ARKIV FÖR MATEMATIK Band 2 nr 3

Communicated 6 June 1951 by F. CARLSON and J. MALMQUIST

On approximation of continuous and of analytic functions

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1) General survey

Let $\{\xi_{\nu, n}\}$ denote a system of points in the interval $(0, 1)$ with the following properties

$$n = 1, 2, 3, \ldots$$
$$\nu = 0, 1, \ldots, (n-1), n.$$
$$\xi_{\nu, n} > \xi_{\mu, n} \quad \text{if} \quad \nu > \mu.$$

With every point $\xi_{\nu, n}$ we associate a real function $\psi_{\nu, n}(x)$, defined for $0 \leqq x \leqq 1$.

A system of the above-mentioned type will be said to solve the approximation problem, if for every continuous function $f(x)$

$$A_n(f) = \sum_{\nu=0}^{n} f(\xi_{\nu, n})\, \psi_{\nu, n}(x)$$

tends to $f(x)$ when n tends to infinity, uniformly for $0 \leqq x \leqq 1$.

In this paper we are going to treat the case when the approximation functions $\psi_{\nu, n}(x)$ are non-negative. We begin in section 2 by stating the necessary and sufficient conditions of a system $\{\xi_{\nu, n}\}$ of points. We proceed in section 3 by stating the necessary and sufficient conditions of a system $\{\xi_{\nu, n}; \psi_{\nu, n}\}$ of points and functions, which solves the approximation problem. Then in section 4 we apply the obtained results on a special system and finally, in section 5, we study the convergence for complex values of x for this same system.

2) Necessary and sufficient conditions of $\{\xi_{\nu, n}\}$.

We shall prove that the conditions

$$\left.\begin{aligned} &\xi_{0, n} \to 0 \\ &\xi_{n, n} \to 1 \\ &\operatorname*{Max}_{\nu} \{\xi_{\nu+1, n} - \xi_{\nu, n}\} \to 0 \end{aligned}\right\}$$

when $n \to \infty$ are necessary and sufficient for $\{\xi_{\nu, n}\}$ in the following meaning.

If the conditions are fulfilled there is a system $\{\psi_{\nu,n}\}$ of functions so that $\{\xi_{\nu,n};\ \psi_{\nu,n}\}$ solves the approximation problem.

If the conditions are not fulfilled there is a continuous function $f(x)$, not identically zero, so that for every system $\{\psi_{\nu,n}\}$ of functions

$$\underline{\lim}_{n\to\infty} A_n(f) = 0$$

i.e. the system $\{\xi_{\nu,n};\ \psi_{\nu,n}\}$ does not solve the approximation problem.

Let us first suppose that the conditions are fulfilled.

We define

$$\psi_{\nu,n}(x) = \begin{cases} 0 & \text{for} \quad x < \xi_{\nu-1,n} \\ \dfrac{x - \xi_{\nu-1,n}}{\xi_{\nu,n} - \xi_{\nu-1,n}} & \text{for} \quad \xi_{\nu-1,n} \leqq x \leqq \xi_{\nu,n} \\ \dfrac{\xi_{\nu+1,n} - x}{\xi_{\nu+1,n} - \xi_{\nu,n}} & \text{for} \quad \xi_{\nu,n} \leqq x \leqq \xi_{\nu+1,n} \\ 0 & \text{for} \quad x > \xi_{\nu+1,n} \end{cases}$$

This definition is also valid for $\psi_{0,n}$ if $\xi_{-1,n}$ is replaced by 0 and for $\psi_{n,n}$ if $\xi_{n+1,n}$ is replaced by 1.

In each sub-interval

$$A_n(f) = \sum f(\xi_{\nu,n})\,\psi_{\nu,n}(x)$$

is then a linear function and in the points $\xi_{m,n}$

$$A_n(f) = \sum f(\xi_{\nu,n})\,\psi_{\nu,n}(\xi_{m,n}) = f(\xi_{m,n})$$

Hence it follows from the continuity of $f(x)$ that $A_n(f) \to f$ uniformly for $0 \leqq x \leqq 1$.

Let us then suppose that the conditions are not fulfilled.

If we denote

$$\operatorname*{Max}_{\nu} \{\xi_{0,n};\ (1 - \xi_{n,n});\ (\xi_{\nu+1,n} - \xi_{\nu,n})\} = d_n$$

the supposition is equivalent to the existence of a constant $\alpha > 0$ so that

$$\overline{\lim}\, d_n = \alpha > 0.$$

Hence there is a sub-sequence d_{n_μ} and a constant μ_0 so that

$$d_{n_\mu} > \frac{\alpha}{2} \quad \text{for} \quad \mu > \mu_0.$$

This statement can also be expressed as follows. There is an infinite set of intervals I_μ, each of a length greater than $\frac{\alpha}{2}$, such that I_μ contains no point of the set $\sum_{\nu} \xi_{n_\mu,\nu}$.

