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On the analytic continuation of Eulerian products

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1. Introduction and summary

1.1. Let $h(z)$ be an analytic function that is regular and takes the value 1 for $z = 0$ and has no limit-point of zeros or singularities in the region $|z| \leq 1$. Consider the formal Eulerian product

$$f(s) = \prod_p h(p^{-s}) \tag{1.1}$$

where p runs through all prime numbers, and

$$s = \sigma + i\tau$$

is a complex variable. We have, e. g.

$$h(z) = (1-z)^{-1} \qquad f(s) = \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \tag{1.2}$$

$$h(z) = (1-z) \qquad f(s) = \zeta(s)^{-1} = \sum_{n=1}^{\infty} \mu(n) \cdot n^{-s} \tag{1.3}$$

$$h(z) = \prod_{\nu=1}^{k} (1-z^{\nu})^{-\beta_\nu} \qquad f(s) = \prod_{\nu=1}^{k} \zeta(\nu s)^{\beta_\nu} \tag{1.4}$$

$$h(z) = e^z \qquad f(s) = e^{P(s)} \tag{1.5}$$

where

$$P(s) = \sum p^{-s} \tag{1.6}$$

p running through all primes.

The main purpose of this paper is to show[1]

Theorem I. *The imaginary axis is a natural boundary of $f(s)$, except for the case in which the functions $h(z)$ and $f(s)$ have the form* (1.4).

A wider class $\{h(z)\}$ is discussed in section 4.2., and in Part 5 the corresponding results are derived for functions of the form

$$\prod_p h(\chi(p) \cdot p^{-s}).$$

[1] I am indebted to Prof. F. CARLSON for suggesting the problem and for his valuable advice.

$\chi(p)$ being a group character mod. k. In order to prove this result, we need the analogue for L-functions of the famous theorem of HARDY and LITTLEWOOD [4] concerning the zeros of $\zeta(s)$ on the critical line. For the sake of completeness, the proof is worked out in detail in Part 6.

1.2. Some subclasses of the class $\{f(s)\}$ have been discussed in earlier papers. The function $P(s)$ has been investigated by KLUYVER and later by LANDAU and WALFISZ [6], who made use of an expansion equivalent to

$$e^{P(s)} = \prod_{\nu=1}^{\infty} \zeta(\nu s)^{\mu(\nu)} \tag{1.7}$$

where $\mu(\nu)$ is the Möbius function also occurring in (1.3). KLUYVER observed that every zero and singularity of $e^{P(s)}$ is a zero or singularity of one of the functions

$$\zeta(s);\ \zeta(2s);\ \zeta(3s);\ \zeta(4s);\ \ldots \tag{1.8}$$

and furthermore that, if the Riemann hypothesis is true, then every point on the imaginary axis is a limit-point of zeros and singularities, the imaginary axis thus forming a natural boundary. His argument is applicable to the case investigated here in Part 3. But if the Riemann hypothesis is not assumed, the problem is not so simple, because it is possible that different factors cancel each other. LANDAU and WALFISZ surmounted this difficulty. They used, however, special properties of the occurring coefficients, which have no counterpart in the general case treated here. The expansion

$$f(s) = \prod_{\nu=1}^{\infty} \zeta(\nu s)^{\beta_\nu}, \tag{1.9}$$

however, is one of the devices used in Part 3, in which $h(z)$ is assumed to have no zeros and singularities inside the unit circle. But the most important of the new difficulties has to be overcome by use of Lemma 3.3 concerning a general property of arbitrary sequences of positive integers.

In Part 4, $h(z)$ is assumed to have zeros or singularities inside the unit circle. In this case, the product (1.9) is divergent in the neighbourhood of the imaginary axis.[1] ESTERMANN has treated the case in which $h(z)$ is a polynomial with integral coefficients. Instead of (1.9) he used a sequence of products of the form

$$f(s) = \prod_{p \le q} h(p^{-s}) \cdot \prod_{\nu=1}^{\infty} \zeta_q(\nu s)^{\beta_\nu} \tag{1.10}$$

where

$$\zeta_q(s) = \zeta(s) \cdot \prod_{p \le q} (1 - p^{-s}). \tag{1.11}$$

We generalize his method here in Part 4.

[1] WINTNER [8] treats the case in which $h(z) = 1 - \alpha \cdot z$ (α is an arbitrary constant). He uses results similar to those of LANDAU and WALFISZ, although their method of proof is not applicable to WINTNER's case. Moreover, his method seems to require the convergence of (1.9) in the half-plane $\sigma > 0$. If $|\alpha| > 1$, however, the product is divergent in the neighbourhood of the imaginary axis.

2. Some transformations

We shall perform some transformations. We start by proving that $h(z)$ can be factorized in the form

$$h(z) = \prod_{\nu=1}^{\infty} (1 - z^{\nu})^{-\beta_{\nu}}. \tag{2.1}$$

The product is absolutely convergent with respect to z and β_{ν} if $|z| < a$, where a is equal to the smaller of the numbers *1* and the least modulus of a zero or singularity of $h(z)$. This definition of a will be used throughout the paper.

Put

$$z \cdot \frac{h'(z)}{h(z)} = \sum_{\nu=1}^{\infty} c_{\nu} z^{\nu}.$$

The series on the right-hand side is convergent if $|z| < a$, and is formally identical with

$$\sum_{\nu=1}^{\infty} \sum_{m=1}^{\infty} \alpha_{\nu} \cdot z^{m\nu} \tag{2.2}$$

if

$$c_n = \sum_{\nu/n} \alpha_{\nu}.$$

According to Möbius' inversion formula [3, theorem 266], we have

$$|\alpha_n| = \left| \sum \mu\left(\frac{n}{\nu}\right) \cdot c_{\nu} \right| \leqq n \cdot \operatorname*{Max}_{\nu \leqq n} |c_{\nu}| = 0\,((a - \varepsilon)^{-n}) \qquad (n \to \infty). \tag{2.3}$$

It follows that the double series (2.2), each term of which is less than the corresponding term of the series

$$k \cdot \sum \sum \frac{|z|^{m\nu}}{(a - \varepsilon)^{\nu}},$$

is uniformly convergent for

$$|z| \leq a - 2\varepsilon.$$

Hence

$$\frac{h'(z)}{h(z)} = \frac{1}{z} \cdot \sum_{\nu=1}^{\infty} \sum_{m=1}^{\infty} = \sum_{\nu=1}^{\infty} \frac{\alpha_{\nu} z^{\nu-1}}{1 - z^{\nu}}.$$

But the second member is the logarithmic derivative of the second member of (2.1), if we put

$$\beta_{\nu} = \frac{\alpha_{\nu}}{\nu} \tag{2.4}$$

and since (2.1) is valid for $z = 0$, this proves (2.1), for $|z| < a$.

