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# Nestinvertible matrices

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## Introduction

In the present paper we will define and characterize a class of unimodular matrices which we call nestinvertible matrices. The defining property is that if $A$ is a nestinvertible matrix and $B$ is its inverse, then each lower right square submatrix of adjacent elements of $B$ is the inverse of the corresponding upper left submatrix of $A$. The nestinvertible matrices are then characterized directly by the properties of their elements. This characterization is proved using formal power series. The hint to use such series in this connexion was given by Prof. O. Hanner.

Except in the final section the matrices are over an arbitrary ring with a unit. This is no vain generality since it will make the formulae valid for block matrices. However, in our example in the last section we will use only real matrices. The intended use of our results and concepts is for constructing test examples of matrix inversion and $LR$-decomposition.

## 1. Definitions and notations

For a given $n\times n$ matrix $A$, let ${}^kA$ denote the $k\times k$ submatrix of adjacent elements that contains the upper left corner of $A$ and let $A_k$ denote the corresponding lower right submatrix.

**Definition 1.** An $n\times n$ matrix $A$ over a ring $\mathfrak{R}$ with unit (denoted by 1) is called *nestinvertible* if it is invertible and if furthermore $({}^kA)^{-1}=(A^{-1})_k$ for $1\leqslant k\leqslant n$.

**Definition 2.** The *nest class* $N(n, \mathfrak{R})$ is the set of all $n\times n$ nestinvertible matrices $A$ over $\mathfrak{R}$ which satisfy ${}^1A=(1)$.

It is easy to see that those matrices that are of the form (exemplified here on $5\times 5$ matrices)

$$A=\begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ a_1 & 1 & 0 & 0 & 0\\ a_2 & a_1 & 1 & 0 & 0\\ a_3 & a_2 & a_1 & 1 & 0\\ a_4 & a_3 & a_2 & a_1 & 1\end{pmatrix}=f(T),\ \text{with}\ T=\begin{pmatrix} 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\end{pmatrix}$$

constitute a subset of the nest class. They may, in fact, be regarded as power series in the nilpotent variable $T$ with coefficients in $\mathfrak{R}$ and with 1 as leading coefficient. Hence this set forms a group, and it is trivial to verify that the conditions of Definition 1 and 2 are satisfied.

**Definition 3.** The group of all $n\times n$ matrices $A=f(T)$, where $f$ denotes a formal power series with coefficients in $\mathfrak{R}$ and with leading coefficients 1 and where $T$ is the $n\times n$ matrix whose elements are 1 in the first subdiagonal and 0 elsewhere, is called the *power series group* $PS(n, \mathfrak{R})$.

**Definition 4.** We denote by $PS'(n, \mathfrak{R})$ the group of matrices which are transposes of matrices in $PS(n, \mathfrak{R})$.

Finally, we need the notion of unimodularity of a matrix. We make the following definition which generalizes the one given for general fields by Dieudonné [1], p 36.

**Definition 5.** The *unimodular group* $SL(n, \mathfrak{R})$ is the group generated by those $n\times n$ matrices $B_{ij}(\lambda)$ which are obtained from the unit matrix by replacing one of its zeros (the one in position $(i, j)$) with an arbitrary element $\lambda\in\mathfrak{R}$.

## 2. Results

The results of this paper are summarized in the following theorem.

**Theorem.** $N(n, \mathfrak{R})=PS(n, \mathfrak{R})PS'(n, \mathfrak{R})\subset SL(n, \mathfrak{R})$.

The theorem states that all products of one element in the power series group and the transpose of another element in this group, taken in that order, belong to the nest class and that all elements of the nest class may be so obtained and are unimodular.

To prove that $N(n,\mathfrak{R})\supset PS(n, \mathfrak{R})PS'(n, \mathfrak{R})$ we suppose that $L\in PS(n, \mathfrak{R})$ and $R\in PS'(n, \mathfrak{R})$. Then the relation ${}^k(LR)={}^kL\,{}^kR$ holds because of the lower triangularity of $L$ and the upper triangularity of $R$. But $({}^kL\,{}^kR)^{-1}=({}^kR^{-1})({}^kL)^{-1}=(R^{-1})_k(L^{-1})_k=((LR)^{-1})_k$, so that the matrix $LR$ is indeed nestinvertible. Hence $LR\in N(n, \mathfrak{R})$ since ${}^1(LR)=({}^1L)({}^1R)=(1)$.

For the proof of the opposite inclusion we need a pair of lemmas on $LR$-decomposition.

**Lemma 1.** If ${}^kA$ is invertible for $1\leqslant k\leqslant n$ then there exists a unique decomposition $A=LR$, where $L$ is a lower triangular matrix, $R$ is upper triangular and all main diagonal elements of $L$ are 1. Also ${}^kA=({}^kL)({}^kR)$ and all ${}^kL$, ${}^kR$ are invertible.

**Lemma 2.** If $A_k$ is invertible for $1\leqslant k\leqslant n$ then there exists a unique decomposition $A=RL$ where $L$ and $R$ are as in Lemma 1. Also $A_k=(R_k)(L_k)$ and all $L_k$, $R_k$ are invertible.