Now choose a number N such that $\frac{1}{N} < \frac{a}{4} \leqq \frac{1}{N-1}$ and divide the interval $(0, 1)$ into N equal sub-intervals $i_1, i_2, \ldots, i_N$. Each I_μ being greater than $\frac{a}{2}$, it covers at least one of the intervals i_ν. As the number of intervals I_μ is infinite, there must be at least one interval i_k which is covered by an infinite number of intervals I_μ. Thus we have found that there is a sub-sequence n_λ and an interval i_k such that i_k contains no point of the set $\sum_\lambda \sum_\nu \xi_{n_\lambda, \nu}$.

Consider now a continuous function $f(x)$ which is different from zero in i_k but zero elsewhere. Let $\{\psi_{\nu,n}\}$ be some system of approximation functions. Then

$$A_n(f) = \sum f(\xi_{\nu,n})\, \psi_{\nu,n}.$$

In particular

$$A_{n_\lambda}(f) \equiv 0 \quad \text{for every } \lambda.$$

Hence $\underline{\lim}\, A_n(f) = 0$.

3) Necessary and sufficient conditions in the case $\psi_{\nu,n} \geqq 0$

In the preceding section we made no assumptions concerning the sign of $\psi_{\nu,n}$. From now, however, we shall always assume that $\psi_{\nu,n}$ is non-negative. The consequences of this restriction are prima facie somewhat unexpected.

We shall give two different necessary and sufficient conditions for a system $\{\xi_{\nu,n};\, \psi_{\nu,n}\}$ of points and non-negative functions that solves the approximation problem.

Condition A

For each $\eta > 0$

$$\sum_n' = \sum_{|\xi_{\nu,n} - x| \geq \eta} \psi_{\nu,n} \to 0$$

$$\sum_n'' = \sum_{|\xi_{\nu,n} - x| < \eta} \psi_{\nu,n} \to 1$$

as $n \to \infty$, uniformly for $0 \leqq x \leqq 1$.

Condition B

$$A_n(1) \to 1$$

$$A_n(x) \to x$$

$$A_n(x^2) \to x^2$$

as $n \to \infty$, uniformly for $0 \leqq x \leqq 1$.

Let us first assume that condition A is fulfilled. If $f(x)$ is a continuous function there is a number M such that

$$|f| < M$$

and an $\eta = \eta(\varepsilon)$ such that

$$|f(x_1) - f(x_2)| < \varepsilon \quad \text{for} \quad |x_1 - x_1| < \eta.$$

Then we get

$$|f - A_n(f)| = |(1 - \sum_n'')f + f\sum_n'' - A_n(f)| <$$
$$< M\,|1 - \sum_n''| + M\sum_n' + \varepsilon\sum_n'' < 2\varepsilon \quad \text{for} \quad n > n_0.$$

Condition A is thus sufficient. In particular it follows, that if condition A is fulfilled, the same is true of condition B.

Secondly, let us assume that condition B is fulfilled. This is evidently a necessary condition.

From the assumption follows

$$x^2 A_n(1) - 2x A_n(x) + A_n(x^2) \to x^2 - 2x^2 + x^2 = 0.$$

On the other hand

$$x^2 A_n(1) - 2x A_n(x) + A_n(x^2) = \sum (x - \xi_{\nu,n})^2 \psi_{\nu,n}(x) \geq \eta^2 \sum_n'.$$

Hence $\sum_n' \to 0$ and as $A_n(1) = \sum_n' + \sum_n'' \to 1$ we have also $\sum_n'' \to 1$. Thus, if condition B is fulfilled, the same is true of condition A.

4) **Application of the obtained results**

Let us consider the system

$$\xi_{\nu,n} = \frac{\nu}{n} \qquad \psi_{\nu,n}(x) = e^{-Nx}\frac{(Nx)^\nu}{\nu!}$$

where $N = N(n)$ is a posititive function of n.

