqslant t_0 r^{-1}} |u_1^\wedge(M)n+v_1^\wedge(M)m|^2|M|^{\beta-2}\leqslant K_\beta r^{\gamma-\beta},\\ \text{and}\\ \sum_{t_0(2r)^{-1}\leqslant|M|\leqslant t_0 r^{-1}} |u_1^\wedge(M)m-v_1^\wedge(M)n|^2|M|^{\beta-2}\leqslant K_\beta r^{\gamma-\beta}.\end{array}\right\} \tag{32}$$

Next, we observe there exists an integer j_0 such that for $j\geqslant j_0$, $t_0 2^{-j}<r_0$. Therefore from (32), it follows that for $j\geqslant j_0$,

$$\sum_{2^{j-1}\leqslant|M|\leqslant 2^j} |u_1^\wedge(M)n+v_1^\wedge(M)m|^2|M|^{\beta-2}\leqslant K_\beta t_0^{\gamma-\beta}2^{(\beta-\gamma)j}$$

and

$$\sum_{2^{j-1}\leqslant|M|\leqslant 2^j} |u_1^\wedge(M)m-v_1^\wedge(M)n|^2|M|^{\beta-2}\leqslant K_\beta t_0^{\gamma-\beta}2^{(\beta-\gamma)j}.$$

However, $\beta<\gamma$: consequently the series $\sum_{j=0}^\infty 2^{(\beta-\gamma)j}<\infty$, and we conclude that for $\beta<\gamma$,

$$\left.\begin{array}{l}\sum_{M\neq 0} |u_1^\wedge(M)n+v_1^\wedge(M)m|^2|M|^{\beta-2}<\infty\\ \text{and}\\ \sum_{M\neq 0} |u_1^\wedge(M)m-v_1^\wedge(M)n|^2|M|^{\beta-2}<\infty.\end{array}\right\} \tag{33}$$

Next, we conclude from (27) and [8; Lemma 2, p. 606] that

$$\left.\begin{array}{l}(-i)\sum_M [u_1^\wedge(M)n+v_1^\wedge(M)m]e^{i(M,X)-|M|t}\to g(X) \text{ as } t\to 0\\ \text{and}\\ i\sum_M [u_1^\wedge(M)m-v_1^\wedge(M)n]e^{i(M,X)-|M|t}\to h(X) \text{ as } t\to 0\\ \qquad\qquad \text{for } X \text{ in } T_2\sim[\bar{D}(0,r_2)\cap E].\end{array}\right\} \tag{34}$$

We next introduce the Fourier series of g and h, that is $S[g]$ and $S[h]$ respectively, and write

$$\left.\begin{aligned} S[g] &= \sum_M g^\wedge(M)\, e^{i(M,X)}, \\ S[h] &= \sum_M h^\wedge(M)\, e^{i(M,X)}. \end{aligned}\right\} \tag{35}$$

Since $g(X)$ and $h(X)$ are continuous periodic functions and therefore in L^2 on T_2 and since $\gamma<2$, we obtain from (35) that

$$\sum_{M\neq 0} |g^\wedge(M)|^2 |M|^{\beta-2} < \infty \quad\text{and}\quad \sum_{M\neq 0} |h^\wedge(M)|^2 |M|^{\beta-2} < \infty$$

$$\text{for every } \beta<\gamma. \tag{36}$$

Also, from [9; p. 56], we obtain that

$$\left.\begin{aligned} &\sum_M g^\wedge(M)\, e^{i(M,X)-|M|t} \to g(X) \ \text{ uniformly on } T_2 \text{ as } t\to 0, \\ &\sum_M h^\wedge(M)\, e^{i(M,X)-|M|t} \to h(X) \ \text{ uniformly on } T_2 \text{ as } t\to 0. \end{aligned}\right\} \tag{37}$$

Next we observe (since the Hausdorff dimension of E is $\leqslant\alpha$ and since $\bar{D}(0, r_2)\cap E$ is a closed set and since, furthermore, the Hausdorff dimension of $\bar{D}(0, r_2)\cap E$ is the same as the capacity dimension of $\bar{D}(0, r_2)\cap E$, [5, p. 90]) that the β-capacity of $\bar{D}(0, r_2)\cap E=0$ for $\alpha<\beta$, i.e.

