ARKIV FÖR MATEMATIK Band 4 nr 36

Communicated 12 April 1961 by O. Frostman

Remarks on potential theory[1]

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Although some of the results of the present note hold for potential theories constructed with a variety of kernels, we are interested only in the applications to Bessel potentials [2], Riesz potentials and the logarithmic potential. Accordingly we state everything in the terminology of Bessel potentials on the space R^n, since the results are then easily extended to other classes.

The kernel $G_\alpha(x,y)$ is defined on R^n for all real α by $G_\alpha(x,y) = G_\alpha(x-y)$, where $G_\alpha(x) = [2^{(n+\alpha-2)/2}\, \pi^{n/2}\, \Gamma(\alpha/2)]^{-1} |x|^{(\alpha-n)/2}\, K_{(n-\alpha)/2}\,(|x|)$, the function $K_\nu(z)$ being the modified Bessel function of the third kind. We remark that the kernel vanishes identically when α is a non-positive even integer. In a neighborhood of the origin $G_\alpha(x)$ is equivalent to $|x|^{\alpha-n}$ if $\alpha < n$, and equivalent to $\log(1/|x|)$ if $\alpha = n$, while for large values of $|x|$ $G_\alpha(x)$ is equivalent to $|x|^{(\alpha-n-1)/2} e^{-|x|}$. The kernel diminishes monotonically with increasing $|x|$, and for all real α and $x \neq 0$ is an analytic function for which $(1-\Delta) G_\alpha = G_{\alpha-2}(x)$. Moreover, for positive α and β the composition law, $G_\alpha \ast G_\beta = G_{\alpha+\beta}$ is valid.

The Bessel potentials of order $\alpha > 0$ form the space P^α of all functions which coincide except for a set of 2α-capacity zero with convolutions of the form $u = G_\alpha \ast f$, where f is in L^2; the integral exists, except, perhaps, for a set of the corresponding capacity zero. The norm of u in P^α equals the L^2 norm of the corresponding f and P^α is a Hilbert space which also appears as the perfect functional completion of the space of all (Bessel) potentials of order 2α of measures of finite 2α energy. In contradistinction to Riesz potentials, the Bessel potentials are always L^2 functions, and we have the following convenient formula for the norm in terms of the Fourier transform:

$$\|u\|_\alpha^2 = \int (1+|\xi|^2)^\alpha |\hat{u}(\xi)|^2 d\xi.$$

For $0 < 2\alpha < n$ the potentials coincide locally with the Riesz potentials of the same order and have exactly the same exceptional sets; similarly, for $2\alpha = n$ the potentials are locally logarithmic potentials, and the sets of n-capacity zero are precisely those of the usual logarithmic capacity zero. For $2\alpha > n$ the potentials are continuous functions, and only the empty set has capacity zero.

The exceptional sets for the functional space P^α form the hereditary sigma-ring $\mathfrak{a}_{2\alpha}$, the sets of 2α-capacity zero. The capacity is defined as usual in potential theory, first for compact sets, after which the inner capacity is defined for an arbitrary set A as the supremum of the capacities of all compacts contained in A. The outer capacity of A is taken as the infimum of the inner capacities of all open sets containing A. A set is capacitable if the inner and outer capacities are equal, and a result of Choquet [4] extended by Aronszajn and Smith [1] guarantees that all analytic sets are capacitable. By the 2α-capacity, denoted by $\gamma_{2\alpha}$ we always understand the outer capacity.

[1] Paper written under Contract Nonr 853(04) with the Office of Naval Research.

It is clear that this is an outer measure. We say that a property holds exc. $\mathfrak{a}_{2\alpha}$ if it holds everywhere, except, perhaps, for a set in the class $\mathfrak{a}_{2\alpha}$.