Our first problem is to determine $N(n)$ so, that the system solves the approximation problem. For this investigation we apply condition B.

$$A_n(1) = e^{-Nx}\sum_{\nu=0}^{n}\frac{(Nx)^\nu}{\nu!}$$

and $\dfrac{d\,A_n(1)}{dx} = -N\,e^{-Nx}\dfrac{(Nx)^n}{n!} \leq 0$

$$\text{for} \quad x = 0 \quad \text{is} \quad A_n(1) = 1$$
$$\text{for} \quad x = 1 \quad \text{is} \quad A_n(1) = e^{-N}\sum_{\nu=0}^{n}\frac{N^\nu}{\nu!}.$$

If we show that the latter expression tends to 1 as n tends to infinity, it is clear that $A_n(1)$ tends to 1, uniformly for $0 \leq x \leq 1$.

$$e^{-N}\sum_{\nu=0}^{n}\frac{N^\nu}{\nu!} = \frac{1}{n!}\int_N^\infty e^{-x}x^n\,dx = 1 - \frac{1}{n!}\int_0^N e^{-x}x^n\,dx.$$

Put $x=n+t\sqrt{n}$

$$\frac{1}{n!}\int_0^N e^{-x}x^n\,dx=\frac{n^{n+\frac{1}{2}}e^{-n}}{n!}\int_{-\sqrt{n}}^{\frac{N-n}{\sqrt{n}}} e^{-t\sqrt{n}}\left(1+\frac{t}{\sqrt{n}}\right)^n dt\sim\frac{1}{\sqrt{2\pi}}\int_{-\sqrt{n}}^{\frac{N-n}{\sqrt{n}}} e^{-t\sqrt{n}}\left(1+\frac{t}{\sqrt{n}}\right)^n dt$$

by Stirlings formula. Now for every fixed t

$$e^{-t\sqrt{n}}\left(1+\frac{t}{\sqrt{n}}\right)^n=e^{-t\sqrt{n}+n\log\left(1+\frac{t}{\sqrt{n}}\right)}\to e^{-\frac{t^2}{2}}$$

and hence the integral tends to zero, if and only if $\dfrac{n-N}{\sqrt{n}}\to+\infty$, and then $A_n(1)\to 1$.

We must also have $A_n(x)\to x$.

$$A_n(x)=e^{-Nx}\sum_{\nu=0}^{n}\frac{\nu}{n}\frac{(Nx)^\nu}{\nu!}=e^{-Nx}\sum_{\nu=0}^{n-1}\frac{N}{n}\,x\,\frac{(Nx)^\nu}{\nu!}=\frac{N}{n}\,x\,A_n(1)-\frac{N}{n}\,x\,\frac{(Nx)^n}{n!}e^{-Nx}.$$

But

$$\frac{(Nx)^n}{n!}e^{-Nx}\sim\frac{1}{\sqrt{2\pi n}}\left(\frac{Nxe}{n}\right)^n e^{-Nx}=\frac{1}{\sqrt{2\pi n}}e^{n\,g_n(x)}$$

where

$$g_n(x)=\log\frac{N}{n}+\log x+1-\frac{N}{n}x$$

$$g_n'(x)=\frac{1}{x}-\frac{N}{n}.$$

For sufficiently large n is $N<n$ as $\dfrac{n-N}{\sqrt{n}}\to+\infty$, and $g_n'(x)>0$ for $0\le x\le 1$.

Then

$$g_n(x)\le g_n(1)=\log\frac{N}{n}+1-\frac{N}{n}<0$$

and hence

$$\frac{1}{\sqrt{2\pi n}}e^{n\,g_n(x)}\to 0\quad\text{for}\quad 0\le x\le 1.$$

Thus a necessary condition for $A_n(x)\to x$ is that $\dfrac{N}{n}\to 1$.

Finally we have to prove that $A_n(x^2)\to x^2$.

$$A_n(x^2)=e^{-Nx}\sum_{\nu=0}^{n}\left(\frac{\nu}{n}\right)^2\frac{(Nx)^\nu}{\nu!}=e^{-Nx}\sum_{\nu=0}^{n}\frac{\nu(\nu-1)+\nu}{n^2}\frac{(Nx)^\nu}{\nu!}=$$

$$=\frac{1}{n}A_n(x)+\left(\frac{N}{n}\right)^2 x^2 e^{-Nx}\sum_{\nu=0}^{n-2}\frac{(Nx)^\nu}{\nu!}.$$

But

$$e^{-Nx}\sum_{\nu=0}^{n-2}\frac{(Nx)^\nu}{\nu!}=A_n(1)-e^{-Nx}\frac{(Nx)^{n-1}}{(n-1)!}-e^{-Nx}\frac{(Nx)^n}{n!}$$

and this expression tends to 1 as n tends to infinity, provided that $\frac{N}{n}\to 1$ and $\frac{n-N}{\sqrt{n}}\to+\infty$.

Thus we have proved that if $\frac{N}{n}\to 1$ and $\frac{n-N}{\sqrt{n}}\to+\infty$ then $A_n(1)\to 1$, $A_n(x)\to x$ *and* $A_n(x^2)\to x^2$. Hence the system $\left\{\frac{\nu}{n};\ e^{-Nx}\frac{(Nx)^\nu}{\nu!}\right\}$ solves the approximation problem.