$$C_\beta[\bar{D}(0, r_2)\cap E]=0 \quad \text{for} \quad \alpha<\beta. \tag{38}$$

We next invoke the following lemma which we shall prove in Section 5 of this paper:

Lemma. *Let F be a closed set contained in $D(0, 1)$ with $C_\beta(F)=0$, $1<\beta<2$. Suppose that*

$$\text{(i)} \sum_{M\neq 0} |c_M|^2 |M|^{\beta-2} < \infty,$$

$$\text{(ii)} \lim_{t\to 0} \sum_M c_M e^{i(M,X)-|M|t} = 0 \ \text{ for } X \text{ in } T_2 \sim F.$$

Then $c_m=0$ for every M.

(The above lemma is the two dimensional analogue of [3; Theorem 5, p. 36]. The proof of the above lemma which we shall give in Section 5 of this paper will have many points in common with this last named reference.)

By selecting a β such that $\alpha<\beta<\gamma$ and recalling that $\gamma<2$, we conclude from (33), (34), (36), (37), (38), Minkowski's inequality, and the lemma that

$$\text{and}\quad \left.\begin{aligned} (-i)[u_1^\wedge(M)\, n + v_1^\wedge(M)\, m] &= g^\wedge(M) \ \text{ for every } M \\ (i)[u_1^\wedge(M)\, m - v_1^\wedge(M)\, n] &= h^\wedge(M) \ \text{ for every } M. \end{aligned}\right\} \tag{39}$$

Next, if $w(X)$ is a function in L^1 on T_2 with Fourier series $S[w]=\sum_m w^\wedge(M)\,e^{i(M,X)}$, we shall set for $t>0$

$$w(X,t)=\sum_M w^\wedge(M)\,e^{i(M,X)-|M|t}.$$

It follows that for $t>0$, $u_1(X,t)$ and $v_1(X,t)$ are functions in class C^∞ on the plane and that their derivatives are obtained by differentiating under the summation sign. We conclude in particular from (25), (35), and (39) that for $t>0$

$$\left.\begin{aligned} -\partial u_1(X,t)/\partial y-\partial v_1(X,t)/\partial x&=g(X,t)\\ \text{and}\qquad \partial u_1(X,t)/\partial x-\partial v_1(X,t)/\partial y&=h(X,t)\end{aligned}\right\} \tag{40}$$

Consequently, for our fixed two simplex τ_0 in (24), we have from (40) that

$$\left.\begin{aligned} \int_{\partial\tau_0} u_1(X,t)\,dx-v_1(X,t)\,dy&=\int_{\tau_0} g(X,t)\,dX\\ \text{and}\qquad \int_{\partial\tau_0} u_1(X,t)\,dy+v_1(X,t)\,dx&=\int_{\tau_0} h(X,t)\,dX.\end{aligned}\right\} \tag{41}$$

Now, from the continuity of $u_1(X)$, $v_1(X)$, $g(X)$, and $h(X)$, from (29) and (35), and from [9, p. 56], we obtain that

$$\left.\begin{aligned} &u_1(X,t)\to u_1(X),\ v_1(X,t)\to v_1(X),\ g(x,t)\to g(X)\\ \text{and}\qquad &h(X,t)\to h(X)\ \text{ uniformly in } X \text{ as } t\to 0.\end{aligned}\right\} \tag{42}$$

We conclude from (41) and (42) that

$$\left.\begin{aligned} \int_{\partial\tau_0} u_1(X)\,dx-v_1(X)\,dy&=\int_{\tau_0} g(X)\,dX\\ \text{and}\qquad \int_{\partial\tau_0} u_1(X)\,dy+v_1(X)\,dx&=\int_{\tau_0} h(X)\,dX.\end{aligned}\right\} \tag{43}$$

But by (19), $\tau_0\subset D(0,r_1)$ and by (26), $g(X)=h(X)=0$ for X in $\bar{D}(0,r_1)$. We conclude from (43) that

$$\left.\begin{aligned} \int_{\partial\tau_0} u_1(X)\,dx-v_1(X)\,dy&=0\\ \text{and}\qquad \int_{\partial\tau_0} u_1(X)\,dy+v_1(X)\,dx&=0.\end{aligned}\right\}$$

Consequently, (24) is established and the proof of the sufficiency will be complete once the lemma is established. We now prove the lemma.