Next we consider the formal identity

$$\frac{f(s)}{\prod\limits_{p \leq q} h(p^{-s})} = \prod_{p>q} h(p^{-s}) = \prod_{p>q} \prod_{\nu=1}^{\infty} (1 - p^{-\nu s})^{-\beta_\nu} =$$

$$= \prod_{\nu} \prod_{p>q} (1 - p^{-\nu s})^{-\beta_\nu} = \prod_{\nu=1}^{\infty} \zeta_q(\nu s)^{\beta_\nu}$$

where $\zeta_q(s)$ is defined by (1.11). We have to justify the inversion of the order of the multiplications. It is obvious that

$$|\log(1-z)| < A \cdot |z| \qquad (|z| < \tfrac{1}{2})$$

where A is an absolute constant, and we have chosen the principal branch of the logarithm. Hence, if $\sigma > 1$, we have

$$\sum_{p>q} |\beta_\nu \cdot \log(1 - p^{-\nu s})| < |\beta_\nu| \cdot A \cdot \sum_{p>q} p^{-\nu\sigma} < |\beta_\nu| \cdot A \cdot \int_q^\infty x^{-\nu\sigma}\, dx < \frac{q}{q^{\nu\sigma}} \cdot O\left(\frac{1}{(a-\varepsilon)^\nu}\right).$$

We obtained the last inequality with the aid of (2.3) and (2.4). If we suppose that

$$q > a^{-\frac{1}{\sigma}}, \text{ i. e. } \sigma > \frac{\log \frac{1}{a}}{\log q} \tag{2.6}$$

we can conclude that the double sum

$$\sum_{\nu} \sum_{p>q} \beta_\nu \cdot \log(1 - p^{-\nu s})$$

converges absolutely. This justifies the inversion, if $\sigma > 1$ and if (2.6) is satisfed.

Next we consider the product

$$\prod_{\nu=\nu_0}^{\infty} \zeta_q(\nu s)^{\beta_\nu}. \tag{2.7}$$

If

$$\nu \geq \nu_0 > \frac{1}{\sigma}$$

then, by the Dirichlet series of the logarithm of $\zeta_q(s)$, we find that

$$|\log \zeta_q(\nu s)| < \sum_{n=q}^{\infty} \frac{1}{n^{\nu\sigma}} < \int_q^\infty x^{-\nu\sigma}\, dx = \frac{q^{1-\nu\sigma}}{\nu\sigma - 1}.$$

Hence, if q satisfies (2.6), the series

$$\sum_{\nu=\nu_0}^{\infty} \beta_\nu \cdot \log \zeta_q(\nu s)$$

represents a regular function in the half-plane $\sigma > \frac{1}{\nu_0}$. Therefore the product (2.7) is regular and different from zero in the same region. From (2.5) it follows that

$$f(s) = \prod_{p \le q} h(p^{-s}) \cdot \prod_{\nu=1}^{\nu_0 - 1} \zeta_q(\nu s)^{\beta_\nu} \cdot \prod_{\nu=\nu_0}^{\infty} \zeta_q(\nu s)^{\beta_\nu}.$$

Taking q and ν_0 large enough we can attain any point in the half-plane $\sigma > 0$. From this formula and the results mentioned above, we get the following

Lemma 2.1. *The product*

$$f(s) = \prod_p h(p^{-s})$$

where $h(z)$ is regular for $z = 0$, and $h(0) = 1$, defines a Dirichlet series convergent in a certain half-plane $\sigma > A$. If $h(z)$ has only a finite number of zeros and singular points in the circle $|z| \le 1$ this function $f(s)$ can be continued into the half-plane $\sigma > 0$ with the aid of products of the form

$$f(s) = \prod_{p \le q} h(p^{-s}) \cdot \prod_{\nu=1}^{\infty} \zeta_q(\nu s)^{\beta_\nu}.$$

Thus $f(s)$ is regular in $\sigma > 0$ except perhaps for the s-values for which a function

$$h(p^{-s}) \qquad p = 2, 3, 5, 7, 11 \ldots$$

has a singular point or the s-values for which a function

$$\zeta(\nu s) \qquad \nu = 1, 2, 3 \ldots$$

has a singular point or a zero.

Remark. *If $h(z)$ is regular and different from zero in the circle $|z| < 1$, i. e. if $a = 1$, then q is arbitrary and there is no need of the factors $h(p^{-s})$. In fact, we may use an expansion of the form*

$$f(s) = \prod_{\nu=1}^{\infty} \zeta(\nu s)^{\beta_\nu}$$

for any s in the half-plane $\sigma > 0$.

3. Proof of theorem I in the case $a = 1$

The function $h(z)$ has no zeros or singular points inside the circle

$$|z| < 1.$$

3.1. We need some results concerning the zeros of $\zeta(s)$. The following results are sufficient here. According to BOHR and LANDAU [1], the number of zeros of $\zeta(s)$ in the region

$$0 < \tau < T; \ |\sigma - \tfrac{1}{2}| > h$$

is $o(T)$. It is also known that $\sigma < 1$ for every zero of $\zeta(s)$. According to HARDY and LITTLEWOOD [4], the number of distinct zeros of $\zeta(s)$ on the line $\sigma = \frac{1}{2}$ between $\tau = T$ and $\tau = T + \eta \cdot T$ is greater than

$$K(\eta) \cdot T$$

for $|T| > T_0(\eta)$.[1]

Definition 3.1. The rectangle

$$0 \leq \sigma \leq 1; \ (1 - \varepsilon) \cdot t \leq \tau \leq (1 + \varepsilon) \cdot t$$

is denoted by $\mathfrak{R}(t; \varepsilon)$.

Lemma 3.1. *For any η and any sufficiently large T, there exists at least one straight line through the origin, containing at least one zero of $\zeta(s)$, inside the rectangle $\mathfrak{R}(T; \eta)$ but no zero outside that rectangle.*

There are more than

$$K(\eta) \cdot T \qquad (T > T_0(\eta))$$

distinct zeros on the line $\sigma = \frac{1}{2}$ in the rectangle $\mathfrak{R}(T; \eta)$, whereas the lines joining the origin and these zeros contain altogether only

$$o(T) \qquad (T \to \infty)$$

zeros outside $\mathfrak{R}(T; \eta)$. Hence, if $T > T_1(\eta)$, there exist lines possessing the properties required.

