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On the growth of minimal positive harmonic functions in a plane region

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1. We consider a plane, open, connected region D, the case of infinite connectivity being most interesting. This paper has its starting-point in an attempt to characterize boundary points in a way more suited for purposes of function theory than is possible with the aid of the purely geometric concept *accessible*. Let us fix an arbitrary accessible boundary point P of D, defined by a system of equivalent curves from some inner point to P. Two continuous curves Γ_1 and Γ_2 in our region, ending at P, are said to be equivalent, if there exist curves in D situated arbitrarily close to P which join a point of Γ_1 with a point of Γ_2.

We now suppose that there exist in D *positive harmonic functions, tending to zero in the vicinity of every boundary point except P;* we denote the class of these functions by U_P. That U_P is non-void evidently implies a certain regularity of the region D; for simplicity we have chosen a definition of U_P somewhat more restrictive than is necessary for the following.

2. One may ask how to generate functions of U_P. A procedure, near at hand, is to start from the (generalized) Green's function $G(M_1, M_2)$ for D and a sequence of inner points $P_0, P_1, \ldots, P_n, \ldots$, converging to P, then form the quotients

$$q_n(M) = \frac{G(M, P_n)}{G(P_0, P_n)}, \qquad n = 1, 2, \ldots, \tag{1}$$

and finally take limit functions of the family $\{q_n(M)\}_1^\infty$, normal in every closed part of D. Every member of U_P can be linearly expressed by such limit functions (MARTIN [4]). On the other hand, as instances of irregularity, we can construct regions where boundary continua are so accumulated towards an accessible boundary point P that it holds true for every limit function of (1) that

(a) it tends to infinity in the vicinity of a whole boundary continuum, or that

(b) it tends to zero in the vicinity of every boundary point except $P' \neq P$.

3. We now turn to a closer examination of the class U_P. If all functions of the class are proportional, we define P as *harmonically simple*, otherwise *multiple*. We shall not examine here how to distinguish between these eventualities, a question which we have had occasion to investigate somewhat in another connection [3]. A trivial case, when P is simple, may be mentioned: if a sufficiently small circle with P as centre cuts off from D a simply connected region with P as a boundary point. If not, we only state the following sufficient condition (proof as in [3], p. 25–26):

Suppose P can be circumscribed by a sequence of closed curves γ_n in D of length l_n such that $l_n \to 0$ as $n \to \infty$. Let d_n denote the shortest distance from γ_n to the boundary of D. *Then P is simple, if*

$$\limsup_{n \to \infty} \frac{d_n}{l_n} > 0.$$

4. Consider now the case when the multiplicity of P is a finite number n, i.e. the class U_P consists of linear combinations of n positive harmonic functions $u_1, u_2, \ldots, u_n$. We normalize them to be $= 1$ at some inner point P_0 of D. We then form the function

$$w = \sum_{\nu=1}^{n} \lambda_\nu u_\nu, \tag{2}$$

where the real constants $\lambda_1, \lambda_2, \ldots, \lambda_n$ satisfy the condition

$$\sum_1^n \lambda_\nu = 1,. \tag{3}$$

and interest ourselves in the values of $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ for which w is positive, i.e. belongs to U_P. The set K of these values is convex; let us show that it also is compact. K is the complement of the open set which corresponds to those w's which change signs in D. Substitute for a moment (3) for $\sum_1^n \lambda_\nu = 0$; then $w(P_0) = 0$ and w changes signs. By continuity we infer that for some $\varepsilon > 0$ it must be true that w changes signs for

$$\begin{cases} \left| \sum_1^n \lambda_\nu \right| < \varepsilon \\ \sum_1^n |\lambda_\nu| = 1, \end{cases} \qquad \text{i.e. also for} \begin{cases} \sum_1^n \lambda_\nu = 1 \\ \sum_1^n |\lambda_\nu| > \frac{1}{\varepsilon}. \end{cases}$$

Hence positive w's, under the condition (3), must correspond to $\sum_1^n |\lambda_\nu| \leq \frac{1}{\varepsilon}$. As image of the normalized functions of U_P we have thus obtained a bounded, closed, convex set K on the plane $\sum_1^n \lambda_\nu = 1$ in the Euclidean space R^n.

In the usual manner we may associate planes of support with K and by "corner" denote a boundary point as being the sole point of K belonging to a plane of support. A corner obviously corresponds to an extremal function of U_P *that can dominate no other positive harmonic functions than its own sub-*

multiples. Our supposition that U_P is a class of functions linearly composed of n elements, also implies that K must be a "polyhedron" with exactly n corners. Each function of U_P is a linear combination with non-negative coefficients of the extremal functions which correspond to these corners.

5. This property: to dominate no other positive harmonic functions than its own submultiples, has been taken by R. S. MARTIN as the definition of a *minimal positive harmonic function.* In a work [4] of great interest he has made an exhaustive study of these functions and showed their set to be sufficiently wide to serve as a basis for representation of every positive harmonic function by a linear process. This holds true for an arbitrary region; not only in the plane but in a space of any number of dimensions. In the case of a circle, for instance, the result is well-known: the representation by means of Poisson-Stieltjes' integral formula, whose kernel is minimal positive.