We first establish a uniqueness theorem for Bessel potentials, that is, we show that different measures give rise to different potentials. Since the kernel decays exponentially, the class of measures for which the potential is not identically infinite contains measures which are not temperate distributions, hence the theorem cannot be obtained from an argument using Fourier transforms of temperate distributions. Our proof is a modification of that given by M. Riesz [7] for Riesz potentials. We show next that when the capacity is regarded as an outer measure, the only sets which are measurable are the sets of capacity zero and their complements. Finally, following Carleson, we extend a theorem of Frostman's concerning the relation of a capacity to the corresponding Hausdorff measure.

Theorem 1. *Let α be positive and μ and ν two positive Borel measures on R^n such that the Bessel potentials $G_\alpha\mu$ and $G_\alpha\nu$ are neither identically infinite; then if those potentials are equal almost everywhere (Lebesgue) the measures μ and ν are the same.*

Proof. Although we do not need it in the proof, we remark that the equality of the potentials almost everywhere implies that they are equal everywhere, in view of the Frostman mean value theorem [6], which holds for Bessel potentials as well as Riesz potentials.

From the composition formula for the kernel, $G_\alpha G_\beta = G_{\alpha+\beta}$, it follows that it is enough to prove the theorem for small values of α; in the sequel we shall take $0<\alpha<1$. Let φ be a C^∞ function with compact support; from an easy argument with the Fourier transforms we find that there exists a unique function $h(y)$ which is C^∞ and vanishes rapidly at infinity such that $G_\alpha h = \varphi$. Our theorem is then an immediate consequence of the following computation, if the interchange of order of integration is justified.

$$\int \varphi(x)\,d\mu(x) = \int G_\alpha h(x)\,d\mu(x) = \int G_\alpha \mu(y)\,h(y)\,dy = \int G_\alpha \nu(y)\,h(y)\,dy = $$
$$= \int \varphi(x)\,d\nu(x).$$

Thus the distributions μ and ν coincide and the measures are equal.

In view of the Fubini–Tonnelli theorem the change in the order of integration is legitimate if the potential $G_\alpha|h|$ is both μ-integrable and ν-integrable, and this will be a consequence of the estimate $G_\alpha|h|(x) \leqslant C G_\alpha(x)$ for large $|x|$ and an appropriate constant C. The remainder of our proof will consist in the setting up of such an estimate.

Let $\beta = 2-\alpha$, from $\varphi = G_\alpha h$ and the composition formula we have $G_\beta\varphi = G_2 h$, whence $h = (1-\Delta)G_\beta\varphi$. Since, for any real β, $(1-\Delta)G_\beta(x) = G_{\beta-2}(x)$ away from the origin, we infer that $h = G_{-\alpha}\varphi$. Of course, potentials are not defined for negative values of the index; here, however, we can either reason with the convolution of distributions, or, since we are interested in the behavior of the smooth $h(y)$ only for large values of $|y|$, we can easily justify the differentiation under the integral sign. It thus becomes clear that for large $|y|$, $|h(y)|$ is bounded by a constant multiple of $G_{-\alpha}(y)$, and this function is equivalent to $|y|^{(-\alpha-n-1)/2}e^{-|y|}$. Thus setting $g_\alpha(y) = |y|^{(\alpha-n-1)/2}$ $e^{-|y|}$ we find the existence of a constant C such that for $|y| \geqslant 1$ both the inequalities $|h(y)| \leqslant C g_{-\alpha}(y)$ and $G_\alpha(y) \leqslant C g_\alpha(y)$ hold.

If χ is the characteristic function of the unit sphere, then

$$G_\alpha|h| = G_\alpha * |h| = G_\alpha * (\chi|h|) + (\chi G_\alpha) * ((1-\chi)|h|) + ((1-\chi)G_\alpha) * ((1-\chi)|h|).$$