5) Convergence for complex values of x

In the previous section we have found that the system

$$\left\{\frac{\nu}{n};\ e^{-Nx}\frac{(Nx)^\nu}{\nu!}\right\}$$

solves the approximation problem. We shall now see whether for this same system it is possible to extend the region of convergence to complex values of x. We begin with the simplest case, $f(x)=1$, for which

$$A_n(1)=e^{-Nz}\sum_{\nu=0}^{n}\frac{(Nz)^\nu}{\nu!}$$

where $z=x+iy=\varrho e^{i\varphi}$.

Let us denote by ω the function

$$\omega=z\,e^{1-z}$$

and consider the curve $|\omega|=1$. The equation of this curve is

$$\varrho\, e^{1-\varrho\cos\varphi}=1$$

and it is easily seen that it divides the z-plane into three different parts.

In Ω_1 is $|\omega|<1$ and $\varrho<1$

In Ω_2 is $|\omega|<1$ and $\varrho>1$

In Ω_3 is $|\omega|>1$.

In accordance with this definition, each region Ω is an open set.

If we put

$$j_n(z)=e^{-nz}\sum_{\nu=0}^{n}\frac{(nz)^\nu}{\nu!}$$

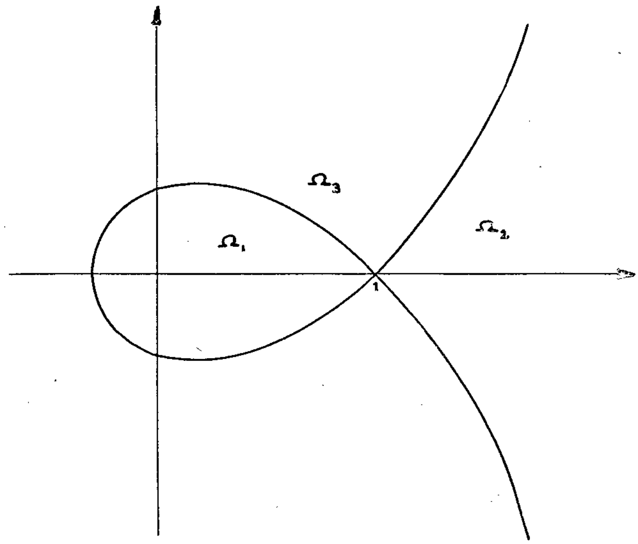

Fig. 1.

we have $j_n\left(\frac{Nz}{n}\right)=A_n(1)$ and if $j_n(z)$ tends to a limit, uniformly in a region D, then also $A_n(1)$ tends to the same limit, uniformly in every bounded, closed region, interior to D. For we know from the previous section that $\frac{N}{n}\to 1$.

Now

$$j_n(z)=e^{-nz}\sum_{\nu=0}^{n}\frac{(nz)^\nu}{\nu!}$$

$$=\frac{(nz)^n}{n!}e^{-nz}\sum_{\nu=0}^{n}\frac{n!}{\nu!\,n^{n-\nu}}\frac{1}{z^{n-\nu}}$$

$$=\frac{\omega^n}{n!}\frac{n^n}{e^n}\sum_{\nu=0}^{n}\frac{n!}{(n-\nu)!\,n^\nu}\frac{1}{z^\nu}$$

but $\frac{n!}{(n-\nu)!\,n^\nu}=\frac{n(n-1)\cdots(n-\nu+1)}{n^\nu}<1$ and tends to 1 for every fixed ν as $n\to\infty$. Hence

$$\sum_{\nu=0}^{n}\frac{n!}{(n-\nu)!\,n^\nu}\frac{1}{z^\nu}\to\sum_{\nu=0}^{\infty}\frac{1}{z^\nu}=\frac{z}{z-1}$$

uniformly for $|z|\geq 1+\eta$ for every $\eta>0$.

Also

$$n!\sim\left(\frac{n}{e}\right)^n\sqrt{2\pi n}.$$

Thus

$$j_n(z) \sim \frac{\omega^n}{\sqrt{2\pi n}} \frac{z}{z-1}$$

for $|z|>1$. From this we obtain that in the region where $|z|>1$ and $|\omega|<1$, i.e. in Ω_2, $j_n(z) \to 0$.

Again, in the region where $|z|>1$ and $|\omega|>1$, $|j_n(z)| \to \infty$. We shall prove that $|j_n(z)| \to \infty$ in Ω_3. This is now proved for the part of Ω_3 where $|z|>1$. It is easily seen that in the part where $|z| \leq 1$ is $R\{z\}<1$. We shall make use of this fact.