To prove Lemma 1 we suppose inductively that its results have been found true for matrices of order up to $n-1$. This is trival for $n=2$. From the block matrix formula

$$A=\begin{pmatrix} {}^{n-1}A & b \\ c & d\end{pmatrix}=\begin{pmatrix} {}^{n-1}L & 0 \\ c\,({}^{n-1}R)^{-1} & 1\end{pmatrix}\begin{pmatrix} {}^{n-1}R & ({}^{n-1}L)^{-1}b \\ 0 & d-c\,({}^{n-1}A)^{-1}b\end{pmatrix}=LR,$$

where $b$ is a column, $c$ is a row and $d$ is an element of $\mathfrak{R}$ follows both the existence and the uniqueness of the $LR$-decomposition for $A$. The inverse of $L$ is

$$L^{-1}=\begin{pmatrix} ({}^{n-1}L)^{-1} & 0 \\ -c\,({}^{n-1}A)^{-1} & 1\end{pmatrix}$$

and $R=L^{-1}A$ is also invertible as the product of two invertible matrices. Hence Lemma 1 is proved. The proof of Lemma 2 is analogous.

Now take $A\in N(n,\mathfrak{R})$ and construct its $LR$-decomposition $A=LR$. By Lemma 1 we have ${}^kA=({}^kL)({}^kR)$. But, as $A^{-1}=(R^{-1})(L^{-1})$ we get from Lemma 2 that $(A^{-1})_k=(R^{-1})_k(L^{-1})_k$. Furthermore, the relations $(R^{-1})_k=(R_k)^{-1}$, $(L^{-1})_k=(L_k)^{-1}$ are evident from the block decompositions of $R$ and $L$ respectively,

$$RR^{-1}=\begin{pmatrix} a & b \\ 0 & R_k\end{pmatrix}\begin{pmatrix} c & d \\ 0 & (R^{-1})_k\end{pmatrix}=R^{-1}R=\begin{pmatrix} E_{n-k} & 0 \\ 0 & E_k\end{pmatrix},$$

where $E_{n-k}$ and $E_k$ denote unit matrices of order $n-k$ and $k$. Now, as

$${}^kA=({}^kL)({}^kR)=((A^{-1})_k)^{-1}=((L^{-1})_k)^{-1}((R^{-1})_k)^{-1}=(L_k)(R_k)$$

we can again appeal to Lemma 1 and deduce that ${}^kL=L_k$, ${}^kR=R_k$. This, however, means that the elements of $L$ and $R$ do not vary along diagonals parallel to the main diagonal. For $L$ this immediately implies that $L\in PS(n,\mathfrak{R})$ but also $R\in PS'(n,\mathfrak{R})$ follows, since by hypothesis ${}^1A=(1)$ and this has for consequence that the first element in the main diagonal of $R$ is 1.

Now we have proved that $N(n,\mathfrak{R})=PS(n,\mathfrak{R})\,PS'(n,\mathfrak{R})$. The assertion $N(n,\mathfrak{R})\subset SL(n,\mathfrak{R})$ is proved if we can show that $PS(n,\mathfrak{R})\subset SL(n,\mathfrak{R})$. But it is easy to prove that all lower triangular matrices whose main diagonals consist of ones are unimodular, and hence also those of the special kind that constitute $PS(n,\mathfrak{R})$. In fact, let $A=(a_{ij})$ be such a matrix, then

$$A=B_{21}(a_{21})B_{31}(a_{31})\dots B_{n1}(a_{n1})B_{32}(a_{32})\dots B_{n2}(a_{n2})\dots B_{n,\,n-1}(a_{n,\,n-1}),$$

whence $A\in SL(n,\mathfrak{R})$ by Definition 5.

## 3. Applications

The nest class $N,(n,\mathfrak{R})$ is a class of matrices with easily determined inverses. Suppose we have a matrix $A\in(n,\mathfrak{R})$ and write it with the power series notation introduced earlier,

$$A=f(T)\,[g(T)]'=f(T)\,g(T')$$

then

$$A^{-1}=(g(T'))^{-1}(f(T))^{-1}.$$

Since there are a certain number of simple functions $f(x)$ for which the power series of both $f(x)$ and $1/f(x)$ are known explicitly, this gives us a method of constructing pairs of reciprocal matrices whose elements may be given explicitly and by "the same formula" for any order $n$. As an example we may take $f(x) = g(x) = (1-x)^2$. With this choice, $A = (a_{ij})$ is given by the formulae

$$\begin{aligned} &a_{11} = 1,\ a_{22} = 5,\ a_{ii} = 6 \text{ for } i \geqslant 3, \\ &a_{12} = a_{21} = -2,\ a_{i,\,i+1} = a_{i+1,\,i} = -4 \text{ for } i \geqslant 2, \\ &a_{i,\,i+2} = a_{i+2,\,i} = 1,\ 1 \leqslant i \leqslant n-2, \\ &\text{and } a_{ij} = 0 \text{ for } |i-j| > 2. \end{aligned}$$

The elements of the inverse $A^{-1} = (b_{ij})$ are given by

$$b_{n+1-p,\,n+1-q} = \begin{cases} \frac{1}{6}\, p\,(p+1)\,(3q-p+1) \text{ for } q \geqslant p-1, \\ \frac{1}{6}\, q\,(q+1)\,(3p-q+1) \text{ for } q \leqslant p+1. \end{cases}$$

Our theorem holds for any ring $\mathfrak{R}$ with a unit and we may utilize this by choosing, e.g. $\mathfrak{R}$ to be a ring of real matrices, thus extending the scope of the applications in a potentially useful way.

## REFERENCE

1. DIEUDONNÉ, J., La géometrie des groupes classiques, Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, Heft 5, Springer, Berlin 1955.

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