5. Proof of the Lemma

We shall suppose from the start that F is a non-empty closed set contained in $D(0, 1)$, for the lemma is already known in the case F is empty, see [9, p. 65]. We first recall that $C_\beta(F)=0$ means that for every probability measure μ on the plane with its support contained in F the following fact obtains:

$$\int_F \int_F |X-P|^{-\beta} d\mu(X)\, d\mu(P) = +\infty. \tag{44}$$

Next, we introduce the function $G_\beta^*(X)$, $1<\beta<2$, defined as follows on the plane:

$$\left.\begin{aligned} G_\beta^*(X) &= |X|^{-\beta} + \lim_{R\to\infty} \sum_{1\leq |M| \leq R} [|X+2\pi M|^{-\beta} - |2\pi M|^{-\beta}] \\ &\qquad \text{for}\quad (2\pi)^{-1}X \neq \text{integral lattice point}, \\ G_\beta^*(X) &= +\infty \quad \text{for}\quad (2\pi)^{-1}X = \text{integral lattice point}. \end{aligned}\right\} \tag{45}$$

(For other approaches to the function $G_\beta^*(X)$, see [2, p. 50] or [11, p. 40] and (49) and (53) below.)

We observe that for S a compact set contained in $D(0, 2\pi R_0)$, the following limit is finite and furthermore

$$\lim_{R\to\infty} \sum_{R_0 \leq |M| \leq R} [|X+2\pi M|^{-\beta} - |2\pi M|^{-\beta}] \quad \text{exists uniformly for } X \text{ in } S. \tag{46}$$

Also, we observe that for $(2\pi)^{-1}X \neq$ integral lattice point,

$$\lim_{R\to\infty} \sum_{|M|\leq R} [|X+2\pi(M+M_0)|^{-\beta} - |X+2\pi M|^{-\beta}] = 0. \tag{47}$$

We conclude from (45), (46), and (47) that

$G_\beta^*(X)$ is a periodic function of period 2π in each variable, and $G_\beta^*(X)$ is continuous in the neighborhood of every point not of the form $2\pi M$. (48)

It follows from (45) and (48) that $G_\beta^*(X)$ assumes its minimum value. We designate this minimum value by η_β and set

$$G_\beta(X) = G_\beta^*(X) - \eta_\beta + 1. \tag{49}$$

Then it follows from (45), (46), (48), and (49) that

$$\left.\begin{aligned} &\text{(i)}\ \ G_\beta(X) \text{ is continuous in the torus sence on } T_2 - 0, \\ &\text{(ii)}\ \ G_\beta(X) \geq 1 \text{ for } x \text{ in } T_2, \\ &\text{(iii)}\ \ G_\beta(X) \text{ is in } L^1 \text{ on } T_2. \end{aligned}\right\} \tag{50}$$

Also, it follows from (44), (45), and (49) that $C_\beta(F)=0$ means that

$$\int_F \int_F G_\beta(X-P)\, d\mu(X)\, d\mu(P) = +\infty \tag{51}$$

for every non-negative Borel measure μ defined on the Borel subsets of T_2 with the property that $\mu(T_2-F)=0$ and $\mu(F)=1$.

