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On a theorem of Hanner

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OLOF HANNER (See reference [4]) has shown that a separable metric space is an absolute neighborhood retract (ANR) for normal spaces if and only if it is both an ANR for separable metric spaces and an absolute G_δ. Using an example given in a recent paper of R. H. BING [1] we show (theorem 1) that if a metric space is an ANR for normal spaces it is separable, and that hence the hypothesis of separability in Hanner's theorem can be dropped. In the same paper Bing defined a class of spaces more restricted than normal called collectionwise normal. We show (theorem 2) that Hanner's theorem extends to non-separable metric spaces if normal is replaced by collectionwise normal. Moreover (corollary 1) this form of Hanner's theorem characterizes collectionwise normal spaces in the same way as Tietze's extension theorem characterizes normal spaces.

1. Given a class τ of spaces, a space Y belonging to τ is called an ANR$_\tau$ [respectively AR$_\tau$] if $Y \in \tau$ and if every map f of a closed set A of a space X of class τ into the space Y can be extended to a map f_1 of an open set U, such that $A \subset U \subset X$, into Y [respectively, to a map f_1 of X into Y]. In particular ANR$_n$, ANR$_{cn}$, ANR$_m$ and ANR$_{sm}$ will mean absolute neighborhood retract for normal, collectionwise normal, metric and separable metric spaces respectively. If a class σ of spaces is contained in τ, if $Y \in \sigma$ and if Y is ANR$_\tau$ then clearly Y is also ANR$_\sigma$. (The above definition of ANR is equivalent in all cases considered below to the usual definition terms of retraction (See for example [4], theorem 3. 2) but we make no use of this equivalence.)

Theorem 1. *A metric space Y is ANR_n [respectively AR_n] if and only if it is ANR_m [respectively AR_m], separable and absolute G_δ.*

Proof. Sufficiency. If Y is separable and ANR$_m$ it is ANR$_{sm}$. If it is also absolute G_δ then, by [4] theorem 4. 2, it is ANR$_n$.

Necessity. Let Y be metric and ARN$_n$. Suppose if possible that Y is not separable. Then there exists $\epsilon > 0$ and a non-countable subset B of Y such that each pair of points of B have distance $> \epsilon$. Bing ([1], page 184, example G) has shown that there exists a normal space X with a closed subset A of arbitrary non-countable cardinal number such that the subspace A has the discrete topology but no collection of mutually non-intersecting neighborhoods of the points of A exists. Choose for A the cardinal number of B and let f be a 1-1 map of A on B. Then $f: A \to Y$ is continuous and, since Y is ANR$_n$, can be extended to a map $f_1: U \to Y$ of a neighborhood U of A. The inverse images by f_1 of the $(\epsilon/2)$-neighborhoods of the points of B form a collection of non-intersecting neighborhoods in X of the points of A, which is impossible. Therefore Y is separable.

Since Y is metric and ANR_n, it is ANR_m. Since Y is separable metric and ANR_n, by [4], theorem 4.2, it is absolute G_δ.

The proof for absolute retracts is similar and is omitted.

2. A set which is the union of a countable number of closed sets is called an F_σ set and its complement is called a G_δ set. If X is a normal space, a subset U of X is an open F_σ set if and only if there exists a continuous real valued function f defined on X such that $f(x)>0$ for $x \in U$ and $f(x)=0$ for $x \in X-U$. If A is a closed subset of a normal space and if U is an open set containing A there exists an open F_σ set V with $A \subset V \subset U$. A normal space is called perfectly normal if every open set is an F_σ set.

A collection $\{A_\alpha\}$ of sets of X is called locally finite in X if every point of X has a neighborhood meeting at most a finite number of the sets A_α. Clearly any subcollection of a locally finite collection is locally finite. The closure of the union of a locally finite collection of sets is the union of the closures;

$$\overline{\bigcup_\alpha A_\alpha} = \bigcup_\alpha \bar{A}_\alpha.$$

The union of a locally finite collection of closed sets is closed, the union of a locally finite collection of F_σ sets is an F_σ set and the union of a locally finite collection of open F_σ sets is an open F_σ set.