Integrating by parts we obtain the formula

$$\frac{n^n z^{n+1}}{n!} \int_0^n e^{zt} \left(1-\frac{t}{n}\right)^n dt = e^{nz} - \sum_{\nu=0}^{n} \frac{(nz)^\nu}{\nu!}$$

$$1 - j_n(z) = \frac{(nz)^n e^{-nz}}{n!} z \int_0^n e^{zt} \left(1-\frac{t}{n}\right)^n dt$$

$$1 - j_n(z) = \frac{\omega^n}{n!} \left(\frac{n}{e}\right)^n z \int_0^n e^{zt} \left(1-\frac{t}{n}\right)^n dt.$$

Now $\left(1-\frac{t}{n}\right)^n < e^{-t}$ so that

$$\int_0^n e^{zt} \left(1-\frac{t}{n}\right)^n dt \to \int_0^\infty e^{t(z-1)}\, dt = \frac{1}{1-z}$$

uniformly for $R\{z\} \leq 1-\eta$ for every $\eta>0$.

Hence

$$1 - j_n(z) \sim \frac{\omega^n}{\sqrt{2\pi n}} \frac{z}{1-z}$$

i.e. for $R\{z\}<1$ and $|\omega|>1$, $|j_n(z)| \to \infty$.

Finally we shall study the convergence in Ω_1.

$$j_n(z) = e^{-nz} \sum_{\nu=0}^{n} \frac{(nz)^\nu}{\nu!}$$

$$1 - j_n(z) = e^{-nz} \sum_{\nu=1}^{\infty} \frac{(nz)^{n+\nu}}{(n+\nu)!}$$

$$= (nz)^n e^{-nz} \sum_{\nu=1}^{\infty} \frac{(nz)^\nu}{(n+\nu)!}$$

$$1 - A_n(1) = 1 - i_n\left(\frac{Nz}{n}\right) = (Nz)^n e^{-Nz} \sum_{\nu=1}^{\infty} \frac{(Nz)^\nu}{(n+\nu)!}.$$

Now in Ω_1 $|1 - A_n(1)|$ is less than its maximum value on the boundary;

$$\left|\sum_{\nu=1}^{\infty} \frac{(N z)^\nu}{(n+\nu)!}\right| \leq \sum_{\nu=1}^{\infty} \frac{N^\nu}{(n+\nu)!}$$

and $|z e^{1-z}| = 1$ on the boundary, so that

$$|N^n z^n e^{-Nz}| = \frac{N^n}{e^N} |z|^{n-N} \leq \frac{N^n}{e^N}$$

as $N < n$ for n sufficiently large.

Hence

$$|1 - A_n(1)| \leq \frac{N^n}{e^N} \sum_{\nu=1}^{\infty} \frac{N^\nu}{(n+\nu)!} = e^{-N} \sum_{\nu=1}^{\infty} \frac{N^{n+\nu}}{(n+\nu)!} = 1 - e^{-N} \sum_{\nu=0}^{n} \frac{N^\nu}{\nu!}$$

and as we have proved in section 4 that

$$e^{-N} \sum_{\nu=0}^{n} \frac{N^\nu}{\nu!} \to 1$$

it follows that $A_n(1) \to 1$ uniformly in Ω_1 and on the boundary.

Summing up our results we have thus found

$A_n(1) \to 1$ uniformly in Ω_1 and on the boundary.
$A_n(1) \to 0$ in Ω_2 and the convergence is uniform in every bounded, closed region interior to Ω_2.
$|A_n(1)| \to \infty$ in Ω_3 and the convergence is uniform in every bounded, closed region interior to Ω_3.

In particular it follows that if $f(z)$ is an arbitrary analytic function, the region of convergence where $A_n(f) \to f$ is at most equal to Ω_1.

We shall now prove the following theorem.

Let $f(z)$ be regular inside Ω_1 and continuous up to and on its contour. Suppose further that $f(1) = 0$ and that $\left|\frac{f(z)}{1-z}\right|$ is bounded on the contour. Then

$$A_n(f) = e^{-Nz} \sum_{\nu=0}^{n} f\left(\frac{\nu}{n}\right) \frac{(N z)^\nu}{\nu!}$$

tends to $f(z)$ as $n \to \infty$, uniformly in every closed region interior to Ω_1

In the proof we shall frequently use the function $\log z$ of a complex number $z = r e^{iv}$. We define this function as $\log r + i v$, where $0 \leq v < 2\pi$.

Let us first notice that the function

$$B_n = e^{-nz} \sum_{\nu=0}^{n} f\left(\frac{\nu}{n}\right) \frac{(n z)^\nu}{\nu!}$$

is equal to $A_n(f)$ at the point $\frac{N}{n} z$. As $N < n$ for n sufficiently large and

$\frac{N}{n} \to 1$, it follows that if $B_n \to f$, the same is true of $A_n(f)$. B_n is more easy to handle than $A_n(f)$, and therefore we choose to prove the convergence for B_n.