3.2. Consider an arbitrary infinite set $\mathfrak{S}$ of different positive integers. Let

$$n = \prod p_j^{x_j} \qquad (x_j \geq 0)$$

be the standard form of a number $n \in \mathfrak{S}$, where

$$p_1 = 2; \ p_2 = 3; \ p_3 = 5; \ p_4 = 7; \ldots .$$

Definition 3.2. A number $n^* = \prod p_j^{x_j^*}$ is a *vertex number* of $\mathfrak{S}$, if there is a sequence of numbers (not necessarily integers)

$$\lambda_1, \lambda_2, \lambda_3 \ldots$$

such that

$$\sum \lambda_j x_j^* > \sum \lambda_j x_j$$

for all $n \in \mathfrak{S}$ satisfying the conditions

$$n < 2 \cdot n^*; \ n \neq n^*.$$

[1] The symbols $K(\eta)$ and $T_0(\eta)$ denote numbers depending on η but not on T.

The sequence (λ_j) may be different for different n^*. We shall prove:

Lemma 3.2. *Every infinite set $\mathfrak{S}$ of positive integers contains an infinity of vertex numbers.*

To this purpose we represent the numbers of $\mathfrak{S}$ by points in a plane. We put e. g.

$$x(n) = \frac{\log n}{\log 2}; \quad y(n) = \sum x_j \left[\frac{\log p_j}{\log 3}\right].$$

We observe that

A. $y(n)$ is an integer.

B. $x(n) - y(n) \to \infty$; as $n \to \infty$.

C. $\limsup y(n) = \infty$.

The condition C is valid, unless $\mathfrak{S}$ is composed of integers for which the product of the odd prime factors is bounded. That case will be considered a little later. By B, there are at most a finite number of points of $\mathfrak{S}$ for which

$$y - x > -q.$$

Now we shall try to remove all numbers that are not vertex numbers. To begin with, if a line

$$x - y = q$$

contains more than one point of $\mathfrak{S}$, we remove all such points except the highest. We arrange the remaining points in order of increasing q, and then we make a new selection. This time we keep only those points having greater ordinate than each point counted before. On account of C an infinity of points will still remain. Let the corresponding numbers be

$$n^{(1)};\ n^{(2)};\ n^{(3)};\ n^{(4)};\ \ldots. \tag{3.1}$$

For the sake of brevity, we shall write x instead of $x(n)$, $x^{(i)}$ instead of $x(n^{(i)})$ etc. Moreover, we do not distinguish between the points and the corresponding numbers.

Because of the construction, it follows that

$$y \geq y^{(i)};\ n \neq n^{(i)} \text{ implies } q > q^{(i)}$$

whence

$$y \geq y^{(i)};\ n \neq n^{(i)} \text{ implies } y + q > y^{(i)} + q^{(i)}, \text{ i. e. } x > x^{(i)}.$$

Hence

$$x < x^{(i)}, \text{ i. e. } n < n^{(i)} \text{ implies } y < y^{(i)}$$

and by A

$$n < n^{(i)} \text{ implies } y \leq y^{(i)} - 1. \tag{3.2}$$

The reader may interpret the argument geometrically. In the same way it is seen that

$$n < 2 \cdot n^{(i)} \text{ implies } y \leq y^{(i)}. \tag{3.3}$$

Before we complete the proof that the numbers of the sequence (3.1) are vertex numbers, we derive results corresponding to (3.2) and (3.3) in the case previously omitted (page 539, line 11). It is easily seen that, in this case, the numbers of $\mathfrak{S}$ has to be of the form $2^{x_1} \cdot u$, where x_1 runs through an infinite set of positive integers, and u runs through a finite set of odd numbers. In this case we represent the number n by the point

$$x = \frac{\log n}{\log 2}; \; y = \sum x_j \left[\frac{\log p_j}{\log 2}\right]$$

in a plane. Now the conditions A and C are satisfied, but B is not valid. It is easily seen that $q(n)$ defined by the equation

$$q(n) = x(n) - y(n)$$

depends only on u, and that q therefore assumes only a finite number of different values. Put

$$\liminf q(n) = g.$$

g is attained in an infinity of points of $\mathfrak{S}$. We denote them by

$$n^{(1)} < n^{(2)} < n^{(3)} < \cdots.$$

The inequality $q < g$ is valid for at most a finite set of numbers of $\mathfrak{S}$ and we can therefore assume that i is so large that

$$y \geq y^{(i)} \text{ implies } q \geq g.$$

It follows that

$$y \geq y^{(i)} \text{ implies } q + y \geq g + y^{(i)}, \text{ i. e. } x \geq x^{(i)}.$$

Hence

$$n < n^{(i)} \text{ implies } y \leq y^{(i)} - 1 \tag{3.4}$$

also in the case previously excluded. In a similar way we find that

$$n < 2 \cdot n^{(i)} \text{ implies } y \leq y^{(i)}. \tag{3.5}$$

Let $\mathfrak{S}$ become an arbitrary infinite set of different positive integers again, and consider

$$\varphi(n) = y(n) - \mu \cdot \log n.$$

$\varphi(n)$ is obviously of the form $\sum \lambda_j x_j$. We have

$$\varphi(n) - \varphi(n^{(i)}) = y - y^{(i)} - \mu \cdot (\log n - \log n^{(i)}).$$

Take

$$0 \leq \mu < \frac{1}{\log n^{(i)}}.$$

Then we find that

$n < n^{(i)}$ implies $\varphi - \varphi^{(i)} < y - y^{(i)} + 1$ and by (3.2) and (3.4): $\varphi < \varphi^{(i)}$,

$n^{(i)} < n < 2\, n^{(i)}$ implies $\varphi - \varphi^{(i)} < y - y^{(i)}$ and by (3.3) and (3.5): $\varphi < \varphi^{(i)}$.

Hence

$$n \neq n^{(i)};\ n < 2 \cdot n^{(i)} \text{ implies } \varphi(n) < \varphi(n^{(i)}).$$

This proves the lemma.

3.3. Consider again formula (1.9)

$$f(s) = \prod_{n=1}^{\infty} \zeta(n\,s)^{\beta_n}.$$

Let $\mathfrak{S}$ be the set of all n for which $\beta_n \neq 0$. Suppose that $\mathfrak{S}$ is an infinite set. We shall show that there is an infinity of zeros or singularities of $f(s)$ in $\Re(u;\eta)$, i. e. in the rectangle

$$0 < \sigma < 1;\ u \cdot (1 - \eta) < \tau < u \cdot (1 + \eta).$$

We may obviously assume, without loss of generality, that $u > 0$ and $0 < \eta < \frac{1}{3}$.