A space is called collectionwise normal [1] if for every locally finite collection $\{F_\alpha\}$ of mutually non-intersecting closed sets there is a collection $\{G_\alpha\}$ of mutually non-intersecting open sets with $F_\alpha \subset G_\alpha$. Metric spaces are collectionwise normal and collectionwise normal spaces are normal. The collection $\{G_\alpha\}$ of open sets may be assumed to be locally finite. For, if it is not, let E be the set of points of X every neighborhood of which meets an infinite number of the sets G_α. Then E is closed, no point of any G_α is in E and hence E and $\bigcup_\alpha F_\alpha$ are non-intersecting closed sets. Since X is normal there exist open sets U and V with

$$E \subset U, \ \bigcup_\alpha F_\alpha \subset V$$

and $UV=0$. Let $H_\alpha = G_\alpha V$. Then $\{H_\alpha\}$ is a collection of mutually non-intersecting open sets with $F_\alpha \subset H_\alpha$. Every point of $X-E$ has a neighborhood meeting at most a finite number of G_α and hence at most a finite number of H_α. Every point of E has a neighborhood, namely U, meeting none of the sets H_α. Hence $\{H_\alpha\}$ is locally finite.

A covering of a space X is a collection $\{U_\alpha\}$ of open sets whose union is X. If the collection $\{U_\alpha\}$ is locally finite it is called a locally finite covering. For each locally finite covering $\{U_\alpha\}$ of a normal space there exists ([6] page 26, proposition 33.4) a covering $\{V_\alpha\}$ with $\bar{V}_\alpha \subset U_\alpha$; hence there is a covering $\{W_\alpha\}$ of X by open F_σ sets W_α with $\bar{V}_\alpha \subset W_\alpha \subset U_\alpha$.

Lemma 1. *Let A be a closed subset of a collectionwise normal space X and let $\{U_\alpha\}$ be a locally finite covering of A. Then there exists a locally finite covering $\{V_\alpha\}$ of X such that, for each α, $V_\alpha A \subset U_\alpha$.*

Proof. Since A is normal there is a covering $\{W_\alpha\}$ of A by open F_α sets such that $W_\alpha < U_\alpha$. Assume the indices α well ordered and let

$$C_\alpha = W_\alpha(A - \bigcup_{\beta<\alpha} W_\beta);$$

then C_α, being the intersection of an F_σ set with a closed set in an F_σ set. Let

$$C_\alpha = \bigcup_r C_{\alpha r}, \qquad r = 1, 2, \ldots,$$

where $C_{\alpha r}$ is closed in A and hence also in X. The sets C_α are mutually non-intersecting and

$$\bigcup_\alpha C_\alpha = \bigcup_\alpha W_\alpha = A.$$

Since the collection $\{U_\alpha\}$ is locally finite in the closed set A it is locally finite in X. Hence, since

$$C_{\alpha r} \subset C_\alpha \subset W_\alpha \subset U_\alpha,$$

$\{C_{\alpha r}\}$ for fixed r is a locally finite collection of mutually non-intersecting closed sets of X. Hence, since X is collectionwise normal, there exists for each r a locally finite collection $\{G_{\alpha r}\}$ of mutually non-intersecting open sets of X such that $C_{\alpha r} \subset G_{\alpha r}$. There exists an open F_σ set $H_{\alpha r}$ containing $C_{\alpha r}$ and contained in the open set

$$G_{\alpha r}(X - (A - U_\alpha)).$$

Let $H = \bigcup_{\alpha, r} H_{\alpha r}$; then H is open and

$$A \subset \bigcup_\alpha \bigcup_r C_{\alpha r} \subset H.$$

Hence there exists an open F_σ set H_0 such that

$$X - H \subset H_0 \subset X - A.$$

Adding H_0 to an arbitrary one of the sets $H_{\alpha r}$ we get a family $\{L_{\alpha r}\}$ of open F_σ sets with

$$\bigcup_{\alpha, r} L_{\alpha r} = X,$$

$$C_{\alpha r} \subset L_{\alpha r} \subset X - (A - U_\alpha)$$

and, for each r, $\{L_{\alpha r}\}$ is locally finite. $L_r = \bigcup_\alpha L_{\alpha r}$; then L_r is an open F_σ set and $\{L_r\}$ is a covering of X. For each L_r there is a continuous real function $\phi_r(x)$, $0 \le \phi_r(x) \le 1$, such that $\phi_r(x) > 0$ if and only if $x \in L_r$. Let F_{rn} be the set of points x of X for which $\phi_r(x) \ge 1/n$, and let

$$V_r = L_r(X - \bigcup_{s<r} F_{sr}).$$

Then $\{V_r\}$ is a locally finite covering ([3] proof of proposition (e)) of X and $V_r \subset L_r$.