From (50), it follows that we can introduce the Fourier series of G_β, which we designate by $S[G]$ and write as

$$S[G_\beta]=\sum_M G_\beta^\wedge(M)\,e^{i(M,X)}. \tag{52}$$

From (45), (46) and (49) we obtain that for $M_0 \neq 0$,

$$\begin{aligned}(2\pi)^2 G_\beta^\wedge(M_0) &= \lim_{R\to\infty} \sum_{|M|\leqslant R} \int_{T_2+2\pi M} |X|^{-\beta} e^{-i(M_0,X)} dX \\ &= \lim_{R\to\infty} \int_{D(0,R)} e^{-i(M_0,X)} |X|^{-\beta}\,dx \\ &= 2\pi\beta \int_0^\infty J_1(r)\,r^{-\beta}\,dr/|M_0|^{2-\beta}.\end{aligned}$$

Observing that $\lim_{t\to 0} \sum_M G_\beta^\wedge(M)e^{-|M|t} = +\infty$, [9, p. 55], we obtain from the above computation that

$$G_\beta^\wedge(M)=K/|M|^{2-\beta} \quad \text{for} \quad M\neq 0 \quad \text{where} \quad K>0. \tag{53}$$

Next, we note from (45), (46), and (49) that if $(2\pi)^{-1}X \neq$ integral lattice point, then there exists a neighborhood of X such that G_β is in class $C^{(\infty)}$ in this neighborhood, and that in this neighborhood all the partial derivatives of G_β can be computed under the summation sign in (45). In particular, with Δ designating the Laplace operator, we infer from (45), (46) and (49) that

$$\Delta G_\beta(X)=\beta^2\{|X|^{-(\beta+2)}+\lim_{R\to\infty} \sum_{1\leqslant (M)\leqslant R} |X+2\pi M|^{-(\beta+2)}\}$$

$$\text{for } (2\pi)^{-1}X \neq \text{integral lattice point}. \tag{54}$$

We conclude from (54) that

$G_\beta(X)$ is subharmonic and in class C^∞ in a neighborhood of every point in T_2-0. (55)

We also conclude from (45), (49), and (50) that

$G_\beta(X)$ is lower semi-continuous on T_2. (56)

Furthermore, from (45), (46), (49) and (50), we obtain that

there exists a constant K_1 such that

$$(\pi r^2)^{-1}\int_{D(0,r)} G_\beta(X+P)\,dP \leqslant K_1 G_\beta(X)$$

for every X and for $0<r\leqslant 1$. (57)

Consequently, if we designate by F_k, the closed set defined as

$$F_k = \{X: \text{distance } (X, F) \leqslant k^{-1}\},$$

where k is a positive integer, we obtain from properties (45) to (57) and from the theory expounded in [5, pp. 24–41] that for each k there exists a non-negative Borel measure μ_k defined on the Borel subsets of T_2 with the following properties:

$$\left.\begin{array}{l}
\text{(i) } \mu_k(T_2 - F_k) = 0 \text{ and } \mu_k(F_k) = 1, \\
\text{(ii) } \int_{F_k}\int_{F_k} G_\beta(X-P)\,d\mu_k(X)\,d\mu_k(P) = V_k, \\
\text{(iii) } U_k(X) = \int_{F_k} G_\beta(X-P)\,d\mu_k(P) \text{ is such that} \\
\qquad 0 \leqslant U_k(X) \leqslant V_k \text{ for every } X \\
\text{and} \\
\qquad U_k(X) = V_k \text{ for } X \text{ in } F_k.
\end{array}\right\} \quad (58)$$

Also, it follows from (51) and the theory expounded in [5, pp. 22–23] that

$$\lim_{k\to\infty} V_k = +\infty. \quad (59)$$

We denote by

$$S[d\mu_k] = \sum_M a_M^k e^{i(M,X)} \quad (60)$$

the Fourier-Stieltjes series of μ_k and obtain from (52), (53), (57), (58) and (60) that

$$\sum_M |a_M^k|^2 G_\beta^\wedge(M) = (4\pi^2)^{-2} V_k. \quad (61)$$

and that

$$S[U_k] = (4\pi^2) \sum_M a_M^k G_\beta^\wedge(M) e^{i(M,X)}. \quad (62)$$

Next, we set

$$f(X,t) = \sum_M c_M e^{i(m,X)-|M|t} \quad \text{for} \quad t > 0, \quad (63)$$

and observe from (i) of the lemma and Schwarz's inequality that

$$\sum_{M\neq 0} |c_M|\,|M|^{-2} < \infty. \quad (64)$$

We consequently obtain from (64), (ii) of the lemma, and [8; Lemma 6, p. 609] that

$$\lim_{t\to 0} \int_{T_2-F_k} |f(X,t)|\,dX = 0 \text{ for every } k. \quad (65)$$