We begin by cutting off the part of Ω_1 which lies to the right of the straight line $R\{z\} = 1 - \frac{1}{2n}$. We obtain a suite of new regions, which are part of Ω_1, with the contours $C'_n + C''_n$, where C'_n denotes the straight line and C''_n the contour to the left of C'_n. The contour of Ω_1 is denoted by C.

Consider now the integral

$$-\frac{n e^{-nz}}{2\pi i} \int\limits_{C'_n + C''_n} f(\zeta)\, e^{-\pi i n \zeta} (nz)^{n\zeta}\, \Gamma(-n\zeta)\, d\zeta = e^{-nz} \sum_{\nu=0}^{n-1} f\left(\frac{\nu}{n}\right) \frac{(nz)^\nu}{\nu!} = B_n$$

where $z^{n\zeta} = e^{n\zeta \log z}$.

The scheme of our proof is a follows.

We prove first that the integral along C'_n tends to zero, then that the integral along C''_n is bounded for z on C. We know that $B_n \to f$ on the real axis for $0 \le z < 1$. Hence $B_n \to f$ inside Ω_1.

Now ve have

$$\Gamma(-n\zeta) = -\frac{e^{\gamma n \zeta}}{n\zeta \prod\limits_{\nu=1}^{\infty} \left(1 - \frac{n\zeta}{\nu}\right) e^{\frac{n\zeta}{\nu}}}.$$

As $|1 - x - iy| \ge |1 - x|$ it is clear that on C'_n where $\zeta = 1 - \frac{1}{2n} + iy$

$$|\Gamma(-n\zeta)| \le |\Gamma(\tfrac{1}{2} - n)|$$

and as

$$|\Gamma(\tfrac{1}{2} - n)\, \Gamma(\tfrac{1}{2} + n)| = \pi$$

Stirlings formula gives

$$|\Gamma(\tfrac{1}{2} - n)| \sim \pi \left(\frac{e}{n + \frac{1}{2}}\right)^{n+\frac{1}{2}} \sqrt{\frac{n}{2\pi}} \sim \frac{\sqrt{2\pi}}{2} \left(\frac{e}{n}\right)^n.$$

On C is $|z e^{1-z}| = 1$ so that for ζ on C'_n

$$\left| \frac{n e^{-nz}}{2\pi i} f(\zeta)\, e^{-\pi i n \zeta} (nz)^{n\zeta}\, \Gamma(-n\zeta) \right| < M n\, |e^{-nz}|\, n^{n-\frac{1}{2}}\, |z|^n \left(\frac{e}{n}\right)^n = M\sqrt{n}$$

and as the length of $C'_n = O\left(\frac{1}{n}\right)$ the integral along C'_n tends to zero.

For the investigation of the integral along C''_n we need an asymptotic expression for $\Gamma(-n\zeta)$ on C''_n. Stirlings formula for complex values of ζ gives us

$$\log \Gamma(-n\zeta) = (n\zeta + \tfrac{1}{2})(\pi i - \log n\zeta) + n\zeta + \tfrac{1}{2}\log 2\pi + \int\limits_0^\infty \frac{[u] - u + \frac{1}{2}}{u - n\zeta}\, du.$$

valid for $0 < \arg\{\zeta\} < 2\pi$.

The function

$$\varphi(x)=\int_0^x([u]-u+\tfrac{1}{2})\,d u$$

is evidently bounded, so that we can write the "remainder term" in the following form.

$$g_n(\zeta)=\int_0^\infty \frac{[u]-u+\frac{1}{2}}{u-n\zeta}\,d u=\int_0^\infty \frac{\varphi'(u)}{u-n\zeta}\,d u=\int_0^\infty \frac{\varphi(u)}{(u-n\zeta)^2}\,d u=\frac{1}{n}\int_0^\infty \frac{\varphi(n u)}{(u-\zeta)^2}\,d u$$

i.e. for any $\varepsilon>0$, $|g_n(\zeta)|\to 0$ uniformly in the region $\varepsilon<\arg\{\zeta\}<2\pi-\varepsilon$.

If we put $\zeta=\xi+i\eta$ we have in the neigbourhood of $\zeta=1$

$$\xi^2+\eta^2=e^{2(\xi-1)}$$

$$\eta^2=e^{2(\xi-1)}-1-2(\xi-1)-(\xi-1)^2$$

$$\eta^2=(\xi-1)^2+\cdots$$

so that $\dfrac{\eta}{1-\xi}\to\pm 1$ as $\zeta\to 1$ along C, i.e. if $1-\xi_n=\dfrac{1}{2n}$

$$|2n\eta_n|\to 1;$$

hence there is a constant a such that $|n\eta_n|>a>0$.