Let $V_{\alpha r} = L_{\alpha r} V_r$ and let $V_\alpha = \bigcup_r V_{\alpha r}$. Each point x of X is in some V_r and hence, since $V_r \subset \bigcup_\alpha L_{\alpha r}$, in some $L_{\alpha r}$ and hence in $L_{\alpha r} V_r = V_{\alpha r}$. Hence x is in some V_α. Since $L_{\alpha r}$ and V_r are open,. V_α is open. Thus $\{V_\alpha\}$ is a covering of X. Each point of X has a neighborhood meeting only a finite number of the sets V_r and has a smaller neighborhood meeting at most a finite number of the sets $L_{\alpha r}$ for each such r. Thus there is a neighborhood meeting only a finite number of $V_{\alpha r}$ and hence only a finite number of V_α. Thus $\{V_\alpha\}$ is a locally finite covering of X. Since

$$V_a \subset \cup_r L_{ar} \subset X - (A - U_a),$$

therefore

$$V_{\dot{a}} A \subset U_a.$$

This completes the proof of the lemma.

3. A covering $\mathfrak{B} = \{V_a\}$ is called a refinement of a covering $\mathfrak{U} = \{U_\beta\}$ if each V_a is contained in some U_β. A space X is called paracompact if every covering of X has a refinement which is locally finite. A generalized Hilbert space (or generalized Euclidean space) is a space having all properties of Hilbert space except separability. A generalized Hilbert space is paracompact ([3] lemma 2 or [7] corollary 1).

Lemma 2. *Let A be a closed subset of a collectionwise normal space X and let f be a map of A into a generalized Hilbert space H. Then f can be extended to a map g of X into H.*

Proof. It is sufficient to construct a sequence of maps $g_n : X \to H$ for $n = 1, 2, \ldots$, such that (1) if $n > 1$, the distance

$$\varrho(g_n(x), g_{n-1}(x)) < 2^{-n+2}$$

and (2) if $x \in A$,

$$\varrho(g_n(x), f(x)) < 2^{-n}.$$

Then the Cauchy sequence $\{g_n(x)\}$ converges to a point $g(x)$ of the complete space H and, since the sequence $\{g_n\}$ is uniformly convergent, the limit function g is continuous. For each x,

$$\varrho(g_n(x), g(x)) < 2^{-n+2}.$$

If $x \in A$,

$$\varrho(f(x), g(x)) < \varrho(f(x), g_n(x)) + \varrho(g_n(x), g(x)) < 2^{-n} + 2^{-n+2} < 2^{-n+3}$$

for each n; hence $\varrho(f(x), g(x)) = 0$ and $g(x) = f(x)$. Thus g is an extension of f.

We construct the sequence $\{g_n\}$ by recursion. Since H is paracompact the covering of H by all open sets of diameter less than 2^{-n} has a locally finite refinement $\mathfrak{U}_n$. Then $f^{-1}\mathfrak{U}_n$, the collection of inverse images of the open sets of $\mathfrak{U}_n$, is a locally finite covering of A. Hence by lemma 1, there is a locally finite covering $\mathfrak{B}_n$ of X such that, for each $V \in \mathfrak{B}_n$, VA is contained in some element of $f^{-1}\mathfrak{U}_n$. When $n > 1$ we assume that $g_{n-1} : X \to H$ has already been defined. Then $g_{n-1}^{-1}\mathfrak{U}_n$ is a locally finite covering of X. Let $\mathfrak{W}_n$ be a locally finite common refinement of $\mathfrak{B}_n$ and $g_{n-1}^{-1}\mathfrak{U}_n$. When $n = 1$ let $\mathfrak{W}_1 = \mathfrak{B}_1$.

Let K_n be the nerve of $\mathfrak{W}_n$ and let ϕ_n be a canonical map ([2] page 202) of X into K_n. For $x \in X$ let $\bar{\sigma}_n(x)$ be the closed simplex of K_n whose vertices correspond to the open sets of W_n containing x. Then $\phi_n(x) \in \bar{\sigma}_n(x)$. Let $\psi_n : K_n \to H$ be the linear map (linear on each simplex) of K_n into H defined for the vertices of K_n as follows. Let $W \in \mathfrak{W}_n$ and let w be the corresponding vertex of K_n. If $WA = 0$ and $n > 1$ choose a point $y \in W$ and let $\psi_n(w) = g_{n-1}(y)$. If $n = 1$ and $WA = 0$ choose $\psi_1(w)$ arbitrarily in H. If $WA \neq 0$ choose $y \in WA$ and let $\psi_n(w) = f(y)$. Let $g_n : X \to H$ be defined by $g_n(x) = \psi_n \phi_n(x)$; then g_n is a continuous ([2] lemma 1.2) map of X into H.