Suppose that n^* is a vertex number of $\mathfrak{S}$ and that

$$\lambda_1,\ \lambda_2,\ \lambda_3,\ \cdots$$

is a sequence of numbers associated with n^* in the sense of definition 3.2. By lemma 3.1., there are straight lines containing at least one zero s' of $\zeta(s)$ inside the rectangle $\Re(n^* u;\eta)$, whereas there are no zeros of $\zeta(s)$ on them outside the rectangle. This is true, if n^* is large enough. Hence

$$\frac{s'}{n^*} \in \Re(u;\eta).$$

This point is a zero or a singularity of $f(s)$, unless the ζ-function of Riemann has another zero s in $\Re(n^* u;\eta)$ such that

$$\frac{s'}{n^*} = \frac{s}{n}. \qquad (n \in \mathfrak{S})$$

In this case it is possible that two factors of (1.9) cancel each other in order to make $f(s)$ regular and different from zero in the point considered. Now consider all zeros of the form $s = s' \cdot r$, where r is a real and rational number. Let the standard form of r be

$$r = \prod p_i^{y_i} \qquad (y_i \text{ integer} \gtreqless 0)$$

and let g be the lower bound of $\sum \lambda_i y_i$, as s runs through these zeros. We assume that the lower bound is attained for the zero

$$s^* = s' \cdot \prod p_i^{y_i^*}.$$

Then we have

$$\sum \lambda_i y_i \geq \sum \lambda_i y_i^* = g.$$

We know that

$$\frac{s^*}{n^*} \in \Re(u; \eta).$$

If $\dfrac{s^*}{n^*}$ were not a zero or a singularity of $f(s)$, there would be a number $n'' \in \mathfrak{S}$ and a zero s'' of $\zeta(s)$ such that

$$\frac{s^*}{n^*} = \frac{s''}{n''}.$$

By the fundamental theorem of arithmetic, we then get

$$y_i^* - x_i^* = y_i'' - x_i''$$

whence

$$\sum \lambda_i \cdot (y_i^* - y_i'') = \sum \lambda_i \cdot (x_i^* - x_i'').$$

According to the definition, the first member is non-positive, whereas by lemma 3.2. the second member is positive, provided that

$$\frac{1+\eta}{1-\eta} < 2, \text{ i. e. } \eta < \frac{1}{3}.$$

This restriction is, however, unessential, as was pointed out earlier. Thus we have obtained a contradiction, showing that $f(s)$ has at least one zero or singularity in $\Re(u; \eta)$, corresponding to each vertex number which is large enough.

3.4. By the Bolzano-Weierstrass theorem these zeros and singularities have at least one limit-point, which must lie on the imaginary axis, according to lemma 2.1. This limit-point is a singularity of $f(s)$. The results of Section 3.3. are valid for any $u > 0$ and for any positive η. Moreover, the same result is true for $u < 0$; because $\zeta(s)$ takes conjugate values in conjugate points. Hence the imaginary axis is a natural boundary of $f(s)$.

Remark. If $\mathfrak{S}$ is a finite set, then $h(z)$ and $f(s)$ are of the form (1.4), and $f(s)$ is regular in the whole s-plane, except perhaps in a finite or infinite set of isolated points.

4. Proof of theorem I in the case $a < 1$

$h(z)$ has $M \geq 1$ zeros or singularities inside the unit circle.

4.1. Let the zeros and singularities of $h(z)$ be

$$z_m = e^{-(b_m + i c_m)} \qquad (m = 1, 2, 3, \ldots M)$$

so arranged, that

$$b_1 \geq b_2 \geq b_3 \cdots \geq b_M > 0.$$

According to Section 1.3. we have to consider products of the form

$$f(s) = \prod_{p \le q} h(p^{-s}) \cdot \prod_{n=1}^{\infty} \zeta_q(n\, s)^{\beta_n}.$$

Let Ω_δ be the region

$$\sigma \le \delta; \quad 0 < u \le \tau \le u + \eta.$$

If the zeros and singularities of $f(s)$ produced by z_m, i.e.

$$s = \frac{b_m + i\, c_m + 2\pi i n}{\log p} \qquad \begin{matrix} (n = 0, \pm 1, \pm 2, \ldots) \\ (p = 2, 3, 5, 7, \ldots) \end{matrix}$$

do not interfere with those produced by the other zeros and singularities of $h(z)$ or by the ζ_q-factors, their number in Ω_δ is given by

$$N_m = \sum_{p < e^{\frac{b_m}{\delta}}} \left(\frac{\eta \log p}{2\pi} + \theta \right)$$

where $|\theta| \le 1$, and p runs through the prime numbers. But according to the prime number theorem

$$\sum_{p \le x} \log p \sim x \qquad (x \to \infty)$$

$$\sum_{p < x} \theta = 0(\pi(x)) = o(x) \qquad (x \to \infty)$$

whence

$$N_m \sim \frac{\eta}{2\pi} e^{bm/\delta}. \qquad (\delta \to 0)$$

A zero or singularity in Ω_δ produced by the ζ_q-factors must have its source in some zero of $\zeta(s)$, situated below the line

$$\tau = \frac{u + \eta}{\delta} \cdot \sigma.$$

The number of such zeros is, according to a classical result

$$0\left(\frac{1}{\delta} \cdot \log \frac{1}{\delta}\right) \qquad (\delta \to 0)$$

and the same zero is used at most

$$0\left(\frac{1}{\delta}\right) \qquad (\delta \to 0)$$

times, since all zeros have $\sigma < 1$, the number of possible ζ_q-factors thus being at most of the order $\frac{1}{\delta}$. Hence the ζ_q-factors produce only

$$0\left(\frac{1}{\delta^2}\cdot\log\frac{1}{\delta}\right) \qquad (\delta\to 0)$$

zeros or singularities of $f(s)$ in Ω_δ.

We shall show that z_1 produces so many zeros or singularities that the other z_ν's and the ζ_q-factors cannot cancel all of them. Two zeros or singularities of $h(z)$ with the same modulus cannot interfere with each other. We may therefore assume that

$$b_2 < b_1$$

without loss of generality. Even if all other zeros and singularities contribute to cancel those produced by z_1, there is still more than

$$\frac{\eta}{2\pi}\cdot\left[e^{\frac{b_1}{\delta}} - o\left(e^{\frac{b_1}{\delta}}\right)\right] - \sum_{b_\nu<b} 0\left(e^{\frac{b_\nu}{\delta}}\right) - 0\left(\frac{1}{\delta^2}\log\frac{1}{\delta}\right) \quad (\delta\to 0) \qquad (4.1)$$

zeros and singularities of $f(s)$ in Ω_δ. But this expression is unbounded when $\delta\to 0$. Hence there is an infinity of zeros or singularities of $f(s)$ in the region

$$\Omega = \lim_{\delta\to 0}\Omega_\delta.$$

By means of the argument used at the end of Section 3.4, we find that the imaginary axis is a natural boundary of $f(s)$, and the proof of Theorem I is complete.