Now for ζ on C_n'' and $\xi>0$ we get the following inequality

$$|g_n(\zeta)|=\left|\int_0^\infty \frac{\varphi(u)}{(u-n\zeta)^2}\,d u\right|<M\int_0^\infty \frac{d u}{|u-n\zeta|^2}<M\int_{-\infty}^{+\infty}\frac{d u}{u^2+n^2\eta^2}<M\int_{-\infty}^{+\infty}\frac{d u}{u^2+a^2}<\infty.$$

Thus there is an upper bound N, independent of n, such that $|g_n(\zeta)|$ is less than N on C_n''.

We now replace $\Gamma(-n\zeta)$ by the obtained expression in the integral I_n

$$I_n=\frac{n e^{-n z}}{2\pi i}\int_{C_n''} f(\zeta)\,e^{-\pi i n\zeta}(n z)^{n\zeta}\,\Gamma(-n\zeta)\,d\zeta=\frac{n}{2\pi i}\int_{C_n''} f(\zeta)\,e^{h_n(\zeta)}\,d\zeta$$

where

$$\begin{aligned} h_n(\zeta)&=-n z-\pi i n\zeta+n\zeta\log n z+\\ &+(n\zeta+\tfrac{1}{2})(\pi i-\log n\zeta)+n\zeta+\tfrac{1}{2}\log 2\pi+g_n(\zeta)=\\ &=n(\zeta-z+\zeta\log z-\zeta\log\zeta)+\frac{\pi i}{2}-\tfrac{1}{2}\log n\zeta+\\ &+\tfrac{1}{2}\log 2\pi+g_n(\zeta)\end{aligned}$$

i.e.

$$I_n = \frac{\sqrt{n}}{\sqrt{2\pi}} \int_{C_n''} \frac{f(\zeta)}{\sqrt{\zeta}}\, e^{g_n(\zeta)}\, e^{n(\zeta - z + \zeta \log z - \zeta \log \zeta)}\, d\zeta$$

$$= \frac{\sqrt{n}}{\sqrt{2\pi}} \int_{\delta_n}^{2\pi-\delta_n} f(\zeta) \frac{\zeta'}{\sqrt{\zeta}}\, e^{g_n(\zeta)}\, e^{n(\zeta - z + \zeta \log z - \zeta \log \zeta)}\, d\varphi$$

where

$$\zeta = \varrho\, e^{i\varphi} \quad \text{and} \quad \zeta' = \frac{d\zeta}{d\varphi}.$$

Now

$$\zeta' = i\zeta + \varrho' e^{i\varphi} = \zeta \left(i + \frac{\varrho'}{\varrho}\right)$$

and

$$\log \varrho = \varrho \cos \varphi - 1$$

$$\frac{\varrho'}{\varrho} = \varrho' \cos \varphi - \varrho \sin \varphi$$

$$\frac{\varrho'}{\varrho} = - \frac{\varrho \sin \varphi}{1 - \varrho \cos \varphi}$$

which is bounded, because the only critical point is $\varphi = 0$, and we know that

$$\left| \frac{\varrho \sin \varphi}{1 - \varrho \cos \varphi} \right| = \left| \frac{\eta}{1 - \xi} \right| \to 1 \text{ as } \zeta \to 1 \text{ along } C.$$

Also $\left| \frac{f(z)}{1-z} \right|$ was supposed to be bounded on C. Hence $\frac{|f(\varrho e^{i\varphi})|}{\varphi(2\pi - \varphi)}$ is bounded.

Thus we have

$$|I_n| < M \sqrt{n} \int_0^{2\pi} \varphi (2\pi - \varphi)\, e^{n R\{\zeta - z + \zeta \log z - \zeta \log \zeta\}}\, d\varphi.$$

Now we put $z = r e^{iv}$, and so

$$R\{\zeta - z + \zeta \log z - \zeta \log \zeta\} =$$

$$= \varrho \cos \varphi - r \cos v + \varrho \cos \varphi (\log r - \log \varrho) + \varrho \sin \varphi (\varphi - v) = \psi(\varphi, v).$$

As

$$\log r = r \cos v - 1$$

$$\log \varrho = \varrho \cos \varphi - 1$$

this may also be written

$$\psi(\varphi, v) = (1 - \varrho \cos \varphi)(\varrho \cos \varphi - r \cos v) + \varrho \sin \varphi (\varphi - v).$$