The sum of the first two terms is obviously of the order of magnitude of G_α at infinity, while the third is bounded by $C^2 g_\alpha * g_{-\alpha}$; our argument is complete if we show the existence of a constant C for which $g_\alpha * g_{-\alpha}(x) \leqslant C G_\alpha(x)$ for large $|x|$. We may write the inequality to be shown in the form

$$\int\limits_{\substack{|y|\geqslant 1\\ |x-y|\geqslant 1}} |x-y|^{(\alpha-n-1)/2} |y|^{(-\alpha-n-1)/2} e^{-[|x-y|+|y|-|x|]} dy \leqslant C|x|^{(\alpha-n-1)/2}$$

and observe that the contribution to the integral made by the integration over the domain $|y| \geqslant |x|$ surely satisfies an inequality of the required type. Since a similar comment holds for the integration over $|x-y| \geqslant |x|$, it is enough to consider the integration over the domain D defined by the inequalities $1 \leqslant |y| \leqslant |x|$ and $1 \leqslant |x-y| \leqslant |x|$. This domain is symmetric relative to the origin and x, and the integrand is larger on the half of the domain nearer the origin. Since, on that half, the inequality $|x-y| \geqslant |x|/2$ holds, the integral over D is bounded by

$$2\left(\frac{|x|}{2}\right)^{(\alpha-n-1)/2} \int\limits_{|y|\geqslant 1} |y|^{(-\alpha-n-1)/2} e^{-[|x-y|+|y|-|x|]} dy.$$

All that remains is to show that the integral above is bounded independently of x; for $n=1$ this is immediate, while for $n \geqslant 2$ we make use of the estimate

$$|x-y| + |y| - |x| \geqslant |y| \sin^2(\theta/2),$$

where θ is the angle between the radius vectors to y and x. Introducing polar coordinates and integrating over the surfaces on which the integrand is constant, we obtain

$$2\,\omega_{n-1} \int_0^{\pi/2} \int_1^{\infty} r^{(-\alpha-n-1)/2} e^{-r\sin^2(\theta/2)} (r\sin\theta)^{n-2} r\,dr\,d\theta,$$

where ω_{n-1} is the area of the surface of the unit sphere in R^{n-1}. Since the order of integration is immaterial, we integrate first with respect to θ to obtain

$$\int_0^{\pi/2} e^{-r\sin^2\theta/2} \sin^{n-2}\theta\,d\theta = 2^{n-2} \int_0^{1/2} e^{-rt} t^{(n-3)/2} (1-t)^{(n-3)/2} dt < 2^n \int_0^{\infty} e^{-rt} t^{(n-3)/2} dt =$$
$$= C r^{(1-n)/2},$$

from which the finiteness of the integral follows immediately.

Theorem 2. *The only sets measurable for the outer measure $\gamma_{2\alpha}$ are the sets of capacity zero and their complements.*

Proof. We recall certain facts which are essentially contained in [2]. For any set of finite 2α-capacity the set of all potentials v in P^α for which $v(x) \geqslant 1$ on A exc. $\mathfrak{a}_{2\alpha}$

forms a closed convex cone; the unique element u_A nearest the origin has the property $\|u_A\|_\alpha^2 = \gamma_{2\alpha}(A)$ and u_A is in fact the 2α-potential of a measure ν_A, a support for which is the closure of A. If $|\nu|$ denotes the total mass of the positive measure γ, and A is bounded, we have $|\mu_A| = \gamma_{2\alpha}(A)$ and $\gamma_{2\alpha}(A) = \inf|\nu|$ over all ν of finite energy for which $G_{2\alpha}\nu(x) \geqslant 1$ on A exc $\mathfrak{a}_{2\alpha}$. Thus we may speak of a capacitary potential and a capacitary distribution for arbitrary sets of finite capacity, in particular for bounded sets.

Suppose M is measurable for the 2α-capacity; then for any set S we must have $\gamma_{2\alpha}(S) = \gamma_{2\alpha}(MS) + \gamma_{2\alpha}(S - MS)$. Take S as a sphere of radius R and let μ_1 be the capacitary distribution for MS, and μ_2 that for $S - MS$. We suppose first that neither measure is zero. Now $\gamma_{2\alpha}(S) = |\mu_1| + |\mu_2|$, and $u_1(x) = G_{2\alpha}\mu_1(x) \geqslant 1$ on MS, except perhaps for a set of capacity zero, while the strictly positive $u_1(x)$ has a positive lower bound on the compact S. Since a similar assertion holds for $u_2(x) = G_{2\alpha}\mu_2(x)$, we find that if $\mu = \mu_1 + \mu_2$, there exists a positive ε such that $G_{2\alpha}\mu(x) \geqslant 1 + \varepsilon$ on S, except for a set of capacity zero, hence, in view of the Frostman mean value theorem, everywhere on S. Thus the measure $(1+\varepsilon)^{-1}\mu$ has total mass smaller than $\gamma_{2\alpha}(S)$ and a potential $\geqslant 1$ on S, a contradiction. It follows that either μ_1 or μ_2 is zero, and therefore that either MS or $S - MS$ has capacity zero. Since R is arbitrary, the theorem is proved.