4.2. In the preceding sections the set of zeros and singularities of $h(z)$ in the circle $|z|\leq 1$ was supposed to be finite. The investigation of the general case, where no such restriction is made, seems to require more delicate methods. The method used above is, however, sufficient with slight modifications, if the number of zeros and singularities in the circle $|z|$, is less than, say, $A\,.\,2^{\frac{b}{1-r}}$, if $b<b_1$. In this rather general case, Theorem I is still valid.

5. A more general class of Dirichlet series.

The essential facts needed in the previous analysis were:

A. Lemma 2.1., giving the formula

$$f(s) = \prod_{p\leq q} h(p^{-s})\cdot\prod_{n=1}^{\infty}\zeta_q(n\,s)^{\beta_n}. \qquad (5.1)$$

B. Lemma 3.1., concerning the distribution of the zeros of $\zeta(s)$.

Consider, for instance

$$f(s) = \prod h(\chi(p)\cdot p^{-s}) \qquad (5.2)$$

instead of the function (1.1). $\chi(p)$ is a group character mod. k. The formula corresponding to (5.1), i.e.

$$f(s) = \prod_{p<q} h(\chi(p)\cdot p^{-s}) \cdot \prod_{n=1}^{\infty} L_q(n\,s;\,\chi)^{\beta_n} \tag{5.3}$$

is proved in exactly the same way as (5.1). The analogue of Lemma 3.1. can also be proved for the functions $L(s;\chi)$ because the analogue of the theorem of BOHR and LANDAU as well as the analogue of the theorem of HARDY and LITTLEWOOD is valid for the functions $L(s;\chi)$. The first of these results was proved by BOHR and LANDAU [1], but the second does not seem to have been proved before. It is, however, merely an exercise in the methods developed by TITCHMARSH [cf. 7, p. 49]. The proof is found in Part 6. We obtain just as above:

Theorem II: *The imaginary axis is a natural boundary of the functions (5.2), if $h(z)$ satisfies the conditions mentioned in Section 1.1 or Section 4.2. Only the cases, in which $h(z)$ is of the form (1.4), are exceptions.*

6. The L-functions on the critical line

Theorem III: *Let $N_0(T)$ be the number of distinct zeros of $L(\frac{1}{2}+i\tau;\chi)$ in the interval $(0, T)$, $\chi(n)$ being a group character mod. k. Then*

$$|N_0(T+\eta T) - N_0(T)| > K(\eta)\cdot|T|$$

for $|T| > T_0(\eta)$, $\eta > 0$.

6.1. Let $\chi(n)$ be a proper character[1] mod. k. If $k=1$, theorem III is a consequence of the theorem of HARDY and LITTLEWOOD. We assume therefore $k>1$, and consider

$$L(s;\chi) = \sum_{m=1}^{\infty} \chi(m)\cdot m^{-s}. \tag{6.1}$$

Let

$$\gamma = \begin{cases} 1, & \text{if} \quad \chi(-1) = -1 \\ 0, & \text{if} \quad \chi(-1) = 1 \end{cases} \tag{6.2}$$

and consider

$$\psi(x;\chi) = 2\sum_{m=1}^{\infty} m^{\gamma}\cdot\chi(m)\cdot e^{-\frac{m^2\pi x}{k}}. \tag{6.3}$$

According to LANDAU [5; § 128, (3)], we have the functional equation

$$\psi(x;\chi) = \frac{\varepsilon(\chi)}{x^{\gamma+\frac{1}{2}}}\cdot\psi\left(\frac{1}{x};\bar{\chi}\right) \tag{6.4}$$

$\bar{\chi}$ denoting the character conjugate to χ, $\varepsilon(\chi)$ having the properties

$$|\varepsilon(\chi)| = 1 \qquad \varepsilon(\bar{\chi}) = \overline{\varepsilon(\chi)}. \tag{6.5}$$

[1] Cf. LANDAU [5]; proper character = eigentlicher Gruppencharakter.

Consider also [cf. 5, § 128, (2)]

$$\xi(s;\chi) = \left(\frac{\pi}{k}\right)^{-\frac{s+\gamma}{2}} \cdot \Gamma\left(\frac{s+\gamma}{2}\right) \cdot L(s;\chi) = \tfrac{1}{2}\int_0^\infty \psi(x;\chi)\, x^{\frac{s+\gamma}{2}-1}\, dx \tag{6.6}$$

and

$$\eta(s;\chi) = \frac{1}{\sqrt{\varepsilon(\chi)}} \cdot \xi(s;\chi)^{1} \tag{6.7}$$

which satisfies the functional equation [5; § 128, (5)]

$$\eta(s;\chi) = \eta(1-s;\bar\chi) = \overline{\eta(1-\bar s;\chi)} \tag{6.8}$$

whence

$$\eta(\tfrac{1}{2}+it;\chi)$$

is a real function. Consider furthermore the real function

$$Q(t) = -\frac{\sqrt{\varepsilon(\chi)}}{2}\cdot\left(\frac{\pi}{k}\right)^{-\frac{1+2\gamma}{4}-\frac{it}{2}} \cdot e^{\left(\frac{\pi}{4}-\frac{\delta}{2}\right)\cdot t} \cdot \Gamma\left(\frac{1+2\gamma}{4}+\frac{it}{2}\right)\cdot L(\tfrac{1}{2}+it;\chi) =$$

$$= -\frac{\sqrt{\varepsilon(\chi)}}{2}\, e^{\left(\frac{\pi}{2}-\delta\right)\cdot\frac{t}{2}} \cdot \xi(s;\chi) \tag{6.9}$$

and consider the integrals, which we shall compare with each other

$$I = \int_t^{t+h} Q(u)\, du \tag{6.10}$$

$$J = \int_t^{t+h} |Q(u)|\, du. \tag{6.11}$$

By Mellin's integral formula applied to (6.6), we have

$$\psi(x;\chi)\cdot x^{\frac{\gamma}{2}} = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \xi(s;\chi)\, x^{-\frac{s}{2}}\, ds \tag{6.12}$$

if $\alpha > 1$. By (6.6), Stirlings formula, and the well-known fact that the L-functions are of finite order, the relation

$$|\xi(s;\chi)| = 0\left(e^{-\frac{\pi|\tau|}{4}}\cdot|\tau|^{A}\right) \qquad (|\tau|\to\infty) \tag{6.13}$$

holds uniformly for $\frac{1}{2} \le \sigma \le \alpha$. Furthermore $\xi(s;\chi)$ is an integral function. By Cauchy's integral theorem

[1] We make a sign convention such that $\sqrt{\varepsilon(\chi)}\cdot\sqrt{\varepsilon(\bar\chi)} = 1$.