When $n > 1$, if $x \in W$ then, since y is also in W and since $g_{n-1}(W)$ has diameter less than 2^{-n},

$$\varrho(g_{n-1}(x), g_{n-1}(y)) < 2^{-n}.$$

If $WA \neq 0, y \in A$ and hence

$$\varrho(g_{n-1}(y), f(y)) < 2^{-n+1}$$

(by the induction hypothesis) and, since $\psi_n(w) = f(y)$,

$$\varrho(g_{n-1}(x), \psi_n(w)) < 2^{-n} + 2^{-n+1} < 2^{-n+2}.$$

If $WA = 0$, $\psi_n(w) = g_{n-1}(y)$ and hence

$$\varrho(g_{n-1}(x), \psi_n(w)) < 2^{-n} < 2^{-n+2}.$$

Thus ψ_n maps each vertex of $\bar{\sigma}_n(x)$ within a spherical neighborhood of $g_{n-1}(x)$ of radius 2^{-n+2}. Hence ψ_n maps $\bar{\sigma}_n(x)$ within this neighborhood and, in particular,

$$\varrho(g_{n-1}(x), \psi_n \phi_n(x)) < 2^{-n+2}.$$

Thus

$$\varrho(g_{n-1}(x), g_n(x)) < 2^{-n+2}.$$

When $x \in A$, if W contains x then $WA \neq 0$ and $\psi_n(w) = f(y)$ with $y \in WA$. Since x and y are both points of WA and since $f(WA)$ has diameter less than 2^{-n}, $\varrho(f(x), f(y)) < 2^{-n}$. Thus

$$\varrho(f(x), \psi_n(w)) < 2^{-n}.$$

Thus ψ_n maps $\bar{\sigma}_n(x)$ into the 2^{-n} neighborhood of $f(x)$. Hence

$$\varrho(f(x), \psi_n \phi_n(x)) < 2^{-n};$$

that is,

$$\varrho(f(x), g_n(x)) < 2^{-n}.$$

Thus g_n has the required properties (1) and (2). This completes the proof of the lemma.

4. For non-separable Hilbert spaces our lemma 2 replaces the Tietze extension theorem. Similarly, for non-separable metric spaces, Hanner's theorem may be replaced by the following:

Theorem 2. *A metric space Y is ANR_{cn} [respectively AR_{cn}] if and only if it is ANR_m [respectively AR_m] and absolute G_δ.*

Proof. Necessity. Let Y be metric and ANR_{cn}. Let M be a metric space with a subset Y_1 homeomorphic to Y and let $g: Y_1 \to Y$ be a homeomorphism. Let the space X be defined thus: the points of X are the points of M and a set U of X is open if it is the union of an open set of M and a subset of $M - Y_1$. Let $h: X \to M$ map each point of X on the same point of M; then h is continuous. Let $A = h^{-1} Y_1$; then A is closed in X.

Let $\{F_\alpha\}$ be a locally finite set of disjoint closed sets of X. Then, if $B_\alpha = h(F_\alpha A)$, $\{B_\alpha\}$ is a locally finite collection of closed sets of Y_1. Let G_α be the set of points of M which are nearer to B_α than to $\bigcup_{\beta\neq\alpha} B_\beta$; then $B_\alpha \subset G_\alpha$, G_α is open and, if $\alpha \neq \beta$, $G_\alpha B_\beta = 0$. Then the sets $h^{-1} G_\alpha$ are mutually non-intersecting. Since $\{F_\alpha\}$ is locally finite, $\bigcup_{\beta\neq\alpha} F_\beta$ is closed. Then the sets

$$U_\alpha = F_\alpha + (h^{-1} G_\alpha - \bigcup_{\beta\neq\alpha} F_\beta)$$

are open mutually non-intersecting sets and $F_\alpha \subset U_\alpha$. Hence X is collectionwise normal.