For $0 < 2\alpha \leqslant n$ there is associated with the capacity $\gamma_{2\alpha}$ a Hausdorff measure determined by the function $h(t) = t^{n-2\alpha}$ if $2\alpha < n$ and $h(t) = -1/\log(t)$ when $2\alpha = n$. For $|x| \leqslant 1$ the function $h(|x|)$ is equivalent to the reciprocal of $G_{2\alpha}(x)$. Frostman [6] has proved that if a set has positive capacity the corresponding Hausdorff measure cannot vanish, and Erdös and Gillis [5] have shown that in the logarithmic case the Hausdorff measure must be infinite. An elegant demonstration of their result has been given by Carleson [3], whose idea we follow here.

Theorem 3. *If A has positive capacity the corresponding Hausdorff measure is infinite.*

Proof. We will suppose the Hausdorff measure finite and deduce a contradiction. The Hausdorff measure is realized by a covering of A by a sequence of open sets, each a covering of A by a countable family of spheres. For A we may therefore substitute the intersection of this sequence of open sets to obtain a larger set which is G_δ with the same Hausdorff measure and possibly larger capacity. Since A may therefore be supposed capacitable, there exists a non-trivial measure μ of finite 2α-energy whose compact support is contained in A.

Let $\chi_n(t)$ be defined on the real axis as the characteristic function of the interval $0 \leqslant t \leqslant 2^{-n}$. The sequence of positive numbers

$$m_n = \int\int \chi_n(|x-y|)\, G_{2\alpha}(x,y)\, d\mu(x)\, d\mu(y)$$

converges monotonically to 0, hence there exists an infinite subsequence m_{n_k} for which $\sum_{k=0}^{\infty} m_{n_k}$ is finite. We set $f(t) = 1 + \sum_{k=0}^{\infty} \chi_{n_k}(t)$ to obtain a monotone decreasing function on the positive axis which converges to infinity as t approaches 0 from the right. Moreover $f(t/2) \leqslant 1 + f(t) \leqslant 2f(t)$. Now let $K(|x-y|) = f(|x-y|)G_{2\alpha}(x-y)$ and $K\mu(x) = \int K(|x-y|)d\mu(y)$; from the construction we have $K\mu(x)$ μ-integrable and lower semi-continuous.

For $m > 0$ the set E_m upon which $K\mu(x) \leqslant m$ is closed, and with increasing m, $\mu(E_m)$ converges to $\mu(A)$, hence there exists a compact F contained in A, such that if ν is the restriction of μ to F, then $K\nu(x)$ is bounded on F and $\nu(F) = \mu(F) > 0$.

It is easy to infer from the behavior of $G_{2\alpha}(x)$ near the origin that there exists a constant C such that $G_{2\alpha}(x/2) \leqslant C G_{2\alpha}(x)$, and in view of the fact that $f(t)$ has a corresponding property it follows that $K(|z-y|/2) \leqslant 2CK(|z-y|)$. This inequality is precisely what is required for the application of an argument of Frostman to show that $K\nu(x)$ is bounded on the whole space. For the sake of completeness we carry out that proof. If m is the bound for $K\nu$ on the set F and x is a point not in F, let z be any nearest point of F to x. Now $2|x-y| \geqslant |z-y|$, for y in F, whence $K(|x-y|) \leqslant K(|z-y|/2) \leqslant 2CK(|z-y|)$ holds for all y in the support of ν. Thus $K\nu(x) \leqslant 2C$. $K\nu(z) \leqslant 2\,Cm = M$.