$$\frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT}\xi(s;\chi)\cdot x^{-\frac{s}{2}}\,ds=\int_{\alpha-iT}^{\frac{1}{2}-iT}+\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}+\int_{\frac{1}{2}+iT}^{\alpha+iT}\equiv I_1+I_2+I_3.$$

From (6.13)

$$I_1\to 0,\quad I_3\to 0$$

when $T\to\infty$. Hence

$$\int_{\alpha-i\infty}^{\alpha+i\infty}=\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}.$$

Both integrals converge by (6.13). By (6.9) and (6.12) we obtain

$$-\frac{\sqrt{\varepsilon(\chi)}}{2}\cdot\psi(x;\chi)\cdot x^{\frac{2\gamma+1}{4}}=\frac{1}{2\pi}\int_{-\infty}^{+\infty}Q(t)\,e^{-\left(\frac{\pi}{2}-\delta\right)\frac{t}{2}}\,x^{-\frac{it}{2}}\,dt. \tag{6.14}$$

Put

$$x=e^{2y}\cdot e^{i\left(\frac{\pi}{2}-\delta\right)}=\lambda\cdot e^{2y}. \tag{6.15}$$

By (6.14)

$$-\sqrt{\varepsilon(\chi)}\cdot\sqrt{\frac{\pi}{2}}\cdot\psi(x(y);\chi)\cdot\{x(y)\}^{\frac{2\gamma+1}{4}}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}Q(t)\,e^{-ity}\,dt.$$

Hence the function

$$F(y)\equiv-\sqrt{\frac{\pi}{2}}\cdot\psi(x(y);\chi\cdot\{x(y)\}^{\frac{2\gamma+1}{4}}\cdot\sqrt{\varepsilon(\chi)}$$

has the Fourier transform $Q(t)$, whence it follows that

$$F(y)\cdot\frac{e^{ihy}-1}{iy}$$

has the Fourier transform

$$I=\int_t^{t+h}Q(u)\cdot du.$$

Then, by Parseval's equation

$$\int_{-\infty}^{+\infty}|I|^2\,dt=\int_{-\infty}^{+\infty}|F(y)|^2\frac{4\cdot\sin^2\frac{hy}{2}}{y^2}\cdot dy=2\pi\int_{-\infty}^{+\infty}|\psi(x;\chi)|^2\cdot|x|^{\frac{2\gamma+1}{2}}\cdot\frac{\sin^2\frac{hy}{2}}{y^2}\cdot dy.$$

Put $x=\lambda z$, i.e. $y=\frac{1}{2}\cdot\log z$. Then

$$\int_{-\infty}^{+\infty}|I|^2\,dt=4\pi\int_0^{\infty}|\psi(\lambda z;\chi)|^2\,z^{\frac{2\gamma-1}{2}}\cdot\frac{\sin^2\left(\frac{h}{4}\log z\right)}{\log^2 z}\cdot dz. \tag{6.16}$$

Put

$$\frac{\sin^2\left(\frac{h}{4}\log z\right)}{\log^2 z} = g(z)$$

and observe that

$$g(z) = g\left(\frac{1}{z}\right).$$

Then, by (6.4)

$$\int_0^1 |\psi(\lambda z;\chi)|^2 z^{\frac{2\gamma-1}{2}}\cdot g(z)\,dz = \int_0^1 \left|\psi\left(\frac{\lambda}{z};\bar\chi\right)\right|^2 z^{-2\gamma-1} z^{\frac{2\gamma-1}{2}} g(z)\,dz =$$

$$= \int_0^1 \left|\psi\left(\frac{\lambda}{z};\chi\right)\right|^2 z^{-\frac{3}{2}-\gamma} g(z)\,dz = \int_1^\infty |\psi(\lambda u;\chi)|\, u^{\frac{2\gamma-1}{2}} g(u)\,du$$

and by (6.16)

$$\int_{-\infty}^{+\infty} |I|^2\,dt = 8\pi\cdot\int_1^\infty |\psi(\lambda z;\chi)|^2\cdot z^{\frac{2\gamma-1}{2}}\cdot g(z)\cdot dz = 0\,(I' + I'') \tag{6.17}$$

where

$$I' = h^2 \int_1^{1+\frac{1}{h}} |\psi|^2\cdot z^{\frac{2\gamma-1}{2}}\,dz \tag{6.18}$$

and

$$I'' = \int_{1+\frac{1}{h}}^\infty |\psi|^2\cdot z^{\frac{2\gamma-1}{2}}\cdot\frac{dz}{\log^2 z}. \tag{6.19}$$

The O-symbol refers here and in the following pages to the limit process ($\delta\to 0$, $h\to\infty$). The letter A will denote numbers, which depend only on k. In order to obtain (6.18) and (6.19) we used the inequalities

$$\frac{\sin^2\left(\frac{h}{4}\log z\right)}{\log^2 z} \le \begin{cases} A\cdot h^2 & z \le 1 + \frac{1}{h} \\ \frac{1}{\log^2 z} & z \ge 1 + \frac{1}{h}. \end{cases}$$

6.2. We shall estimate I' and I''. First, by using (6.15) we get

$$\tfrac{1}{4}|\psi|^2 = \sum_{m=1}^\infty m^{2\gamma} e^{-\frac{2m^2\pi z}{k}\cdot\sin\delta} +$$

$$+ \sum_{m\neq n}\sum m^\gamma n^\gamma\cdot\chi(m)\,\bar\chi(n)\,e^{-(m^2+n^2)\frac{\pi z}{k}\sin\delta + i(m^2-n^2)\frac{\pi z}{k}\cos\delta}. \tag{6.20}$$

$$\sum_{m \neq n} \sum \text{ may be replaced by } 2\boldsymbol{R}\Big(\sum_{m>n}\sum\Big).\text{[1]} \tag{6.21}$$

The first sum is less than

$$A \int_1^\infty x^{2\gamma} e^{-x^2 \delta A z} \, dx < A\,(z\,\delta)^{-\frac{1+2\gamma}{2}}.$$

Its contribution to I' is

$$I_1' = 0\left[h^2 \int_1^{1+\frac{1}{h}} (z\,\delta)^{-\frac{1+2\gamma}{2}} z^{\gamma-\frac{1}{2}} \, dz\right] = 0\left(h \cdot \delta^{-\frac{1+2\gamma}{2}}\right). \tag{6.22}$$

Its contribution to I'' is

$$I_1'' < A \int_{1+\frac{1}{h}}^\infty \delta^{-\frac{1+2\gamma}{2}} z^{-\frac{1}{2}-\gamma+\gamma-\frac{1}{2}} \frac{dz}{\log^2 z} = A \cdot \delta^{-\frac{1+2\gamma}{2}} \int_{1+\frac{1}{h}}^\infty \frac{dz}{z \cdot \log^2 z}$$

i.e.