As an immediate consequence of the definition of minimal positive harmonic functions, we prove

Theorem I. *Let $v_1, v_2, \ldots, v_n$ be minimal positive harmonic functions in a region D and u an arbitrary positive harmonic function. Suppose that the relation $u < \sum_1^n v_\nu$ holds throughout D. Then $u \equiv \sum_1^n c_\nu v_\nu$, where the constants $c_\nu \geq 0$.*

Proof. We form a sequence of regions regular for Dirichlet's problem, whose sum is D: $D_1 \subset D_2 \subset \ldots \subset D_k \subset \ldots \to D$. It is then possible, in each D_k, to represent u as a sum: $u = \sum_{\nu=1}^n h_{\nu k}$, where $h_{\nu k}$ are positive harmonic functions such that $h_{\nu k} < v_\nu$, $\nu = 1, 2, \ldots, n$. Let, for instance, $h_{\nu k}$ be the solution of Dirichlet's problem for D_k with the boundary values $\dfrac{v_\nu}{\sum_1^n v_\nu} \cdot u$. As the sequences $\{h_{\nu k}\}_1^\infty$ constitute normal families, a selection process gives $u = \sum_1^n h_\nu$, where h_ν is a harmonic function, satisfying $0 \leq h_\nu \leq v_\nu$. Because v_ν is minimal positive, we have $h_\nu \equiv c_\nu v_\nu$, and the theorem is proved.

6. In the following study of the growth of functions we locate the boundary point P at infinity. We measure the growth of a function $u(z)$ in the region D by the order

$$\varrho = \limsup_{r \to \infty} \frac{\log M(r)}{\log r}, \quad \text{where} \quad M(r) = \underset{|z|=r}{\text{l.u.b.}}\, u(z).$$

We now prove

Theorem II. *In a region D, let $v_1, v_2, \ldots, v_n$, $n \geq 2$, be minimal positive harmonic functions of which no two are proportional, tending to zero in the vicinity* *of every finite boundary point. The order of v_ν being denoted by ϱ_ν, it then holds true that*

$$\sum_1^n \frac{1}{\varrho_\nu} \leq 2. \tag{4}$$

Proof. We form the harmonic functions

$$w_\nu = v_\nu - \sum_{k \neq \nu} v_k, \qquad \nu = 1, 2, \ldots, n.$$

According to theorem I there exist partial regions W_ν of D, where $w_\nu > 0$. These regions $W_1, W_2, \ldots, W_n$, where one of the functions v_ν exceeds the sum of the others, have no common points. Hence there are n separate regions, stretching out towards infinity; in each one a positive harmonic function is defined, vanishing at every finite boundary point. The situation is analogous to that appearing in the proof of the Denjoy-Carleman-Ahlfors theorem about the number of asymptotic paths of an integral function of given order. Here we need only refer to one of the well-known proofs of this theorem to obtain (4); [1], or [2], p. 105.

Let us now return to the exposition in no. 4 and suppose that our boundary point ∞ has finite multiplicity n, yet that $n \geq 2$. The orders of the linearly independent positive harmonic functions $u_1, u_2, \ldots, u_n$ are denoted by $\varrho_1', \varrho_2', \ldots, \varrho_n'$. As mentioned, each u_ν can be written as a linear combination with non-negative coefficients of the minimal positive $v_1, v_2, \ldots, v_n$, with orders $\varrho_1, \varrho_2, \ldots, \varrho_n$. From this we conclude

$$\sum_1^n \frac{1}{\varrho_\nu'} \leq \sum_1^n \frac{1}{\varrho_\nu},$$

i.e. relation (4) *holds true also for* $u_1, u_2, \ldots, u_n$.

The condition $n \geq 2$ is essential; if the boundary is sufficiently "weak", the order of a $u \in U_P$ may be 0; for instance, in the trivial extreme case of D being $1 < |z| < \infty$, then $u = \log |z|$.

7. Consider finally a region D where there exist an infinity of minimal positive harmonic functions, vanishing at every finite boundary point. Of course, we here mean functions of which no two are proportional. It is easy to construct regions with a continuum of such functions: by choosing as boundary of D appropriate segments of the lines $\arg z = \varkappa\pi$ for rational $\varkappa$'s, we may assign a function to every direction corresponding to an irrational $\varkappa$.

For every particular choice of n minimal positive harmonic functions theorem II holds true; especially, the number of functions of order $\leq \varrho$ is $\leq 2\varrho$. More completely, we can state

Theorem III. *Those minimal positive harmonic functions, vanishing at every finite boundary point, which are of finite order, form a countable set. For the orders ϱ_ν of these functions it holds true that*

$$\sum_1^\infty \frac{1}{\varrho_\nu} \leq 2.$$

REFERENCES: [1]. **L. Ahlfors,** Über die asymptotischen Werte der ganzen Funktionen endlicher Ordnung. Ann. Acad. Sci. Fenn. Serie A, vol. 32, 1929. — [2]. **A. Beurling,** Études sur un problème de majoration. Thèse, Upsal 1933. — [3]. **B. Kjellberg,** On certain integral and harmonic functions. A study in minimum modulus. Thesis, Uppsala 1948. — [4]. **R. S. Martin,** Minimal positive harmonic functions. Trans. A.M.S. vol. 49, 1941.

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