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On the ideal structure of group algebras

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Let G be a locally compact abelian group with dual group $\hat{G}$, and denote by $\mathfrak{L}$ the group algebra of G, consisting of the functions summable on G for Haar measure dx. The maximal regular ideals of $\mathfrak{L}$ are in correspondence with the points of $\hat{G}$; the functions belonging to such an ideal are exactly those whose Fourier transforms vanish at the given point. Now let $\mathfrak{J}$ be any closed ideal in $\mathfrak{L}$. The Tauberian theorem states that the Fourier transforms of the functions of $\mathfrak{J}$ have at least one common zero. Differently stated, $\mathfrak{J}$ is contained in at least one regular maximal ideal. The central problem about the ideal structure of $\mathfrak{J}$ is to determine under what conditions $\mathfrak{J}$ is the intersection of the regular maximal ideals which contain it; and this is the same as to decide whether $\mathfrak{J}$ necessarily contains all functions whose transforms vanish on the set of zeros of the transforms of $\mathfrak{J}$.

L. Schwartz [4] has given an example in the three-dimensional Euclidean group of a closed ideal which is actually smaller than the intersection of the maximal regular ideals containing it. Positive results can be given by restricting the ideals considered. We refer to [2] for these results and to references to the literature (the problem is there discussed in $\mathfrak{L}^\infty$ from the dual point of view). But our present point of departure is the fact that there are groups where distinct ideals determine the same set of zeros. We shall show in such a case that infinitely many ideals determine that set of zeros, and we can exhibit a few of them. The crucial tool is a theorem of Godement on unitary representations of abelian groups.

The reader should be familiar with the theory of Fourier transforms on abelian groups, in particular with the content of [1]. The theorem of Godement referred to is also proved in this paper.

For a closed set $\hat{E}$ in $\hat{G}$, let $\mathfrak{J}(\hat{E})$ be the closed ideal of summable functions on G whose transforms vanish on $\hat{E}$. Let $\mathfrak{J}_0(\hat{E})$ be the closure of the ideal of summable functions whose transforms vanish on some open set containing $\hat{E}$ (depending on the function). Then any closed ideal whose set of zeros is $\hat{E}$ lies between $\mathfrak{J}_0$ and $\mathfrak{J}$. We assume that $\mathfrak{J}_0$ and $\mathfrak{J}$ are distinct, and we are going to show that there are indeed a great many ideals between.

If $\mathfrak{U}$ and $\mathfrak{B}$ are closed ideals such that

$$\mathfrak{J}_0 \subset \mathfrak{U} \subset \mathfrak{B} \subset \mathfrak{J}$$

(equality is not excluded), we can form the Banach algebra $\mathfrak{B}/\mathfrak{U}$ with norm

$$\|f\| = \inf_g \|g\| \qquad (f-g \in \mathfrak{U}).$$

Translation by a group element leaves $\mathfrak{U}$ and $\mathfrak{B}$ invariant, and so defines an operator in the quotient algebra which is evidently unitary. Together these translation operators are a unitary representation of G in $\mathfrak{B}/\mathfrak{U}$. We have to show that this representation is reducible.

Recall that the closed ideals in $\mathfrak{L}$ are characterized as the closed subspaces invariant under translation. Presumably a closed ideal in the quotient algebra need not be invariant under translation. We are interested in the closed invariant subspaces lying between $\mathfrak{J}_0$ and $\mathfrak{J}$. It is easy to see that these are the same as the subspaces of $\mathfrak{J}/\mathfrak{J}_0$, invariant with respect to the induced translation operators, and closed in the quotient norm. So we shall speak of invariant subspaces instead of ideals in quotient algebras, and we know that we can restrict attention to the quotient norm in discussing the closure of subspaces.

For any function f on G, let f_a be defined by the formula

$$f_a(x) = f(a^{-1}x)$$

for all x. The first lemma states that the natural representation of G in $\mathfrak{B}/\mathfrak{U}$ cannot be one-dimensional.

Lemma. Suppose $\mathfrak{U}$ and $\mathfrak{B}$ are closed invariant subspaces of $\mathfrak{L}$ with

$$\mathfrak{J}_0 \subset \mathfrak{U} \subset \mathfrak{B} \subset \mathfrak{J}.$$

If $f_a - \hat{y}(a)f$ belongs to $\mathfrak{U}$ for every $a \in G$, $f \in \mathfrak{B}$, and some $\hat{y} \in \hat{G}$, then $\mathfrak{U} = \mathfrak{B}$.

Take any summable function h with Fourier transform $\hat{h}$. For almost all a, $h(a)f_a - \hat{y}(a)h(a)f$ is a function in $\mathfrak{U}$; integrating this vector-valued function of a gives $h * f - \hat{h}(\hat{y})f$, which must belong to $\mathfrak{U}$ since $\mathfrak{U}$ is closed. This holds for all $h \in \mathfrak{L}$ and $f \in \mathfrak{B}$. It follows that $h * f \in \mathfrak{U}$ if and only if $\hat{h}(\hat{y}) = 0$ (unless $\mathfrak{U} = \mathfrak{B}$).

Suppose the fixed point $\hat{y}$ does not belong to $\hat{E}$. For any summable h we can find a function $g \in \mathfrak{J}_0$ such that $\hat{h}(\hat{y}) = \hat{g}(\hat{y})$ (where $\hat{g}$ is the Fourier transform of g), since $\hat{E}$ is a closed set not containing $\hat{y}$. By what has been shown, $(h-g) * f \in \mathfrak{U}$ for each $f \in \mathfrak{B}$. Now $g * f \in \mathfrak{J}_0 \subset \mathfrak{U}$, so $h * f \in \mathfrak{U}$ for each $f \in \mathfrak{B}$. Hence $\hat{h}(\hat{y}) = 0$, which is absurd, since h was arbitrary. So $\hat{y} \in \hat{E}$.