$$I_1'' = 0\,(h \cdot \delta^{-\frac{1+2\gamma}{2}}). \tag{6.23}$$

The double sum (6.21) contributes to I' and I'' with terms of the form

$$I_2'(m;\, n) = h^2 \cdot \int_1^{1+\frac{1}{h}} e^{-(M\delta - iN)z} \cdot z^{\frac{2\gamma-1}{2}} \, dz. \tag{6.24}$$

$$I_2''(m;\, n) = \int_{1+\frac{1}{h}}^\infty e^{-(M\delta - iN)z} \cdot \frac{z^{\frac{2\gamma-1}{2}}}{\log^2 z} \cdot dz \tag{6.25}$$

where

$$\left.\begin{aligned} M &= (m^2 + n^2) \cdot \frac{\pi}{k} \cdot \frac{\sin\delta}{\delta} \\ N &= (m^2 - n^2) \cdot \frac{\pi}{k} \cdot \cos\delta;\ N > 0. \end{aligned}\right\} \tag{6.26}$$

Furthermore

$$|I_2'(m;\, n)| \le h^2 \cdot e^{-M\delta} \left| \int_0^{1/h} e^{-(M\delta - iN)z} \cdot (1+z)^{\frac{2\gamma-1}{2}} \cdot dz \right| < \frac{A\,h^2\,e^{-M\delta}}{N} \tag{6.27}$$

by the Riemann-Lebesgue theorem, since the functions

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[1] **R** denotes the real part.

$$e^{-M\delta z}\cdot(1+z)^{\frac{2\gamma-1}{2}}$$

are of bounded variation in $\left(0, \frac{1}{h}\right)$ uniformly with respect to δ and M.

$$|I_2''(m;n)| = e^{-M\delta}\left|\int_{\frac{1}{h}}^{\infty} e^{iNz}e^{-M\delta z}\cdot\frac{(1+z)^{\frac{2\gamma-1}{2}}}{\log^2(1+z)}\cdot dz\right|.$$

Turning the line of integration by the angle $\frac{\pi}{2}$ round the point $\frac{1}{h}$, we find

$$I_2''(m;n)| = e^{-M\delta\left(1+\frac{1}{h}\right)}\cdot\left|\int_0^{\infty}\frac{e^{-(N+iM\delta)y}\cdot\left(1+\frac{1}{h}+iy\right)^{\frac{2\gamma-1}{2}}}{\log^2\left(1+\frac{1}{h}+iy\right)}\cdot dy\right| <$$

$$< A h^2\cdot e^{-M\delta}\int_0^{\infty} e^{-Ny}(1+y^{\gamma})\,dy$$

since

$$\left|\log^2\left(1+\frac{1}{h}+iy\right)\right| > \frac{A}{h^2}$$

$$\left|1+\frac{1}{h}+iy\right|^{\frac{2\gamma-1}{2}} < A(1+|y|^{\gamma}).$$

Thus

$$|I_2''(m;n)| < \frac{A\cdot h^2\cdot e^{-M\delta}}{N};\ N\geq 0. \tag{6.28}$$

It follows from (6.27) and (6.28) that each term of the double sum

$$S = A\sum_{m>n}\sum m^{\gamma} n^{\gamma}\cdot\frac{h^2 e^{-M\delta}}{N}$$

is greater than the corresponding term in the integral from *1* to ∞ of the series (6.21). The case $\gamma = 0$ has been discussed by TITCHMARSH [7, p. 51] with the result

$$S = 0\left(h^2\cdot\log^2\frac{1}{\delta}\right) < A\cdot h\cdot\delta^{-\frac{1}{2}} \tag{6.29}$$

if

$$\delta < \delta_0(h).$$

In the case $\gamma = 1$, we find

$$S < A h^2 \cdot \sum_{m=2}^{\infty} \sum_{n=1}^{m-1} m \cdot e^{-m^2 \delta} \cdot \frac{n e^{-n^2 \delta}}{m^2 - n^2} <$$

$$< A h^2 \cdot \sum_{m=2}^{\infty} m e^{-m^2 \delta} \sum_{n=1}^{m-1} \frac{1}{m - n} <$$

$$< A h^2 \sum_{m=2}^{\infty} m \cdot \log m \cdot e^{-m^2 \delta} < A h^2 \int_2^{\infty} x^{1+\varepsilon} e^{-x^2 \delta} dx <$$

$$< A h^2 \delta^{-1-\varepsilon}.$$

Hence

$$S < A h \delta^{-3/2} \tag{6.30}$$

if

$$\delta < \delta_1(h); \ \gamma = 1.$$

Hence by (6.17), (6.22), (6.23), (6.29), and (6.30)

$$\int_T^{2T} |I|^2 dt < \int_{-\infty}^{+\infty} |I|^2 dt = 0\left(h \cdot \delta^{-\frac{1+2\gamma}{2}}\right). \tag{6.31}$$

6.3. Now we shall show that, if

$$\delta = \frac{2}{T} \tag{6.32}$$

then

$$J > (A h + U) \cdot T^{\frac{\gamma}{2} - \frac{1}{4}} \tag{6.33}$$

where

$$\int_T^{2T} |U|^2 \cdot dt < A \cdot T. \tag{6.34}$$

We now suppose that $t > 0$. For any fixed σ, we have, by Stirlings formula

$$|\Gamma(\sigma + i\tau)| \sim e^{-\frac{1}{2}\pi \cdot |\tau|} \cdot |\tau|^{\sigma - \frac{1}{2}} \sqrt{2\pi} \qquad (|\tau| \to \infty)$$

whence

$$|Q(t)| > A \cdot e^{-\frac{\delta}{2} t} \cdot t^{\frac{2\gamma - 1}{4}} \cdot |L(\tfrac{1}{2} + it; \chi)|$$

and using (6.32), we find

$$T^{\frac{1-2\gamma}{4}} \cdot |Q(t)| > A \cdot \mathbf{R} \left\{ 1 + \sum_{n=2}^{\infty} \chi(n) \cdot n^{-\frac{1}{2} - it} \right\}$$

for

$$0 < T < t < 2T.$$

By integration, we find

$$T^{\frac{1-2\gamma}{4}} \cdot J > A \cdot \left(h + \mathbf{R}\left[i \cdot g(s)\right]_{\frac{1}{2}+it}^{\frac{1}{2}+i(t+h)}\right) \tag{6.35}$$

where

$$g(s) = \sum \frac{\chi(n)}{\log n} \cdot n^{-s}.$$

It remains to prove (6.34). By (6.35) it is seen that it is sufficient to prove that