In particular $\psi(\varphi, \varphi) = 0$

$$\frac{\partial \psi}{\partial v} = (1 - \varrho \cos \varphi)(r \sin v - r' \cos v) - \varrho \sin \varphi$$

or as $r' = \dfrac{dr}{dv} = -\dfrac{r^2 \sin v}{1 - r \cos v}$

$$\frac{\partial \psi}{\partial v} = (1 - \varrho \cos \varphi)\frac{r \sin v}{1 - r \cos v} - \varrho \sin \varphi$$

so that also $\dfrac{\partial \psi}{\partial v} = 0$ for $v = \varphi$

$$\frac{\partial^2 \psi}{\partial v^2} = (1 - \varrho \cos \varphi)\frac{(r' \sin v + r \cos v)(1 - r \cos v) + (r' \cos v - r \sin v) r \sin v}{(1 - r \cos v)^2}$$

$$= (1 - \varrho \cos \varphi)\frac{r' \sin v + r \cos v - r^2}{(1 - r \cos v)^2}$$

$$= (1 - \varrho \cos \varphi)\frac{(1 - r \cos v)(r \cos v - r^2) - r^2 \sin^2 v}{(1 - r \cos v)^3}$$

$$= -(1 - \varrho \cos \varphi)\frac{2r^2 - (r + r^3) \cos v}{(1 - r \cos v)^3}.$$

The function $\dfrac{2r^2 - (r + r^3)\cos v}{(1 - r \cos v)^3}$ is > 0 for all values of v. To prove this we put $1 - r \cos v = t$, $r = e^{-t}$. The function may then be written

$$s(t) = \frac{2e^{-2t} - (e^{-2t} + 1)(1 - t)}{t^3} =$$

$$= \frac{e^{-2t}(1 + t) - 1 + t}{t^3}.$$

When $t \to 0$ is

$$s(t) = \frac{(1 - 2t + 2t^2 - \frac{4}{3}t^3 + \cdots)(1 + t) - 1 + t}{t^3}$$

$$= \frac{\frac{2}{3}t^3 + \cdots}{t^3} \to \frac{2}{3}$$

for $t > 0$ we consider

$$\frac{d}{dt}[e^{-2t}(1 + t) - 1 + t] = 1 - e^{-2t}(1 + 2t) = 1 - \frac{1 + 2t}{e^{2t}} > 0$$

and so $e^{-2t}(1 + t) - 1 + t > 0$ for $t > 0$. Hence there is a constant a such that

$$\frac{2r^2 - (r + r^3)\cos v}{(1 - r \cos v)^3} > a > 0.$$

Now we expand $\psi(\varphi, v)$, considered as a function of v, in its Taylor's series, using the first three terms only. Then for some ϑ between v and φ we have

$$\psi(\varphi, v) = \psi(\varphi, \varphi) + (v-\varphi)\frac{\partial \psi(\varphi, \varphi)}{\partial v} + \frac{(v-\varphi)^2}{2}\frac{\partial^2 \psi(\varphi, \vartheta)}{\partial v^2}$$

$$= \frac{(v-\varphi)^2}{2}\frac{\partial^2 \psi(\varphi, \vartheta)}{\partial v^2} < -\frac{a}{2}(v-\varphi)^2(1-\varrho\cos\varphi).$$

Again there is a constant b such that

$$\frac{1-\varrho\cos\varphi}{2\varphi(2\pi-\varphi)} > b > 0$$

and so

$$\psi(\varphi, v) < -ab(\varphi-v)^2\varphi(2\pi-\varphi).$$

If we make use of this inequality in the expression for $|I_n|$ we obtain

$$|I_n| < M\sqrt{n}\int_0^{2\pi} \varphi(2\pi-\varphi)\, e^{-nab(\varphi-v)^2\varphi(2\pi-\varphi)}\, d\varphi$$

$$< M\sqrt{n}\int_0^{2\pi} \frac{\varphi(2\pi-\varphi)}{1+nab(\varphi-v)^2\varphi(2\pi-\varphi)}\, d\varphi$$

and as $\varphi(2\pi-\varphi) \leq \pi^2$ for $0 \leq \varphi \leq 2\pi$

$$|I_n| < M\sqrt{n}\int_0^{2\pi} \frac{\pi^2}{1+nab\pi^2(\varphi-v)^2}\, d\varphi.$$

Putting $\varphi = v + \frac{t}{\sqrt{n}}$ this becomes

$$|I_n| < \int_{-\infty}^{+\infty} \frac{M\pi^2}{1+ab\pi^2t^2}\, dt < \infty.$$

The theorem is thus proved and I conclude this paper by expressing my gratitude to Professor F. Carlson who suggested the problem.

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Tryckt den 10 december 1951

Uppsala 1951. Almqvist & Wiksells Boktryckeri AB