Let $f: A \to Y$ be defined by $f(x) = gh(x)$. Then, since Y is ANR_{cn}, there is a neighborhood U of A in X and an extension $f_1: U \to Y$ of f. For $x \in U$ let

$$\phi(x) = \varrho(h(x), g^{-1} f_1(x));$$

then ϕ is continuous and $\phi(x) = 0$ if and only if $x \in A$. Therefore A is a G_δ set in U. Let $A = \bigcap_n U_n$, $n = 1, 2, \ldots$, with U_n open in U and hence open in X. Then $h(U_n) = V_n + C_n$ where V_n is open in M and $C_n \subset M - Y_1$. Then $Y_1 \subset V_n$ and $Y_1 = \bigcap_n (V_n + C_n)$; hence $Y_1 = \bigcap_n V_n$ and Y_1 is a G_δ set in M. Hence Y is an absolute G_δ.

Since metric spaces are collectionwise normal, Y is ANR_m.

Sufficiency. Let Y be ANR_m and absolute G_δ. Replacing Y by a homeomorphic space if necessary we may assume ([7] corollary 1 and [3] lemma 1) that Y is a subspace of a generalized Hilbert space H. Let X be a collectionwise normal space, A a closed subset of X and $f; A \to Y$ a map of A into Y. By lemma 2 there exists an extension $f_1: X \to H$ of f.

Since Y is an absolute G_δ there exist open sets W_n, $n = 1, 2, \ldots$, of H such that $Y = \bigcap_n W_n$. Let h_n be a real continuous function on X such thas $0 \leq h_n(x) \leq 1$, $h_n(x) = 0$ if $x \in A$ and $h_n(x) = 1$ if

$$x \in X - f_1^{-1} W_n,$$

and let

$$h(x) = \sum_{n=1}^{\infty} 2^{-n} h_n(x).$$

Then h is continuous, $h(x) = 0$ if $x \in A$ and $h(x) > 0$ if $f_1(x) \in H - Y$. Let I be the closed segment $[0, 1]$ and let

$$f_2: X \to H \times I - (H - Y) \times 0$$

be defined by $f_2(x) = (f_1(x), h(x))$. Let $g: Y \times 0 \to Y$ be the homeomorphism for which $g(y, 0) = y$. Since $Y \times 0$ is closed in the metric space $H \times I - (H - Y) \times 0$ and since Y is ANR_m there is an extension $g_1: V \to Y$ of g to a neighborhood V of $Y \times 0$. Let $U = f_2^{-1} V$; then U is a neighborhood of A in X and $g_1 f_2: U \to Y$ is an extension of f. Hence Y is ANR_{cn}.

The proof for absolute retracts is similar and is omitted.

Corollary 1. *A space X is collectionwise normal if and only if for each closed set A of X, each map f of A into a complete ANR_m can be extended over a neighborhood of A in X.*

Proof. Necessity follows immediately from theorem 2 and the fact that a complete metric space an absolute G_δ.

Sufficiency. Let $\{F_\alpha\}$ be a locally finite set of disjoint closed sets of X. Let Y be a metric space whose points y_α are in $1-1$ correspondence with the sets F_α and let each pair of distinct points of Y have distance 1. Then Y is ANR_m and complete. Let $A = \bigcup_\alpha F_\alpha$ and let $f: A \to Y$ be defined by $f(x) = y_\alpha$ for all $x \in F_\alpha$. Then f is continuous and hence can be extended to a map $g: U \to Y$ where U is a neighborhood of A in X. Then the inverse images $g^{-1}(y_\alpha)$ form a collection of mutually non-intersecting open sets of X with $F_\alpha \subset g^{-1}(y_\alpha)$. Hence X is collectionwise normal.

Remark. The above space Y can be imbedded as a neighborhood retract in a suitable generalized Hilbert space H. If each map of a closed set A of X into a generalized Hilbert space H can be extended over X then each map of A into Y can be extended over a neighborhood of A and hence, as above, X is collectionwise normal. This is the converse of lemma 2. It follows that the converse of lemma 1 is also true.

Let ANR_{cnpn} mean absolute neighborhood retract for collectionwise normal perfectly normal spaces.

Corollary 2. *A metric space Y is ANR_{cnpn} if and only if it is ANR_m.*

Proof. Necessity follows from the fact that metric spaces are collectionwise normal and perfectly normal.

Sufficiency. Let Y be ANR_m. If A is a closed set of a perfectly normal space X, there exists a continuous function k defined on X, $0 \le k(x) \le 1$, such that $k(x) = 0$ if and only if $x \in A$. In the proof of sufficiency in theorem 2 above one may replace the function $h(x)$, whose existence depended on Y being an absolute G_δ, by this function $k(x)$. The details are omitted.

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