Let M_0 be a fixed integral lattice point. The proof of the lemma will be complete if we show

$$c_{M_0} = 0. \quad (66)$$

To establish (66), we observe from (iii) of (58), (62), (63), and (65) that for every k,

$$4\pi^2 c_{M_0} = \lim_{t\to 0}\int_{T_2} f(X,t)\, e^{-i(M_0,X)}\, dX$$

$$= \lim_{t\to 0}\int_{F_k} f(X,t)\, e^{-i(M_0,X)}\, dX$$

$$= V_k^{-1} \lim_{t\to 0}\int_{F_k} f(X,t)\, U_k(X)\, e^{-i(M_0,X)}\, dX$$

$$= V_k^{-1} \lim_{t\to 0}\int_{T_2} f(X,t)\, U_k(X)\, e^{-i(M_0,X)}$$

$$= V_k^{-1} \lim_{t\to 0}\sum_M c_M\, a^k_{M_0-M}\, G^{\wedge}_{\beta}(M_0-M)(4\pi^2)^2 e^{-|M|t}.$$

But then it follows from Schwarz's inequality and (61) that

$$|c_{M_0}| \leqslant V_k^{-\frac{1}{2}} \{\sum_M |c_M|^2 G^{\wedge}_{\beta}(M_0-M)\}^{\frac{1}{2}}. \tag{67}$$

But it follows from (i) of the lemma and (53) that the sum on the right side of the inequality in (67) is finite. Consequently, there is a constant K_2 such that

$$|c_{M_0}| \leqslant K_2/V_k^{\frac{1}{2}} \quad \text{for} \quad k=1,2,\ldots .$$

But then it follows immediately from (59) that $c_{M_0}=0$. (66) is established, and the proof of the lemma is complete.

ACKNOWLEDGEMENT

This research was sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under AFOSR Grant No. AF-AFOSR 694-64.

University of California, Riverside, California

BIBLIOGRAPHY

1. Besicovitch, A. S., On sufficient conditions for a function to be analytic, and on behavior of analytic functions in the neighborhood of non-isolated singular points. *Proc. London Math. Soc.* (2) *32*, 1–9 (1931).
2. Bochner, S., Zeta functions and Green's functions for linear partial differential operators of elliptic type with constant coefficients. *Ann. of Math.* (2) *57*, 32–36 (1953).
3. Broman, A., On two classes of trigonometric series. Ph. D. Dissertation, Uppsala, 1947.
4. Frostman, O., Potentiel d'équilibre et capacité des ensembles avec quelques applications à la théorie des fonctions. *Comm. of the Math. Lund 3*, 1–118 (1935).
5. Hausdorff, F., Dimension und äusseres Mass. *Math. Ann. 79*, 157–179 (1919).
6. Hewitt, E., and Stromberg, K., *Real and abstract analysis.* Springer-Verlag, New York, Inc., New York, 1965.
7. Saks, S., *Theory of the integral.* Monografie Matematyczne, Vol. *7*, Warsaw, 1937.

8. SHAPIRO, V. L., The divergence theorem for discontinuous vector fields. *Ann. of Math.* (2) *68*, 604–624 (1958).
9. —— Fourier series in several variables. *Bull. Amer. Math. Soc. 70*, 48–93 (1964).
10. —— Topics in Fourier and geometric analysis. *Mem. Amer. Math. Soc.*, No. *39* (1961).
11. WAINGER, S., Special trigonometric series in k-dimensions, *Mem. Amer. Math. Soc.*, No. *59* (1965).

Tryckt den 23 mars 1967

Uppsala 1967. Almqvist & Wiksells Boktryckeri AB