$$\int_{T}^{2T} |g(\tfrac{1}{2} + i(t+h))|^2 \, dt = 0(T)$$

holds uniformly for $0 \leq h \leq T$. But the uniformity follows easily from the particular case $h = 0$. We use a formula of TITCHMARSH [cf. 7; 2.31, eq. (4)], and find

$$\int_{0}^{\infty} |g(\tfrac{1}{2} + it)|^2 e^{-2\delta t} \, dt = \int_{0}^{\infty} \left| \sum_{m=2}^{\infty} \frac{\chi(m) \cdot w^m}{\log m} \right|^2 dx + 0(1) \tag{6.36}$$

where we have put

$$w = e^{-ixe^{-i\delta}} = e^{-x \sin \delta} \cdot e^{-ix \cos \delta}.$$

We first note that the integrand on the right of (6.36) is bounded for small x, uniformly in δ. In order to see this, we consider the identity

$$\sum_{n=1}^{kN+l} \chi(n) \cdot w^n = \frac{1 - w^{kN}}{1 - w^k} \cdot \sum_{n=0}^{k-1} \chi(n) \cdot w^n + w^{kN} \sum_{n=1}^{l} \chi(n) \, w^n. \qquad (0 < l < k) \tag{6.37}$$

Since

$$\sum_{n=0}^{k-1} \chi(n) \cdot w^n = 0$$

when $w = 1$, the right-hand side of (6.37) has an upper bound, independent of w, N, l in the sector

$$|w| \leq 1 \qquad -\frac{\pi}{k} \leq \arg w \leq \frac{\pi}{k}$$

and hence it is bounded for small x, uniformly in δ. But

$$\frac{1}{\log m}$$

tends to zero steadily, when $m \to \infty$. Hence

$$\sum_{m=2}^{\infty} \frac{\chi(m) \cdot u^m}{\log m}$$

is uniformly bounded for $|x| \leq \frac{\pi}{k}$.

We may then consider

$$\int_b^{\infty} \left| \sum_{n=2}^{\infty} \chi(m) \frac{\cdot e^{-imxe^{-i\delta}}}{\log m} \right|^2 \cdot dx, \qquad b = \frac{\pi}{k}.$$

But the integrand is equal to

$$\sum_{m=2}^{\infty} \frac{e^{-2mx \cdot \sin \delta}}{\log^2 m} + \sum_{m \neq n} \sum \frac{\chi(m) \cdot \bar{\chi}(n)}{\log m \cdot \log n} \cdot e^{ix(ne^{i\delta} - me^{-i\delta})}.$$

We may obviously integrate these series from b to ∞ for any $\delta > 0$. It follows that

$$\int_0^{\infty} |g|^2 \cdot e^{-2\delta t}\, dt = 0\left(\sum \frac{e^{-2mb \cdot \sin \delta}}{m \cdot \log^2 m \cdot \sin \delta}\right) + 0\left(\sum \sum' \frac{e^{-b(m+n) \sin \delta}}{\log m \cdot \log n \cdot (m-n)}\right) =$$

$$= 0\left(\frac{1}{\delta}\right) \qquad (6.38)$$

the last result according to TITCHMARSH [7, 3.43]. From (6.32), (6.38), and (6.35), we deduce (6.34).

6.4. Suppose still that $T > 0$. Let $\mathfrak{S}$ be the sub-set of the interval $(T, T + \eta T)$, where $|I| = J$. Then

$$\int_{\mathfrak{S}} |I| \cdot dt = \int_{\mathfrak{S}} J \cdot dt.$$

Put $\delta = \frac{2}{T}$ in (6.31). Then

$$\int_{\mathfrak{S}} |I|\, dt \leq \int_T^{T+\eta T} |I|\, dt \leq \left\{\eta T \cdot \int_T^{2T} |I|^2\, dt\right\}^{\frac{1}{2}} < A\, \eta^{\frac{1}{2}}\, T^{\frac{1}{2}+\frac{1}{4}+\frac{\gamma}{2}} \cdot h^{\frac{1}{2}}$$

whence

$$\int_{\mathfrak{S}} |I|\, dt < A \cdot \eta^{\frac{1}{2}} \cdot T^{\frac{3+2\gamma}{4}} \cdot h^{\frac{1}{2}}.$$

Furthermore by (6.33) and (6.34)

$$\int_{\mathfrak{S}} J\, dt > T^{\frac{2\gamma-1}{4}} \cdot \left(A\, h \cdot m(\mathfrak{S}) - \int_T^{T+\eta T} |U|\, dt\right) > T^{\frac{2\gamma-1}{4}} \left(A\, h \cdot m(\mathfrak{S}) - A \cdot T \cdot \eta^{\frac{1}{2}}\right)$$

where $m(\mathfrak{S})$ is the measure of $\mathfrak{S}$. Hence, for h sufficiently large

$$m(\mathfrak{S}) < A \cdot T \cdot \eta^{\frac{1}{2}} \cdot h^{-\frac{1}{2}}.$$

Now divide the interval $(T, T + \eta T)$ into $[T/2h]$ pairs of abutting intervals j_1, j_2 each, except the last j_2 of length ηh, and each j_2 lying immediately to the right of the corresponding j_1. Then either j_1 or j_2 contains a zero of $Q(t)$, unless j_1 consists entirely of points of $\mathfrak{S}$. Suppose the latter occurs for $\nu\, j_1$'s. Then

$$\nu \eta h \leq m(\mathfrak{S}) < A \cdot \eta^{\frac{1}{2}} \cdot h^{-\frac{1}{2}} \cdot T.$$

Hence there are in $(T, T + \eta T)$ at least

$$\left[\frac{T}{2h}\right] - \nu > \frac{T}{h}\left(\tfrac{1}{3} - A\,\eta^{-\frac{1}{2}}\,h^{-\frac{1}{2}}\right) > \frac{T}{4h} > k(\eta) \cdot T$$

zeros, if h is large enough. This proves the theorem for positive T. If T is negative, the considered zeros of $L(s; \chi)$ are conjugate to the zeros of $L(s; \bar{\chi})$, which have positive imaginary part, and hence, theorem III is valid also for $T < 0$.

To this point, the proof is valid only for proper characters $\chi(n)$. If, however, $\chi(n)$ is not a proper character mod. k, we can write (cf. LANDAU [5; § 125])

$$L(s; \chi) = \prod_{\nu=1}^{c} \left(1 - \frac{\varepsilon_\nu}{p_\nu^s}\right) \cdot L(s; X) \qquad (|\varepsilon_\nu| = 1)$$

where $X(n)$ is a proper character mod. q where q is a divisor of k, which sometimes may be equal to 1. Since the first factors are regular, theorem III is valid, also for improper characters.

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