To finish the proof we have to find a directed system of functions h_γ such that $\hat{h}_\gamma(\hat{y}) = 0$ for each γ, and $h_\gamma * f$ converges to f, for arbitrary $f \in \mathfrak{B}$. Then it follows that $f \in \mathfrak{U}$, since $\mathfrak{U}$ is closed, and hence $\mathfrak{U} = \mathfrak{B}$ after all.

The existence of such a system is implied by a theorem of Kaplansky on primary ideals [3]. Without loss of generality assume that $\hat{y}$ is the identity of $\hat{G}$. It is known that we can find a system $\{e_\alpha\}$ of summable functions of norm one with $\hat{e}_\alpha(\hat{0}) = 1$ for each α, such that $e_\alpha * f$ converges to zero for any summable f whose transform vanishes at $\hat{0}$ (and hence for all functions of $\mathfrak{B}$, since $\hat{y} \in \hat{E}$) ([2], § 3). Let $\{g_\beta\}$ be an approximate identity for $\mathfrak{L}$ with $\hat{g}_\beta(\hat{0}) = 1$ for each β. Then $\{g_\beta - g_\beta * e_\alpha\}$ is the required system $\{h_\gamma\}$.

We shall need to know the form of the linear functionals on quotient algebras of the form $\mathfrak{B}/\mathfrak{J}_0$. They are the functionals on $\mathfrak{B}$ which vanish on $\mathfrak{J}_0$. A functional on $\mathfrak{B}$ can be extended to all of $\mathfrak{L}$, and so can be represented by a function φ of $\mathfrak{L}^\infty$, whose value for $f \in \mathfrak{B}$ is

$$\varphi * f(0).$$

Of course, two bounded functions may determine the same functional in $\mathfrak{B}$. The condition that the functional φ vanish on $\mathfrak{J}_0$ is exactly the requirement that the spectral set Λ_φ of φ lie in $\hat{E}$. So the linear functionals on $\mathfrak{B}/\mathfrak{J}_0$ are the functions of $\mathfrak{L}^\infty$ whose spectral sets are contained in $\hat{E}$, with certain identifications allowed depending on $\mathfrak{B}$.

Godement [1] has shown that if a unitary representation of a locally compact abelian group in a Banach space is not one-dimensional, then it cannot be irreducible; and in fact there must be non-trivial invariant subspaces of a special type which he calls *spectral varieties*. We are given a natural unitary representation of G in $\mathfrak{B}/\mathfrak{J}_0$, and by the lemma it cannot be one-dimensional unless $\mathfrak{U} = \mathfrak{B}$. To obtain our theorem we only have to interpret the notion of spectral variety in this case.

Let $\hat{F}$ be a closed subset of $\hat{E}$. The spectral variety $\mathfrak{B}_{\hat{F}}$ in $\mathfrak{B}/\mathfrak{J}_0$ is the set of functions $f \in \mathfrak{B}$ (more precisely, cosets of functions) such that the spectral set $\Lambda_{\varphi * f}$ is contained in $\hat{F}$ for every $\varphi \in \mathfrak{L}^\infty$ whose spectral set is contained in $\hat{E}$. (Such φ are the linear functionals on $\mathfrak{B}/\mathfrak{J}_0$, and this is the purpose for which we determined them.)

Theorem. Suppose $\mathfrak{B}$ is a closed invariant subspace of $\mathfrak{J}$, distinct from and containing $\mathfrak{J}_0$. There is a closed subset $\hat{F}$ of $\hat{E}$ such that $\mathfrak{B}$ contains the spectral variety $\mathfrak{B}_{\hat{F}}$, and $\mathfrak{B}_{\hat{F}}$ is distinct from $\mathfrak{B}$ and from $\mathfrak{J}_0$.

Applying the theorem to the case $\mathfrak{B} = \mathfrak{J}$, the existence of a non-trivial invariant subspace between $\mathfrak{J}$ and $\mathfrak{J}_0$ is shown. Evidently an infinite descending chain of invariant subspaces can be so obtained, and there is no minimal invariant subspace containing $\mathfrak{J}_0$.

The characterization of the linear functionals on $\mathfrak{B}/\mathfrak{J}_0$ was only used to find the form of the spectral varieties. Applying the lemma and Godement's theorem to the spaces $\mathfrak{J}/\mathfrak{U}$ (where $\mathfrak{U}$ is assumed to be different from $\mathfrak{J}$ and to contain $\mathfrak{J}_0$) shows that no closed subspace is maximal, but the spectral varieties are less easy to describe. The same reasoning shows that there are invariant closed subspaces between any pair $\mathfrak{U}$, $\mathfrak{B}$.

It is an open question whether $\mathfrak{B}$ is characterized by the $\mathfrak{B}_{\hat{F}}$ which it contains. There is no positive evidence and experience with the problem recommends pessimism. One would like to use the notion of spectral variety to define a notion of zero of higher order for transforms of functions of $\mathfrak{J}$. For $\varphi \in \mathfrak{L}^\infty$, with spectral set contained in $\hat{E}$, we know that no point of Λ_φ survives in the spectral set of $\varphi * f$ if $f \in \mathfrak{J}_0$. Now if $\mathfrak{B}$ contains a spectral variety $\mathfrak{B}_{\hat{F}}$, then $\mathfrak{B}$ contains all the functions which annihilate spectral points lying outside $\hat{F}$. If a maximal such set $\hat{F}$ could be defined in some way, presumably its complement in $\hat{E}$ could be used in place of $\hat{E}$ in a more exact discussion of $\mathfrak{B}$, and so through finitely or infinitely many stages. As yet the idea has had no success.

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