Let $S = S(x,r)$ be a sphere with center at x and radius r. We have $M \geqslant K\nu(x) \geqslant \int_S K(|x-y|)\,d\nu(y) \geqslant \nu(S)\,K(r)$, whence $\nu(S)/M \leqslant 1/K(r)$. If F is covered by a family of spheres $S_i = S(x_i, r_i)$, where $r_i \leqslant \varrho \leqslant 1$, then $\Sigma 1/K(r_i) \geqslant \Sigma \nu(S_i)/M \geqslant \nu(F)/M$, while, on the other hand, $\Sigma 1/K(r_i) \leqslant [1/f(\varrho)]\Sigma 1/G_{2\alpha}(r_i)$. Since $h(r)$ is equivalent to the reciprocal of $G_{2\alpha}(r)$ for these small values of r, there exists a constant C such that $\Sigma h(r_i) \geqslant Cf(\varrho)\nu(F)/M$, and since $f(\varrho)$ tends to infinity as ϱ decreases, the Hausdorff measure of F is infinite, and therefore also the Hausdorff measure of A.

Using the notations and mechanics of the proof of Theorem 3, where A has a positive inner capacity and $S = S(x,\varrho)$ we have

$$M \geqslant \int_S K(|x-y|)\,d\nu(y) = \int_S f(|x-y|)\,G_{2\alpha}(x-y)\,d\nu(y) \geqslant f(\varrho)\int_S G_{2\alpha}(x-y)\,d\nu(y),$$

whence $\int_S G_{2\alpha}(x-y)\,d\nu(y) \leqslant M/f(\varrho)$, an estimate independent of x. If $G_{2\alpha}^{(\varrho)}(x-y) = \inf\,(G_{2\alpha}(x-y), 1/\varrho)$, this kernel is continuous and so also is its "potential" $G_{2\alpha}^{(\varrho)}\nu(x)$. As ϱ approaches 0, the functions $G_{2\alpha}^{(\varrho)}\nu(x)$ converge monotonically to $G_{2\alpha}\nu(x)$, and this convergence is uniform, since $G_{2\alpha}\nu(x) - G_{2\alpha}^{(\varrho)}\nu(x) \leqslant \int_S G_{2\alpha}(x-y)\,d\nu(y) \leqslant M/f(\varrho)$ and $f(\varrho)$ converges to infinity. Thus $G_{2\alpha}\nu(x)$ is not only bounded but continuous, and since it vanishes exponentially at infinity, it is uniformly continuous on R^n. Following Carleson, we have therefore established

Theorem 4. *If A is a set of positive inner capacity of order 2α, there exists a non-trivial measure ν of finite 2α-energy concentrated on a compact subset of A for which the potential $G_{2\alpha}\nu$ is uniformly continuous on R^n.*

We remark that the theorem is trivial if $2\alpha > n$; for smaller values of 2α, however, the hypothesis that the inner capacity be positive is essential, for a set of zero inner capacity supports no mass of finite energy. Choquet [4] has shown the existence of a wide class of non-capacitable sets for sub-additive capacities having the property that the capacity of a point is zero; his theorem applies then to $\gamma_{2\alpha}$ when $2\alpha \leqslant n$. It is interesting to note that this is not the case for $2\alpha > n$, as the following theorem shows.

Theorem 5. *If $2\alpha > n$ every set is capacitable.*

Proof. If a set A is not capacitable, its inner and outer capacities are different, and there exists a sequence K_n of compacts contained in A whose capacities converge to the (necessarily finite) inner capacity of A. The union F of these compacts is a capacitable subset of A with capacity equal to the inner capacity of A. Thus the set $A-F$ has positive outer capacity and zero inner capacity. Since a point is a compact set of positive capacity when $2\alpha > n$, it follows that A equals F and therefore that A is capacitable.

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Tryckt den 29 januari 1962

Uppsala 1962. Almqvist & Wiksells Boktryckeri AB